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OVERLAPPING ADDITIVE SCHWARZ PRECONDITIONERS FOR BOUNDARY ELEMENT METHODS

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Dedicated to Prof. Ian H. Sloan on the occasion of his 60th birthday

ABSTRACT. We study overlapping additive Schwarz preconditioners for the Galerkin boundary element method when used to solve Neumann problems for the Laplacian. Both the h and p versions of the Galerkin scheme are considered. We prove that the condition number of the additive Schwarz operator is bounded by $O(1+\log^2(H/\delta))$ for the h version, where H is the size of the coarse mesh and δ is the size of the overlap, and bounded independently of the mesh size and the polynomial order for the p version.

1. Introduction. We consider in this paper the Neumann problem for the Laplace equation in the exterior of a curve in the plane. Via the standard fundamental solution, we reformulate the problem as a boundary integral equation of the first kind with a symmetric kernel. The Galerkin method when used to solve this equation results in solving a linear system, the coefficient matrix $\mathcal{A}$ of which is *dense*. If N is the size of $\mathcal{A}$, then the Gauss solver requires $O(N^3)$ operations for computation of the coefficients giving the corresponding Galerkin approximate solution. Hence when N is large, one resorts to iterative methods. The matrix being symmetric and positive definite, the conjugate gradient method is among the most practical and efficient iterative methods. Since $\mathcal{A}$ is ill-conditioned, in the sense that its condition number increases with N, the convergence rate of the conjugate gradient method will deteriorate, which leads to a huge number of iterations needed to achieve accuracy.

Several authors have investigated preconditioners for this equation. Multigrid preconditioners, wavelet approximation and matrix com-

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pression, multiplicative Schwarz methods, multilevel additive Schwarz methods, and hierarchical basis preconditioning have been studied [**37**], [**38**], [**3**], [**8**], [**9**], [**10**], [**36**], [**22**], [**16**], [**18**], [**17**], [**29**], [**31**], [**33**], [**34**]. In [**23**], [**28**] a different approach was suggested: the Dirichlet stiffness matrix is used to precondition the Neuman stiffness matrix, and vice versa.

In spite of the abundance of results on preconditioners for first kind integral equations, the simple two-level additive Schwarz methods have not been thoroughly studied. In [**30**] a nonoverlapping method was suggested for the h version, with an open question on overlapping methods. In [**32**] a nonoverlapping and an overlapping method for the p version were suggested and the condition numbers were proved to be bounded by $O(1+\log^2 p)$ for both preconditioners, even though the numerical results in that paper seem to suggest a constant bound for the overlapping method. It is reasonable to expect that an overlapping method yields a bounded condition number (at least when the overlap is generous enough), implying fewer iterations in the solutions as compared to a nonoverlapping method.

In this paper we aim at two goals. Our first task is to discuss an overlapping method for the h version and prove that the condition number of the additive Schwarz operator is bounded by $O(1+\log^2(H/\delta))$ where H is the mesh size of the coarse level and δ is the overlap size. We will report on several numerical experiments with this method. The numbers not only support the theoretical result but also suggest that if the coarse mesh is properly chosen, this overlapping method performs even better than the multilevel method proposed in [**31**].

Our second task is to give a sharper estimate for the condition number of the overlapping method for the p version discussed in [**32**], namely, we will prove that the condition number is bounded independently of h and p. Thus our theoretical result matches the numerical results reported in that paper.

It should be noted that overlapping methods for finite element discretization of partial differential equations have been discussed by several authors. A nonexhaustive list includes [**5**], [**11**], [**12**], [**13**], [**20**], [**21**], [**25**], [**27**]. We refer to [**6**], [**26**] for comprehensive surveys and fairly complete lists of references. Since integral operators are *nonlocal*, which results in *dense* stiffness matrices when discretization methods are used, a study of the method for integral equations is important.

In the analysis for both versions, it is crucial that the boundedness of some interpolation operators, namely, the linear interpolation in the case of the h version and an interpolation which yields a polynomial of degree p from a polynomial of degree $p+1$ in the case of the p version, be proved in the $\tilde{H}^{1/2}$ norm, which is the energy norm for the equation under consideration. Even though the boundedness of these operators are known in the H^1 norm (for the finite element method), the corresponding result for the $\tilde{H}^{1/2}$ norm is not obvious and must be carefully checked to establish bounds that are independent of h and p.

The article is organized as follows. Section 2 describes the boundary integral formulation and serves to introduce our notation. Section 3 gives an abstract description of additive Schwarz methods. In Section 4 we prove some technical lemmas necessary for the analysis. We discuss in Section 5 a two-level overlapping method for the h version and prove in Section 6 a sharper bound for the overlapping method suggested in [**32**]. Section 7 is devoted to numerical experiments for the h version, the results of which confirm our theoretical result. The reader is referred to [**32**] for the numerical results for the p version.

2. The boundary-integral equation. We recall some standard facts from [**7**], [**39**] about boundary integral reformulations of boundary value problems. For brevity, we discuss in this paper the Neumann problem in the exterior of an oriented open arc Γ in $\mathbf{R}^2$. A generalization to a polygonal curve is straightforward. The analysis for the Dirichlet problem can be conducted by using the result for the Neumann problem in the same manner as in [**31**], [**32**] and therefore will not be considered further in this paper.

