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THE THIRD PROBLEM FOR THE LAPLACE EQUATION ON A PLANAR CRACKED DOMAIN WITH MODIFIED JUMP CONDITIONS ON CRACKS

DAGMAR MEDKOVÁ

ABSTRACT. The paper studies the third problem for the Laplace equation on a cracked bounded planar domain with multiply connected Lipschitz boundary and boundary conditions from L^p. It is shown that, for $1 < p \leq 2$, there is a unique solution of the problem. This solution is constructed for a domain, which boundary is formed by curves with bounded rotation.

1. Introduction. Several boundary value problems for the Laplace equation in a cracked planar domain has been studied by the integral equation method recently, see [**12–15**]. Krutitskii studied in [**11**] the boundary value problem for the Laplace equation outside several smooth cuts in the plane. Two boundary conditions were given on the cuts. One of them specified the jump of the unknown function. Another one of the type of the Robin condition contained the jump of the normal derivative of an unknown function and the one-side limit of this function on the cuts. He looked for a solution of the problem in the form of the sum of a single layer potential and an angular potential. He has reduced the problem to solving an integral equation which turns out to have a unique solution. Therefore, he proved the unique solvability of the problem under the assumption that the boundary conditions are smooth. We remark that he studied the same problem for the Helmholtz equation by the same method, see [**10**].

This paper deals with the boundary value problem for the Laplace equation on a bounded multiply connected planar domain G with Lipschitz boundary and cracks. The cracks are arbitrary closed subsets of Lipschitz arcs and can touch the boundary. The Robin condition

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is given on the boundary of the domain. The same conditions as in [**11**] are on the cuts. The boundary conditions are from $L^p(\partial G)$, and they are fulfilled in the sense of the nontangential limit. We found the necessary and sufficient conditions for the solvability of the problem. We compared an L^p-solution of the problem and a weak solution in $W^{1,2}(G)$. We proved that the L^p-solution is a weak solution in $W^{1,2}(G)$. (The L^p-solution is a strong solution in some sense.) On the other hand, we proved that for $1 < p \leq 2$ the weak solution with L^p data is an L^p-solution. From this fact we deduced Hölder regularity results for a weak solution.

The Robin problem was studied for noncracked domains with connected Lipschitz boundary in R^m, where $m > 2$, by Lanzani and Shen in [**17**]. They looked for a solution in the form of a single layer potential $\mathcal{S}g$ where $g \in L^p(\partial G)$. Hence, the authors were led to solving the integral equation $(1/2)g + K^*g + h\mathcal{S}g = f$ instead of the original problem with the boundary condition $\partial u/\partial n + hu = f$. They proved for $1 < p \leq 2$ that $(1/2)I + K^* + h\mathcal{S}$ is a Fredholm operator with index 0 in $L^p(\partial G)$. Since the kernel of this operator is trivial, it gives that the operator $(1/2)I + K^* + h\mathcal{S}$ is continuously invertible in $L^p(\partial G)$. This article includes the Robin problem for a noncracked domain as a special case. Unfortunately, it is not possible to use the approach of Lanzani and Shen for planar domains because the operator $(1/2)I + K^* + h\mathcal{S}$ is not injective. We looked for a solution of the problem in the form of the sum of a modified single layer potential with an unknown density $\varphi \in L^p(\partial G)$ and a double layer potential corresponding to the jump on the crack. (If G has no cracks then we look for a solution in the form of a modified single layer potential.) We reduced the problem to the integral equation $\tau\varphi = f$ on ∂G.

In the last part of the paper we constructed the solution of the corresponding integral equation $\tau\varphi = f$. Fabes, Sand and Seo studied in [**4**] this problem for the Neumann problem in noncracked domains. They proved for G convex and $f \in L^2(\partial G)$, $\int f \, d\mathcal{H}_1 = 0$ that

$$\varphi = 2\sum_{j=0}^{\infty}(I - 2\tau)^j f$$

is a solution of the equation $\tau\varphi = f$. Unfortunately, this series does not converge for multiply connected sets. We expressed the solution of the equation $\tau\varphi = f$ as a series for $f \in L^p(\partial G)$, $1 < p \leq 2$, on a wide class of domains including convex domains with cracks and domains with piecewise C^1 boundary and arbitrary cracks.

2. Formulation of the problem. We say that a bounded open set $H \subset R^2$ has Lipschitz boundary ∂H if there exist a finite number of ("local") coordinate systems (x_k, y_k), $k = 1, \dots, m$, and a finite number of Lipschitz functions φ_k, $k = 1, 2, \dots, m$, defined on $(-\delta, \delta)$, where $\delta > 0$, such that

1. $(x_k, y_k) \in H$ for $|x_k| < \delta$, $\varphi_k(x_k) - \delta < y_k < \varphi_k(x_k)$,

2. $(x_k, y_k) \notin \operatorname{cl} H$, the closure of H, for $|x_k| < \delta$, $\varphi_k(x_k) < y_k < \varphi_k(x_k) + \delta$,

3. for every $z \in \partial H$ there exists k, $k = 1, \dots, m$, and $x_k \in (-\delta, \delta)$, such that $z = (x_k, \varphi_k(x_k))$ in the corresponding coordinate system.

(We will say that H has (piecewise) C^α boundary if φ_k above are (piecewise) C^α.)

Let Γ be a rectifiable Jordan curve in R^2. Let Γ be parametrized by the arc length $s : \Gamma = \{\varphi(s);\ s \in [a, b]\}$. Extend φ as a periodical function in R^1 with the period $b - a$. Let f and $\partial f/\partial\tau$ be functions defined on Γ. We say that $\partial f/\partial\tau$ is the tangential derivative of f on Γ if

$$\int_c^d h'(t) f(\varphi(t))\, dt = -\int_c^d h(t)\, \frac{\partial f}{\partial\tau}\, (\varphi(t))\, dt$$

holds for all infinitely differentiable h supported in (c, d) with $d - c < b - a$.

Let $H \subset R^2$ be a bounded open set with Lipschitz boundary, $1 < p < \infty$. We say that $g \in W^{1,p}(\partial H)$ if there is $\partial g/\partial\tau \in L^p(\partial H)$ such that $\partial g/\partial\tau$ is the tangential derivative of g on each curve from ∂H. Define for such a g the norm

$$\|g\|_{W^{1,p}(\partial H)} = \left[\int_{\partial H} \left(|g|^p + \left|\frac{\partial g}{\partial\tau}\right|^p\right) d\mathcal{H}_1\right]^{1/p}.$$

(Here $\mathcal{H}_k$ is the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in R^k.) $W^{1,p}(\partial H)$ endowed with this norm is a Banach space.

Let $1 < p < \infty$, G an open subset of R^2, k a positive integer. Denote by $W^{k,p}(G)$ the space of all functions $f \in L^p(G)$ such that $\partial^\alpha f \in L^p(G)$ in the sense of distributions for each multiindex α with $|\alpha| \le k$. (Here $|\alpha|$ is the length of α.) For $f \in W^{k,p}(G)$, denote

$$\|f\|_{W^{k,p}(G)} = \left[\sum_{|\alpha| \le k} \int_G |\partial^\alpha f|^p \, d\mathcal{H}_2 \right]^{1/p}.$$

For $K \subset R^2$, denote by $C^\alpha(K)$ the space of all α-Hölder functions on K with the norm

$$\|g\|_{C^\alpha(K)} = \sup_{x \in K} |g(x)| + \sup_{\substack{x,y \in K \\ x \ne y}} \frac{|g(x) - g(y)|}{|x-y|^\alpha}.$$

For a bounded open set H, $x \in \partial H$ and $\alpha > 0$, denote

$$\Gamma_\alpha(x, H) = \{y \in H;\ |x - y| < (1 + \alpha) \operatorname{dist}(y, \partial H)\}$$

the nontangential approach regions of opening α corresponding to H and x. Here $\operatorname{dist}(y, M)$ denotes the distance of the point y from the set M. If u is a function on H, we denote

$$N_\alpha(u, H)(x) = \sup\{|u(y)|;\ y \in \Gamma_\alpha(x, H)\}$$

the *nontangential maximal function* of u with respect to H. If $x \in \operatorname{cl} \Gamma_\beta(x, H)$ and

$$c = \lim_{\substack{y \to x \\ y \in \Gamma_\alpha(x,H)}} u(y)$$

for each $\alpha > \beta$, we say that c is the *nontangential limit* of u at x with respect to H.

Let $H, H_+ \subset R^2$ be bounded open sets with Lipschitz boundaries, $H_+ \subset H$. (The boundaries of H and H_+ consist of a finite number of closed curves. We do not suppose that ∂H and ∂H_+ are disjoint.) Put $H_- = H \setminus \operatorname{cl} H_+$. Suppose that H_- is an open set with Lipschitz boundary and H is connected. Then the outward unit normal $n(x)$, $(n^+(x), n^-(x))$, to H, (H_+, H_-), exists at almost any point x of ∂H, $(\partial H_+, \partial H_-)$, respectively. For a fixed closed subset γ of $\partial H_+ \cap \partial H_-$, put $G = H \setminus \gamma$. We will study the Robin problem for the Laplace equation on the cracked open set G. (We remark that γ is an arbitrary closed subset of $\partial H_+ \cap \partial H_-$. If $\mathcal{H}_1(\gamma) = 0$, then our problem reduces to a problem without a crack, see below.)

Since H_+, H_- are open sets with Lipschitz boundary and $H_+ \cup H_- \subset G$, $\partial G \subset (\partial H_+ \cup \partial H_-)$, there is an $\alpha > 0$ such that for each $x \in \partial G \setminus \gamma$ and $y \in \gamma$ we have $x \in \mathrm{cl}\, \Gamma_\alpha(x, G)$, $y \in \mathrm{cl}\, \Gamma_\alpha(y, H_+)$, $y \in \mathrm{cl}\, \Gamma_\alpha(y, H_-)$. If u is a function in G, $x \in \partial G \setminus \gamma$ ($x \in \partial H_+, x \in \partial H_-$) denote by $u(x)$, $(u_+(x), u_-(x))$ the nontangential limit of u at x with respect to G (H_+, H_-), respectively.

Let $1 < p < \infty$. Denote by $W_0^{1,p}(\gamma)$ the space of all $g \in L^p(\gamma)$ $(\equiv L^p(\mathcal{H}_1 \mid \gamma))$ for which there is an extension $g \in W^{1,p}(\partial H_+)$ such that $g = 0$ on $\partial H_+ \cap \partial H_- \setminus \gamma$. Denote

$$\|g\|_{W_0^{1,p}(\gamma)} = \left[\int_\gamma \left(|g|^p + \left| \frac{\partial g}{\partial \tau} \right|^p \right) d\mathcal{H}_1 \right]^{1/p}.$$

Fix $p \in (1, \infty)$. Let now $h \in L_p(\partial G)$ $(\equiv L^p(\mathcal{H}_1 | \partial G))$, $h \geq 0$, $f \in L^p(\partial G)$, $g \in W_0^{1,p}(\gamma)$. We say that a function u on G is an L^p-solution of the problem

$$\Delta u = 0 \quad \text{in} \quad G, \tag{1}$$

$$\frac{\partial u}{\partial n} + hu = f \quad \text{on} \quad \partial G \setminus \gamma, \tag{2}$$

$$u_+ - u_- = g \quad \text{on} \quad \gamma, \tag{3}$$

$$\left[\frac{\partial u}{\partial n^+} \right]_+ - \left[\frac{\partial u}{\partial n^+} \right]_- + hu_+ = f \quad \text{on} \quad \gamma \tag{4}$$

if 1. u is harmonic in G.

2. $N_\alpha(|\nabla u|, H_+) \in L^p(\mathcal{H}_1 | \partial \mathcal{H}_+)$, $N_\alpha(|\nabla u|, H_-) \in L^p(\mathcal{H}_1 | \partial H_-)$.

3. The nontangential limits of u and ∇u with respect to G exist at $\mathcal{H}_1$-almost any $x \in \partial G \setminus \gamma$ and $n(x) \cdot \nabla u(x) + h(x) u(x) = f(x)$ almost everywhere on $\partial G \setminus \gamma$.

4. The nontangential limits of u with respect to H_+ and with respect to H_- exist $\mathcal{H}_1$-almost everywhere on γ and $u_+(x) - u_-(x) = g(x)$ almost everywhere on γ.

5. The nontangential limits of ∇u with respect to H_+ and with respect to H_- exist $\mathcal{H}_1$-almost everywhere on γ and $n^+(x) \cdot [\nabla u]_+(x) - n^+(x) \cdot [\nabla u]_-(x) + h(x)u_+(x) = f(x)$ almost everywhere on γ.

If $h = 0$ $\mathcal{H}_1$-almost everywhere on ∂G, we shall refer to (1)–(4) as being a Neumann problem. In the opposite case we shall refer to (1)–(4) as a Robin problem. If $\gamma = \varnothing$, then this definition coincides with the common definition of an L^p-solution of the Neumann or the Robin problem, compare [**9, 17**].

