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A NOTE ON VOLTERRA INTEGRAL EQUATIONS WITH POWER NONLINEARITY

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ABSTRACT. In the paper the existence of nontrivial solutions to integral Volterra equations with the power nonlinearity is studied. The behavior of nontrivial solutions near origin is considered.

1. Introduction. The nonlinear Volterra integral equation

$$u(x) = \int_0^x k(x-s)g(u(s))\,ds, \quad x \in [0,\delta], \quad \delta > 0, \tag{1.1}$$

has been studied in the modeling of some problems in nonlinear diffusion and shock-wave propagation [**6**]. In these problems the kernel k is nonnegative and g is an increasing continuous function such that $g(0) = 0$. Moreover, g does not satisfy a Lipschitz condition near the origin. A typical example of such a function is $g(u) = u^p$, $p \in (0,1)$. Obviously, $u \equiv 0$ is the trivial solution to (1.1). The physically interesting solutions of (1.1) are continuous functions u such that $u(x) > 0$ for $x > 0$. Some particular answers concerning the existence of nontrivial solutions can be found in different works, e.g., [**3, 5**]. In this paper we study nontrivial solutions to the equation

$$u(x) = \int_0^x k(x-s)u^p(s)\,ds, \quad x \in [0,\delta], \quad p \in (0,1). \tag{1.2}$$

Using necessary and sufficient conditions for the existence of nontrivial solutions in the form of function series [**5**], we derive them in the integral form. These last conditions are the same as those obtained in [**4**]. With respect to applications, the behavior of nontrivial solutions near the origin is also interesting. We present lower and upper estimates of nontrivial solutions to (1.2) at the vicinity of zero. For the case

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$k(x) = x^{\alpha-1}$, $\alpha > 0$, some estimates are given in [**2**]. But the knowledge concerning the behavior of nontrivial solutions near the origin for small increasing kernels as, e.g., $k(x) = x^{-\alpha-1}\exp(-1/x^{\alpha})$, $\alpha > 1$, is still very poor [**1**]. In this work we give new results for this last case. Some examples are also included.

2. Preliminary information. First we recall some results from [**5**]. We consider (1.1). We assume that

$$k : (0,\delta] \longrightarrow [0,+\infty),\ \delta > 0,\ \text{is an integrable function such that } k > 0 \text{ a.e.;} \tag{2.1}$$

$$g : [0,+\infty) \longrightarrow [0,+\infty),\ \text{is an increasing, continuous function such that } g(0) = 0. \tag{2.2}$$

Let K^{-1} denote the inverse function to $K(x) = \int_0^x k(s)\,ds$, which is strictly increasing for k satisfying (2.1). Let g^{-1} denote the inverse function to g satisfying (2.1). For a given function f, we define the sequence of functions f^n, $n = 0, 1, \dots$, as follows: $f^0(x) = x$, $f^{n+1}(x) \equiv (f^n \circ f)(x)$, $n = 0, 1, \dots$. As proven in [**5**], we have

Theorem 2.1. *Let the kernel k satisfy* (2.1) *and g satisfy* (2.2)*. If equation* (1.1) *has a nontrivial solution on an interval, then the series*

$$\sum_{n=0}^{\infty} K^{-1}\left(\left(g^{-1}\right)^n(x)\right) \tag{2.3}$$

is convergent on $[0,\delta]$ $(\delta > 0)$.

Theorem 2.2. *Let the kernel k satisfy* (2.1) *and g satisfy* (2.2)*. Let φ be a continuous function on $[0,\delta]$, $\delta > 0$, such that $x < \varphi(x) < g(x)$ for $(0,\delta]$ and $x/\varphi(x) \to 0$ as $x \to 0^+$. If the series*

$$\sum_{n=0}^{\infty} K^{-1}\left(\left(g^{-1}\circ\varphi\right)^n(x)/\varphi\left(\left(g^{-1}\circ\varphi\right)^n(x)\right)\right) \tag{2.4}$$

converges uniformly on $[0,\delta]$, then equation (1.1) *has a nontrivial solution on an interval.*

