

ON THE ESSENTIAL NORM OF THE DOUBLE
LAYER POTENTIAL ON POLYHEDRAL
DOMAINS AND THE STABILITY
OF THE COLLOCATION METHOD

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ABSTRACT. In this article we study the double layer operator on a polyhedral domain D in \mathbf{R}^3 . We construct weighted norms for a class of nonconvex cones, such that the double layer operator has a norm smaller than $1/2$ with respect to this weighted norm. Applied to the rectangular example of Král and Wendland, our method reproduces their weight function. For certain domains we are able to extend this local weighted norm to the whole boundary S of D in such a way that the double layer operator has an essential norm smaller than $1/2$ on S . This allows us to prove stability for the collocation method with piecewise polynomial trial functions of arbitrary degree if we introduce some modification near the vertices and if the grid respects the weighted norm in some given sense.

1. Introduction. In this paper we consider polyhedral domains $D \subset \mathbf{R}^3$ with boundary $S = \partial D$. In the space $\mathcal{C}(S)$ we study the equation

$$(1.1) \quad \left(\frac{1}{2}I - \overline{W}\right)u = f, \quad f \in \mathcal{C}(S),$$

where the double layer operator \overline{W} is defined by

$$(1.2) \quad (\overline{W}u)(x) = Wu(x) + \left(\frac{1}{2} - d(x)\right)u(x), \quad x \in S,$$

with

$$(1.3) \quad d(x) := \lim_{r \searrow 0} \frac{\text{volume}(B_r(x) \cap D)}{\text{volume}(B_r(x))}$$

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and

$$(1.4) \quad (Wu)(x) := \frac{1}{4\pi} \int_S \frac{n(y) \cdot (y-x)}{\|x-y\|^3} u(y) dy,$$

see [1], [11], [12], [13]. For the following we denote the kernel of the above integral operator by k

$$(1.5) \quad k(x, y) := \frac{1}{4\pi} \frac{n(y) \cdot (y-x)}{\|x-y\|^3}, \quad y \in S, \quad x \in \mathbf{R}^3 \setminus \{y\}.$$

The kernel k is well defined if y does not lie on an edge. If y lies on an edge, we define $k(x, y) = 0$, $x \in S$. Equation (1.1) is well studied, and it is used, for example, in applications for the solution of the Dirichlet problem, see [2], [4],

$$(1.6) \quad \begin{cases} \Delta v(x) = 0 & x \in D, \\ v(x) = f(x) & x \in S. \end{cases}$$

Despite the fact that (1.1) has been studied for a long time, some properties of the operator \overline{W} are still not clear for the case of polyhedral domains. Rathsfeld proved in his article [14, Theorem 0.1, Corollary 0.2] that the spectral radius in the space of continuous functions is smaller than 1/2 on polyhedral domains, and he proved the invertibility of equation (1.1). Further results on the invertibility can be found in [9], [10]. Here we will prove that the essential norm, see (1.7), of the double layer operator is smaller than 1/2 on certain polyhedral domains, see Theorem 3.1 for a precise description. This result is not included in the above mentioned articles, and this result is particularly useful for the proof of the stability of the collocation method, see Section 4.

One of the important results in the above-cited papers which is important for second kind integral equations like (1.1) is the Fredholm property of these operators. In the case of nonsmooth boundaries S , the operator \overline{W} is not compact, and so it is not clear if the Fredholm alternative holds for equation (1.1). A sufficient condition would be

(1.7)

$$(1.8) \quad \begin{aligned} \|\overline{W}\|_{\mathcal{C}(S), \text{ess}} &:= \inf \{ \|\overline{W} - T\|_{\mathcal{C}(S)} \mid T \in \mathcal{L}(\mathcal{C}(S), \mathcal{C}(S)), T \text{ compact} \}, \\ &< \frac{1}{2} \end{aligned}$$

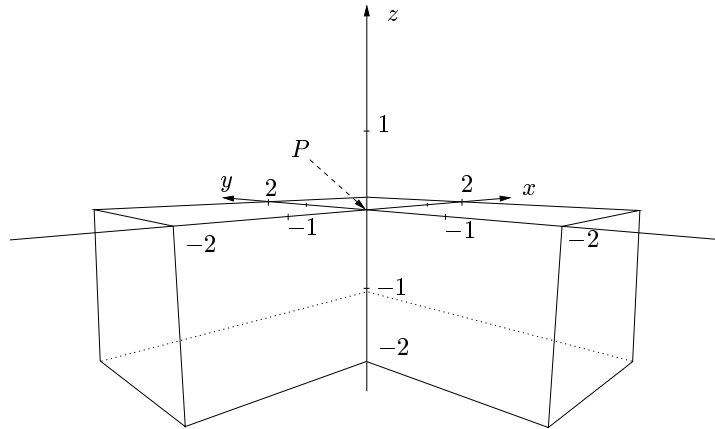


FIGURE 1. An example of a domain D with $\|\overline{W}\|_{\mathcal{C}(S_1), ess} > 0.5$.

But Král and Wendland [12] have shown that, even in the case of the simple domain D_1 , with boundary S_1 (see Figure 1), the condition (1.8) is not fulfilled.

To study this it is sufficient to consider the operator \overline{W} only locally around P . For this localized consideration we introduce the infinite cone K_P

$$(1.9) \quad K_P := \{P + \lambda(x - P) \mid x \in B_\varepsilon(P) \cap S, \lambda \geq 0\},$$

$\varepsilon > 0$ sufficiently small. We will always assume that $P = 0$ holds and define the spherical polygon γ_P by

$$(1.10) \quad \gamma_P := K_P \cap B_1(0).$$

To substitute for the lack of (1.7)–(1.8), Král and Wendland introduced a weighted norm on $\mathcal{C}(S_1)$ which is compatible with the usual supremum norm. If $w \in L^\infty(S)$ denotes a function with $0 < c < w(x) < C$, $x \in S$, almost everywhere, the norm

$$(1.11) \quad \|u\|_{\mathcal{C}(S), w} := \sup \left\{ \frac{|u(x)|}{w(x)} \mid x \in S \right\}$$

is equivalent to $\|\cdot\|_{\mathcal{C}(S)}$, and the set of compact operators does not change.

How does the weight function of Král and Wendland look near the vertex P , respectively on the cone K_P ? In their situation we have the picture given in Figure 2. The spherical polygon $\gamma = \cup_{j=1}^5 \gamma_j$ is partitioned according to the above picture and the cone K_P is partitioned in a similar way

$$(1.12) \quad K_P = \bigcup_{j=1}^5 K_j$$

$$(1.13) \quad = \bigcup_{j=1}^5 \{\lambda x \mid \lambda \geq 0, x \in \gamma_j\}.$$

The weight function of Král and Wendland is constant on every K_j ,

$$(1.14) \quad w(x) = \begin{cases} 1 & x \in K_4 \cup K_5, \\ \approx 0.783612 & x \in K_1 \cup K_3, \\ \approx 0.261204, & x \in K_2. \end{cases}$$

This weight function is found by some combinatoric estimates and the solution of a quadratic equation. The result is

$$(1.15) \quad \|\overline{W}\|_{\mathcal{C}(K_P),w} \lesssim 0.478553.$$

This result proves (1.8) and can also be used as a starting point for the stability analysis of the collocation method. This connection to the stability of the collocation method makes the study of the essential norm of the operator \overline{W} especially interesting.