3. Single layer potentials. Fix $R > 0$. For $f \in L^p(\partial G)$, $p > 1$, define

$$\mathcal{S}_R f(x) = \frac{1}{2\pi} \int_{\partial G} f(y) \ln \frac{R}{|x-y|} \, d\,\mathcal{H}_1(y)$$

the modified single layer potential with density f. In the case of several sets, we will write $\mathcal{S}_R^G f$. (For $R = 1$ we get the usual single layer potential.) We remark that $\mathcal{S}_R f$ for different R's differ by constants. The function $\mathcal{S}_R f$ is harmonic in $R^2 \setminus \partial G$. Now we prove an auxiliary lemma which will be used later.

Lemma 3.1. *Let $1 < p < \infty$, $R > 0$, $h \in L^p(\partial G)$. The operator $\mathcal{S}_R : f \mapsto \mathcal{S}_R f$ is a bounded linear operator from $L^p(\partial G)$ to $W^{1,2p}(G)$ and from $L^p(\partial G)$ to $C^{(p-1)/p}(\mathrm{cl}\, G)$. If we denote $V_h f = h S_R f$, then V_h is a compact linear operator in $L^p(\partial G)$.*

Proof. Suppose first that $\gamma = \varnothing$. Let $f \in L^p(\partial G)$. We will show that $\mathcal{S}_R f \in W^{1,2p}(G)$. Denote by $B^{p,q}_\alpha(G)$ the classical Besov space. If $p \le 2$, then $\mathcal{S}_R f \in B^{p,2}_{1+1/p}(G) \subset B^{2p,2}_1(G) \subset W^{1,2p}(G)$ by [**21**, Theorem 7.4], [**2**, Theorem 6.5.1] and [**2**, Theorem 6.4.4]. Suppose now that $2 < p$. According to [**21**, Theorem 7.4], we have $\mathcal{S}_R f \in B^{p,p}_{1+1/p}(G)$. Using [**8**, Theorem 4.1], [**8**, Theorem 4.2] and the imbedding theorem, see [**1**, Theorem 7.63], we have $\mathcal{S}_R f \in W^{1+1/p,p}(G) \subset W^{1,2p}(G)$.

Denote

$$\Omega_r(x) = \{y \in R^2;\ |x-y| < r\}.$$

Fix $r > 0$ such that cl $G \subset \Omega_R(0)$. Since G has Lipschitz boundary, there is $u \in W^{1,2p}(\Omega_r(0))$ such that $u = \mathcal{S}_R f$ in G, see [**27**, Remark 2.5.2]. According to [**7**, Theorem 4.5.13], there is $\tilde{u} \in C^{(p-1)/p}(\mathrm{cl}\, G)$ such that $\tilde{u} = u = \mathcal{S}_R f$ almost everywhere in G. Since $\mathcal{S}_R f$ is a continuous function in cl G, we conclude $\mathcal{S}_R f \in C^{(p-1)/p}(\mathrm{cl}\, G)$.

Let now $\gamma \neq \varnothing$. Fix $f \in L^p(\partial G)$. Denote $f_1(x) = f(x)$ for $x \in \gamma$, $f_1(x) = 0$ elsewhere, $f_2(x) = f(x)$ for $x \in \partial H$, $f_2(x) = 0$ elsewhere. We have proved that $\mathcal{S}_R f_2 \in W^{1,2p}(G) \cap C^{(p-1)/p}(\mathrm{cl}\, G)$, $\mathcal{S}_R f_1 \in W^{1,2p}(H_+) \cap C^{(p-1)/p}(\mathrm{cl}\, H_+)$, $\mathcal{S}_R f_1 \in W^{1,2p}(H_-) \cap C^{(p-1)/p}(\mathrm{cl}\, H_-)$. Hence $\mathcal{S}_R f = \mathcal{S}_R f_1 + \mathcal{S}_R f_2 \in W^{1,2}(G) \cap C^{(p-1)/p}(\mathrm{cl}\, G)$.

If $f_n \to f$ in $L^p(\partial G)$, then $\mathcal{S}_R f_n(x) \to \mathcal{S}_R f(x)$ for each $x \in G$. The closed graph theorem, see [**26**, Chapter II, Section 6, Theorem 1], gives that S_R is a bounded liner operator from $L^p(\partial G)$ to $W^{1,2}(G)$ and from $L^p(\partial G)$ to $C^{(p-1)/p}(\mathrm{cl}\, G)$.

Since $\mathcal{S}_R$ is a continuous linear operator from $L^p(\partial G)$ to $C^{(p-1)/p}(\mathrm{cl}\, G)$, the operator $\mathcal{S}_R$ maps each bounded subset of $L^p(\partial G)$ onto the set of equibounded equicontinuous functions on ∂G. Since such a set is precompact in $C(\partial G)$, see [**26**, Chapter III, Section 3], the operator $\mathcal{S}_R$ is a compact linear operator from $L^p(\partial G)$ to $C(\partial G)$. Denote $Hg = hg$ for $g \in C(\partial G)$. Then H is a bounded linear operator from $C(\partial G)$ to $L_p(\partial G)$. Since V_h is the composition of the bounded linear operator H and the compact linear operator $\mathcal{S}_R$, it is a compact linear operator, see [**26**, Chapter X, Section 2, Theorem]. □

4. Properties of solutions. Verchota proved the following lemma, see [**25**, Theorem 1.12].

Lemma 4.1. *If V is a bounded open set with Lipschitz boundary, then there is a sequence of C^∞ domains V_j with the following properties*:

1. cl $V_j \subset V$.

2. *There are homeomorphisms $\Lambda_j : \partial V \to \partial V_j$, such that* $\sup\{|y - \Lambda_j(y)|;\ y \in \partial V\} \to 0$ *as $j \to \infty$, and there is an $\alpha > 0$ such that $\Lambda_j(y) \in \Gamma_\alpha(y, V)$ for each j and each $y \in \partial V$.*

3. *There are positive functions ω_j on ∂V bounded away from zero and infinity uniformly in j such that, for any measurable set $E \subset$*

∂V, $\int_E \omega_j \, d\mathcal{H}_1 = \mathcal{H}_1(\Lambda_j(E))$, and so that $\omega_j \to 1$ pointwise almost everywhere and in every $L^q(\mathcal{H}_1)$, $1 \le q < \infty$.

4. *The normal vectors to V_j, $n(\Lambda_j(y))$, converge pointwise almost everywhere and in every $L^q(\partial V)$, $1 \le q < \infty$, to $n(y)$.*

Proposition 4.2. *Let u be an L^p-solution of the problem* (1)–(4)*, $1 < p < \infty$. Then $u \in W^{1,2}(G)$ and u can be continuously extended onto* cl H_+ *and onto* cl H_-.

Proof. Put $q = \min(p, 2)$. Fix a component V of H_+. Define $F = n^+ \cdot [\nabla u]_+$ on ∂V. Then u is an L^q solution of the Neumann problem for the Laplace equation on V with the boundary condition F. According to [**20**, Theorem 5.1] and [**20**, Theorem 5.2] there is a $\varphi \in L^q(\mathcal{H}_1|\partial V)$ such that $\mathcal{S}_1^V \varphi$ is an L^q solution of the Neumann problem for the Laplace equation in V with the boundary condition F. Lemma 3.1 yields that $\mathcal{S}_1^V \varphi \in C(\text{cl } V) \cap W^{1,2}(V)$. According to [**9**, Corollary 2.1.12], there is a constant c such that $u = \mathcal{S}_1^V \varphi + c$ in V. Therefore $u \in C(\text{cl } V) \cap W^{1,2}(V)$. Since V is arbitrary, we have $u \in C(\text{cl } H_+) \cap W^{1,2}(H_+)$. Similarly, $u \in C(\text{cl } H_-) \cap W^{1,2}(H_-)$. Since $u \in C^\infty(G)$ and $\partial^\alpha u \in L^2(G)$ for each multi-index α with $|\alpha| \le 1$, we deduce $u \in W^{1,2}(G)$. □

Proposition 4.3. *Let $1 < p < \infty$, $h \equiv 0$, $f \in L^p(\mathcal{H}_1|\partial G)$. If there exists an L^p-solution of the problem* (1)–(4)*, then*

(5)
$$\int_{\partial G} f \, d\mathcal{H}_1 = 0.$$

Proof. Denote $F_+ = n^+ \cdot [\nabla u]_+$ in ∂H_+ and $F_- = n^- \cdot [\nabla u]_-$ in ∂H_-. Let V_j be a sequence of open sets from Lemma 4.1 corresponding to H_+. Since u is a classical solution of the Neumann problem for the Laplace equation in V_j with the boundary condition $n \cdot \nabla u$, the Lebesque lemma yields

$$0 = \lim_{j \to \infty} \int_{\partial V_j} n \cdot \nabla u \, d\mathcal{H}_1 = \int_{\partial H_+} F_+ \, d\mathcal{H}_1.$$

Similarly,

$$0 = \int_{\partial H_-} F_- \, d\mathcal{H}_1.$$

Since $n^- = -n^+$ on $\partial H_+ \cap \partial H_-$, we have

$$0 = \int_{\partial H_-} F_- \, d\mathcal{H}_1 + \int_{\partial H_+} F_+ \, d\mathcal{H}_1 = \int_{\partial G} f \, d\mathcal{H}_1. \qquad \square$$

5. Uniqueness.

Theorem 5.1. *Let $1 < p < \infty$, $h \in L^p(\partial G)$, $h \geq 0$, $f \equiv 0$, $g \equiv 0$, u be an L^p-solution of the problem* (1)–(4). *Then u is constant in G. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then $u = 0$ in G.*

Proof. Let V_j be the sets from the Lemma 4.1 corresponding to the set H_+. Since $g \equiv 0$, the function u can be extended, to the function continuous on the closure of G, see Proposition 4.2. Using Green's formula and Lebesgue's lemma, we get

$$\begin{aligned}\int_{\partial H_+} u(n^+ \cdot [\nabla u]_+) \, d\mathcal{H}_1 &= \lim_{j \to \infty} \int_{\partial V_j} u(n \cdot \nabla u) \, d\mathcal{H}_1 \\ &= \lim_{j \to \infty} \int_{V_j} |\nabla u|^2 \, d\mathcal{H}_2 = \int_{H_+} |\nabla u|^2 \, d\mathcal{H}_2.\end{aligned}$$

Similarly,

$$\int_{\partial H_-} u(n^- \cdot [\nabla u]_-) \, d\mathcal{H}_1 = \int_{H_-} |\nabla u|^2 \, d\mathcal{H}_2.$$

Since $n^- = -n^+$ on $\partial H_+ \cap \partial H_-$, we get using the continuity of u on cl G,

$$\begin{aligned}0 &= \int_{\partial G \setminus \gamma} [u(n \cdot \nabla u) + hu^2] \, d\mathcal{H}_1 \\ &\quad + \int_{\gamma} u\{n^+ \cdot [\nabla u]_+ - n^+ \cdot [\nabla u]_- + hu_+\} \, d\mathcal{H}_1 \\ &= \int_{\partial G} hu^2 \, d\mathcal{H}_1 + \int_{\partial H_+} u(n^+ \cdot [\nabla u]_+) \, d\mathcal{H}_1 + \int_{\partial H_-} u(n^- \cdot [\nabla u]_-) \, d\mathcal{H}_1 \\ &= \int_{\partial G} hu^2 \, d\mathcal{H}_1 + \int_{G} |\nabla u|^2 \, d\mathcal{H}_2.\end{aligned}$$

Since $h \geq 0$, we have

$$\int_{\partial G} hu^2 \, d\mathcal{H}_1 = 0, \quad \int_G |\nabla u|^2 \, d\mathcal{H}_2 = 0.$$

Since $\nabla u = 0$ on G, the function u is constant in each component of G. Since u is continuous in cl G, there is a constant c such that $u = c$ on cl G. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then

$$c^2 = \left[\int_{\partial G} h \, d\mathcal{H}_1\right]^{-1} \int_{\partial G} hu^2 \, d\mathcal{H}_1 = 0. \qquad \square$$

6. Double layer potentials. If V is a bounded open set with Lipschitz boundary, $g \in L^p(\partial V)$ and $n^V(y)$ denotes the outward unit normal to V at y, define

$$\mathcal{D}_V g(x) = \frac{1}{2\pi} \int_{\partial V} \frac{n^V(y) \cdot (y - x)}{|x - y|^2} \, g(y) \, d\mathcal{H}_1(y),$$

the double layer potential corresponding to V with density g. If $V = H_+$, we write $\mathcal{D}g$ instead of $\mathcal{D}_V g$. The following lemma is an easy generalization of known results.