Remark 2.3. If the assumptions of Theorem 2.1 and Theorem 2.2 are satisfied then on the basis of results from [**5**] we can write

$$\begin{aligned}\sum_{n=0}^{\infty} K^{-1}\left(\left(g^{-1}\right)^{n}(x)\right) &\le u^{-1}(x)\\ &\le \sum_{n=0}^{\infty} K^{-1}\left(\left(g^{-1}\circ\varphi\right)^{n}(x)/\varphi\left(\left(g^{-1}\circ\varphi\right)^{n}(x)\right)\right)\end{aligned} \tag{2.5}$$

for $x \in [0,\delta]$, where u^{-1} denotes the inverse function to the nontrivial solution u of (1.1).

3. Studies concerning the power nonlinearity. In this part we shall study equation (1.2). If we designate $g(x) = x^p$, $0 < p < 1$, then Theorem 2.1 can be written as follows.

Theorem 3.1. *Let the kernel k satisfy* (2.1). *If equation* (1.2) *has a nontrivial solution on an interval, then the series $\underline{S}(x)$ defined by*

$$\underline{S}(x) = \sum_{n=0}^{\infty} K^{-1}\left(x^{(1/p)^n}\right) \tag{3.1}$$

is convergent on $[0,\delta]$, $\delta > 0$.

We also modify Theorem 2.2 by using $g(x) = x^p$, $0 < p < 1$, and $\varphi(x) = x^q$, $p < q < 1$. We get

Theorem 3.2. *Let the kernel k satisfy* (2.1). *If the series $\overline{S}(x)$ defined by*

$$\overline{S}(x) = \sum_{n=0}^{\infty} K^{-1}\left(x^{(1-q)(q/p)^n}\right). \tag{3.2}$$

converges uniformly on $[0,\delta]$, then equation (1.2) *has a nontrivial solution on an interval.*

We can reformulate Remark 2.3 as follows.

Remark 3.3. If (1.2) has a nontrivial solution, then

$$\underline{S}(x) \le u^{-1}(x) \le \overline{S}(x) \tag{3.3}$$

for $x \in [0, \delta]$, where u^{-1} denotes the inverse function to the nontrivial solution.

Let

$$I(x) = \int_0^x K^{-1}(s) \, \frac{ds}{s\,(-\ln s)}, \tag{3.4}$$

where $x > 0$. We can formulate the following theorem (see also [**4**]).

Theorem 3.4. *If equation* (1.2) *has a nontrivial solution, then* $I(x) < \infty$ *for* $x \in (0, \delta]$, $\delta > 0$.

Proof. Let $\Phi(x) = x^{1/p} < x < 1$. Let x_n be a strictly decreasing and convergent to zero sequence for all $0 < x \le \delta$ such that

$$x_0 = x, \; x_{n+1} = \Phi(x_n), \quad n = 0, 1, 2, \dots .$$

We notice that

$$\int_{x_{i+1}}^{x_i} K^{-1}(s) \, \frac{ds}{s\,(-\ln s)} \le -\ln p \;\; K^{-1}(x_i) = -\ln p \;\; K^{-1}\left(x^{(1/p)^i}\right), \tag{3.5}$$

for $i = 0, 1, 2, \dots$. Adding inequalities of the form (3.5) from $i = 0$ to n we get

$$\int_{x_{n+1}}^{x} K^{-1}(s) \, \frac{ds}{s\,(-\ln s)} \le -\ln p \;\; \sum_{k=0}^{n} K^{-1}\left(x^{(1/p)^k}\right). \tag{3.6}$$

Now, taking $n \to \infty$, we obtain

$$\sum_{n=0}^{\infty} K^{-1}\left(x^{(1/p)^n}\right) \ge \frac{1}{-\ln p} \; \int_0^x K^{-1}(s) \, \frac{ds}{s\,(-\ln s)} = \frac{1}{-\ln p} \; I(x). \tag{3.7}$$

On the basis of Theorem 3, we have $I(x) < \infty$ for $0 < x \le \delta$. $\square$

Theorem 3.5. *If $I(x) < \infty$ for $x > 0$, then equation* (1.2) *has a nontrivial solution.*