In Section 2 we use the above ideas as a starting point for finding suitable weights for a larger class of spherical cones. Given a cone K , the generating spherical polygon γ_P and a partition $\mathcal{P}(\gamma_P) = \{\gamma_1, \dots, \gamma_n\}$ of γ_P , we will construct a matrix $M = M(\mathcal{P}(\gamma_P)) \in \mathbf{R}^{n,n}$. The maximal eigenvalue α is always positive, and if this eigenvalue is smaller than $1/2$, we can construct a weight function w such that

$$\|\overline{W}\|_{\mathcal{C}(K_P),w} \leq \alpha.$$

We are also able to show that this weight function is unique for this partition in a particular sense. This result already gives the invertibility

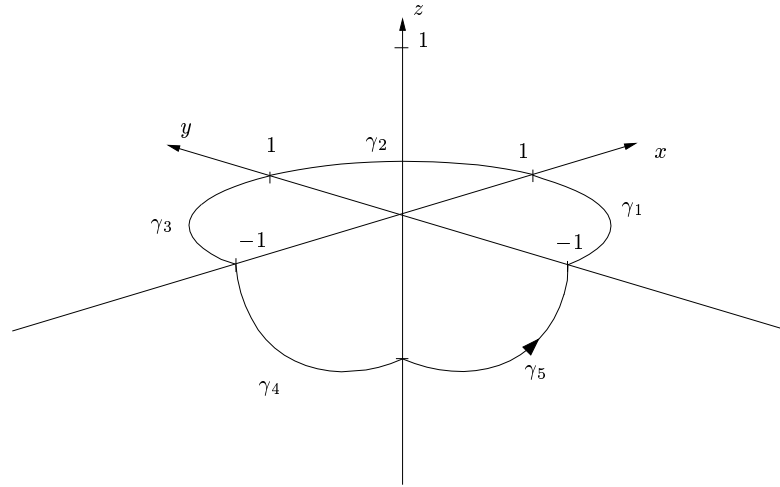


FIGURE 2. The spherical polygon γ and its partition.

of the operator $1/2I - \overline{W}$ on suitable cones, see Definition 2.3. But if we want to estimate the essential norm of the double layer operator on S we have to glue together the local weighted norms. The corresponding sufficient condition (2.18) is given in Definition 2.3.

The above condition enables us in Section 3 to construct a weighted norm on S such that the essential norm of the double layer operator \overline{W} on S is smaller than $1/2$ with respect to this norm. The splitting of the operator $\overline{W} = \overline{W}_\varepsilon + \overline{K}_\varepsilon$, where \overline{W}_ε has a norm smaller than $1/2$ and a compact operator \overline{K}_ε , is so explicit that we can use this splitting in Section 4 to prove the stability of the collocation method. Here we allow an arbitrary degree for the trial functions, but we must use some i^* modification, see [5], near the vertices (it will be called ε^* modification) to guarantee the stability. The triangulations we consider have to respect the weighted norm in some sense, see (4.13), but otherwise they can be chosen arbitrarily.

2. On polyhedral cones and their corresponding matrices.

In this section we consider an arbitrary infinite polygonal cone K , the spherical polygon $\gamma = K \cap B_1(0)$ and a given partition $\mathcal{P}(\gamma) = \{\gamma_1, \dots, \gamma_n\}$ of γ , where every γ_j , $j = 1(1)n$ should lie in a plane.

Connected to the partition of γ is a partition $K = \cup_{j=1}^n K_j$, according to equation (1.13). We are looking for a weight function $w : K \rightarrow \mathbf{R}$,

$$(2.1) \quad w(x) = \omega_j > 0, \quad x \in K_j,$$

such that

$$(2.2) \quad \|\overline{W}\|_{C(S),w} < \frac{1}{2}.$$

First we consider a sector $\Delta(\varphi_0) \subset \mathbf{R}^3$, $\varphi_0 \in (0, \pi)$, defined by

$$(2.3) \quad \Delta(\varphi_0) = \left\{ r \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} : |\varphi| \leq \varphi_0, r \geq 0 \right\}.$$

For $x \in \mathbf{R}^n \setminus \Delta(\varphi_0)$ we define

$$(2.4) \quad W_{E,\varphi_0}(x) := \int_{\Delta(\varphi_0)} |k(x,y)| dy, \quad x \in \mathbf{R}^3 \setminus \Delta(\varphi_0),$$

the integral of the absolute value of the kernel of the double-layer potential. A scaling argument gives

$$(2.5) \quad W_{E,\varphi_0}(x) = W_{E,\varphi_0}\left(\frac{x}{\|x\|}\right),$$

and therefore it is enough to consider

$$(2.6) \quad W_{E,\varphi_0} : S^2 \setminus \Delta(\varphi_0) \longrightarrow \mathbf{R}.$$

Here we recall the fact that W_{E,φ_0} is nothing other than the integral of the absolute value of the double layer potential with constant layer function for the sector $\Delta(\varphi_0)$. This also implies that $W_{E,\varphi_0}(x)$ is equal to the normalized solid angle of $\Delta(\varphi_0)$ seen from x . So it is natural to extend W_{E,φ_0} to S^2 by

$$(2.7) \quad W_{E,\varphi_0}(x) := \frac{1}{2}, \quad x \in S^2 \cap \Delta(\varphi_0),$$

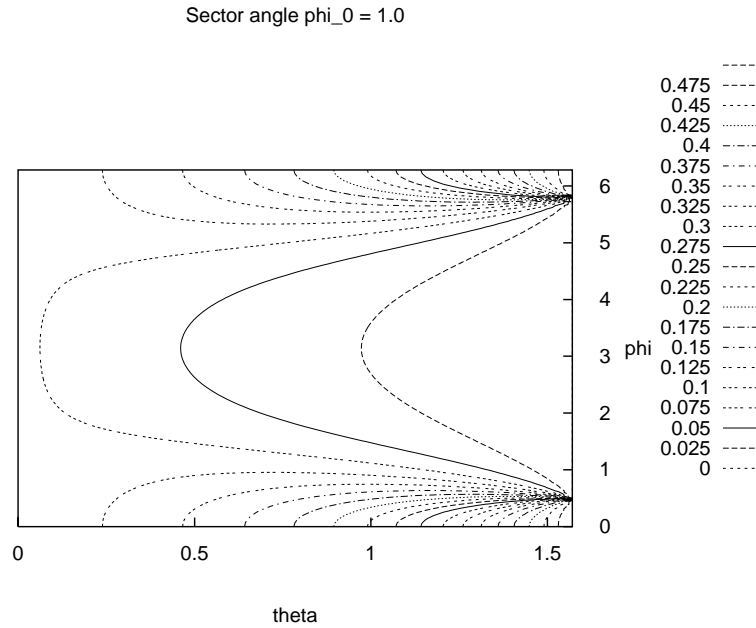


FIGURE 3. The contour of $W_{E,1.0}$ on $S^+ \cong [0, \pi/2] \times [0, 2\pi]$.

because the (normalized) solid angle approaches $1/2$ if one comes closer and closer to the sector $\Delta(\varphi_0)$, see also [13]. Then W_{E,φ_0} is continuous (jump relation of the double layer potential) in

$$S^2 \setminus \{(\cos(\varphi_0), -\sin(\varphi_0), 0)^T, (\cos(\varphi_0), \sin(\varphi_0), 0)^T\}.$$

The value of W_{E,φ_0} is equal to

$$(2.8) \quad W_{E,\varphi_0}(\theta, \varphi) = \frac{1}{4\pi} \int_{-\varphi_0/2}^{\varphi_0/2} \frac{|\cos(\theta)|}{1 - \sin(\theta) \cos(\tau - \varphi)} d\tau,$$

which follows by an easy calculation. Here we used the coordinates $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$ for S^2 given by $(\theta, \varphi) \rightarrow (\sin(\theta) \cos(\varphi), \sin(\theta) \sin(\varphi), \cos(\theta))^T$. Using these coordinates we get, for example, Figures 3 and 4 of W_{E,φ_0} . The contour lines are for constant $W_{E,\varphi_0}(\theta, \varphi)$.