Lemma 6.1. *Let $1 < p < \infty$, $g \in W^{1,p}(\partial H_+)$. Then $\mathcal{D}g$ is a harmonic function in G, $N_\alpha(|\nabla \mathcal{D}g|, H_+) \in L^p(\mathcal{H}_1|\partial H_+)$, $N_\alpha(|\nabla \mathcal{D}g|, H_-) \in L^p(\mathcal{H}_1|\partial H_-)$, there are the nontangential limits of $\nabla \mathcal{D}g$ with respect to H_+ and with respect to H_- $\mathcal{H}_1$-almost everywhere in γ and $n^+(x) \cdot [\nabla \mathcal{D}g]_+(x) - n^+(x) \cdot [\nabla \mathcal{D}g]_-(x) = 0$ almost everywhere in γ. There is the nontangential limit of $\nabla \mathcal{D}g$ with respect to G $\mathcal{H}_1$-almost everywhere in $\partial G \setminus \gamma$ and*

$$\| |\nabla \mathcal{D}g| \|_{L^p(\partial G \setminus \gamma)} \leq C \|g\|_{W^{1,p}(\partial H_+)},$$

where C is a constant depending only on G and p. The function $\mathcal{D}g$ can be extended onto $[\mathcal{D}g]_+ \in C^{(p-1)/p}(\mathrm{cl}\ H_+)$ and onto $[\mathcal{D}g]_- \in$

$C^{(p-1)/p}(\mathrm{cl}\ H_-)$. *Moreover,* $[\mathcal{D}g]_+ - [\mathcal{D}g]_- = g$ $\mathcal{H}_1$*-almost everywhere in* $\partial H_+ \cap \partial H_-$ *and* $\mathcal{D}g \in W^{1,2p}(G)$,

$$\|\mathcal{D}g\|_{W^{1,2p}(G)} + \|\mathcal{D}g\|_{L^\infty(G)} \le \widetilde{C}\|g\|_{W^{1,p}(\partial H_+)},$$

where $\widetilde{C}$ *is a constant depending only on* G *and* p.

Proof. Since $g \in W^{1,p}(\partial H_+)$, we can suppose that g is continuous, see [**1**, Theorem 5.4]. Since $\partial H_+ \cap \partial H_-$ is formed by finitely many arcs and there is a continuous extension operator from $W^{1,p}((a,b))$ to $W^{1,p}(R^1)$, see [**1**, Theorem 4.26], we can extend g onto $g \in W^{1,p}(\partial H_-)$ so that

$$\|g\|_{W^{1,p}(\partial H_-)} \le M\|g\|_{W_0^{1,p}(\gamma)},$$

with a constant M depending only on G and p. We can again suppose that g is continuous. By the characterization of $W^{1,p}$ on an interval, we see that $g \in W^{1,p}(\partial H)$.

The boundary of H_+ is formed by finitely many Jordan curves. Fix one of these curves Γ. Denote $g_\Gamma = g$ on Γ, $g_\Gamma = 0$ elsewhere. Let Γ be parametrized by the arc length s: $\Gamma = \{\varphi(s);\ s \in [a,b]\}$ and H_+ is to the right when the parameter s increases on Γ. Put $f(\varphi(s)) = -[g(\varphi)]'(s)$. Then $f \in L^p(\Gamma)$ because $g \in W^{1,p}(\partial H_+)$. For $x \in R^2 \setminus \Gamma$ and $s \in [a,b]$, denote by $v(x,\varphi(s))$ the increment of the argument of $y - x$ along the curve $\{y = \varphi(t);\ t \in [a,s]\}$, and

$$Vf(x) = \frac{1}{2\pi}\int_\Gamma v(x,y)f(y)\,d\mathcal{H}_1(y)$$

the angular potential corresponding to f. (The angular potential was introduced and studied by Gabov in [**5**].) Define $f = 0$ on $R^2 \setminus \Gamma$. Since Vf is a conjugate function to $-\mathcal{S}_1^{H_+} f$, see [**19**, pp. 226–227], we have

$$\frac{\partial Vf}{\partial x_1} = -\frac{\partial \mathcal{S}_1^{H_+} f}{\partial x_2}, \qquad \frac{\partial Vf}{\partial x_2} = \frac{\partial \mathcal{S}_1^{H_+} f}{\partial x_1}.$$

Using boundary properties of single layer potentials, see [**9**, Theorem 2.2.13], we can deduce that $N_\alpha(|\nabla Vf|, H_+) \in L^p(\mathcal{H}_1|\partial H_+)$, there are the nontangential limits of ∇Vf with respect to H_+ and with respect to H_- $\mathcal{H}_1$-almost everywhere in γ and $n^+(x)\cdot[\nabla Vf]_+(x)-n^+(x)\cdot[\nabla Vf]_-(x)=0$ almost everywhere in γ and

$$\begin{aligned}\||[\nabla Vf]_+|\|_{L^p(\partial H_+)} &= \||[\nabla \mathcal{S}_1^{H_+} f]_+|\|_{L^p(\partial H_+)}\\ &\le C_1\|f\|_{L^p(\partial G)} \le C_2\|g_\Gamma\|_{W^{1,p}(\partial H_+)},\end{aligned}$$

where C_1, C_2 are constants depending only on G and p. Using Lemma 3.1 we get that $\partial_1 Vf, \partial_2 Vf \in L^{2p}(G)$. Since Vf is continuously extendible onto cl H_+ by [**19**, Lemma 5], we obtain $Vf \in W^{1,2p}(H_+)$.

Put $\tilde g = g - g(\varphi(a))$, $\hat g = g(\varphi(a))$ on Γ, $\tilde g, \hat g = 0$ elsewhere. Since

$$g_\Gamma(\varphi(s)) - g_\Gamma(\varphi(a)) = \int_a^s (-f(t))\,dt = \int_s^b f(t)\,dt,$$

we have according to [**19**, p. 226],

$$\mathcal{D}_{H_+}\tilde g = Vf.$$

Therefore, $N_\alpha(|\nabla \mathcal{D}_{H_+}\tilde g|, H_+) \in L^p(\mathcal{H}_1|\partial H_+)$, there are the nontangential limits of $\nabla\mathcal{D}_{H_+}\tilde g$ with respect to H_+ and with respect to H_- $\mathcal{H}_1$-almost everywhere in γ and $n^+(x)\cdot[\nabla\mathcal{D}_{H_+}\tilde g]_+(x) - n^+(x)\cdot[\nabla\mathcal{D}_{H_+}\tilde g]_-(x) = 0$ almost everywhere in γ. Moreover, $\mathcal{D}_{H_+}\tilde g \in W^{1,2p}(H_+)$ and $\mathcal{D}_{H_+}\tilde g$ can be continuously extendible onto cl H_+. It is well known that the double layer potential with constant density corresponding to the Jordan curve Γ is constant in the interior of Γ and in the exterior of Γ as well. (We can prove it using the expression of a harmonic function as the sum of the single layer potential corresponding to the normal derivative and the double layer potential corresponding to the boundary value of the function.) Thus $\nabla\mathcal{D}_{H_+}\hat g = 0$ in $R^2\setminus\Gamma$. So, there are the nontangential limits of $\nabla\mathcal{D}g_\Gamma = \nabla\mathcal{D}_{H_+}\tilde g$ with respect to H_+ and with respect to H_- $\mathcal{H}_1$-almost everywhere in γ, $N_\alpha(|\nabla\mathcal{D}g_\Gamma|, H_+) \in L^p(\mathcal{H}_1|\partial H_+)$, $n^+(x)\cdot[\nabla\mathcal{D}g_\Gamma]_+(x) - n^+(x)\cdot[\nabla\mathcal{D}g_\Gamma]_-(x) = 0$ almost everywhere in γ and

$$\||[\nabla\mathcal{D}g_\Gamma]_+|\|_{L^p(\partial H_+)} \le C_2\|g_\Gamma\|_{W^{1,p}(\partial H_+)}.$$

Summing over all Γ we get $N_\alpha(|\nabla\mathcal{D}g|, H_+) \in L^p(\mathcal{H}_1|\partial H_+)$; there are the nontangential limits of $\nabla\mathcal{D}g$ with respect to H_+ and with respect to H_- $\mathcal{H}_1$-almost everywhere in γ, $n^+(x)\cdot[\nabla\mathcal{D}g]_+(x) - n^+(x)\cdot[\nabla\mathcal{D}g]_-(x) = 0$ almost everywhere in γ and

$$\| |[\nabla\mathcal{D}g]_+| \|_{L^p(\partial H_+)} \le C_2 \|g\|_{W^{1,p}(\partial H_+)}.$$

Moreover, $\mathcal{D}g \in W^{1,2p}(H_+)$. Fix $r > 0$ such that $\mathrm{cl}\, H_+ \subset \Omega_r(0)$. Since H_+ has Lipschitz boundary, there is a $u \in W^{1,2p}(\Omega_r(0))$ such that $u = \mathcal{D}g$ in H_+, see [**27**, Remark 2.5.2]. According to [**7**, Theorem 4.5.13], there is a $\tilde u \in C^{(p-1)/p}(\mathrm{cl}\, H_+)$ such that $\tilde u = u = \mathcal{D}g$ almost everywhere in H_+. Since $\mathcal{D}g \in C(H_+)$, we have $\tilde u = \mathcal{D}g$ in H_+.

Similarly, $N_\alpha(|\nabla\mathcal{D}_{H_-}g|, H_-) \in L^p(\mathcal{H}_1|\partial H_-)$, $N_\alpha(|\nabla\mathcal{D}_H g|, H) \in L^p(\mathcal{H}_1|\partial H)$, and

$$\| |[\nabla\mathcal{D}_{H_-}g]_-| \|_{L^p(\partial H_-)} \le C_3 \|g\|_{W^{1,p}(\partial H_-)},$$
$$\| |\nabla\mathcal{D}_H g| \|_{L^p(\partial H)} \le C_3 \|g\|_{W^{1,p}(\partial H)}$$

where C_3 is a constant depending only on G and p. Moreover, $\mathcal{D}_H g \in W^{1,2p}(H)$, $\mathcal{D}_{H_-}g \in W^{1,2p}(H_-)$, $\mathcal{D}_H g$ can be extended onto $[\mathcal{D}_H g]_- \in C^{(p-1)/p}(\mathrm{cl}\, H_-)$ and $\mathcal{D}_{H_-}g$ can be extended onto $[\mathcal{D}_{H_-}g]_- \in C^{(p-1)/p}(\mathrm{cl}\, H_-)$. Since $\mathcal{D}_H g = \mathcal{D}g + \mathcal{D}_{H_-}g$, we deduce that $\mathcal{D}g \in W^{1,2p}(G)$, and $\mathcal{D}g$ can be extended onto $[\mathcal{D}g]_- \in C^{(p-1)/p}(\mathrm{cl}\, H_-)$. Since $N_\alpha(|\nabla\mathcal{D}_{H_-}g|, H_-)$, $N_\alpha(|\nabla\mathcal{D}_H g|, H_-) \in L^p(\partial H_- \setminus \partial H_+)$, we have $N_\alpha(|\nabla\mathcal{D}_H g|, H_-) \in L^p(\partial H_- \setminus \partial H_+)$ and

$$\| |\nabla\mathcal{D}g| \|_{L^p(\partial G\setminus\gamma)} \le (C_2 + C_3 + 2C_3 M)\|g\|_{W^{1,p}(\partial H_+)}.$$

Fix $R > 0$ such that $\mathrm{cl}\, H \subset \Omega_R(0)$. Put $g = 0$ on $\partial\Omega_R(0)$ and $U = \Omega_R(0) \setminus \mathrm{cl}\, H_+$. We have shown that $N_\alpha(|\nabla\mathcal{D}_U g|, U) \in L^p(\partial U)$. If $g_n \to g$ in $W^{1,p}(\partial H_+)$, then $\mathcal{D}g_n(x) \to \mathcal{D}g(x)$ for each $x \in G$. The closed graph theorem, see [**26**, Chapter II, Section 6, Theorem 1], gives that $\mathcal{D} : g \mapsto \mathcal{D}g$ is a bounded linear operator from $W^{1,p}(\partial H_+)$ to $W^{1,2p}(G)$ and from $W^{1,p}(\partial H_+)$ to $L^\infty(G)$.

According to [**9**, Theorem 2.2.13], we have $[\mathcal{D}g]_+(x) - [\mathcal{D}g]_-(x) = g(x)$ almost everywhere in $\partial H_+ \cap \partial H_-$. $\square$

7. Reduction of the problem. If $1 < p < \infty$, $\varepsilon > 0$, we define for $w \in L^p(\partial G)$, $x \in \partial G \setminus \gamma$,

$$K^*_{G,\varepsilon} w(x) = \frac{1}{2\pi}\int_{\partial G\setminus\Omega_\varepsilon(x)} w(y)\, \frac{n(x)\cdot(y-x)}{|x-y|^2}\, d\mathcal{H}_1(y).$$

(We repeat that $\Omega_\varepsilon(x)$ is the open ball with the center x and the radius ε.) Denote $w_1(x) = w(x)$ for $x \in \gamma$, $w_1(x) = 0$ elsewhere, $w_2(x) = w(x)$ for $x \in \partial H \cap \partial H_+$, $w_2(x) = 0$ elsewhere, $w_3(x) = w(x)$ for $x \in \partial H \cap \partial H_-$, $w_3(x) = 0$ elsewhere. If we use the properties of the single layer potentials with densities w_1, w_2 and w_3 corresponding to open sets H_+, H_- and H, see [**9**, Theorem 2.2.13], we get that for $\mathcal{H}_1$ almost any $x \in \partial G \setminus \gamma$ there is

$$K_G^* w(x) = \lim_{\varepsilon \to 0_+} K_{G,\varepsilon}^* w(x)$$

and K_G^* is a bounded linear operator from $L^p(\partial G)$ to $L^p(\partial G \setminus \gamma)$.