Proof. Let $\Psi(x) = x^{q/p} < x < 1$. Let x_n be a strictly decreasing and convergent to zero sequence for all $0 < x \le \delta$ such that

$$x_0 = x,\ x_{n+1} = \Psi(x_n), \quad n = 0, 1, 2, \dots .$$

Let $K_0^{-1}(x) \doteq K^{-1}(x^{1-q})$. We have

$$\begin{aligned}\int_{x_{i+1}}^{x_i} K_0^{-1}(s)\,\frac{ds}{s\,(-\ln s)} &\ge -\ln\frac{p}{q}\quad K_0^{-1}(x_{i+1})\\ &= -\ln\frac{p}{q}\quad K^{-1}\left(x^{(1-q)(q/p)^{i+1}}\right),\end{aligned} \tag{3.8}$$

for $i = 0, 1, 2, \dots$. In a similar manner as in the proof of Theorem 3.4, we obtain

$$\int_0^x K^{-1}\left(s^{1-q}\right)\frac{ds}{s\,(-\ln s)} \ge -\ln\frac{p}{q}\quad \sum_{n=1}^{\infty} K^{-1}\left(x^{(1-q)(q/p)^n}\right). \tag{3.9}$$

The left-hand side of the last inequality is equal to $I(x^{1-q})$. Since $I(y) < \infty$ for $y = x^{1-q} < 1$, then the series (3.2) is convergent. By Theorem 3.2 we get the existence of a nontrivial solution. $\square$

With the help of Theorems 3.4 and 3.5, we can formulate

Corollary 3.6. *Equation* (1.2) *has a nontrivial solution u if and only if $I(x) < \infty$ where $x \in (0, \delta]$, $\delta > 0$.*

On the basis of Remark 3.3 and proofs of Theorems 3.4 and 3.5, we can write:

Lemma 3.7. *If equation* (1.2) *has a nontrivial solution, then*

$$c_1 I(x) \le u^{-1}(x) \le c_2 I\left(x^a\right), \tag{3.10}$$

where $x \in (0,\delta]$, $a = (1-q)\,p/q$, $c_1 = (-\ln p)^{-1}$ *and* $c_2 = (-\ln(p/q))^{-1}$ *are constants.*

Proof. On the basis of the proof of Theorem 3.4, we get the left-hand side of (3.11). The proof of the right-hand side of the inequality is similar to the proof of Theorem 3.5. Combining inequalities (3.3) and (3.10), we obtain

$$u^{-1}(x) \le K^{-1}(x^{1-q}) + c_2 \int_0^x K^{-1}\left(s^{1-q}\right) \frac{ds}{s\,(-\ln s)}. \tag{3.11}$$

Since

$$K^{-1}(x^{1-q}) \le c_2 \int_x^{x^{p/q}} K^{-1}\left(s^{1-q}\right) \frac{ds}{s\,(-\ln s)}, \tag{3.12}$$

then we get finally the right-hand side inequality

(3.13)

$$u^{-1}(x) \le c_2 \int_0^{x^{(1-q)p/q}} K^{-1}\left(s\right) \frac{ds}{s\,(-\ln s)} = c_2 I\left(x^{(1-q)p/q}\right). \qquad \square$$

We also obtain the following corollary.

Corollary 3.8. *If equation* (1.2) *has a nontrivial solution* u*, then*

$$\left(I^{-1}(-(\ln(p/q))\;x)\right)^{q/p(1-q)} \le u(x) \le I^{-1}(-(\ln p)\;x) \tag{3.14}$$

where $x \in (0,\delta]$ *and* I^{-1} *denotes the inverse function to* I*.*

Proof of Corollary 3.8. Since I is strictly increasing and continuous function on $x \in (0,\delta]$, then on the basis of inequality (3.10), we can easily get inequality (3.14). $\square$

4. Some results concerning the power nonlinearity and small kernels. In this part we shall present some interesting results based on inequality (3.10). Assume, additionally, that the function k is nondecreasing, $k(0) = 0$, and the following condition is satisfied:

$$1/\ln K(x) \quad \text{is a concave function on interval} \quad (0,\delta), \quad \delta > 0. \tag{4.1}$$

Remark 4.1. Let us note that the condition (4.1) is satisfied for functions as for example $K(x) = \exp(-1/x^\alpha)$, $\alpha \geq 1$ and $K(x) = \exp(-\exp(1/x^\alpha))$, $\alpha > 0$, but it is not satisfied for $K(x) = x^\alpha$, $\alpha > 0$.