For every $j \in \{1, \dots, n\}$ we denote by R_j a rotation which maps K_j bijectively onto $\Delta(\varphi_j)$, $\varphi_j \in (0, \pi)$; here we recall the fact that the

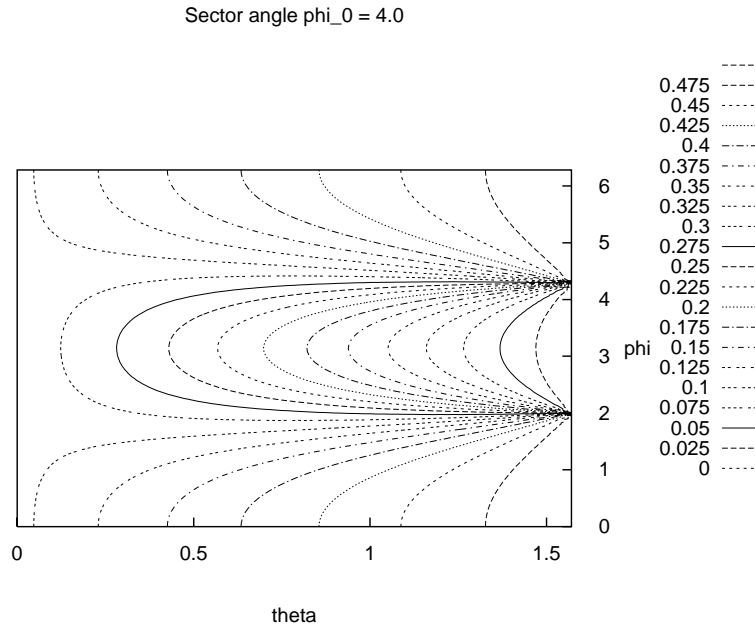


FIGURE 4. The contour of $W_{E,4.0}$ on $S^+ \cong [0, \pi/2] \times [0, 2\pi]$.

vertex of K is at zero. The matrix $M = M(\mathcal{P}(\gamma)) := (m_{i,j})_{i,j=1(1)n}$ is defined by

$$(2.9) \quad m_{i,j} := \begin{cases} \sup\{W_{E,\varphi_j}(x) \mid x \in R_j K_i \setminus \Delta(\varphi_j)\}, & i \neq j \\ 0, & \text{otherwise.} \end{cases}$$

All entries of M (if possible we suppress the dependence on the partition) outside the diagonal are greater than zero, and the theorem of Perron and Frobenius proves the existence of a one-dimensional subspace $\langle \vec{\omega} \rangle$ in \mathbf{R}^n , such that

$$(2.10) \quad \begin{cases} \vec{\omega} = (\omega_1, \dots, \omega_n), \omega_i > 0, \\ \text{and} \\ M\vec{\omega} = \rho(M)\vec{\omega}, \rho(\omega) > 0. \end{cases}$$

In the following we will always assume that the weight function w is defined by (2.1), $\omega_j, j = 1(1)n$, given by (2.9), (2.10) and $\max\{\omega_j \mid j = 1(1)n\} = 1$.

Lemma 2.1. *Given a cone K with a partition $K = \cup_{j=1}^n K_j$ corresponding to the partition $\mathcal{P}(\gamma) = \{\gamma_1, \dots, \gamma_n\}$ of the spherical polygon γ , we get*

$$(2.11) \quad \|\overline{W}\|_{\mathcal{C}(K),w} \leq \rho(M(\mathcal{P}(\gamma))),$$

where the weight function w on K is given by (2.1), (2.9) and (2.10).

Proof. Let $u \in L^\infty(K)$, $\|u\|_{L^\infty(K),w} \leq 1$ and $x \in K_i$. We can assume $x \notin K_j$, $i \neq j$. Then we have

$$\begin{aligned} |(\overline{W}u)(x)| &\leq \sum_{j=1, i \neq j}^n \int_{K_j} |k(x, y)| |u(y)| dy \\ &\leq \sum_{j=1, i \neq j}^n \omega_j \int_{K_j} |k(x, y)| dy \\ &= \sum_{j=1, i \neq j}^n \omega_j \int_{R_j K_j} |k(R_j x, y)| dy \\ &= \sum_{j=1, i \neq j}^n \omega_j \int_{\Delta(\varphi_j)} |k(R_j x, y)| dy \\ &\stackrel{(2.9)}{\leq} \sum_{j=1, i \neq j}^n \omega_j m_{i,j} \\ &\stackrel{(2.10)}{=} \rho(M)\omega_i. \end{aligned}$$

Note, $k(x, y) = 0$ for $x, y \in K_i$. \square

Remark. After reading the proof of Lemma 2.1, one immediately asks if there are other vectors $\vec{\nu} = (\nu_1, \dots, \nu_n)^T$, $\nu_i > 0$, $i = 1(1)n$, such that

$$M\vec{\nu} \leq \alpha\vec{\nu}, \quad \alpha \in (0, \rho(M)).$$

This would imply that we can reduce the essential norm of \overline{W} by choosing the ν_i instead of the ω_i as our weight function. But here we have the result of Woodbury [15, Theorem 1] which states that if M is positive, $\vec{\nu} \geq 0$, $\lambda \in \mathbf{R}$ and

$$M\vec{\nu} \leq \lambda\vec{\nu},$$

then we get $\lambda \geq \rho(M)$. This shows that the choice of the eigenvector $\vec{\omega}$ is optimal in this sense.

First we take a look at the example of Král and Wendland in our context. We denote by K_{KW} the corresponding cone. The partition of γ_{KW} is then given by

$$\gamma_{KW} = \{\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3\}, \quad \tilde{\gamma}_1 := \gamma_1 \cup \gamma_3, \quad \tilde{\gamma}_2 := \gamma_2, \quad \tilde{\gamma}_3 := \gamma_4 \cup \gamma_5,$$

see Figure 2. We remark here that this partition does not fit into our general scheme (2.9), but it is easy to extend our scheme to this special partition, but now we get a nonzero diagonal element, because $\tilde{\gamma}_3$ does not lie in a plane. The matrix M_{KW} is given by

$$M_{KW} = \begin{pmatrix} 0 & 0 & 0.375 \\ 0 & 0 & 0.125 \\ 0.250 & 0.125 & 0.250 \end{pmatrix},$$

the spectral radius is given by $\rho(M_{KW}) \approx 0.478553$ and the corresponding eigenvector $\vec{\omega}_{KW}$ corresponds exactly to the weights in equation (1.14). So our method works after a slight extension to nonplane partitions. But the question is if one can find suitable partitions for a larger class of cones. We propose the following.

Given a cone K with a starting partition of plane sectors K_j , $j = 1(1)n$, we define a partition $\mathcal{P}^N(\gamma)$ by dividing each γ_j , $j = 1(1)n$, into N equal plane sectors $\gamma_{j,l}^N$, $l = 1(1)N$. Correspondingly we have $K = \cup_{j=1}^n \cup_{l=1}^N K_{j,l}^N$. We will also denote the corresponding matrix with an upper index N .

In addition to the Král-Wendland example with 5 starting partitions (see Figure 2), we consider the nonconvex cone K_A with a spherical polygon γ_A defined by the corners $(0^\circ, 0^\circ)$, $(90^\circ, 0^\circ)$, $(80^\circ, 45^\circ)$ and $(90^\circ, 90^\circ)$ on the unit sphere. The spherical polygon evidently has a partition with 4 plane sections. Now we can calculate with a program the matrices M_{KW}^N , respectively M_A^N , for a sequence of N values, and as soon as we get $\rho(M_{KW}^N) < 0.5$, respectively $\rho(M_A^N) < 0.5$, we can stop; then we know that there exists a suitable weight function such that (2.2) is fulfilled. We get the results shown in Figure 5.

These results show that by our very simple strategy we find that both cones have a weight function for which the essential norm of \overline{W}

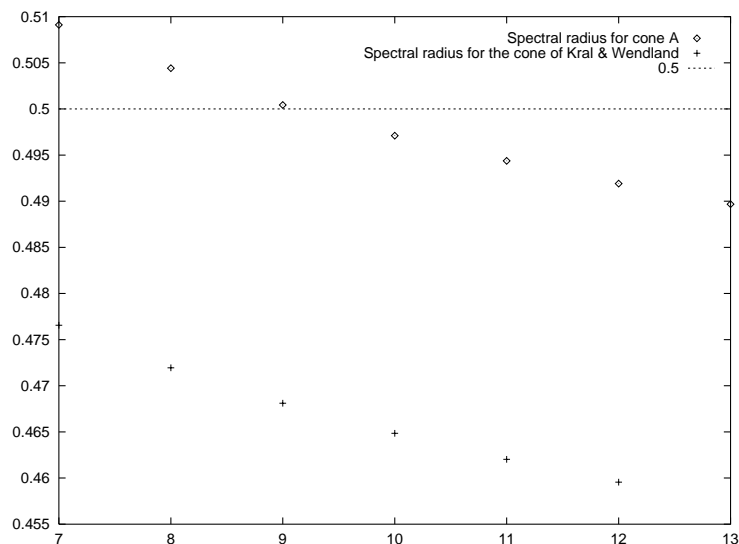


FIGURE 5. The spectral radii for different values of N and the two cones K_{KW} and K_A .

is smaller than 0.5. In their paper [12] Král and Wendland consider a further rectangular cone; here also our method works and we find suitable weights. The cone K_A is only ‘a little bit’ nonconvex. What happens if one considers more nonconvex cones? We denote by K_B the cone whose spherical polygon is given by the corners $(0^\circ, 0^\circ)$, $(90^\circ, 0^\circ)$, $(45^\circ, 45^\circ)$ and $(90^\circ, 90^\circ)$ on the unit sphere; see Figure 6. Here we find no suitable weight function with fewer than 100 partitions of each of the faces of K_B , indicated in Figure 7.