Let $1 < p < \infty$, $f \in L^p(\partial G)$, $h \in L^p(\partial G)$, $h \geq 0$, $g \in W_0^{1,p}(\gamma)$. We will look for a solution u of the problem (1)–(4) in the form

$$u = \mathcal{D}g + v. \tag{6}$$

According to Lemma 6.1 and [**9**, Theorem 2.2.13], the function u is an L^p solution of the problem (1)–(4) if and only if the function v is an L^p solution of the problem

$$\Delta v = 0 \quad \text{in} \quad G, \tag{7}$$

$$\frac{\partial v}{\partial n} + hv = F \quad \text{on} \quad \partial G \setminus \gamma, \tag{8}$$

$$v_+ - v_- = 0 \quad \text{on} \quad \gamma, \tag{9}$$

$$\left[\frac{\partial v}{\partial n^+}\right]_+ - \left[\frac{\partial v}{\partial n^+}\right]_- + hv_+ = F \quad \text{on} \quad \gamma \tag{10}$$

where

$$F = f - \frac{\partial \mathcal{D}g}{\partial n} - h\mathcal{D}g \quad \text{on} \quad \partial G \setminus \partial H_+, \tag{11}$$

$$F = f - n \cdot [\nabla \mathcal{D}g]_+ - h[\mathcal{D}g]_+ \quad \text{on} \quad \partial G \cap \partial H_+ \setminus \gamma, \tag{12}$$

$$F(x) = f(x) - h(x)\left(\frac{1}{2}\, g(x) + K^*_{H_+} g(x)\right), \quad x \in \gamma. \tag{13}$$

We will look for an L^p solution of the problem (7)–(10) in the form of a modified single layer potential $\mathcal{S}_R w$ where $w \in L^p(\partial G)$ and $R > \operatorname{diam} G$, the diameter of G. Denote $w_1(x) = w(x)$ for $x \in \gamma$, $w_1(x) = 0$ elsewhere, $w_2(x) = w(x)$ for $x \in \partial H \cap \partial H_+$, $w_2(x) = 0$ elsewhere, $w_3(x) = w(x)$ for $x \in \partial H \cap \partial H_-$, $w_3(x) = 0$ elsewhere. If we use the properties of the single layer potentials with densities w_1, w_2 and w_3 corresponding to open sets H_+, H_- and H, see [**9**, Theorem 2.2.13], we get that $N_\alpha(|\nabla \mathcal{S}_R w|, H_+) \in L^p(\partial H_+)$ and $N_\alpha(|\nabla \mathcal{S}_R w|, H_-) \in L^p(\partial H_-)$, the nontangential limit of $\nabla \mathcal{S}_R w$ with respect to G exists almost everywhere in $\partial G \setminus \gamma$ and

$$n(x) \cdot \nabla \mathcal{S}_R w(x) = \frac{1}{2}\, w(x) + K^*_G w(x) \quad \text{a.e. in} \quad \partial G \setminus \gamma, \tag{14}$$

the nontangential limits of $\nabla \mathcal{S}_R w$ with respect to H_+ and with respect to H_- exist almost everywhere in γ and

$$n^+(x) \cdot [\nabla \mathcal{S}_R w]_+(x) - n^+(x) \cdot [\nabla \mathcal{S}_R w]_-(x) = w(x) \quad \text{a.e. in} \quad \gamma. \tag{15}$$

Since the modified single layer potential $\mathcal{S}_R w$ is a harmonic function in G, we get using Lemma 3.1 that $\mathcal{S}_R w$ is an L^p solution of the problem (7)–(10) if and only if

$$T_{h,R} w = F \tag{16}$$

where

$$T_{h,R} w = \frac{1}{2}\, w + K^*_G w + h \mathcal{S}_R w \quad \text{on} \quad \partial G \setminus \gamma, \tag{17}$$

$$T_{h,R} w = w + h \mathcal{S}_R w \quad \text{on} \quad \gamma. \tag{18}$$

Since K^*_G is a bounded linear operator from $L^p(\partial G)$ to $L^p(\partial G \setminus \gamma)$, Lemma 3.1 gives that $T_{h,R}$ is a bounded linear operator in $L^p(\partial G)$.

8. Solvability of the problem.

Lemma 8.1. *Let* $1 < p < \infty$*,* $g \in L^p(\partial G)$*,* $\mathcal{H}_1(\{x \in \partial G;\ |g(x)| > 0\}) > 0$*,* $R > \operatorname{diam} G$*. Then*

$$0 < \int_{\partial G} g(x) S_R\, g(x)\, d\,\mathcal{H}_1(x) < \infty.$$

Proof. Fix x_0 in ∂G. Put $P(x) = (x - x_0)/R$, $P_{-1}(x) = Rx + x_0$ for $x \in R^2$, $\widetilde{G} = P(G)$. For $x \in \partial\widetilde{G}$ put $\tilde{g}(x) = g(P_{-1}(x))$. Then $\tilde{g} \in L^p(\partial\widetilde{G})$ and $\mathcal{S}_R g(x) = R\mathcal{S}_1^{\widetilde{G}}\tilde{g}(P(x))$ for $x \in G$. Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_1$ onto $\partial\widetilde{G}$. Since $\mathcal{S}_1^{\widetilde{G}}|\tilde{g}|$ is continuous in R^2 by [**19**, Lemma 4] we conclude

$$\int |\tilde{g}|\mathcal{S}_1^{\widetilde{G}}\, |\tilde{g}|\, d\,\mathcal{H} < \infty$$

and thus the real measure $\tilde{g}\mathcal{H}$ has finite energy, see [**16**, Chapter 1, Section 4]). Since $\mathcal{H}_1(\{x \in \partial\widetilde{G};\ |\tilde{g}(x)| > 0\}) > 0$, [**16**, Theorem 1.16] yields

$$\int \tilde{g}\mathcal{S}_1^{\widetilde{G}}\, \tilde{g}\, d\,\mathcal{H} > 0.$$

Now we use the fact that

$$\int_{\partial G} g(x) S_R\, g(x)\, d\,\mathcal{H}_1(x) = R^2 \int_{\partial\widetilde{G}} \tilde{g}(x) S_1^{\widetilde{G}}\, \tilde{g}(x)\, d\,\mathcal{H}_1(x). \qquad \square$$

Lemma 8.2. *Let* $1 < p < \infty$*,* $h \in L^p(\partial G)$*,* $h \geq 0$*,* $R > \operatorname{diam} G$*,* $\varphi \in L^p(\partial G)$*. If* $T_{h,R}^2\varphi = 0$*, then* $T_{h,R}\varphi = 0$*.*

Proof. According to Section 7 we have $T_{h,R}\varphi \in L^p(\partial G)$, and the modified single layer potential $\mathcal{S}_R T_{h,R}\varphi$ is an L^p-solution of the problem (1)–(4) with $g \equiv 0$ and $f = T_{h,R}^2\varphi = 0$. According to Theorem 5.1 and Lemma 3.1, there is a constant c such that $\mathcal{S}_R T_{h,R}\varphi = c$ in $\operatorname{cl} G$. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then $c = 0$.

According to Section 7 the modified single layer potential $\mathcal{S}_R\varphi$ is an L^p-solution of the problem (1)–(4) with $g \equiv 0$ and $f = T_{h,R}\varphi$. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) = 0$, then

$$\int_{\partial G} T_{h,R}\varphi \, d\mathcal{H}_1 = 0.$$

Thus

$$\int_{\partial G} (T_{h,R}\varphi)\mathcal{S}_R T_{h,R}\varphi \, d\mathcal{H}_1 = c \int_{\partial G} (T_{h,R}\varphi) \, d\mathcal{H}_1 = 0.$$

According to Lemma 8.1 we have $T_{h,R}\varphi = 0$ almost everywhere in ∂G. □

Definition 8.3. The bounded linear operator T on the Banach space X is called Fredholm if $\alpha(T)$, the dimension of the kernel of T, is finite, the range $T(X)$ of T is a closed subspace of X and $\beta(T)$, the codimension of $T(X)$, is finite. The number $i(T) = \alpha(T) - \beta(T)$ is the index of T.

Proposition 8.4. *Let $1 < p < \infty$, $h \in L^p(\partial G)$, $h \geq 0$, $R >$ diam G, $T_{h,R}$ be a Fredholm operator with index 0 in $L^p(\partial G)$. Denote by $L^p_0(\partial G)$ the set of all $f \in L^p(\partial G)$ for which (5) holds. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) = 0$ then $T_{h,R}$ is continuously invertible in $L^p_0(\partial G)$. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then $T_{h,R}$ is continuously invertible in $L^p(\partial G)$.*

Proof. Suppose first that $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$. If $\varphi \in L^p(\partial G)$, $T_{h,R}\varphi = 0$, then $\mathcal{S}_R\varphi$ is an L^p-solution of the problem (1)–(4) with $f \equiv 0$, $g \equiv 0$, see Section 7, and Theorem 5.1, Lemma 3.1 give that $\mathcal{S}_R\varphi = 0$ on cl G. Since

$$\int_{\partial G} \varphi \mathcal{S}_R \varphi \, d\mathcal{H}_1 = 0,$$

Lemma 8.1 shows that $\varphi = 0$ almost everywhere in ∂G. Since $\alpha(T_{h,R}) = 0$ and $i(T_{h,R}) = 0$, we deduce that $\beta(T_{h,R}) = 0$. Since $T_{h,R}(L^p(\partial G)) = L^p(\partial G)$ and the kernel of $T_{h,R}$ is trivial, the operator

$T_{h,R}$ is continuously invertible in $L^p(\partial G)$ by the closed graph theorem, see [**26**, Chapter II, Section 6, Theorem 1].

Suppose now that $h = 0$ almost everywhere in ∂G. If $\varphi \in L^p(\partial G)$, then $\mathcal{S}_R\varphi$ is an L^p-solution of the problem (1)–(4) with $g \equiv 0$ and $f = T_{h,R}\varphi$, see Section 7. Thus, $T_{h,R}\varphi \in L^p_0(\partial G)$ by Proposition 4.3. Since $T_{h,R}(L^p(\partial G)) \subset L^p_0(\partial G)$, we have $\beta(T_{h,R}) \geq 1$.

If $\varphi \in L^p(\partial G)$, $T_{h,R}\varphi = 0$,then $\mathcal{S}_R\varphi$ is an L^p-solution of the problem (1)–(4) with $f \equiv 0$, $g \equiv 0$, see Section 7. According to Theorem 5.1, Lemma 3.1 there is a constant c such that $\mathcal{S}_R\varphi = c$ on cl G. If $c = 0$, then Lemma 8.1 yields that $\varphi = 0$ almost everywhere on ∂G. This means that $\alpha(T_{h,R}) \leq 1$. Since $1 \leq \beta(T_{h,R}) = \alpha(T_{h,R}) \leq 1$, we deduce that $\beta(T_{h,R}) = \alpha(T_{h,R}) = 1$. Since $T_{h,R}(L^p(\partial G)) \subset L^p_0(\partial G)$, $\beta(T_{h,R}) = 1$ shows that $T_{h,R}(L^p(\partial G)) = L^p_0(\partial G)$.

The kernel of the operator $T_{h,R}$ in $L^p_0(\partial G)$ is trivial by Lemma 8.2. Since $L^p_0(\partial G)$ is a $T_{h,R}$-invariant closed linear subspace of finite codimension in $L^p(\partial G)$ and $T_{h,R}$ is a Fredholm operator with index 0, the restriction of $T_{h,R}$ onto $L^p_0(\partial G)$ is a Fredholm operator with index 0 by [**18**, Proposition 3.7.1]. Since the kernel of the operator $T_{h,R}$ in $L^p_0(\partial G)$ is trivial, we deduce that $T_{h,R}(L^p_0(\partial G)) = L^p_0(\partial G)$. The closed graph theorem, see [**26**, Chapter II, Section 6, Theorem 1], gives that $T_{h,R}$ is continuously invertible in $L^p_0(\partial G)$. $\square$

Notation 8.5. Let X be a real Banach space. Denote compl $X = \{x + iy;\ x, y \in X\}$ the complexification of X. If T is a linear operator in X we define $T(x + iy) = Tx + iTy$ the linear extension of T onto compl X.