The orientation of Γ permits the definition of the normal vector n and the identification of Γ as an obstacle with sides Γ_1 and Γ_2. The problem then consists in finding U satisfying

$$\begin{aligned} \Delta U &= 0 && \text{in } \Omega_\Gamma := \mathbf{R}^2 \backslash \overline{\Gamma}, \\ \frac{\partial U}{\partial n}|_{\Gamma_i} &= f_i && \text{for } i = 1, 2, \\ U(z) &= o(1) && \text{as } |z| \to \infty. \end{aligned}$$

We shall not pursue classical solutions in this paper; the solutions will be functions in Sobolev spaces prompting us to define the following spaces.

Let $\tilde{\Gamma}$ be a closed curve which contains Γ and is the boundary of a regular plane domain. We define, as in [**14**], [**19**],

$$H^{1/2}(\tilde{\Gamma}) := \{v = U|_{\tilde{\Gamma}} : U \in H^1_{\mathrm{loc}}(\mathbf{R}^2)\},$$
$$H^{1/2}(\Gamma) := \{v = w|_{\Gamma} : w \in H^{1/2}(\tilde{\Gamma})\},$$

and

$$\tilde{H}^{1/2}(\Gamma) := \{v \in H^{1/2}(\Gamma) : \tilde{v} \in H^{1/2}(\tilde{\Gamma})\},$$

where

$$\tilde{v} := \begin{cases} v & \text{on } \Gamma, \\ 0 & \text{on } \tilde{\Gamma}\backslash\overline{\Gamma}. \end{cases}$$

The spaces $H^{-1/2}(\Gamma)$ and $\tilde{H}^{-1/2}(\Gamma)$ are defined as the dual spaces of $\tilde{H}^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively. The space $\tilde{H}^{1/2}(\Gamma)$ is denoted as $H^{1/2}_{00}(\Gamma)$ in [**19**]. Here we use the notation which is commonly seen in the literature of boundary element methods.

The weak solution of the above Neumann problem belongs to the Sobolev space $H^1_{\mathrm{loc}}(\Omega_\Gamma)$. The trace of a function $v \in H^1_{\mathrm{loc}}(\Omega_\Gamma)$ is a function in $H^{1/2}(\Gamma)$. Moreover, if $\Delta v = 0$, the weak normal derivative $\partial v/\partial n \in \tilde{H}^{-1/2}(\Gamma)$ can be defined. Thus it is natural to assume that $f_1, f_2 \in H^{-1/2}(\Gamma)$ and $f_1 - f_2 \in \tilde{H}^{-1/2}(\Gamma)$; see [**39**] for more details.

It was proved in [**39**, Theorem 1.3] that if U is a solution to the above boundary value problem, then the jump u of U across Γ belongs to $\tilde{H}^{1/2}(\Gamma)$ and satisfies the hypersingular integral equation

$$-\frac{1}{\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\log|x-y|u(y)\,ds_y = f(x), \quad x \in \Gamma, \tag{2.1}$$

where

$$f(x) := -\frac{1}{2}[f_1(x) + f_2(x)] - \int_\Gamma [f_1(y) - f_2(y)]\frac{\partial}{\partial n_x}\log|x-y|\,ds_y.$$

On the other hand, if $u \in \tilde{H}^{1/2}(\Gamma)$ is a solution of (2.1), then U defined by

$$U(z) := \int_\Gamma u(x)\frac{\partial}{\partial n_x}\log|z-x|\,ds_x$$
$$- \int_\Gamma \log|z-x|[f_1(x) - f_2(x)]\,ds_x, \quad z \in \Omega_\Gamma,$$

which belongs to $H^1_{\mathrm{loc}}(\Omega_\Gamma)$, is a solution to the Neumann problem.

It was shown in [**7**], [**39**] that if

$$Du(x) := -\frac{1}{\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\log|x-y|u(y)\,ds_y, \quad x\in\Gamma,$$

then

$$a(v,w) := \langle Dv, w\rangle \quad \forall\, v,w \in \tilde{H}^{1/2}(\Gamma)$$

is a positive-definite and symmetric bilinear form on $\tilde{H}^{1/2}(\Gamma)$ satisfying

$$a(v,v) \sim \|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \quad \forall\, v \in \tilde{H}^{1/2}(\Gamma). \tag{2.2}$$

A weak form of equation (2.1) is the problem of finding

$$u \in \tilde{H}^{1/2}(\Gamma): \quad a(u,v) = \langle f, v\rangle \quad \forall\, v \in \tilde{H}^{1/2}(\Gamma). \tag{2.3}$$

Here $\langle f, v\rangle$ denotes the duality pairing which coincides with the L^2 inner product on Γ if $f, v \in L^2(\Gamma)$. The problem (2.3) will be approximated by first constructing a finite-dimensional subspace $\mathcal{S} \subset \tilde{H}^{1/2}(\Gamma)$ and then finding

$$u_{\mathcal{S}} \in \mathcal{S}: \quad a(u_{\mathcal{S}}, v) = \langle f, v\rangle \quad \forall\, v \in \mathcal{S}. \tag{2.4}$$

For ease of presentation we assume, without loss of generality, that Γ is an interval in $\mathbf{R}$. If Γ is an arbitrary interval (a,b), then it is known that, cf. [**2**], [**4**], [**14**], [**19**]

$$\|v\|^2_{H^{1/2}(a,b)} \sim \|v\|^2_{L^2(a,b)} + |v|^2_{H^{1/2}(a,b)} \tag{2.5}$$

and

$$\|v\|^2_{\tilde{H}^{1/2}(a,b)} \sim |v|^2_{H^{1/2}(a,b)} + \int_a^b \frac{v^2(x)}{b-x}\,dx + \int_a^b \frac{v^2(x)}{x-a}\,dx, \tag{2.6}$$

where

$$|v|^2_{H^{1/2}(a,b)} := \int_a^b \int_a^b \frac{|v(x)-v(y)|^2}{|x-y|^2}\,dx\,dy. \tag{2.7}$$