The above considerations naturally lead to the question of whether a function $w \in L^\infty(\gamma)$, $0 < c \leq w(x) \leq C$, $x \in \gamma$, exists almost everywhere, and a $\lambda \in (0, 1/2)$, such that

$$(2.12) \quad \lambda w(x) \leq \int_K |k(x, y)| w(y) dy, \quad x \in \gamma.$$

We remark that w actually only depends on the norm of x . Then the operator norm of \overline{W} with respect to $\|\cdot\|_{C(K), w}$ would be smaller than

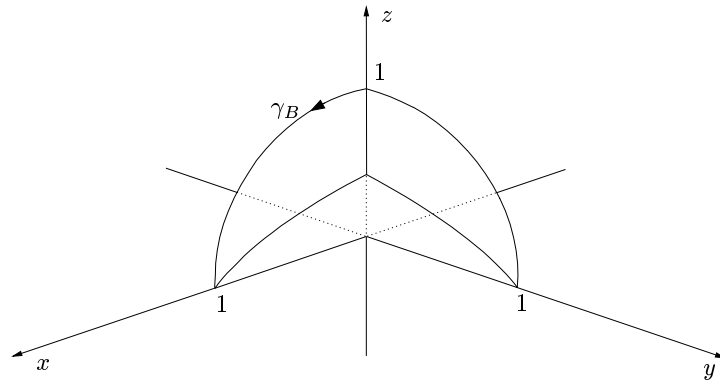


FIGURE 6. The cone K_B .

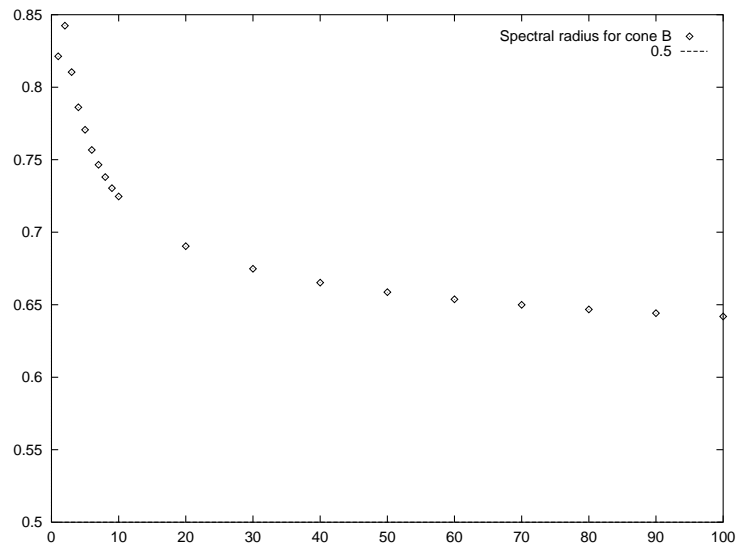


FIGURE 7. The spectral radii for M_B^N , with different N values below 100.

1/2. Now we take a look at the operator on the righthand side of equation (2.12). We introduce the following notations $x = rq$, $q \in \gamma$, $r > 0$.

$$\begin{aligned}
 (2.13) \quad & \int_K |k(x, y)|w(y) dy \\
 &= \frac{1}{4\pi} \int_\gamma \int_0^\infty \frac{|n(p) \cdot (rq - sp)|}{\|rq - sp\|^3} s w(p) ds dp \\
 &= \frac{1}{4\pi} \int_\gamma w(p) \int_0^\infty \frac{|n(p) \cdot (rq - sp)|s}{(r^2 - 2rs(p, q) + s^2)^{3/2}} ds dp \\
 &= \frac{1}{4\pi} \int_\gamma \frac{w(p)}{r} \int_0^\infty \frac{|n(p) \cdot (q - (s/r)p)|}{(1 - 2(s/r)(p, q) + (s/r)^2)^{3/2}} \frac{s}{r} ds dp \\
 &= \frac{1}{4\pi} \int_\gamma w(p) \int_0^\infty \frac{|n(p) \cdot (q - \tau p)|}{(1 - 2\tau(p, q) + \tau^2)^{3/2}} \tau d\tau dp \\
 &\stackrel{n(p) \cdot p=0}{=} \int_\gamma w(p) |n(p) \cdot q| \int_0^\infty \frac{\tau}{(1 - 2\tau(p, q) + \tau^2)^{3/2}} d\tau dq \\
 &= \frac{1}{4\pi} \int_\gamma \frac{|n(p) \cdot q|}{1 - (p, q)} w(p) dp \\
 &=: (Vw)(q).
 \end{aligned}$$

With this integral operator with positive kernel V on γ we can reformulate (2.12) in the following way. Does a positive function $w \in L^\infty(\gamma)$ and a constant $\lambda \in (0, 1/2)$ exist such that

$$(2.14) \quad (Vw)(x) \leq \lambda w(x), \quad x \in \gamma.$$

The theorem of Woodbury suggests that $\lambda < \rho(V)$ is impossible, but this is not clear. If V would be a compact operator the theorem of Krein and Rutman would imply the existence of a positive function w such that

$$(2.15) \quad Vw = \rho(V)w,$$

but the next lemma shows that the operator is not compact.

Lemma 2.2. *Let $\gamma \subset S^2$ be a spherical polygon and V be defined by (2.13). The operator V is locally around each vertex ξ of γ , isomorphic*

to a Mellin convolution operator L_α on a finite interval with kernel

$$(2.16) \quad l_\alpha(x) = \frac{|\sin(\alpha)|}{2\pi} \frac{x}{1 + 2 \cos(\alpha)x + x^2},$$

where $\alpha \in (-\pi, \pi)$ is the angle of γ at ξ .

Proof. Let $\xi \in \gamma$ be a vertex of γ . We can assume that $\xi = (0, 0, 1)^T$ and that we have a parametrization β of γ such that $\beta|_{[-\delta_0, \delta_0]}$, $\delta_0 > 0$, is given by

$$\beta(\tau) = \begin{pmatrix} -\cos(\alpha) \sin(\tau) \\ -\sin(\alpha) \sin(\tau) \\ \cos(\tau) \end{pmatrix}, \quad \tau \in [-\delta_0, 0],$$

$$\beta(\tau) = \begin{pmatrix} \sin(\tau) \\ 0 \\ \cos(\tau) \end{pmatrix}, \quad \tau \in [0, \delta_0].$$

For $\tau \in [-\delta_0, 0]$, $w \in L^\infty([-\delta_0, \delta_0])$, $(Vw)(\tau)$ is given by

$$\begin{aligned} (Vw)(\tau) &= \frac{1}{4\pi} \int_0^{\delta_0} \frac{|\sin(\alpha)| \sin(-\tau) w(\nu)}{1 - (-\cos(\alpha) \sin(\tau) \sin(\nu) + \cos(\tau) \cos(\nu))} d\nu \\ &= -\frac{|\sin(\alpha)|}{4\pi} \int_0^{\delta_0} (2 \sin(\tau/2) \cos(\tau/2) w(\nu)) \\ &\quad \cdot \left(2 \cos(\tau/2)^2 \sin(\nu/2)^2 + 2 \sin(\tau/2)^2 \cos(\nu/2)^2 \right. \\ &\quad \left. + 4 \cos(\alpha) \sin(\tau/2) \cos(\tau/2) \sin(\nu/2) \cos(\nu/2) \right)^{-1} d\nu \\ &= -\frac{|\sin(\alpha)|}{4\pi} \\ &\quad \cdot \int_0^{\delta_0} \frac{\tan(\tau/2) w(\nu)}{\tan(\nu/2)^2 + \tan(\tau/2)^2 + 2 \cos(\alpha) \tan(\nu/2) \tan(\tau/2)} \frac{d\nu}{\cos(\nu/2)^2} \end{aligned}$$