Lemma 8.6. *Let $1 < p < \infty$, $h \in L^p(\partial G)$, $h \geq 0$, $R >$ diam G, $\lambda \neq 1$ be a complex number. Then $T_{h,R} - \lambda I$ is a Fredholm operator in* compl $L^p(\partial G)$ *if and only if $((1/2) - \lambda)I + K^*_H$ is a Fredholm operator in* compl $L^p(\partial H)$. *Moreover, $i(T_{h,R} - \lambda I) = i(((1/2) - \lambda)I + K^*_H)$. (Here I denotes the identity operator.)*

Proof. Let V_h be the operator from Lemma 3.1. Since V_h is a compact linear operator in $L^p(\partial G)$, the operator $T_{h,R} - \lambda I$ is a Fredholm operator in compl $L^p(\partial G)$ if and only if $T_{0,R} - \lambda I$ is a Fredholm operator in compl $L^p(\partial G)$ and $i(T_{h,R} - \lambda I) = i(T_{0,R} - \lambda I)$, see [**24**, Theorem 5.10].

Denote

$$Tf(x) = (1/2 - \lambda)f(x) + K_H^* f(x) \quad \text{for} \quad x \in \partial H,$$
$$Tf(x) = (1 - \lambda)f(x) \quad \text{for} \quad x \in \gamma.$$

Then T is a bounded linear operator in compl $L^p(\partial G)$ which is a Fredholm operator in compl $L^p(\partial G)$ if and only if $(1/2 - \lambda)I + K_H^*$ is a Fredholm operator in compl $L^p(\partial H)$ and $i(T) = i((1/2-\lambda)I + K_H^*)$. Easy calculation yields that $T_{0,R} - \lambda I - T$ is a compact operator in $L^p(\partial G)$. Thus, $T_{0,R} - \lambda I$ is a Fredholm operator in compl $L^p(\partial G)$ if and only if T is a Fredholm operator in compl $L^p(\partial G)$ and $i(T) = i(T_{0,R} - \lambda I)$ by [**24**, Theorem 5.10]. □

Theorem 8.7. *There is* $2 < p_0 \le \infty$ *depending only on* G *such that following holds*:

1. *If* $p_0 < p < \infty$, $h \in L^p(\partial G)$, $h \ge 0$, $R > 0$, *then* $T_{h,R}$ *is not a Fredholm operator with index 0 in* $L^p(\partial G)$.

2. *Let* $1 < p < p_0$, $h \in L^p(\partial G)$, $h \ge 0$, $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, $R >$ diam G. *Then* $T_{h,R}$ *is continuously invertible in* $L^p(\partial G)$. *If* $f \in L^p(\partial G)$, $g \in W_0^{1,p}(\gamma)$ *then there is a unique* L^p*-solution of the problem* (1)–(4). *This solution is given by*

$$u = \mathcal{D}g + \mathcal{S}_R T_{h,R}^{-1} F \tag{19}$$

where F *is given by* (11)–(13).

3. *Let* $1 < p < p_0$, $h \equiv 0$, $R >$ diam G. *Denote by* $\widetilde{T}_{0,R}$ *the restriction of* $T_{0,R}$ *onto* $L_0^p(\partial G)$. *Then* $\widetilde{T}_{0,R}$ *is continuously invertible. If* $f \in L^p(\partial G)$, $g \in W^{1,p}(\gamma)$, *then there is an* L^p*-solution of the problem* (1)–(4) *if and only if equation* (5) *is fulfilled. The general form of a solution is*

$$u = \mathcal{D}g + \mathcal{S}_R(\widetilde{T}_{0,R})^{-1} F + c \tag{20}$$

where F *is given by* (11)–(13) *and* c *is arbitrary constant.*

Proof. Suppose first that $1 < p < \infty$ is such that $(1/2)I + K_H^*$ is a Fredholm operator with index 0 in $L^p(\partial H)$. If $h \in L^p(\partial G)$, $h \geq 0$, $R > \operatorname{diam} G$, then $T_{h,R}$ is a Fredholm operator with index 0 in $L^p(\partial G)$ by Lemma 8.6. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then $T_{h,R}$ is continuously invertible; if $h \equiv 0$ then $\widetilde{T}_{0,R}$ is continuously invertible, see Proposition 8.4. Let $f \in L^p(\partial G)$, $g \in W_0^{1,p}(\gamma)$ and $f \in L_0^p(\partial G)$ in case $h \equiv 0$. Let F be given by (11)–(13). Suppose first that $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$. According to Section 7 the function u given by (19) is an L^p-solution of the problem (1)–(4). According to Theorem 5.1 this solution is unique. Suppose now $h \equiv 0$. According to Section 7 the function u given by (20) is an L^p-solution of the problem (1)–(4). Theorem 5.1 shows that (20) gives the general form of a solution. Proposition 4.3 gives that $f \in L_0^p(\partial G)$ is a necessary condition for the solvability of the problem (1)–(4).

According to [**20**, Theorem 5.1], there is an $\varepsilon > 0$ such that $(1/2)I + K_H^*$ is a Fredholm operator with index 0 in $L^p(\partial G)$ for $1 < p < 2 + \varepsilon$. Put $p_0 = \sup\{p;\ p > 1,\ (1/2)I + K_H^*$ is a Fredholm operator with index 0 in $L^p(\partial H)\}$. Then $2 < p_0 \leq \infty$. If $p_0 < p < \infty$, $h \in L^p(\partial G)$, $h \geq 0$, $R > 0$, then $T_{h,R}$ is not a Fredholm operator with index 0 in $L^p(\partial G)$ by Lemma 8.6. Let now $2 \leq p < p_0$. Then there is a $q \in (p, p_0)$ such that $(1/2)I + K_H^*$ is a Fredholm operator with index 0 in $L^q(\partial H)$. Put $h \equiv 1$. Then there is continuous $T_{h,R}^{-1}$ in $L^q(\partial G)$ and in $L^2(\partial G)$. The Riesz-Thorin interpolation theorem, see [**23**, Theorem 6.1.1], yields that there is continuous $T_{h,R}^{-1}$ in $L^p(\partial G)$. Since $T_{h,R}$ is a Fredholm operator with index 0 in $L^p(\partial G)$, Lemma 8.6 gives that $(1/2)I + K_H^*$ is a Fredholm operator with index 0 in $L^p(\partial H)$. □

Corollary 8.8. *Let $2 < p_0 \leq \infty$ be the constant from Theorem* 8.7. *Let $1 < p < p_0$, $h \in L^p(\partial G)$, $h \geq 0$, $f \in L^p(\partial G)$, $g \in W_0^{1,p}(\gamma)$ and u be an L^p solution of the problem* (1)–(4). *Then $u \in W^{1,2p}(G)$, there are $u_+ \in C^{(p-1)/p}(\mathrm{cl}\ H_+)$ and $u_- \in C^{(p-1)/p}(\mathrm{cl}\ H_-)$ such that $u = u_+$ in H_+, $u = u_-$ in H_-. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then*

$$|u|_{L^\infty(G)} + \|u\|_{W^{1,2p}(G)} \leq C[\|f\|_{L^p(\partial G)} + \|g\|_{W_0^{1,p}(\gamma)}].$$

If $h \equiv 0$, then there is a constant c such that

$$|u - c|_{L^\infty(G)} + \|u - c\|_{W^{1,2p}(G)} \leq C[\|f\|_{L^p(\partial G)} + \|g\|_{W_0^{1,p}(\gamma)}].$$

Here C is a constant depending only on G, p and h. If $g \equiv 0$, then $u \in C^{(p-1)/p}(\mathrm{cl}\, G)$ and

$$\sup_{x,y\in G;\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{(p-1)/p}} \leq \widetilde{C}\|f\|_{L^p(\partial G)}$$

where $\widetilde{C}$ is a constant depending only on G, p and h.

Proof. If we define $g(x) = 0$ for $x \in \partial H_+ \cap \partial H_- \setminus \gamma$, then $g \in W^{1,p}(\partial H_+ \cap \partial H_-)$. Since $\partial H_+ \cap \partial H_-$ is formed by finitely many arcs and there is a continuous extension operator from $W^{1,p}((a,b))$ to $W^{1,p}(R^1)$, see [**1**, Theorem 4.26], we can extend g onto ∂H_+ so that

$$\|g\|_{W^{1,p}(\partial H_+)} \leq M\|g\|_{W_0^{1,p}(\gamma)} \tag{21}$$

with a constant M depending only on G and p.

Fix $R > \mathrm{diam}\, G$. Let F be given by (11)–(13). If $h = 0$ almost everywhere on ∂G put $Q = \widetilde{T}_{0,R}^{-1}F$, in the opposite case put $Q = T_{h,R}^{-1}F$. According to Theorem 8.7 there is a constant c such that $u = \mathcal{D}g + \mathcal{S}_R Q + c$. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then $c = 0$. Lemma 6.1 shows that $\mathcal{D}g \in W^{1,2p}(G)$; there are $[\mathcal{D}g]_+ \in C^{(p-1)/p}(\mathrm{cl}\, H_+)$, $[\mathcal{D}g]_- \in C^{(p-1)/p}(\mathrm{cl}\, H_-)$ such that $\mathcal{D}g = [\mathcal{D}g]_+$ in H_+, $\mathcal{D}g = [\mathcal{D}g]_-$ in H_- and there is a constant C_1 depending on G and p such that

$$\|\mathcal{D}g\|_{L^\infty(G)} + \|\mathcal{D}g\|_{W^{1,2p}(G)} \leq C_1\|g\|_{W_0^{1,p}(\gamma)}. \tag{22}$$

$F = f - h[\mathcal{D}g]_+$ on γ by [**9**, Theorem 2.2.13]. Thus Lemma 6.1 gives that there is a constant C_2 depending only on G, p and h such that

$$\|F\|_{L^p(\partial G)} \leq C_2[\|f\|_{L^p(\partial G)} + \|g\|_{W_0^{1,p}(\gamma)}]. \tag{23}$$

According to Theorem 8.7 there is a constant C_3 depending on G, p, h and R so that

$$\|Q\|_{L^p(\partial G)} \leq C_3\|F\|_{L^p(\partial G)}. \tag{24}$$

According to Lemma 3.1 we have $\mathcal{S}_R Q \in W^{1,2p}(G) \cap C^{(p-1)/p}(\mathrm{cl}\, G)$ and

$$\|\mathcal{S}_R Q(x)\|_{C^{(p-1)/p}(\mathrm{cl}\, G)} + \|\mathcal{S}_R Q\|_{W^{1,2p}(G)} \leq C_4\|\, Q\|_{L^p(\partial G)} \tag{25}$$

where C_4 is a constant depending only on G, R and p.

Since $u - c = \mathcal{D}g + \mathcal{S}_R Q$, we obtain from (22), (25), (24) and (23)

$$|u-c|_{L^\infty(G)} + \|u-c\|_{W^{1,2p}(G)} \le (C_1 + C_2C_3C_4)[\|f\|_{L^p(\partial G)} + \|g\|_{W_0^{1,p}(\gamma)}].$$

If $g \equiv 0$, then

$$\sup_{\substack{x,y\in G\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{(p-1)/p}} \le C_4C_3C_2\|f\|_{L^p(\partial G)}. \qquad \square$$

9. Weak solution.

Notation 9.1. Let $u \in W^{1,2}(G)$. Denote by u_+ the restriction of u onto H_+ and by u_- the restriction of u onto H_-. Since $u_+ \in W^{1,2}(H_+)$ there is the trace u_+ of u_+ almost everywhere on ∂H_+. Similarly, there is the trace u_- of u_- almost everywhere on ∂H_-. We can write u instead of u_+ or u_- on $\partial G \setminus \gamma$. If $u_+ = u_-$ on γ, we can write u instead of u_+ or u_- on γ.

Remark 9.2. If u is an L^p-solution of the problem (1)–(4) with $1 < p < \infty$, then $u \in W^{1,2}(G)$ by Corollary 8.8. Then u (u_+, u_-) denotes the nontangential limit of u with respect to G (H_+, H_-) on $\partial G \setminus \gamma$ (γ) and the trace of u with respect to G (H_+, H_-), respectively. But it is well known that if the trace and the nontangential limit exist then they are the equal.

Remark 9.3. If $u \in W^{1,2}(G)$, then $u_+ \in L^p(\partial H_+)$, $u_- \in L^p(\partial H_-)$, for each $p \in \langle 1, \infty)$, see [**22**, Théorème 4.6].

Definition 9.4. Let $f, h \in L^p(\partial G)$, $g \in L^p(\gamma)$, $1 < p < \infty$. We say that $u \in W^{1,2}(G)$ is a weak solution of the problem (1)–(4) if $u_+ - u_- = g$ almost everywhere in γ and

$$\int_G \nabla u \cdot \nabla \varphi \, d\mathcal{H}_2 + \int_{\partial G\setminus\gamma} hu\varphi \, d\mathcal{H}_1 + \int_\gamma hu_+\varphi \, d\mathcal{H}_1 = \int_{\partial G} f\varphi \, d\mathcal{H}_1 \tag{26}$$

for each $\varphi \in W^{1,2}(G)$ for which $\varphi_+ = \varphi_-$ almost everywhere in γ.