3. Abstract framework of additive Schwarz methods. Additive Schwarz methods provide fast solutions to equation (2.4) by solving, at the same time, problems of smaller size. Let $\mathcal{S}$ be decomposed as

$$\mathcal{S} = \mathcal{S}_0 + \cdots + \mathcal{S}_J, \tag{3.1}$$

where $\mathcal{S}_i$, $i = 0, \dots, J$, are subspaces of $\mathcal{S}$, and let $P_i : \mathcal{S} \to \mathcal{S}_i$, $i = 0, \dots, J$, be projections defined by

$$a(P_i v, w) = a(v, w) \quad \forall\, v \in \mathcal{S},\ \forall\, w \in \mathcal{S}_i.$$

If we define $P := P_0 + \cdots + P_J$, then the additive Schwarz method for equation (2.4) consists in solving, by an iterative method, the equation

$$Pu_{\mathcal{S}} = g, \tag{3.2}$$

where the righthand side is given by $g = \sum_{i=0}^{J} g_i$, with $g_i \in \mathcal{S}_i$ being solutions of

$$a(g_i, w) = \langle f, w \rangle \quad \text{for any } w \in \mathcal{S}_i. \tag{3.3}$$

The equivalence of (2.4) and (3.2) was discussed in [**31**]. For a detailed description of the implementation of the method, the reader is referred to [**41**].

Bounds for the minimum and maximum eigenvalues of the additive Schwarz operator P can be obtained by using the following lemma, see [**24**], [**25**], [**40**].

Lemma 3.1. (i) *If there exists a constant C_0 such that, for any $u \in \mathcal{S}$ and $u_i \in \mathcal{S}_i$ satisfying $u = \sum_{i=0}^{J} u_i$, the following inequality*

$$a(u, u) \le C_0^2 \sum_{i=0}^{J} a(u_i, u_i) \tag{3.4}$$

holds, then

$$\lambda_{\max}(P) \le C_0^2.$$

(ii) *If there exists a constant* C_1 *such that any* $u \in \mathcal{S}$ *has a decomposition* $u = \sum_{i=0}^{J} u_i$ *satisfying*

$$\sum_{i=0}^{J} a(u_i, u_i) \le C_1^2 a(u, u), \tag{3.5}$$

then

$$\lambda_{\min}(P) \ge C_1^{-2}.$$

4. Technical lemmas. In this section we will present some technical lemmas. The first lemma provides a bound in the $\tilde{H}^{1/2}$-norm for the linear interpolation operator.

Lemma 4.1. *Let* Γ *be partitioned into subintervals* Γ_i, $i = 1, \dots, J$, *by a mesh of maximum size* h. *If* $u \in C(\overline{\Gamma}) \cap \tilde{H}^{1/2}(\Gamma)$ *is defined so that its restrictions on* Γ_i *are polynomials of degree* p, *then the linear interpolant* $\Pi_h u$ *of* u *at the mesh points of* Γ *satisfies*

$$\|\Pi_h u\|_{\tilde{H}^{1/2}(\Gamma)} \le C'(1 + \log p)\|u\|_{\tilde{H}^{1/2}(\Gamma)},$$

where C' *is independent of* u, h *and* p.

Proof. It was proved in [**32**, Lemma 3.5] that

$$\sum_{i=1}^{J} |\Pi_h u|^2_{H^{1/2}(\Gamma_i)} \le c_1(1 + \log p) \sum_{i=1}^{J} |u|^2_{H^{1/2}(\Gamma_i)},$$

where c_1 is independent of u, h and p. Let $w := u - \Pi_h u$. Then

$$\sum_{i=1}^{J} |w|_{\Gamma_i}|^2_{H^{1/2}(\Gamma_i)} \le 2(c_1 + 1)(1 + \log p) \sum_{i=1}^{J} |u|^2_{H^{1/2}(\Gamma_i)}. \tag{4.1}$$

Moreover, since w vanishes at the mesh points, $w|_{\Gamma_i} \in \tilde{H}^{1/2}(\Gamma_i)$ for $i = 1, \dots, J$. It follows from [**2**, Theorem 6.6] that for any $\alpha \in \mathbf{R}$, noting that $\|\cdot\|_{\tilde{H}^{1/2}(\Gamma_i)}, |\cdot|_{H^{1/2}(\Gamma_i)}$ and $\|\cdot\|_{L^\infty(\Gamma_i)}$ are invariant under scaling,

$$\begin{aligned}\|w|_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} &\le |w|_{\Gamma_i}|^2_{H^{1/2}(\Gamma_i)} + c_2(1+\log p)\|w|_{\Gamma_i}\|^2_{L^\infty(\Gamma_i)} \\ &\le |w|_{\Gamma_i}|^2_{H^{1/2}(\Gamma_i)} \\ &\quad + 2c_2(1+\log p)(\|u-\alpha\|^2_{L^\infty(\Gamma_i)} + \|\Pi_h u - \alpha\|^2_{L^\infty(\Gamma_i)}) \\ &\le |w|_{\Gamma_i}|^2_{H^{1/2}(\Gamma_i)} + 4c_2(1+\log p)\|u-\alpha\|^2_{L^\infty(\Gamma_i)},\end{aligned}$$

where c_2 is a constant independent of p and of the length of Γ_i, i.e., h. Taking α to be the value of u at one endpoint of Γ_i so that $u-\alpha$ vanishes at at least one point in $\overline{\Gamma}_i$ we can use [**32**, Lemma 3.3] to infer