Define the operator

$$T : L^\infty([-\delta_0, \delta_0]) \longrightarrow L^\infty([-\tan(\delta_0/2), \tan(\delta_0/2)]),$$

$$(Tw)(\mu) := w(2 \arctan(\mu)),$$

and the substitution

$$\mu = \tan(\nu/2),$$

see [7]. So we get

$$(Vw)(\tau) = -\frac{|\sin(\alpha)|}{2\pi} \int_0^{\delta_0} \frac{\tan(\tau/2)w(2\arctan(\mu))}{\mu^2 + 2\cos(\alpha)\mu\tan(\tau/2) + \tan(\tau/2)^2} d\mu$$

and therefore

$$\begin{aligned} (TVw)(\kappa) &= -\frac{|\sin(\alpha)|}{2\pi} \int_0^{\delta_0} \frac{\kappa}{\mu^2 + 2\cos(\alpha)\mu\kappa + \kappa^2} (Tw)(\kappa) d\mu \\ &= -\int_0^{\delta_0} l_\alpha\left(\frac{\kappa}{\mu}\right) (Tw)(\mu) \frac{d\mu}{\mu}. \end{aligned}$$

This finally implies

$$V|_{[-\delta_0, \delta_0]} = T^{-1} \begin{pmatrix} 0 & L_\alpha \\ L_\alpha & 0 \end{pmatrix} T,$$

where we have identified $L^\infty([-a, a]) \cong L^\infty([0, a])^2$, $a > 0$. \square

Because the Mellin convolution

$$\begin{aligned} \hat{l}_\alpha(z) &:= \int_0^\infty l_\alpha(x)x^{z-1} dx \\ &= \frac{1}{2} \frac{\sin(\alpha z)}{\sin(\pi z)} \end{aligned}$$

exists for $\operatorname{Re}(z) \in (-1, 1)$, we know that L_α is a bounded operator on $L^p([0, \delta_0])$, $p \in [1, \infty]$, and because the function l_α is positive we get

$$\|L_\alpha\|_{L^p} \leq \hat{l}_\alpha\left(\frac{1}{p}\right).$$

But L_α is not a compact operator. For L^∞ we get the result

$$\|L_\alpha\|_{L^\infty} \leq \frac{1}{2} \frac{|\alpha|}{\pi} < \frac{1}{2},$$

see [6] for these properties of the Mellin convolution operator. But this does not imply $\|V\|_{L^\infty(\gamma)} < 1/2$, which would be a hint that a function w with (2.14) exists. Only the local parts of V around each corner fulfill this condition, but the remaining compact parts are not controlled.

Now we see that the above method does not always work, but if we are able to find a partition such that the spectral radius of the corresponding matrix is smaller than $1/2$, we can calculate a suitable weight function w which fulfills $\|\bar{W}\|_{\mathcal{C}(S),w} < 1/2$. This leads us to the following definition.

Definition 2.3. Given a polyhedral domain $D \subset \mathbf{R}^3$ with boundary S ,

$$(2.17) \quad S = \bigcup_{j=1}^N S_j, \quad S_j \text{ triangles,}$$

and n vertices \mathcal{V}_j , $j = 1(1)n$. We will say that D fulfills the condition **(F)** if we find for every vertex \mathcal{V}_j a partition $\mathcal{P}(\gamma_j) = \{\gamma_i^{(j)} \mid i = 1(1)m_j\}$ of the corresponding spherical polygon γ_j such that the matrix $M(\mathcal{P}(\gamma_j))$ has a spectral radius smaller than $1/2$. We denote by $w_i^{(j)}$, $i = 1(1)m_j$, the weights around vertex \mathcal{V}_j and by $K_i^{(j)}$, $i = 1(1)m_j$, we denote the plane sectors around vertex \mathcal{V}_j , where the weight function has the value $w_i^{(j)}$. If $K_i^{(j)}$ and $K_{i+1}^{(j)}$ are two sectors with a common edge and if β is the angle which is enclosed by these sectors (the smaller one) then we assume

$$(2.18) \quad \frac{1}{2} \left(1 - \frac{\beta}{\pi}\right) \max \left\{ \frac{1}{w_i^{(j)}}, \frac{1}{w_{i+1}^{(j)}} \right\} < \frac{1}{2}.$$

Remark. The reason for the assumption (2.18) will become clear in the proof of Lemma 3.1. This assumption is sufficient for the construction of a global weighted norm on S , such that the essential norm of the double layer operator is smaller than $1/2$ on $\mathcal{C}(S)$. If (2.18) is true, we can glue together the weighted norms of different vertices. Actually condition (2.18) is too strong, see Lemma 3.1. The factor $(1 - \beta/\pi)/2$ in formula (2.18) is the maximal normalized solid angle

under which a point on $K_i^{(j)}$ sees $K_{i+1}^{(j)}$, and this is symmetric between i and $i + 1$. We would like to mention that (2.18) is fulfilled in the Král and Wendland example and for the cone K_A . For every convex cone we can choose the weight function $w(x) = 1$, and (2.18) is automatically fulfilled.

3. The essential norm of the double layer potential. In this section we construct a weighted norm for $\mathcal{C}(S)$ for polyhedral domains which fulfill condition (F), such that the essential norm of the double layer operator with respect to this norm is smaller than $1/2$. Here we denote the weight function on S by Ω in contrast to the weight function ω on the local cones. In the next section the following theorem will be the basis for the study of the stability of the collocation method. First we define a splitting of the operator \overline{W} in a noncompact part \overline{W}_ε and a compact part \overline{C}_ε .

$$(3.1) \quad (\overline{W}_\varepsilon u)(x) := \left(\frac{1}{2} - d(x)\right)u(x) + (W_\varepsilon u)(x),$$

where

$$(3.2) \quad (W_\varepsilon u)(x) = \int_S \chi_\varepsilon(\|x - y\|) k(x, y) u(y) dy,$$

and

$$(3.3) \quad \chi_\varepsilon(t) := \begin{cases} 1, & t \leq \varepsilon, \\ 0, & t > \varepsilon. \end{cases}$$

The compact operator \overline{C}_ε is defined by

$$(3.4) \quad \overline{C}_\varepsilon := \overline{W} - \overline{W}_\varepsilon,$$

see, i.e., [12]. Now we can formulate the main result of this section.

Theorem 3.1. *Let the polyhedral domain D fulfill condition (F). There exists a weight function $\Omega : S \rightarrow (0, \infty)$, Ω has only finite many values, and an $\varepsilon_0 > 0$ such that*

$$(3.5) \quad \|\overline{W}_\varepsilon\|_{\mathcal{C}(S), \Omega} < \frac{1}{2},$$

for $\varepsilon \in (0, \varepsilon_0)$.

Proof. First we recall that the norm $\|u\|_{\mathcal{C}(S), \Omega}$ for a continuous function u is defined by

$$\|u\|_{\mathcal{C}(S), \Omega} := \sup_{x \in S} \frac{|u(x)|}{\Omega(x)}.$$

Denote by $\varepsilon_1 > 0$ a constant with

$$B_{2\varepsilon_1}(\mathcal{V}_i) \cap B_{2\varepsilon_1}(\mathcal{V}_j) = \emptyset, \quad i \neq j,$$

so that the vertices are separated. For each cone K_j , corresponding to the vertex \mathcal{V}_j , we have a partition

$$K_j = \bigcup_{i=1}^{m_j} K_i^{(j)},$$

and weights $w_i^{(j)} \in (0, 1]$, $i = 1(1)m_j$. Each $K_i^{(j)}$ is a plane sector, and we can assume (by further partition) that all $K_i^{(j)}$ who have an edge with S in common have the same opening angle $\alpha > 0$.