Lemma 9.5. *Let $h \in L^p(\partial G)$, $h \geq 0$, $1 < p < \infty$. If u is a weak solution of the problem* (1)–(4) *with $f \equiv 0$, $g \equiv 0$, then there is a constant c such that $u = c$ on* cl G. *If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then $c = 0$.*

Proof. Since $g \equiv 0$ we have $u_+ = u_-$ almost everywhere in γ. We get from (26) for $\varphi = u$

$$\int_G |\nabla u|^2 \, d\mathcal{H}_2 + \int_{\partial G} hu^2 \, d\mathcal{H}_1 = 0.$$

Since $h \geq 0$ we deduce that $\nabla u = 0$ in G. Therefore, u is constant in each component of G. Since $u_+ = u_-$ almost everywhere in γ and H is connected, there is a constant c such that $u = c$ in G and hence on cl G. Suppose now that $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$. Then,

$$0 = \left[\int_G |\nabla u|^2 \, d\mathcal{H}_2 + \int_{\partial G} hu^2 \, d\mathcal{H}_1\right] \left[\int_{\partial G} h \, d\mathcal{H}_1\right]^{-1} = c^2. \qquad \square$$

Lemma 9.6. *Let $f \in L^p(\partial G)$, $g \in L^p(\gamma)$, $1 < p < \infty$, $h \equiv 0$. If there is a weak solution of the problem* (1)–(4), *then $f \in L^p_0(\partial G)$.*

Proof. We get this proposition from (26) for $\varphi \equiv 1$. $\square$

Proposition 9.7. *Let $1 < p < \infty$, $f, h \in L^p(\partial G)$, $h \geq 0$, $g \in W_0^{1,p}(\gamma)$. If u is an L^p-solution of the problem* (1)–(4), *then u is a weak solution of the problem* (1)–(4). *Suppose, moreover, that $p < p_0$, where p_0 is the constant from Theorem* 8.7. *If u is a weak solution of the problem* (1)–(4), *then u is an L^p-solution of the problem* (1)–(4).

Proof. Suppose first that u is an L^p-solution of the problem (1)–(4). Proposition 4.2 gives that $u \in W^{1,2}(G)$. Let φ be an infinitely differentiable function in R^2. Let V_j, $j = 1, \dots,$ be sets from Lemma 4.1 for H_+. Using Green's formula and Lebesque's lemma, we get

$$\begin{aligned}\int_{H_+} \nabla\varphi \cdot \nabla u \, d\mathcal{H}_2 &= \lim_{j\to\infty} \int_{V_j} \nabla\varphi \cdot \nabla u \, d\mathcal{H}_2 \\ &= \lim_{j\to\infty} \int_{\partial V_j} \varphi n \cdot \nabla u \, d\mathcal{H}_1 = \int_{\partial H_+} \varphi n^+ \cdot [\nabla u]_+ \, d\mathcal{H}_1.\end{aligned}$$

Put $p' = p/(p-1)$. Since there is a bounded imbedding from $W^{1,2}(H_+)$ to $L^{p'}(\partial H_+)$, see [**22**, Theorem 4.6], and $n\cdot[\nabla u]_+ \in L^p(\partial H_+)$, Hölder's inequality yields that

$$\varphi \longmapsto \int_{H_+} \nabla\varphi\cdot\nabla u\, d\mathcal{H}_2, \qquad \varphi \longmapsto \int_{\partial H_+} [\varphi]_+ n^+ \cdot [\nabla u]_+\, d\mathcal{H}_1$$

are bounded linear functionals on $W^{1,2}(H_+)$. Since the space of all infinitely differentiable functions in R^2 is dense in $W^{1,2}(H_+)$, see [**1**, Theorem 3.18], we get from the continuity of these functionals that

$$\int_{H_+} \nabla\varphi\cdot\nabla u\, d\mathcal{H}_2 = \int_{\partial H_+} \varphi n^+ \cdot [\nabla u]_+\, d\mathcal{H}_1 \tag{27}$$

holds for each $\varphi \in W^{1,2}(H_+)$. Similarly,

$$\int_{H_-} \nabla\varphi\cdot\nabla u\, d\mathcal{H}_2 = \int_{\partial H_-} \varphi n^- \cdot [\nabla u]_-\, d\mathcal{H}_1 \tag{28}$$

for each $\varphi \in W^{1,2}(H_-)$. Let $\varphi \in W^{1,2}(G)$ for which $\varphi_+ = \varphi_-$ almost everywhere in γ. Using (27) and (28), we get (26).

Suppose now that $p < p_0$ and u is a weak solution of the problem (1)–(4). According to Lemma 9.6 and Theorem 8.7, there is an L^p-solution v of the problem (1)–(4). This solution is a weak solution of the problem (1)–(4). If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then $u = v$ by Lemma 9.5. Suppose now that $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) = 0$. According to Lemma 9.5 there is a constant c such that $u = v + c$. Since v is an L^p-solution of the problem (1)–(4), the function $u = v + c$ is an L^p-solution of the problem (1)–(4). □

Corollary 9.8. *Let* $2 < p_0 \le \infty$ *be the constant from Theorem* 8.7. *Let* $1 < p < p_0$, $f, h \in L^p(\partial G)$, $h \ge 0$, $g \in W_0^{1,p}(\gamma)$. *If* u *is a weak solution of the problem* (1)–(4), *then* $u \in W^{1,2p}(G)$, $u_+ \in C^{(p-1)/p}(\mathrm{cl}\ H_+)$, $u_- \in C^{(p-1)/p}(\mathrm{cl}\ H_-)$.

Proof. The corollary is a consequence of Proposition 9.7 and Corollary 8.8. □

10. Solution of the problem.

Lemma 10.1. *Let $1 < p < \infty$, $R >$ diam G, $f, h \in$ compl $L^p(\partial G)$, $\mathcal{H}_1(\{x \in \partial G;\ |f(x)| > 0\}) > 0$, $h \geq 0$. If λ is a complex number such that $T_{h,R} f = \lambda f$, then $\lambda \geq 0$.*

Proof. Take $f_1, f_2 \in L^p(\partial G)$ such that $f = f_1 + i f_2$. Let V_j be the sets from the Lemma 4.1 for H_+. Since $\mathcal{S}_R(f_1 - i f_2)$ is continuous in R^2 by Lemma 3.1 and there is the nontangential limit of $\nabla \mathcal{S}_R f$ with respect to H_+ almost everywhere in ∂H_+, see Section 7, we get using Fubini's theorem, Green's formula and Lebesgue's lemma

$$\begin{aligned}
\int_{\partial H_+} & [\mathcal{S}_R(f_1 - i f_2)] n^+ \cdot [\nabla \mathcal{S}_R(f_1 + i f_2)]_+ \\
&= \lim_{j \to \infty} \int_{\partial V_j} [\mathcal{S}_R(f_1 - i f_2)] n \cdot [\nabla \mathcal{S}_R(f_1 + i f_2)] \\
&= \lim_{j \to \infty} \int_{V_j} [|\nabla \mathcal{S}_R f_1|^2 + |\nabla \mathcal{S}_R f_2|^2] \, d\mathcal{H}_2 \\
&= \int_{H_+} [|\nabla \mathcal{S}_R f_1|^2 + |\nabla \mathcal{S}_R f_2|^2] \, d\mathcal{H}_2.
\end{aligned}$$

Similarly,

$$\int_{\partial H_-} [\mathcal{S}_R(f_1 - i f_2)] n^- \cdot [\nabla \mathcal{S}_R(f_1 + i f_2)]_- = \int_{H_-} [|\nabla \mathcal{S}_R f_1|^2 + |\nabla \mathcal{S}_R f_2|^2] \, d\mathcal{H}_2.$$

Hence,

$$\begin{aligned}
\lambda & \int_{\partial G} (f_1 \mathcal{S}_R f_1 + f_2 \mathcal{S}_R f_2) \, d\mathcal{H}_1 \\
&= \int_{\partial G} [\mathcal{S}_R(f_1 - i f_2)] T_{h,R}(f_1 + i f_2) \, d\mathcal{H}_1 \\
&= \int_{\partial H_+} [\mathcal{S}_R(f_1 - i f_2)] n^+ \cdot [\nabla \mathcal{S}_R(f_1 + i f_2)]_+ \\
&\quad + \int_{\partial H_-} [\mathcal{S}_R(f_1 - i f_2)] n^- \cdot [\nabla \mathcal{S}_R(f_1 + i f_2)]_-
\end{aligned}$$

$$+\int_{\partial G} h[(\mathcal{S}_R f_1)^2 + (\mathcal{S}_R f_2)^2]\, d\mathcal{H}_1$$
$$= \int_G [|\nabla \mathcal{S}_R f_1|^2 + |\nabla \mathcal{S}_R f_2|^2]\, d\mathcal{H}_1 + \int_{\partial G} h[(\mathcal{S}_R f_1)^2 + (\mathcal{S}_R f_2)^2]\, d\mathcal{H}_1.$$

Since
$$0 < \int_{\partial G} (f_1 \mathcal{S}_R f_1 + f_2 \mathcal{S}_R f_2)\, d\mathcal{H}_1 < \infty,$$
by Lemma 8.1 and $h \geq 0$ we get
$$\lambda = \frac{\int_G [|\nabla \mathcal{S}_R f_1|^2 + |\nabla \mathcal{S}_R f_2|^2]\, d\mathcal{H}_1 + \int_{\partial G} h[(\mathcal{S}_R f_1)^2 + (\mathcal{S}_R f_2)^2]\, d\mathcal{H}_1}{\int_{\partial G} (f_1 \mathcal{S}_R f_1 + f_2 \mathcal{S}_R f_2)\, d\mathcal{H}_1}$$
$$\geq 0. \qquad \square$$

Lemma 10.2. *Let* $1 < p < \infty$*,* $R >$ diam G*,* $f \in$ compl $L^p(\partial G)$*,* $\mathcal{H}_1(\{x \in \partial G;\ |f(x)| > 0\}) > 0$*. If* λ *is a complex number such that* $T_{0,R} f = \lambda f$*, then* $0 \leq \lambda \leq 1$*.*

Proof. We can suppose that $\lambda \neq 0$. Lemma 10.1 yields $\lambda > 0$ and we thus can suppose that $f \in L^p(\partial G)$. Since $\mathcal{S}_R \lambda^{-1} f$ is an L^p-solution of the problem (1)–(4) with $h \equiv 0$, $g \equiv 0$ (see Section 7), Proposition 4.3 gives (5). Since $T_{0,R} f = f$ on γ, we deduce from $T_{0,R} f = \lambda f$ that $\lambda = 1$ or $f = 0$ almost everywhere on γ. We can restrict ourselves to the case when $f = 0$ almost everywhere on γ.

Fix $r > 0$ such that $\partial G \subset \Omega_r(0)$, and put $V = \Omega_r(0) \setminus \text{cl}\, G$. Let V_j be the sets from Lemma 4.1. Then

$$\begin{aligned}\int_V |\nabla \mathcal{S}_R f|^2\, d\mathcal{H}_1 &= \lim_{j\to\infty} \int_{V_j} |\nabla \mathcal{S}_R f|^2\, d\mathcal{H}_1\\ &= \lim_{j\to\infty} \int_{\partial V_j} (\mathcal{S}_R f)(n \cdot \nabla \mathcal{S}_R f)\, d\mathcal{H}_1\\ &= \int_{\partial H} (\mathcal{S}_R f)\left(\frac{1}{2} I - K_H^*\right) f \delta\, \mathcal{H}_1 + \int_{\partial \Omega_r(0)} \frac{\partial \mathcal{S}_R f}{\partial n} \mathcal{S}_R f\, d\mathcal{H}_1\\ &= (1-\lambda) \int_{\partial G} f \mathcal{S}_R f\, d\mathcal{H}_1 + \int_{\partial \Omega_r(0)} \mathcal{S}_R f \,\frac{\partial \mathcal{S}_R f}{\partial n}\, d\mathcal{H}_1.\end{aligned}$$

Since (5) forces $\mathcal{S}_R f(x) = o(1)$, $\nabla \mathcal{S}_R f(x) = O(1/|x|)$ as $|x| \to \infty$, we get for $r \to \infty$,

$$\int_{R^2 \setminus \mathrm{cl}\, G} |\nabla \mathcal{S}_R f|^2 \, d\mathcal{H}_1 = (1-\lambda) \int_{\partial G} f \mathcal{S}_R f \, d\mathcal{H}_1.$$

Using Lemma 8.1, we get

$$(1-\lambda) = \frac{\int_{R^2 \setminus \mathrm{cl}\, G} |\nabla \mathcal{S}_R f|^2 \, d\mathcal{H}_1}{\int_{\partial G} f \mathcal{S}_R f \, d\mathcal{H}_1} \geq 0.$$

So, $\lambda \leq 1$. □

Notation 10.3. Let X be a complex Banach space, and let T be a bounded linear operator in X. Denote by $\sigma(T)$ the spectrum of T, $r(T) = \sup\{|\lambda|;\ \lambda \in \sigma(T)\}$ the spectral radius of T and $r_e(T) = \sup\{|\lambda|;\ \lambda I - T$ is not a Fredholm operator with index $0\}$ the essential spectral radius of T.