$$\|w|_{\Gamma_j}\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \le |w|_{\Gamma_j}|^2_{H^{1/2}(\Gamma_j)} + c_3(1+\log p)^2|u|^2_{H^{1/2}(\Gamma_j)},$$

where c_3 is a constant independent of p and h. This inequality and (4.1) imply

$$\begin{aligned}\|w\|^2_{\tilde{H}^{1/2}(\Gamma)} &\le \sum_{i=1}^{J}\|w|_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \\ &\le 2(c_1+1)(1+\log p)|u|^2_{H^{1/2}(\Gamma)} + c_3(1+\log p)^2|u|^2_{H^{1/2}(\Gamma)} \\ &\le (2+2c_1+c_3)(1+\log p)^2\|u\|^2_{\tilde{H}^{1/2}(\Gamma)}.\end{aligned}$$

Here the first inequality is a standard property of the $\tilde{H}^{1/2}$ norm which was first proved in [**35**]. A proof can also be found in [**1**] or [**32**]. The result now follows by using the triangular inequality, completing the proof. □

Results on the extension of a polynomial on the boundary of a rectangular element onto the element itself are well known; see, e.g., [**2**]. The following lemma yields an extension of a *piecewise* polynomial function on a boundary onto the domain.

In the following, by a square, say Ω, we mean the closed set. However, for notational simplicity we still write $H^s(\Omega)$ instead of $H^s(\overset{\circ}{\Omega})$, where $\overset{\circ}{\Omega}$ is the interior of Ω.

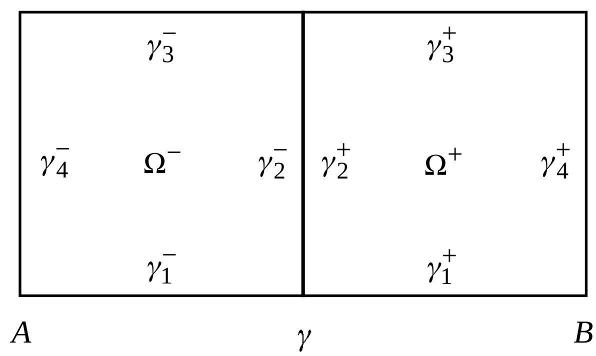


FIGURE 1. $\Omega = \Omega^+ \cup \Omega^-$, $\gamma^\pm = \gamma_1^\pm \cup \cdots \cup \gamma_4^\pm$, $\gamma = \gamma_1^- \cup \gamma_1^+$.

Lemma 4.2. *Let $\Omega^\pm$ be two squares with sides $\gamma_1^\pm, \dots, \gamma_4^\pm$ as in Figure 1. Let $\Omega := \Omega^+ \cup \Omega^-$, $\gamma^\pm := \gamma_1^\pm \cup \cdots \cup \gamma_4^\pm$ and $\gamma := \gamma_1^+ \cup \gamma_1^-$. Assume that f is a continuous function on γ such that $f(A) = f(B) = 0$ and $f^\pm := f|_{\gamma_1^\pm}$ are polynomials of degree p. Then $F \in H^1(\Omega)$ exists such that*

$$F^\pm := F|_{\Omega^\pm} \in \mathcal{P}_p(\Omega^\pm), \qquad F|_{\gamma_3^\pm} = F|_{\gamma_4^\pm} = 0, \qquad F|_\gamma = f,$$

and

$$\|F\|_{H^1(\Omega)} \le c\|f\|_{\tilde{H}^{1/2}(\gamma)}, \tag{4.2}$$

where c is a constant independent of f and p but may depend on the size $|\Omega|$ of Ω. Here $\mathcal{P}_p(\Omega^\pm)$ is the space of polynomials in $\Omega^\pm$ of degree at most p in each variable.

Proof. Let $\tilde{f}^\pm$ be functions defined on $\gamma^\pm$ as

$$\tilde{f}^\pm := \begin{cases} f^\pm & \text{on } \gamma_1^\pm, \\ f^+ \circ \eta & \text{on } \gamma_2^\pm, \\ 0 & \text{on } \gamma_3^\pm \cup \gamma_4^\pm, \end{cases}$$

where η is the rotation transformation that maps γ_2^+ onto γ_1^+. We note that $\tilde{f}^+$ and $\tilde{f}^-$ take the same values on $\gamma_2^\pm$ and are continuous on

$\gamma^{\pm}$, respectively. By [**2**, Theorem 7.5] $F^{\pm} \in \mathcal{P}_p(\Omega^{\pm})$ exists such that $F^{\pm}|_{\gamma^{\pm}} = \tilde{f}^{\pm}$ and

$$\|F^{\pm}\|_{H^1(\Omega^{\pm})} \le c\|\tilde{f}^{\pm}\|_{H^{1/2}(\gamma^{\pm})}, \tag{4.3}$$

where c is independent of f and p but may depend on $|\Omega^{\pm}|$. Since $F^{+}|_{\gamma_2^{+}} = F^{-}|_{\gamma_2^{-}}$, if we define

$$F := \begin{cases} F^{+} & \text{on } \Omega^{+}, \\ F^{-} & \text{on } \Omega^{-}, \end{cases}$$

then F is continuous on Ω; thus $F \in H^1(\Omega)$. In view of (4.3), (4.2) will follow if

$$\|\tilde{f}^{\pm}\|_{H^{1/2}(\gamma^{\pm})} \le c\|f\|_{\tilde{H}^{1/2}(\gamma)}. \tag{4.4}$$

It is clear from the definition of $\tilde{f}^{\pm}$ that

$$\|\tilde{f}^{-}\|_{H^{1/2}(\gamma^{-})} \sim \|f\|_{\tilde{H}^{1/2}(\gamma)}$$

and

$$\|\tilde{f}^{+}\|_{H^{1/2}(\gamma^{+})} \sim \|f^{+}\|_{H^{1/2}(\gamma_1^{+})} \le c\|f\|_{\tilde{H}^{1/2}(\gamma)}.$$

Thus (4.4) holds and the lemma is proved. □

The next lemma is essential to prove a bound for the minimum eigenvalue of P, see (3.5), for both versions.