Remark 4.2. With respect to Remark 4.1 all functions k satisfying (4.1) will be called small kernels.

Lemma 4.3. *If function K satisfies condition* (4.1), *then for any $a \in (0,1)$ the following inequality holds*

$$\frac{K^{-1}(x^a)}{K^{-1}(x)} \leq \frac{1}{a} \quad \textit{for all} \quad x \in (0,\delta]. \tag{4.2}$$

Proof. Let $y(z) = \exp(-1/z)$ and $z(x) = -1/(\ln x)$. It easy to notice that the following equality holds

$$\frac{K^{-1}\,(x^a)}{K^{-1}(x)} = \frac{K^{-1} \circ y \circ z(x^a)}{K^{-1} \circ y \circ z(x)} = \frac{\widetilde{K} \circ z(x^a)}{\widetilde{K} \circ z(x)}, \tag{4.3}$$

where $\widetilde{K} = K^{-1} \circ y$. Taking the property $z(x^a) = (1/a)z(x)$ we can write quotient (4.3) as follows

$$\frac{K^{-1}\,(x^a)}{K^{-1}(x)} = \frac{\widetilde{K}(z/a)}{\widetilde{K}(z)}. \tag{4.4}$$

Notice that $z(x) \to 0^+$ as $x \to 0^+$ and, on the basis of assumption, function $\widetilde{K}$ is concave. Hence we obtain

$$\widetilde{K}\left(\frac{z}{a}\right) \leq \frac{1}{a}\,\widetilde{K}(z), \tag{4.5}$$

and inequality (4.2) holds at the vicinity of zero. □

Notice that for any $p \in (0,1)$ and any q such that $0 < p < q < 1$ we have $a = ((1-q)p)/q \in (0,1)$. Hence we reformulate Lemma 3.7 and Corollary 3.8.

Lemma 4.4. *If equation* (1.2) *has a nontrivial solution, then*

$$c_1\ I(x) \le u^{-1}(x) \le c\ I(x), \tag{4.6}$$

where $x \in (0,\delta]$, $c_1 = (-\ln p)^{-1}$ *and* $c = -q/(p(1-q)\ln(p/q))$ *are constants.*

Proof. On the basis of Lemma 4.3, the following inequality holds

$$I(x^a) \le \frac{1}{a}\ I(x), \quad a \in (0,1). \tag{4.7}$$

Hence, from Lemma 3.7, we obtain inequality (4.7). □

Corollary 4.5. *If function* K *satisfies condition* (4.1) *at the vicinity of zero, then* $u^{-1}(x) = O(I(x)/(-\ln p))$ *as* $x \to 0^+$.

Proof. With the help of Lemma 4.4 we can easily obtain the following inequality

$$1 \le \frac{u^{-1}(x)}{(-\ln p)^{-1} I(x)} \le \frac{q(-\ln p)}{p(1-q)(-\ln p + \ln q)}. \tag{4.8}$$

Hence, from the right-hand side of the above inequality it follows that the function $u^{-1}(x) = O(I(x)/(-\ln p))$ as $x \to 0^+$. □

Corollary 4.6. *If function* K *satisfies* (4.1) *for* $x \in (0,\delta]$, *then*

$$\ln u^{-1}(x) \sim \ln I(x), \qquad x \to 0^+. \tag{4.9}$$

Proof. Taking the logarithmic form of inequality (4.6) and using the property of the logarithm we have

$$1 + \frac{\ln c_1}{\ln I(x)} \ge \frac{\ln u^{-1}(x)}{\ln I(x)} \ge 1 + \frac{\ln c}{\ln I(x)} \quad \text{for} \quad x \in (0,\delta). \tag{4.10}$$