Lemma 10.4. *Let $1 < p < \infty$, $R > \mathrm{diam}\, G$, $h \in L^p(\partial G)$, $h \geq 0$. Put $V_h f = h \mathcal{S}_R f$ for $f \in \mathrm{compl}\, L^p(\partial G)$. Then*

$$r(V_h) \leq \sup_{x \in \partial G} S_R h(x). \tag{29}$$

Proof. If $f \in \mathrm{compl}\, L^1(\partial G)$, then Fubini's theorem yields

$$\int_{\partial G} |h \mathcal{S}_R f| \, d\mathcal{H}_1 \leq \int_{\partial G} |f| \mathcal{S}_R h \, d\mathcal{H}_1 \leq \int_{\partial G} |f| \, d\mathcal{H}_1 \cdot \sup_{x \in \partial G} \mathcal{S}_R h(x).$$

Hence, V_h is a bounded linear operator in $\mathrm{compl}\, L^1(\partial G)$ and

$$\|V_h\|_{\mathrm{compl}\, L^1(\partial G)} \leq \sup_{x \in \partial G} \mathcal{S}_R h(x).$$

According to [**26**, Chapter VIII, Section 2, Theorem 3] and [**26**, Chapter VIII, Section 2, Theorem 4],

$$r(V_h) \leq \|V_h\| \leq \sup_{x \in \partial G} S_R h(x) \tag{30}$$

in $\mathrm{compl}\, L^1(\partial G)$.

Let now $\lambda \in \sigma(V_h)$ in compl $L^p(\partial G)$, $\lambda \neq 0$. Since V_h is a compact linear operator in compl $L^p(\partial G)$, by Lemma 3.1 there is an $f \in$ compl $L^p(\partial G)$ such that $\mathcal{H}_1(\{x \in \partial G;\ |f(x)| > 0\}) > 0$ and $V_h f = \lambda f$, see [**26**, Chapter X, Section 5, Theorem 2]. Thus $\lambda \in \sigma(V_h)$ in compl $L^1(\partial G)$. According to (30),

$$|\lambda| \leq \sup_{x \in \partial G} S_R h(x).$$

This forces (29). $\square$

Lemma 10.5. *Let T be a bounded continuously invertible linear operator on the complex Banach space X. If α is a nonzero complex number such that $\sigma(T) \subset \{\beta \in C;\ |\beta - \alpha| < |\alpha|\}$, then there are constants $q \in (0,1)$, $M \in (1,\infty)$ such that*

$$\|(I - \alpha^{-1}T)^n\| \leq M q^n \tag{31}$$

for each nonnegative integer and

$$T^{-1} = \alpha^{-1} \sum_{n=0}^{\infty} (I - \alpha^{-1}T)^n. \tag{32}$$

Proof. $\sigma(\alpha^{-1}T - I) \subset \{\beta \in C;\ |\beta| < 1\}$ by the spectral mapping theorem, see [**26**, Chapter VIII, Section 7]. Since $\sigma(\alpha^{-1}T - I)$ is compact, see [**26**, Chapter VIII, Section 2, Theorem 1], we deduce $r(\alpha^{-1}T - I) < 1$. Fix $q \in (r(\alpha^{-1}T - I), 1)$. Since $r(\alpha^{-1}T - I) = \lim[\|(\alpha^{-1}T - I)^n\|]^{1/n}$ as $n \to \infty$, see [**26**, Chapter VIII, Section 2], there is a constant $M \in (1,\infty)$ such that (31) holds. So, the series (32) converges. Easy calculation yields

$$\begin{aligned} T\alpha^{-1} \sum_{n=0}^{\infty} (I - \alpha^{-1}T)^n = -(I - \alpha^{-1}T) &\sum_{n=0}^{\infty} (I - \alpha^{-1}T)^n \\ &+ \sum_{n=0}^{\infty} (I - \alpha^{-1}T)^n = I, \end{aligned}$$

and (32) holds. $\square$

Theorem 10.6. *Let* $1 < p < \infty$, $h \in L^p(\partial G)$, $h \geq 0$. *Suppose* $r_e(K_H^*) < (1/2)$ *in* compl $L^p(\partial H)$, $R >$ diam G. *If* $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, *put* $\tilde{T}_{h,R} = T_{h,R}$. *Put*

$$\alpha_0 \equiv 1 + \sup_{x \in \partial G} \mathcal{S}_R h(x). \tag{33}$$

If $\alpha > \alpha_0/2$, *there are constants* $q \in (0,1)$, $M \in (1,\infty)$ *dependent on* G, h, p *and* α *such that*

$$\|(I - \alpha^{-1}\tilde{T}_{h,R})^n\| \leq M q^n \tag{34}$$

for each nonnegative integer, the operator $\widetilde{T}_{h,R}$ *is continuously invertible and*

$$\widetilde{T}_{h,R}^{-1} = \alpha^{-1} \sum_{n=0}^{\infty} (I - \alpha^{-1}\widetilde{T}_{h,R})^n. \tag{35}$$

Proof. $r_e(T_{0,R} - (1/2)I) \leq 1/2$ in compl $L^p(\partial H)$ by Lemma 8.6. If $\lambda \in \sigma(T_{0,R} - (1/2)I)$, $|\lambda| > (1/2)$, then λ is an eigenvalue of $T_{0,R} - (1/2)I$. Since $|\lambda| \leq (1/2)$ for each eigenvalue of $T_{0,R} - (1/2)I$ by Lemma 10.2 we get $r(T_{0,R} - (1/2)I) \leq 1/2$ in compl $L^p(\partial G)$. Put $V_h f = h\mathcal{S}_R f$ for $f \in$ compl $L^p(\partial G)$. According to Lemma 10.4, [**26**, Chapter VIII, Section 2] and [**6**, Satz 45.1], we have $r(T_{h,R} - (1/2)I) \leq r(T_{0,R} - (1/2)I) + r(V_h) \leq \alpha_0 - (1/2)$ in compl $L^p(\partial G)$.

If $|\lambda| \geq (1/2)$, $\lambda \neq (1/2)$, then $\lambda I - (T_{h,R} - (1/2)I)$ is a Fredholm operator with index 0 by Lemma 8.6. If, moreover, $\lambda \in \sigma((T_{h,R} - (1/2)I))$, then λ is an eigenvalue of $(T_{h,R} - (1/2)I)$. Since $\lambda + (1/2)$ is an eigenvalue of the operator $T_{h,R}$, Lemma 10.1 yields $0 \leq \lambda + (1/2)$. Hence $\sigma((T_{h,R} - (1/2)I)) \subset \{\lambda \in C;\ |\lambda| < (1/2)\} \cup \{1/2\} \cup \langle -1/2, r((T_{h,R} - (1/2)I))\rangle \subset \{\lambda \in C;\ |\lambda| < (1/2)\} \cup \langle -1/2, \alpha_0 - (1/2)\rangle$. Using the spectral mapping theorem, see [**26**, Chapter VIII, Section 7], we get $\sigma(T_{h,R}) \subset \{\lambda \in C;\ |\lambda - (1/2)| < (1/2)\} \cup \langle 0, \alpha_0\rangle$.

Let $\lambda \in C \setminus \sigma(T_{h,R})$. If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, put $L^p_h(\partial G) = L^p(\partial G)$. Since compl $L^p_h(\partial G)$ is a $T_{h,R}$-invariant closed linear subspace of finite codimension in compl $L^p(\partial G)$ and $T_{h,R} - \lambda I$ is a Fredholm operator with index 0, the operator $\widetilde{T}_{h,R} - \lambda I$ is a Fredholm operator with index 0 in compl $L^p_h(\partial G)$ by [**18**, Proposition 3.7.1]. Since the kernel of the operator $\widetilde{T}_{h,R} - \lambda I$ is trivial the operator $(\widetilde{T}_{h,R} - \lambda I)$ is onto and the closed graph theorem, see [**26**, Chapter II, Section 6, Theorem 1], gives that $\widetilde{T}_{h,R} - \lambda I$ is continuously invertible. Thus, $\sigma(\widetilde{T}_{h,R}) \subset \sigma(T_{h,R}) \subset \{\lambda \in C;\ |\lambda - (1/2)| < 1/2\} \cup \langle 0, \alpha_0 \rangle$. Since $T_{h,R}$ is a Fredholm operator with index 0, Proposition 8.4 shows that $\widetilde{T}_{h,R}$ is continuously invertible. Therefore, $\sigma(\widetilde{T}_{h,R}) \subset \{\lambda \in C;\ |\lambda - (1/2)| < 1/2\} \cup (0, \alpha_0\rangle \subset \{\lambda \in C;\ |\lambda - \alpha| < \alpha\}$. The rest is a consequence of Lemma 10.5.

Lemma 10.7. *Let ∂H be formed by closed curves $\Gamma_1, \dots, \Gamma_k$. Let $1 < p < \infty$. For $f \in L^p(\Gamma_j)$, $y \in \Gamma_j$, $j = 1, \dots, k$ define*

$$K_j^* f(y) = \lim_{\varepsilon \to 0_+} \frac{1}{2\pi} \int_{\Gamma_j \setminus \Omega_\varepsilon(y)} f(x) \, \frac{n(y) \cdot (x - y)}{|x - y|^2} \, d\mathcal{H}_1(x). \tag{36}$$

Then K_j^* is a bounded linear operator in $L^p(\Gamma_j)$. Let λ be a complex number. Then $\lambda I - K_H^*$ is a Fredholm operator in compl $L^p(\partial H)$ *if and only if $\lambda I - K_j^*$ is a Fredholm operator in* compl $L^p(\Gamma_j)$ *for $j = 1, \dots, k$. Moreover, $i(\lambda I - K_H^*) = i(\lambda I - K_1^*) + \cdots + i(\lambda I - K_k^*)$.*

Proof. For $f \in L^p(\partial H)$, define

$$Lf(y) = K_j^* f(y) \quad \text{for} \quad y \in \Gamma_j.$$

Then L is a bounded operator in $L^p(\partial H)$. Since f is in the kernel of $\lambda I - L$ if and only if $f|\Gamma_j$ is in the kernel of $\lambda I - K_j^*$ for $j = 1, \dots, k$, and g is in the range of $(\lambda I - L)$ if and only if $g|\Gamma_j$ is in the range of $(\lambda I - K_j^*)$ for $j = 1, \dots, k$, we deduce that $\lambda I - L$ is a Fredholm operator in compl $L^p(\partial H)$ if and only if $\lambda I - K_j^*$ is a Fredholm operator in compl $L^p(\Gamma_j)$ for $j = 1, \dots, k$ and $i(\lambda I - L) = i(\lambda I - K_1^*) + \cdots + i(\lambda I - K_k^*)$.

Since $K_H^* - L$ is a compact linear operator from $L^p(\partial H)$ to $C(\partial H)$ by Arzelà-Ascoli's theorem and the imbedding $C(\partial H)$ into $L^p(\partial H)$ is a bounded linear operator, the operator $K_H^* - L$ is a compact linear operator in $L^p(\partial H)$, see [**26**, Chapter X, Section 2]. Since the operator $K_H^* - L$ is compact, the operator $\lambda I - K_H^*$ is a Fredholm operator in compl $L^p(\partial H)$ if and only if the operator $\lambda I - L$ is a Fredholm operator in compl $L^p(\partial H)$ and $i(\lambda I - K^*_H) = i(\lambda I - L)$, see [**24**, Chapter V, Theorem 3.1]. ◻

Lemma 10.8. *Let $1 < p < \infty$. If $(1/2)I + K^*_H$ is a Fredholm operator in* compl $L^p(\partial H)$*, then $i((1/2)I + K^*_H) \le 0$. Let ∂H be formed by closed curves $\Gamma_1, \dots, \Gamma_k$. Let K^*_j be the operators in $L^p(\Gamma_j)$ given by* (36)*. Then $(1/2)I + K^*_H$ is a Fredholm operator in* compl $L^p(\partial H)$ *with index* 0 *if and only if $(1/2)I + K^*_j$ is a Fredholm operator in* compl $L^p(\Gamma_j)$ *with index* 0 *for $j = 1, \dots, k$.*

Proof. We can suppose that $\gamma = \emptyset$. Fix $R > \operatorname{diam} H$. Let $(1/2)I + K^*_H$ be a Fredholm operator in compl $L^p(\partial H)$. Put $h \equiv 1$. Then $T_{h,R}$ is a Fredholm operator in compl $L^p(\partial H)$ and $i(T_{h,R}) = i((1/2)I + K^*)$ by Lemma 8.6. If $\varphi \in L^p(\partial H)$, $T_{h,R}\varphi = 0$, then $\mathcal{S}_R\varphi$ is an L^p-solution of the problem (1)–(4) with $f \equiv 0$. Thus, $\mathcal{S}_R\varphi = 0$ in H by Theorem 5.1. Since $\mathcal{S}_R\varphi$ is continuous on cl H by Lemma 3.1, we deduce $\mathcal{S}_R\varphi = 0$ on ∂H. Lemma 8.1 yields $\varphi = 0$ almost everywhere on ∂H. Thus, $\alpha(T_{h,R}) = 0$ and $i((1/2)I + K^*_H) = i(T_{h,R}) = -\beta(T_{h,R}) \le 0$.