Lemma 4.3. *Let $\{\theta_l : l = 1, \dots, L\}$ be a partition of unity on $\Gamma := (a, b)$ such that*

$$0 \le \theta_l \le 1, \quad \sum_{l=1}^{L} \theta_l = 1, \quad \left|\frac{d\theta_l}{dx}\right| \le \frac{C}{\delta}, \tag{4.5}$$

for some constants C and δ, and let $I_l := [a_l, b_l] = \operatorname{supp}\theta_l$.

(i) *For any $w \in \tilde{H}^{1/2}(\Gamma)$, the following holds*

$$\begin{aligned} \sum_{l=1}^{L} \|\theta_l w\|^2_{\tilde{H}^{1/2}(I_l)} \le 4C^2 \sum_{l=1}^{L} \frac{b_l - a_l}{\delta^2}\|w\|^2_{L^2(I_l)} + 2\sum_{l=1}^{L} |w|^2_{H^{1/2}(I_l)} \\ + \int_{\Gamma} \frac{|w(x)|^2}{b - x}\, dx + \int_{\Gamma} \frac{|w(x)|^2}{x - a}\, dx. \end{aligned} \tag{4.6}$$

(ii) *There exist a positive constant* $\tilde{C}$ *and a partition of unity* $\{\theta_l : l = 1, \dots, L\}$ *composed of piecewise-linear functions on a mesh of size* $\delta \le \min_l (b_l - a_l)/2$ *such that, for any* $w \in \tilde{H}^{1/2}(\Gamma)$*, the following holds*

(4.7)
$$\sum_{l=1}^{L} \|\theta_l w\|^2_{\tilde{H}^{1/2}(I_l)} \le \tilde{C} \sum_{l=1}^{L} \left(1 + \log \frac{b_l - a_l}{\delta}\right)^2 \left(\frac{1}{b_l - a_l}\|w\|^2_{L^2(I_l)} + |w|^2_{H^{1/2}(I_l)}\right) + \int_\Gamma \frac{|w(x)|^2}{b-x}\,dx + \int_\Gamma \frac{|w(x)|^2}{x-a}\,dx.$$

Proof. In view of (2.6) we will estimate

$$T_1 := \int_{I_l}\int_{I_l} \frac{|(\theta_l w)(x) - (\theta_l w)(y)|^2}{|x-y|^2}\,dx\,dy,$$
$$T_2 := \int_{I_1} \frac{|(\theta_l w)(x)|^2}{b_l - x}\,dx \quad \text{and} \quad T_3 := \int_{I_l} \frac{|(\theta_l w)(x)|^2}{x - a_l}\,dx.$$

By using (4.5) we have

$$\begin{aligned} T_1 &\le 2\int_{I_l}\int_{I_l} \frac{|\theta_l(x) - \theta_l(y)|^2}{|x-y|^2}|w(x)|^2\,dx\,dy \\ &\quad + 2\int_{I_l}\int_{I_l} \frac{|w(x) - w(y)|^2}{|x-y|^2}|\theta_l(y)|^2\,dx\,dy \\ &\le 2C^2 \frac{b_l - a_l}{\delta^2}\|w\|^2_{L^2(I_l)} + 2|w|^2_{H^{1/2}(I_l)}. \end{aligned}$$

For the term T_2 we note that $\theta(b_l) = 0$ when $l = 1, \dots, L-1$ and use (4.5) again to obtain

$$T_2 = \int_{I_l} \frac{|\theta_l(b_l) - \theta_l(x)|^2}{b_l - x}|w(x)|^2\,dx \le C^2 \frac{b_l - a_l}{\delta^2}\|w\|^2_{L^2(I_l)}.$$

When $l = L$, i.e., $b_l = b$, $\theta_L(b) = 0$ may not hold, but we can estimate T_2 as

$$T_2 = \int_{I_L} \frac{|(\theta_L w)(x)|^2}{b-x}\,dx \le \int_\Gamma \frac{|w(x)|^2}{b-x}\,dx.$$

Similarly, for T_3 we have

$$T_3 \le C^2 \frac{b_l - a_l}{\delta^2} \|w\|^2_{L^2(I_l)} \quad \text{when } 2 \le l \le L$$

and

$$T_3 \le \int_\Gamma \frac{|w(x)|^2}{x-a}\, dx \quad \text{when } l = 1.$$

Summing over $l = 1, \dots, L$, we obtain (4.6).