Now we define Ω for $x \in S$.

a. $x \in B_{\varepsilon_1}(\mathcal{V}_j) \cap K_i^{(j)}$. Then we define

$$\Omega(x) := w_i^{(j)}.$$

If x is on the boundary of two $K_i^{(j)}$, we can assign one of the two weights.

b. If x is not in an ε_1 neighborhood of any vertex, but has a distance smaller than $\sin(\alpha)\varepsilon_1$ from an edge e , then we assign the value corresponding to Figure 8.

c. If x is not in an ε_1 neighborhood of one of the vertices and has a bigger distance than $\sin(\alpha)\varepsilon_1$ from any edge, we define

$$\Omega(x) := 1.$$

Now we define $\varepsilon_0 := \sin(\alpha)\varepsilon_1$. For $\varepsilon \in (0, \varepsilon_0)$ and $u \in \mathcal{C}(S)$ with $\|u\|_{\mathcal{C}(S), \Omega} \leq 1$ we will now show

$$(3.6) \quad |(\overline{W}_\varepsilon u)(x)| < \frac{\Omega(x)}{2}.$$

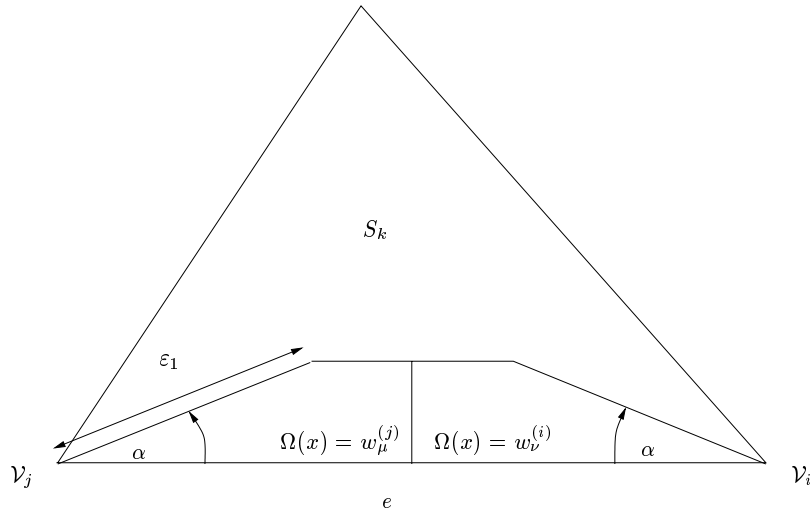


FIGURE 8. $\Omega(x)$ near the edge connecting \mathcal{V}_j and \mathcal{V}_i .

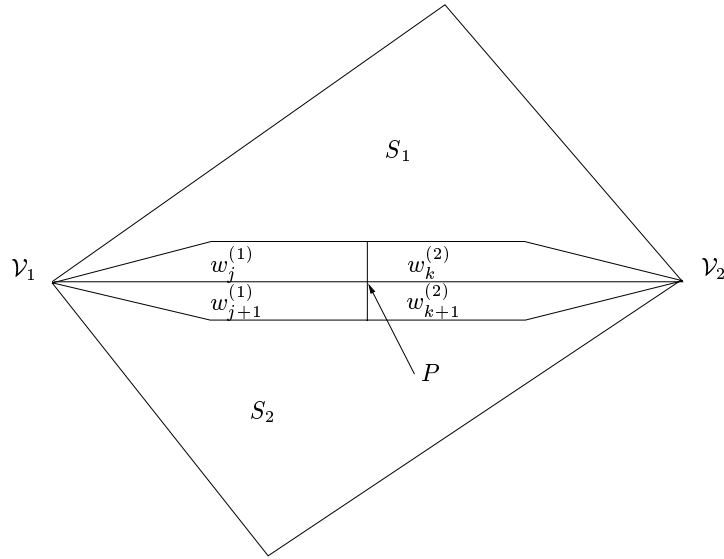
If $x \in S$ has a distance bigger than ε_0 from any edge we get

$$(\overline{W}_\varepsilon u)(x) = 0,$$

which proves (3.6). If $B_\varepsilon(x) \subset B_{\varepsilon_1}(\mathcal{V}_i)$, $i \in \{1, \dots, n\}$, we get (3.6) by the construction of our weights in Lemma 2.1. Now let x have a distance smaller than ε_0 from an edge e and denote by S_1 and S_2 the two triangles which are adjacent to e . By β we again denote the angle enclosed by S_1 and S_2 , see Definition 2.3. We assume that the two vertices at the ends of e are \mathcal{V}_1 and \mathcal{V}_2 . Then it becomes clear that it is enough to estimate $(\overline{W}_\varepsilon u)(x)$ near the point P of Figure 9.

In order to get (3.6) we must have

$$\begin{aligned} \frac{1}{2} \frac{1}{w_j^{(1)}} \left(1 - \frac{\beta}{\pi}\right) \left(\frac{w_{j+1}^{(1)}}{2} + \frac{w_{k+1}^{(2)}}{2}\right) &< \frac{1}{2}, \\ \frac{1}{2} \frac{1}{w_{j+1}^{(1)}} \left(1 - \frac{\beta}{\pi}\right) \left(\frac{w_j^{(1)}}{2} + \frac{w_k^{(2)}}{2}\right) &< \frac{1}{2}, \\ \frac{1}{2} \frac{1}{w_k^{(2)}} \left(1 - \frac{\beta}{\pi}\right) \left(\frac{w_{j+1}^{(1)}}{2} + \frac{w_{k+1}^{(2)}}{2}\right) &< \frac{1}{2}, \end{aligned}$$

FIGURE 9. The weights along the edge e .

$$\frac{1}{2} \frac{1}{w_{k+1}^{(2)}} \left(1 - \frac{\beta}{\pi}\right) \left(\frac{w_j^{(1)}}{2} + \frac{w_k^{(2)}}{2}\right) < \frac{1}{2}.$$

Because of condition (F) we get

$$\frac{1}{2} \left(1 - \frac{\beta}{\pi}\right) \max \left\{ \frac{w_j^{(1)}}{w_{j+1}^{(1)}}, \frac{w_{j+1}^{(1)}}{w_j^{(1)}}, \frac{w_k^{(2)}}{w_{k+1}^{(2)}}, \frac{w_{k+1}^{(2)}}{w_k^{(2)}} \right\} < \frac{1}{2}.$$

We further have the condition (2.18) which, together with $w_i^{(j)} \leq 1$, for all i, j proves (3.6). \square

The above theorem together with the compactness of $\overline{\mathcal{C}}_\varepsilon$ implies that the essential norm of \overline{W} in the weighted space $(\mathcal{C}(S), \|\cdot\|_{\mathcal{C}(S), \Omega})$ is smaller than $1/2$. Because the weighted norm and the usual maximum norm are equivalent, we also get

Corollary 3.2. *The above theorem implies that the operator*

$$\frac{1}{2} - \overline{W}$$

is a Fredholm operator in $C(S)$.

4. The stability of the collocation method. In this section we will investigate the stability of the collocation method for equation (1.1). Before we define exactly the collocation method, we need one more technical lemma which will be important for the stability proof.