Since $I(x) \to 0$ as $x \to 0^+$ then $\ln I(x) \to -\infty$ at the origin, hence

$$\ln u^{-1}(x) \sim \ln I(x), \quad x \to 0^+. \tag{4.11}$$

□

Corollary 4.7. *If equation* (1.2) *has a nontrivial solution, then*

$$I^{-1}\left(\frac{p(q-1)\ln(p/q)}{q}\ x\right) \le u(x) \le I^{-1}(-(\ln p)\ x) \tag{4.12}$$

where $x \in (0,\delta]$ *and* I^{-1} *denotes the inverse function to* I.

Remark 4.8. In fact we need inequality (4.2) to prove all results of this part. However, in our opinion, condition (4.1) is more readable than (4.2).

5. Some examples.

Example 5.1. We consider equation (1.2) with $k(x) = K'(x)$, where $K(x) = \exp\left(-1/x^{\alpha}\right)$, $\alpha > 0$. In this case $K^{-1}(x) = 1/\left(\ln(1/x)\right)^{1/\alpha}$. The convergence of integral (3.4) is a necessary and sufficient condition for existence of nontrivial solutions. The integral (3.4) becomes

$$I(x) = \int_0^x \frac{ds}{\left(\ln(1/s)\right)^{1/\alpha} s(-\ln s)} = \frac{\alpha}{\left(\ln(1/x)\right)^{1/\alpha}}. \tag{5.1}$$

This implies that equation (1.2) has nontrivial solutions for $\alpha > 0$. After some calculations, on the basis of Corollary 4.5, we obtain the following estimate of function u,

$$\exp\left(-\left(\frac{p(q-1)\ln(p/q)\alpha}{qx}\right)^{\alpha}\right) \ \le\ u(x) \le \exp\left(-\left(\frac{\alpha}{(-\ln p)x}\right)^{\alpha}\right). \tag{5.2}$$

Moreover, function K satisfies condition (4.1) only for $\alpha \ge 1$. Since, for all $a > 0$,

$$K^{-1}(x^a) = \frac{1}{a^{1/\alpha}}\ K^{-1}(x), \quad \text{for} \quad a > 0, \tag{5.3}$$

then inequality (4.2) holds. On the basis of Corollary 4.6, we notice that $u^{-1}(x) = O\left(\alpha/\left((-\ln p)\left(\ln(1/x)\right)^{1/\alpha}\right)\right)$ as $x \to 0^+$.

Example 5.2. We consider equation (1.2) with $k(x) = K'(x)$, where $K(x) = \exp(-\exp(1/x^\alpha))$, $\alpha > 0$. In this case $K^{-1}(x) = 1/(\ln\ln(1/x))^{1/\alpha}$ and

$$I(x) = \int_0^x \frac{ds}{(\ln\ln(1/s))^{1/\alpha}\, s(-\ln s)} = \frac{1-\alpha}{\alpha}\, (\ln\ln(1/x))^{(\alpha-1)/\alpha}. \tag{5.4}$$

Since for $\alpha \geq 1$ this integral is divergent for $x \in [0,\delta]$, then equation (1.2) has no nontrivial solutions for $\alpha \geq 1$. If $\alpha \in (0,1)$, then the integral is convergent for small x. This implies that equation (1.2) has nontrivial solutions for $\alpha \in (0,1)$. On the basis of Corollary 4.5, estimation of the function u has the following form

$$\exp\left(\exp\left(\left(\frac{p(q-1)\ln(p/q)}{q}\ Ax\right)^{-1/A}\right)\right) \leq u(x) \leq \exp\left(-\exp\left(((-\ln p)\ Ax)^{-1/A}\right)\right), \tag{5.5}$$

where $A = (1-\alpha)/\alpha$. Since the function K satisfies condition (4.1), then, with the help of Corollary 4.6, we notice that $u^{-1}(x) = O\left((1-\alpha)(\ln\ln(1/x))^{(\alpha-1)/\alpha}/(\alpha(-\ln p))\right)$ as $x \to 0^+$.

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