The proof of (4.7) mainly follows [**13**]. For simplicity of notation we assume, without loss of generality, $I_l = [0, \beta]$ and write I and θ instead of I_l and θ_l. We will prove that each of T_1, T_2, when $1 \le l \le L-1$, and T_3, when $2 \le l \le L$, is bounded by

$$\left(1 + \log \frac{\beta}{\delta}\right)^2 \left(\frac{1}{\beta}\|w\|^2_{L^2(I)} + |w|^2_{H^{1/2}(I)}\right). \tag{4.8}$$

We define θ to be linear in $[0, \delta]$ and $[\beta-\delta, \beta]$ and equal to 1 in $[\delta, \beta-\delta]$, i.e.,

$$\theta(x) := \begin{cases} x/\delta & 0 \le x \le \delta, \\ 1 & \delta < x < \beta - \delta, \\ (\beta - x)/\delta & \beta - \delta \le x \le \beta. \end{cases}$$

Splitting the integral over I into three integrals over $I_1 := [0, \delta]$, $I_2 := [\delta, \beta - \delta]$ and $I_3 := [\beta - \delta, \beta]$, and denoting for any v

$$A_{ij}(v) := \iint_{I_i \times I_j} \frac{|v(x) - v(y)|^2}{|x-y|^2}\, dx\, dy,$$

we observe that, by symmetry, in order to estimate T_1 it suffices to consider $A_{11}(\theta w)$, $A_{12}(\theta w)$, $A_{13}(\theta w)$, $A_{22}(\theta w)$, $A_{23}(\theta w)$ and $A_{33}(\theta w)$. There is no need to consider $A_{22}(\theta w)$ because $A_{22}(\theta w) = A_{22}(w)$. The symmetric shape of θ implies the similarity of $A_{11}(\theta w)$ and $A_{33}(\theta w)$, and of $A_{12}(\theta w)$ and $A_{23}(\theta w)$. Three terms remain to be considered: $A_{11}(\theta w)$, $A_{12}(\theta w)$ and $A_{13}(\theta w)$.

From the definition of θ it is easy to see that

$$A_{11}(\theta w) \le \frac{2}{\delta}\|w\|^2_{L^2(0,\delta)} + 2A_{11}(w).$$

It was proved in [**13**, Lemma 3.4] that

$$\frac{1}{\delta}\|w\|^2_{L^2(0,\delta)} \le \tilde{c}\left(1+\log\frac{\beta}{\delta}\right)\left(\frac{1}{\beta}\|w\|^2_{L^2(I)} + |w|^2_{H^{1/2}(I)}\right), \tag{4.9}$$

where $\tilde{c}$ is independent of w, β and δ. Thus $A_{11}(\theta w)$ is bounded by (4.8). For the term $A_{12}(\theta w)$ we have

$$\begin{aligned}
A_{12}(\theta w) &= \frac{1}{\delta^2}\int_0^\delta \int_\delta^{\beta-\delta} \frac{|xw(x)-\delta w(y)|^2}{|x-y|^2}\,dy\,dx \\
&\le \frac{2}{\delta^2}\int_0^\delta \left(\int_\delta^{\beta-\delta} \frac{dy}{|x-y|^2}\right)|x-\delta|^2|w(x)|^2\,dx + 2A_{12}(w) \\
&\le \frac{2}{\delta}\|w\|^2_{L^2(0,\delta)} + 2A_{12}(w).
\end{aligned}$$

Using again (4.9) we obtain the estimate for $A_{12}(\theta w)$. Finally, for $A_{13}(\theta w)$, since the assumption $\delta \le \beta/2$ implies that $\beta/2 - x \ge 0$ for $x \in I_1$ and $y-(\beta/2) \ge 0$ for $y \in I_3$, we find

$$\begin{aligned}
A_{13}(\theta w) &= \frac{1}{\delta^2}\int_0^\delta \int_{\beta-\delta}^{\beta} \frac{|xw(x)-(\beta-y)w(y)|^2}{|x-y|^2}\,dy\,dx \\
&\le \frac{2}{\delta^2}\int_0^\delta \int_{\beta-\delta}^{\beta} \frac{|x+y-\beta|^2}{|x-y|^2}|w(x)|^2\,dy\,dx \\
&\quad + \frac{2}{\delta^2}\int_0^\delta \int_{\beta-\delta}^{\beta} |\beta-y|^2\frac{|w(x)-w(y)|^2}{|x-y|^2}\,dy\,dx \\
&\le \frac{2}{\delta^2}\int_0^\delta \int_{\beta-\delta}^{\beta} \frac{|(y-(\beta/2))-((\beta/2)-x)|^2}{|(y-(\beta/2))+((\beta/2)-x)|^2}|w(x)|^2\,dy\,dx + 2A_{13}(w) \\
&\le \frac{2}{\delta}\|w\|^2_{L^2(0,\delta)} + 2A_{13}(w).
\end{aligned}$$

Inequality (4.9) yields the estimate for $A_{13}(\theta w)$, implying T_1 is bounded by (4.8). The proof for T_2 follows by using again the definition of θ:

$$\begin{aligned}
T_2 &\le \int_0^{\beta-\delta} \frac{|w(x)|^2}{\beta-x}\,dx + \frac{1}{\delta^2}\int_{\beta-\delta}^{\beta} (\beta-x)|w(x)|^2\,dx \\
&\le \int_0^{\beta-\delta} \frac{|w(x)|^2}{\beta-x}\,dx + \frac{1}{\delta}\|w\|^2_{L^2(\beta-\delta,\beta)}.
\end{aligned}$$

It was proved in [**13**, Lemma 3.5] that

$$\int_0^{\beta-\delta} \frac{|w(x)|^2}{\beta - x}\,dx \le \bar{c}\Big(1+\log\frac{\beta}{\delta}\Big)^2\Big(\frac{1}{\beta}\|w\|^2_{L^2(I)} + |w|^2_{H^{1/2}(I)}\Big).$$

This inequality and an analogue to (4.9) yield the bound (4.8) for T_2. Similar arguments hold for T_3 which completes the proof. □

We finish this section by a result concerning the invariance of the $\tilde{H}^{1/2}$-norm under a special transformation which will be used in the analysis of the p-version.