If $(1/2)I + K^*_j$ is a Fredholm operator in compl $L^p(\Gamma_j)$ with index 0 for $j = 1, \dots, k$, then $(1/2)I + K^*_H$ is a Fredholm operator in compl $L^p(\partial H)$ with index 0 by Lemma 10.7. Suppose now that $(1/2)I + K^*_H$ is a Fredholm operator in compl $L^p(\partial H)$ with index 0. Then the operator $(1/2)I + K^*_j$ is a Fredholm operator in compl $L^p(\Gamma_j)$ for $j = 1, \dots, k$ by Lemma 10.7. Fix j. If H is a subset of the interior of Γ_j, then put V the interior of Γ_j. If H is a subset of the exterior of Γ_j, fix $R > 0$ such that $\Gamma_j \subset \Omega_R(0)$ and put V the bounded domain which boundary is formed by Γ_j and $\partial\Omega_R(0)$. In both cases, Lemma 10.7, [**3**, Theorem 1.9] and [**24**, Chapter IV, Theorem 2.2] yield that $(1/2)I + K^*_V$ is a Fredholm operator in compl $L^p(\partial V)$ and $i\big((1/2)I + K^*_V\big) = i((1/2)I + K^*_j)$. Thus, $i((1/2)I + K^*_j) = i\big((1/2)I + K^*_V\big) \le 0$. Since $0 = i((1/2)I + K^*_H) = i((1/2)I + K^*_1) + \cdots + i((1/2)I + K^*_k)$ by Lemma 10.7 and $i((1/2)I + K^*_j) \le 0$ for $j = 1, \dots, k$, we conclude that $i((1/2)I + K^*_j) = 0$ for $j = 1, \dots, k$. ◻

Definition 10.9. Let S be a rectifiable curve, and let s denote the arc length on S, $0 \le s \le l$. If the angle $\theta(s)$ made by the positively oriented tangent and the abscissa is a function of bounded variation on $\langle 0, l\rangle$, the curve S is said to be a curve with bounded rotation.

We remark that piecewise $C^{1+\alpha}$ bounded curves with $\alpha > 0$ and the boundary of a convex bounded set are curves with bounded rotation. On the other hand, there are C^1 bounded curves which are not curves with bounded rotation.

Notation 10.10. For $x \in R^2$, denote

$$d_G(x) = \lim_{r \to 0_+} \frac{\mathcal{H}_2(\Omega_r(x) \cap G)}{\mathcal{H}_2(\Omega_r(x))}$$

the density of G at x.

Proposition 10.11. *Let ∂H be formed by finitely many curves with bounded rotation. Let p_0 have the meaning of Theorem* 8.7. *Then*

$$p_0 = 1 + \left(\sup_{x \in \partial H} |1 - 2d_H(x)| \right)^{-1} \tag{37}$$

and $r_e(K_H^*) < (1/2)$ in compl *$L^p(\partial H)$ for each $1 < p < p_0$. (If $d_H(x) = (1/2)$ for all $x \in \partial H$, then $p_0 = \infty$.) If $p_0 \le p < \infty$, $h \in L^p(\partial G)$, $h \ge 0$, $R >$* diam *G then $T_{h,R}$ is not a Fredholm operator with index* 0 *in $L^p(\partial G)$.*

Proof. Let $\Gamma_1, \dots, \Gamma_k$ be curves with bounded rotation which form ∂H. Put

$$p_j = 1 + \left(\sup_{x \in \Gamma_j} |1 - 2d_H(x)| \right)^{-1}$$

for $j = 1, \dots, k$. Let K_j^* be given by (36) for $j = 1, \dots, k$. If $1 < p < \infty$, then $(1/2) + K_j^*$ is a Fredholm operator with index 0 in compl $L^p(\Gamma_j)$ if and only if $p < p_j$, see [**19**, Lemma 24]. Using Lemma 10.8 we get that $(1/2) + K_H^*$ is a Fredholm operator with index 0 in compl $L^p(\partial H)$ if and only if $p < \min\{p_j;\ j = 1, \dots, k\}$. According to Lemma 8.6, the operator $T_{h,R}$ is a Fredholm operator with index 0 in compl $L^p(\partial G)$ if and only if $p < \min\{p_j;\ j = 1, \dots, k\}$. Therefore, p_0 is given by (37). $\square$

We use the following result proved in [**19**, Lemma 27]:

Lemma 10.12. *Let $1 < p < \infty$. Suppose that, for each $x \in \partial H$ there are $r > 0$, an open set D with compact Lipschitz boundary such that $r_e(K_D^*) < (1/2)$ in* compl $L^p(\partial D)$, *and there is a coordinate system and Lipschitz functions Ψ_1, Ψ_2 defined in a neighborhood of 0 such that $\Psi_1(0) = \Psi_2(0)$, $\Psi_1 - \Psi_2$ is of class C^1, $(\Psi_1 - \Psi_2)'(0) = 0$ and $U \cap H = \{[t,s];\ |t| < r,\ s > \Psi_1(t)\}$, $U \cap D = \{[t,s];\ |t| < r,\ s > \Psi_2(t)\}$ for some neighborhood U of the point $x = [0,0]$. Then $r_e(K_H^*) < 1/2$ in* compl $L^p(\partial H)$.

Theorem 10.13. *Let H have piecewise C^1 boundary,*

$$1 < p < 1 + \left(\sup_{x \in \partial H} |1 - 2d_H(x)| \right)^{-1}. \tag{38}$$

Then $p < p_0$ (see Theorem 8.7) and $r_e(K_H^*) < 1/2$ in compl $L^p(\partial H)$. *Let $h \in L^p(\partial G)$, $h \geq 0$, $R >$* diam G. *Fix $\alpha > \alpha_0/2$ where α_0 is given by (33). Let $f \in L^p(\partial G)$, $g \in W_0^{1,p}(\gamma)$. If $h = 0$ almost everywhere in ∂G, suppose moreover (5). Let F be given by (11)–(13). Put*

$$\varphi = \alpha^{-1} \sum_{n=0}^{\infty} (I - \alpha^{-1} T_{h,R})^n F.$$

If $\mathcal{H}_1(\{x \in \partial G;\ h(x) > 0\}) > 0$, then $\mathcal{D}g + \mathcal{S}_R\varphi$ is the general form of an L^p solution of the problem (1)–(4). If $h = 0$ almost everywhere in ∂G, then the general form of an L^p solution of the problem (1)–(4) is $\mathcal{D}g + \mathcal{S}_R\varphi + c$, where c is an arbitrary constant.

Proof. Fix $x \in \partial H$. We can choose such a coordinate system, a Lipschitz function Ψ_1 and $r > 0$ that $x = [0,0]$, $\Psi_1 \in C^1((-r,0\rangle)$, $\Psi_1 \in C^1(\langle 0,r))$ and $U \cap H = \{[t,s];\ |t| < r,\ s > \Psi_1(t)\}$ for some neighborhood U of $[0,0]$. If $[0,0]$ is not an angle point of ∂H, then there is a bounded domain D with C^1 boundary such that $D \cap U = H \cap U$. Since K_D^* is a compact operator in compl $L^p(D)$, see [**3**, Theorem 1.9], [**24**, Chapter IV, Theorem 2.2], yields that $r_e(K_D^*) < 1/2$ in compl $L^p(\partial D)$.

Suppose now that $[0,0]$ is an angle point of ∂H. Define $\Psi_2(t) = t(\Psi_1)'_+(0)$ for $0 \le t < r$ and $\Psi_2(t) = t(\Psi_1)'_-(0)$ for $-r < t < 0$. We can choose a bounded domain D with connected Lipschitz boundary such that $D \cap U = \{[t,s];\ |t| < r, s > \Psi_2(t)\} \cap U$, and for each $y \in \partial D \setminus [0,0]\}$ there is a coordinate system, a neighborhood V of y, $\delta > 0$ and a function $\varphi \in C^\infty(-\delta,\delta)$ such that $D \cap V = \{[t,s];\ |t| < \delta,\ s > \varphi(t)\} \cap V$. Clearly, $\Psi_1(0) = \Psi_2(0)$, $\Psi_1 - \Psi_2$ is of class C^1 and $(\Psi_1 - \Psi_2)'(0) = 0$. Since ∂D is a curve with bounded rotation and

$$p < 1 + \left(\sup_{x \in \partial H} |1 - 2d_H(x)|\right)^{-1} \le 1 + \left(\sup_{x \in \partial D} |1 - 2d_D(x)|\right)^{-1}$$

[**19**, Lemma 24] yields that $r_e(K_D^*) < 1/2$ in compl $L^p(\partial D)$.

According to Lemma 10.12 we have $r_e(K_H^*) < 1/2$ in compl $L^p(\partial H)$. Now we use Theorem 8.7 and Theorem 10.6. □

11. Successive approximation method. Let $1 < p < \infty$ be such that $r_e(K_H^*) < 1/2$ in compl $L^p(\partial H)$. (This is fulfilled if H has piecewise C^1 boundary and (38) holds.) Let $f, h \in L^p(\partial G)$, $h \ge 0$, $g \in W_0^{1,p}(\gamma)$ be such that an L^p-solution of the problem (1)–(4) exists. Let F be given by (11)–(13). If $h = 0$ almost everywhere, then $\int F \, d\mathcal{H}_1 = 0$. Fix $R > \operatorname{diam} G$. If $\varphi \in L^p(\partial G)$ is a solution of the equation $T_{h,R}\varphi = F$, then $\mathcal{D}g + \mathcal{S}_R\varphi$ is an L^p-solution of the problem (1)–(4). We construct φ by the successive approximation method.

Fix $\alpha > \alpha_0$ where α_0 is given by (33). (If $h \equiv 0$ we can take $\alpha = 1$.) We can rewrite the equation $T_{h,R}\varphi = F$ as $\varphi = (I - \alpha^{-1}T_{h,R})\varphi + \alpha^{-1}F$. Put

$$\varphi_0 = \alpha^{-1}F,$$
$$\varphi_{n+1} = (I - \alpha^{-1}T_{h,R})\varphi_n + \alpha^{-1}F$$

for nonnegative integers n. Then

$$\varphi_{n+1} = \alpha^{-1} \sum_{k=0}^{n} (I - \alpha^{-1}T_{h,R})^k F.$$

According to Theorem 10.6, there is the limit φ of the sequence φ_n in $L^p(\partial G)$ and $T_{h,R}\varphi = F$. Since

$$\varphi = \alpha^{-1}\sum_{k=0}^{\infty}(I-\alpha^{-1}T_{h,R})^k F,$$
$$\varphi - \varphi_n = \alpha^{-1}\sum_{k=n+1}^{\infty}(I-\alpha^{-1}T_{h,R})^k F$$

there are constants $q \in (0,1)$, $C \in (1,\infty)$, depending only on G, p, h, R and α such that

$$\|\varphi - \varphi_m\|_{L^p(\partial G)} \le Cq^m\Big[\|f\|_{L^p(\partial G)} + \|g\|_{W_0^{1,p}(\gamma)}\Big],$$
$$\|\varphi\|_{L^p(\partial G)} \le C\Big[\|f\|_{L^p(\partial G)} + \|g\|_{W_0^{1,p}(\gamma)}\Big],$$

see Theorem 10.6.

Put $u = \mathcal{D}g + \mathcal{S}_R\varphi$, $u_n = \mathcal{D}g + \mathcal{S}_R\varphi_n$. Then u is an L^p-solution of the problem (1)–(4) and there is a constant $\widetilde{C}$ depending only on G, p, h, R and α such that

$$|u-u_m|_{L^\infty(G)} + \|u-u_m\|_{W^{1,2}(G)} \le \tilde{C}q^m\Big[\|f\|_{L^p(\partial G)} + \|g\|_{W_0^{1,p}(\gamma)}\Big],$$
$$|u|_{L^\infty(G)} + \|u\|_{W^{1,2}(G)} \le \tilde{C}\Big[\|f\|_{L^p(\partial G)} + \|g\|_{W_0^{1,p}(\gamma)}\Big],$$

see Lemma 3.1 and Lemma 6.1.

Example 11.1. Suppose that the boundary of G is formed by segments $C_1,\dots,C_k$ of the lengths $l_1,\dots,l_k$, and h is bounded. The calculation of p_0 using (37) is easy. For solving the problem (1)–(4) by the method described above, we need an estimation of α_0. Denote by x_j the center of C_j for $j=1,\dots,k$. If $x \in \partial G$, then

$$\mathcal{S}_R h(x) \le \|h\|_{L^\infty(\partial G)}\sum_{j=1}^{k}\frac{1}{2\pi}\int_{L_j}\ln(R/|x_j-y|)\,d\mathcal{H}_1(y)$$

and thus

$$\alpha_0 \le 1 + \|h\|_{L^\infty(\partial G)}\frac{1}{2\pi}\sum_{j=1}^{k} l_j[1-\ln(l_j/2R)].$$

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Mathematical Institute of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic
E-mail address: medkova@math.cas.cz