Lemma 4.1. *Let $\varepsilon \in (0, \varepsilon_0]$, $u \in L^\infty(S)$, piecewise continuous, and define*

$$f(x) := (\overline{W}_\varepsilon u)(x), \quad x \in S.$$

If \mathcal{E} denotes the union of all edges of S and

$$\mathcal{E}_\delta := \{x \in S \mid \text{dist}(x, \mathcal{E}) \in [0, \delta)\},$$

it follows that f is Lipschitz continuous on $S \setminus \mathcal{E}_\delta$ for every $\delta > 0$, and the Lipschitz constant $L = L(\|u\|_{L^\infty(S)}, \delta, \varepsilon)$ depends only on the norm of u , δ and ε ,

$$L \leq C_L(\varepsilon) \frac{\|u\|_{L^\infty(S)}}{\delta}.$$

Proof. Because

$$f(x) = \sum_{i=1}^n \int_{S_i} k(x, y) \chi_\varepsilon(\|x - y\|) u(y) dy$$

it is enough to prove the assumption only for $x \in S_j$, $j \in \{1, \dots, n\}$, and

$$f(x) = \int_{S_i} k(x, y) \chi_\varepsilon(\|x - y\|) u(y) dy.$$

If S_i and S_j have a positive distance, we get $f \in C^\infty(S_j)$. So we can assume that S_i and S_j are adjacent and that

$$S_i \subset \{(x_1, x_2, x_3) \mid x_3 = 0\}$$

$$S_j \subset \{(x_1, x_2, x_3) \mid x_3 > 0\}$$

and their common edge is, for example, a subset of $\{(0, x_2, 0) \mid x_2 \geq 0\}$. If $x \in S_j$, $x_3 \geq \varepsilon$, we get

$$(\overline{W}_\varepsilon u)(x) = 0,$$

so f is a \mathcal{C}^∞ function. If we denote by χ_i the characteristic function of S_i , we get for $x_3 \in (0, \varepsilon]$

$$\begin{aligned} f(x) &= \int_{S(i)} k(x, y) (\chi_\varepsilon \chi_i u)(y) dy \\ &= \int_{B(x)} k(x, y) v(y) dy, \end{aligned}$$

where now $\chi_i u$ is again a piecewise continuous function and

$$B(x) = \{(y_1, y_2, 0) \mid (x_1 - y_1)^2 + (x_2 + y_2)^2 \leq \varepsilon^2 - x_3^2\}$$

This representation already implies the continuity of $f(\cdot)$ for $x_3 > 0$, where we consider $f(\cdot)$ also as a function in $\{(x_1, x_2, x_3) \mid x_3 > 0\}$. We denote by

$$c(x, \cdot) : [0, 2\pi] \longrightarrow \mathbf{R}^3,$$

a parametrization of the boundary of $B(x)$ given by

$$c(x, s) := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sqrt{\varepsilon^2 - x_3^2} \begin{pmatrix} \cos(s) \\ \sin(s) \end{pmatrix}.$$

One can compute that for $\nu \in \mathbf{R}^3$, the directional derivative $(\partial f / \partial \nu)(x)$, $x_3 \in (0, \varepsilon)$, is given by

$$\begin{aligned} \frac{\partial f}{\partial \nu}(x) &= \int_{B(x)} \frac{\partial k}{\partial \nu_x}(x, y) v(y) dy \\ &\quad + \int_0^{2\pi} k(x, c(x, s)) v(c(x, s)) \det \left(\frac{\partial c(x + \tau \nu, \sigma)}{\partial (\tau, \sigma)} \Big|_{\tau=0} \right) d\sigma. \end{aligned}$$

Remembering

$$k(x, c(x, s)) = \frac{1}{4\pi} \frac{x_3}{\sqrt{\varepsilon^2 - x_3^2}}$$

we see that the second integral is a smooth function near $x_3 = 0$ and only the first integral has to be considered near $x_3 = 0$. We have (if $n(y) = (0, 0, -1)^T$, $y \in S_i$)

$$\frac{\partial k}{\partial \nu_x} = -\frac{1}{4\pi} \left(\frac{\nu_3}{\|x-y\|^3} - 3x_3 \frac{\nu \cdot (x-y)}{\|x-y\|^5} \right).$$

We estimate the first integral by (assume $\|\nu\| = 1$)

$$\begin{aligned} & \int_{B(x)} \left| \frac{\partial k}{\partial \nu_x}(x, y) v(y) \right| dy \\ & \leq \frac{\|v\|_{L^\infty}}{4\pi} \int_0^\infty \int_0^{2\pi} \left[\frac{1}{(x_3^2 + r^2)^{3/2}} \right. \\ & \quad \left. + 3x_3 \left(\frac{|\nu_1|r|\cos(\varphi)| + |\nu_2|r|\sin(\varphi)| + |\nu_3|x_3|}{(x_3^2 + r^2)^{5/2}} \right) \right] r d\varphi dr \\ & \leq 3\|v\|_{L^\infty} \int_0^\infty \frac{1}{x_3^2} \frac{r/x_3}{(1 + (r/x_3)^2)^{3/2}} + \frac{1}{x_3^2} \frac{(r/x_3)^2}{(1 + (r/x_3)^2)^{5/2}} \\ & \quad + \frac{1}{x_3^2} \frac{r/x_3}{(1 + (r/x_3)^2)^{5/2}} dr \\ & = 3 \frac{\|v\|_{L^\infty}}{x_3} \int_0^\infty \frac{s}{(1 + s^2)^{3/2}} + \frac{s^2}{(1 + s^2)^{5/2}} + \frac{s}{(1 + s^2)^{5/2}} ds \\ & \leq \|v\|_{L^\infty} C_1 \frac{1}{\text{dist}(x, S_i)}. \end{aligned}$$

where the constant C_1 also depends on the angle between S_i and S_j , but we only have finite many triangles and angles in S . So we get that $f(\cdot)$ is continuous in $x_3 > 0$, constant to zero in $x_3 > \varepsilon$, and f is differentiable in $x_3 \in (0, \varepsilon)$ with the bound for the gradient given in the last inequality. This proves our lemma. \square

Now we define a collocation method for the solution of (1.1) with piecewise continuous trial functions. We assume that we have a sequence of grids G_j , $j \in \mathbf{N}$,

$$\begin{aligned} (4.1) \quad G_j &= \{\Delta_1^{(j)}, \dots, \Delta_{k_j}^{(j)}\}, \\ S &= \bigcup_{i=1}^{k_j} \Delta_i^{(j)}, \\ \Delta_i^{(j)} \cap \Delta_k^{(j)} &= \emptyset, \quad i \neq k. \end{aligned}$$

Define further

$$(4.2) \quad d_{j,i} := \text{diam}(\Delta_i^{(j)}), \quad \bar{d}_j := \max_{i=1}^{k_j} \{d_{j,i}\}.$$

The G_j may be, for example, some graded grids in order to get high convergence rates, but we will not assume any special properties besides

$$(4.3) \quad \lim_{j \rightarrow \infty} \bar{d}_j = 0.$$

For the reference triangle $\Delta_{\text{Ref}} := \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x, y; x+y \leq 1\}$ we consider two different collocation projectors for continuous functions f on Δ_{Ref}

$$(4.4) \quad (P_0 f)(x) := f\left(\frac{1}{3}, \frac{1}{3}\right), \quad x \in \Delta_{\text{Ref}},$$

and

$$(4.5) \quad (P_1 f)(x) := \sum_{i,j \geq 0, i+j \leq r} c_{i,j} x_1^i x_2^j, \quad x \in \Delta_{\text{Ref}}$$

such that

$$(4.6) \quad (P_1 f)(\xi_{i,j}) = f(\xi_{i,j}), \quad 0 \leq i, j; i+j \leq r; \xi_{i,j} \in \Delta_{\text{Ref}}.$$

So the second interpolation formula has $f_r = (r+1)(r+2)/2$ coefficients, degree r , and we get

$$(4.7) \quad \|(P_1 f) - f\|_{C(\Delta_{\text{Ref}})} \leq \|P_1\| \sup_{x,y \in \Delta_{\text{Ref}}} \|f(x) - f(y)\|,$$

see [2, Lemma 5.1.1], and it is clear that the $\xi_{i,j}$ could not be chosen arbitrarily. We can also find a Lagrange basis for the interpolation

$$(4.8) \quad (P_1 f)(x) = \sum_{i,j \geq 0, i+j \leq r} f(\xi_{i,j}) l_{i,j}(x).$$

For the triangulation we will further assume some regularity; this means if

$$(4.9) \quad T_i^{(j)} : \Delta_{\text{Ref}} \xrightarrow{1:1} \Delta_i^{(j)}$$

are the parameterizations of the triangles $\Delta_i^{(j)}$ then

$$(4.10) \quad d_{j,i} \| [T_i^{(j)}]^{-1} \| \leq C_T, \quad \forall i, j, C_T \text{ a constant.}$$