Lemma 4.4. *Let $a < b < c$, and let $A : u \mapsto \tilde{u}$ be a mapping from $\tilde{H}^{1/2}(a,c)$ onto $\tilde{H}^{1/2}(-1,1)$ defined as*

$$\tilde{u} := \begin{cases} \tilde{u}_1 & \text{on } [-1,0], \\ \tilde{u}_2 & \text{on } [0,1], \end{cases}$$

where $\tilde{u}_1$ and $\tilde{u}_2$ are the affine images of $u_1 := u|_{[a,b]}$ and $u_2 := u|_{[b,c]}$ on $[-1,0]$ and $[0,1]$, respectively. Then

$$\frac{1}{\mu}\|u\|^2_{\tilde{H}^{1/2}(a,c)} \le \|Au\|^2_{\tilde{H}^{1/2}(-1,1)} \le \mu\|u\|^2_{\tilde{H}^{1/2}(a,c)},$$

where $\mu := \max\{(c-b)/(b-a), (b-a)/(c-b)\}$.

Proof. Assume without loss of generality that $\mu = (c-b)/(b-a)$. For any $t \in [a,b]$ and $\tau \in [b,c]$, if

$$s = \frac{t-b}{b-a} \in [-1,0] \quad \text{and} \quad \sigma = \frac{\tau - b}{c-b} \in [0,1], \tag{4.10}$$

then $\tilde{u}_1(s) = u_1(t)$ and $\tilde{u}_2(\sigma) = u_2(\tau)$. To prove the lemma, in view of (2.6) we consider three terms

$$T_1 := \int_{-1}^1 \int_{-1}^1 \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^2}\,dx\,dy,$$

$$T_2 := \int_{-1}^1 \frac{|\tilde{u}(x)|^2}{1-x}\,dx \quad \text{and} \quad T_3 := \int_{-1}^1 \frac{|\tilde{u}(x)|^2}{1+x}\,dx.$$

By splitting the integral over $[-1,1]$ into two integrals over $[-1,0]$ and $[0,1]$ and noting the symmetry, we consider, instead of T_1, three integrals

$$T_{11} := \int_0^1 \int_0^1 \frac{|\tilde{u}_2(\sigma) - \tilde{u}_2(\sigma')|^2}{|\sigma - \sigma'|^2} \, d\sigma \, d\sigma',$$
$$T_{12} := \int_{-1}^0 \int_{-1}^0 \frac{|\tilde{u}_1(s) - \tilde{u}_1(s')|^2}{|s - s'|^2} \, ds \, ds',$$

and

$$T_{13} := \int_0^1 \int_{-1}^0 \frac{|\tilde{u}_1(s) - \tilde{u}_2(\sigma)|^2}{|s - \sigma|^2} \, ds \, d\sigma.$$

Noting (4.10) we have

$$T_{11} = \int_b^c \int_b^c \frac{|u_2(\tau) - u_2(\tau')|^2}{|\tau - \tau'|^2} \, d\tau \, d\tau',$$
$$T_{12} = \int_a^b \int_a^b \frac{|u_1(t) - u_1(t')|^2}{|t - t'|^2} \, dt \, dt',$$

and

$$T_{13} = \mu \int_b^c \int_a^b \frac{|u_1(t) - u_2(\tau)|^2}{|\tau - \mu t + (\mu - 1)b|^2} \, dt \, d\tau.$$

Since $a \le t \le b \le \tau \le c$ and $\mu \ge 1$, the following holds

$$|\tau - \mu t + (\mu - 1)b| = \tau - t + (\mu - 1)(b - t),$$

which implies

$$\tau - t \le |\tau - \mu t + (\mu - 1)b| \le \mu(\tau - t).$$

Therefore,

$$\frac{1}{\mu} \int_a^b \int_b^c \frac{|u_2(\tau) - u_1(t)|^2}{|\tau - t|^2} \, d\tau \, dt \le T_{13} \le \mu \int_a^b \int_b^c \frac{|u_2(\tau) - u_1(t)|^2}{|\tau - t|^2} \, d\tau \, dt.$$

Thus

$$\frac{1}{\mu}|u|^2_{H^{1/2}(a,c)} \le T_1 \le \mu |u|^2_{H^{1/2}(a,c)}.$$

Similarly we have

$$T_2 = \int_a^b \frac{|u_1(t)|^2}{2b-a-t}\,dt + \int_b^c \frac{|u_2(\tau)|^2}{c-\tau}\,d\tau.$$

Since for $t \in [a,b]$ the following holds, under the assumption $b-a \le c-b$,

$$\frac{c-t}{\mu} \le 2b-a-t \le c-t,$$

we deduce

$$\int_a^c \frac{|u(z)|^2}{c-z}\,dz \le T_2 \le \mu \int_a^c \frac{|u(z)|^2}{c-z}\,dz.$$

Finally, a similar argument yields

$$\frac{1}{\mu}\int_a^c \frac{|u(z)|^2}{z-a}\,dz \le T_3 \le \int_a^c \frac{|u(z)|^2}{z-a}\,dz,$$

completing the proof of the lemma. □

5. An overlapping method for the h version. In this section we design an overlapping method for the h version. We do this by first introducing a two-level mesh.

The coarse mesh. Assuming, without loss of generality, that $\Gamma = (-1,1)$, we first divide Γ into disjoint subdomains Γ_i with length H_i, $i = 1,\dots,J$, so that $\overline{\Gamma} = \cup_{i=1}^J \overline{\Gamma}_i$ and denote by H the maximum value of H_i.