Now we introduce a parameter ε^* to define the trial space for the collocation method. First we split the triangles in G_j :

$$(4.11) \quad G_j = G_{j,0}(\varepsilon^*) \cup G_{j,1}(\varepsilon^*)$$

where

$$(4.12) \quad \Delta_i^{(j)} \in \begin{cases} G_{j,1}(\varepsilon^*), & \text{if } \frac{d_{j,i}}{\text{dist}(\Delta_i^{(j)}, \mathcal{E})} \leq \varepsilon^*, \\ G_{j,0}(\varepsilon^*), & \text{otherwise.} \end{cases}$$

The last assumption on our triangulation is

$$(4.13) \quad \Omega|_{\Delta_i^{(j)}} = \text{const}, \quad \forall \Delta_i^{(j)} \in G_{j,0}.$$

This last assumption can be achieved by starting with a triangulation G_1 which already respects the condition (4.13) and by further refinement. The trial space for the collocation method is given by

$$(4.14) \quad \begin{aligned} \Pi_j(\varepsilon^*) := & \{ \chi_{\Delta_i^{(j)}} \mid i \in G_{j,0}(\varepsilon^*) \} \\ & \cup \{ \chi_{\Delta_i^{(j)}} l_{k,l}([T_i^{(j)}]^{-1}x) \mid i \in G_{j,1}(\varepsilon^*), 0 \leq k, l; k+l \leq r \}, \end{aligned}$$

where χ_M denotes the characteristic function of a set M . The definition of $\Pi_j(\varepsilon^*)$ looks very complicated but it just means the following: Given a sequence of triangulations $(G_j)_{j \in \mathbf{N}}$, a number $r \in \mathbf{N}_0$ and a positive number ε^* , we use piecewise constant interpolation functions on all triangles of G_j which are relatively close to the edges, see (4.12), and polynomials of maximal degree r on all other triangles. The collocation points are defined by

$$(4.15) \quad \begin{aligned} \Phi_j(\varepsilon^*) := & \left\{ [T_i^{(j)}] \left(\frac{1}{3}, \frac{1}{3} \right) \mid i \in G_{j,0}(\varepsilon^*) \right\} \\ & \cup \{ [T_i^{(j)}] (\xi_{k,l}) \mid i \in G_{j,1}(\varepsilon^*), 0 \leq k, l; k+l \leq r \}, \end{aligned}$$

and the interpolation projector

$$(4.16) \quad \begin{cases} P_{j,\varepsilon^*} : L^\infty(S) \rightarrow \Pi_j(\varepsilon^*) \\ (P_{j,\varepsilon^*}u)(x) = u(x), \quad \forall x \in \Phi_j(\varepsilon^*), \quad u \in \mathcal{C}(S) \end{cases}$$

See [3] for the extension of the interpolation projector to $L^\infty(S)$. The collocation method is given by: Find $u_j \in \text{span}(\Pi_j)(\varepsilon^*)$ such that

$$(4.17) \quad \left(\frac{1}{2}I - \overline{W}\right)u_j(x) = f(x), \quad \forall x \in \Phi_j(\varepsilon^*).$$

After all these definitions we can formulate our stability result.

Theorem 4.2. *Let $D \subset \mathbf{R}^3$ be a polyhedral domain with boundary S which fulfills condition (F). Given a sequence G_j , $j \in \mathbf{N}$, of triangulations of S with (4.3), (4.10) and (4.13) and the interpolation projectors P_0 and P_1 . Then there is an $\overline{\varepsilon}^*$, such that for all $\varepsilon^* \in (0, \overline{\varepsilon}^*]$ and sufficiently large j we get*

$$(4.18) \quad \left\| P_{j,\varepsilon^*} \left(\frac{1}{2}I - \overline{W} \right) u \right\|_{L^\infty(S),\Omega} \geq c_S \|u\|_{L^\infty(S),\Omega},$$

$$\forall u \in \text{span}(\Pi_j(\varepsilon^*)).$$

Proof. Choosing $\varepsilon > 0$ sufficiently small, we know that there exists a function $\Omega : S \rightarrow \mathbf{R}$, such that

$$\|\overline{W}_\varepsilon\|_{L^\infty(S),\Omega} \leq q < \frac{1}{2},$$

$$0 < c_\Omega \leq \Omega(x) \leq C_\Omega.$$

The operator $I/2 - \overline{W}$ is invertible, see [14], and the operator \overline{C}_ε is compact. By [8] we also know that we have to show (4.18) only for $I/2 - \overline{W}_\varepsilon$ and then (4.18) follows for sufficiently large j . Now choose

an $\varepsilon^* > 0$ and consider the interpolation error on a triangle in $G_{j,1}(\varepsilon^*)$

$$\begin{aligned}
 & |[P_{j,\varepsilon^*}(\overline{W}_\varepsilon u)](x) - (\overline{W}_\varepsilon u)(x)| \\
 &= \left| [P_1((\overline{W}_\varepsilon u) \circ [T_i^{(j)}]^{-1})](x) - ((\overline{W}_\varepsilon u) \circ [T_i^{(j)}]^{-1})(x) \right| \\
 &\stackrel{(4.7)}{\leq} \|P_1\| \sup_{x,y \in \Delta_{\text{Ref}}} \left| ((\overline{W}_\varepsilon u) \circ [T_i^{(j)}]^{-1})(x) - ((\overline{W}_\varepsilon u) \circ [T_i^{(j)}]^{-1})(y) \right| \\
 &\leq \|P_1\| \sup_{x,y \in \Delta_i^{(j)}} |(\overline{W}_\varepsilon u)(x) - (\overline{W}_\varepsilon u)(y)| \\
 &\stackrel{\text{Lemma 4.1}}{\leq} \|P_1\| \|u\|_{L^\infty} C_L(\varepsilon) \frac{d_{j,i}}{\text{dist}(\Delta_i^{(j)}, \mathcal{E})} \\
 &\stackrel{(4.12)}{\leq} \|P_1\| C_T C_L(\varepsilon) \varepsilon^* \|u\|_{L^\infty}.
 \end{aligned}$$

Define $q' := (1/2 + q)/2 < 1/2$. Then we get for $x \in \Delta_i^{(j)} \in G_{j,1}$ and $\|u\|_{L^\infty(S), \Omega} = 1$

$$\begin{aligned}
 \frac{|[P_{j,\varepsilon^*}(\overline{W}_\varepsilon u)](x)|}{\Omega(x)} &\leq \frac{|[P_{j,\varepsilon^*}(\overline{W}_\varepsilon u)](x) - (\overline{W}_\varepsilon u)(x)| + |(\overline{W}_\varepsilon u)(x)|}{\Omega(x)} \\
 &\stackrel{\text{Theorem 3.1}}{\leq} \frac{\|P_1\|}{c_\Omega} C_T C_L(\varepsilon) C_\Omega \varepsilon^* + q \\
 &\leq q'
 \end{aligned}$$

if $\varepsilon^* \in (0, \bar{\varepsilon}^*)$, if $\bar{\varepsilon}^* > 0$ is sufficiently small. If $x \in \Delta_i^{(j)} \subset G_{j,0}$, with collocation point ξ we get

$$\begin{aligned}
 \frac{|[P_{j,\varepsilon^*}(\overline{W}_\varepsilon u)](x)|}{\Omega(x)} &= \frac{|(\overline{W}_\varepsilon u)(\xi)|}{\Omega(\xi)} \\
 &\leq q < q'.
 \end{aligned}$$

But the last two inequalities show

$$\begin{aligned}
 \sup_{x \in S} \frac{|u(x)/2 - [P_{j,\varepsilon^*}(\overline{W}_\varepsilon u)](x)|}{\Omega(x)} &\geq \frac{1}{2} - q', \\
 u \in \text{span}(\Pi_j(\varepsilon^*)), \quad \|u\|_{L^\infty, \Omega} &= 1.
 \end{aligned}$$

This proves our theorem and the stability of the collocation method.

□

Remark. It is clear that the modification of the collocation method near the edges can be omitted if the collocation projector has norm 1. This means that the panel method, see [14], is stable without any modification.

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