The fine mesh. Each Γ_i is further divided into disjoint subintervals Γ_{ij}, $j = 1,\dots,N_i$, so that $\overline{\Gamma}_i = \cup_{j=1}^{N_i} \overline{\Gamma}_{ij}$. The maximum length of the subintervals Γ_{ij} in Γ_i is denoted by h_i, and the maximum value of h_i is denoted by h.

It is assumed that the meshes are quasi-uniform. To simplify our arguments, we assume that the lengths of the subdomains and subintervals are on the order of H and h, respectively.

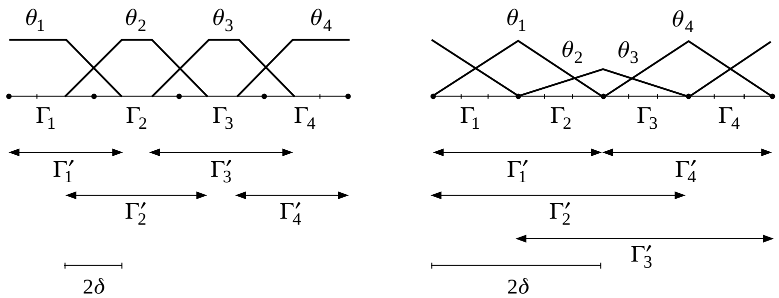


FIGURE 2. Partitions of unity with smallest (left) and largest (right) possible overlaps. Note $\theta_2 = \theta_3$ on $\overline{\Gamma}'_2 \cap \overline{\Gamma}'_3 = \overline{\Gamma}_2 \cup \overline{\Gamma}_3$ on the right picture.

We now extend each subdomain Γ_i on both sides, except for the left side of Γ_1 and the right side of Γ_J which are the endpoints of Γ, so that the length of the overlap between two extended subdomains Γ'_i and Γ'_{i+1} is on the order of δ for some $\delta \in (0, H]$. The smallest possible overlap is of length $2h$ whereas the largest is of length $2H$, which implies

$$|\Gamma'_i| \sim H. \tag{5.1}$$

We note that the endpoints of Γ'_i coincide with fine-mesh points: see Figure 2.

The finite-dimensional space $\mathcal{S}$, see (2.4), is defined as the space of continuous piecewise-linear functions on the fine mesh, vanishing at the endpoints of Γ. The decomposition (3.1) is then performed by letting $\mathcal{S}_0$ be the space of continuous piecewise-linear functions on the coarse mesh, vanishing at the endpoints of Γ, and $\mathcal{S}_i = \mathcal{S} \cap \tilde{H}^{1/2}(\Gamma'_i)$, $i = 1, \ldots, J$.

This overlapping decomposition completely defines the additive Schwarz operator P, introduced in Section 3. In the following we prove a bound for the condition number $\kappa(P)$ of P. In view of Lemma 3.1 we prove the following lemmas.

Lemma 5.1. *There exists a positive constant C_2 independent of h,*

δ and H such that for any $u \in \mathcal{S}$ if $u = \sum_{i=0}^{J} u_i$ for some $u_i \in \mathcal{S}_i$, then

$$a(u,u) \le C_2 \sum_{i=0}^{J} a(u_i,u_i).$$

Proof. By construction there are at most three subdomains $\overline{\Gamma}_i'$ to which any $x \in \Gamma$ can belong. (The utmost case happens when $\delta = H$.) A standard coloring argument (see, e.g., [**15**]) yields

$$\begin{aligned}\|u\|^2_{\tilde{H}^{1/2}(\Gamma)} &\le 2\bigg(\|u_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + \bigg\|\sum_{i=1}^{J} u_i\bigg\|^2_{\tilde{H}^{1/2}(\Gamma)}\bigg) \\ &\le 2\bigg(\|u_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + 3\sum_{i=1}^{J} \|u_i\|^2_{\tilde{H}^{1/2}(\Gamma)}\bigg).\end{aligned}$$

The desired result now comes from (2.2). □

Lemma 5.2. *For any $u \in \mathcal{S}$ there exists $u_i \in \mathcal{S}_i$ satisfying $u = \sum_{i=0}^{J} u_i$ and*

$$\sum_{i=0}^{J} a(u_i,u_i) \le C_3\bigg(1 + \log^2 \frac{H}{\delta}\bigg) a(u,u),$$

where C_3 is a positive constant independent of u, H, h and δ.

Proof. To define a decomposition for $u \in \mathcal{S}$ we need a projection defined as follows. Since the operator $-d^2/dx^2$ with domain of definite $\tilde{H}^1(\Gamma) = H_0^1(\Gamma)$ is positive definite and self-adjoint, we can define $\Lambda = \sqrt{-d^2/dx^2}$ which in turn is self-adjoint as an operator from $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$. Moreover, see [**4**],

$$\langle \Lambda\xi, \xi\rangle \simeq \|\xi\|^2_{\tilde{H}^{1/2}(\Gamma)} \quad \forall\, \xi \in \tilde{H}^{1/2}(\Gamma).$$

Let $P_H : \tilde{H}^{1/2}(\Gamma) \to \mathcal{S}_0$ be the projection defined by the inner product $\langle \Lambda\cdot, \cdot\rangle$, i.e.,

$$\langle \Lambda P_H v, w\rangle = \langle \Lambda v, w\rangle \quad \forall\, v \in \tilde{H}^{1/2}(\Gamma), w \in \mathcal{S}_0.$$

Using standard arguments one can prove that there exists a constant $C_4 > 0$ independent of H such that for any $v \in \tilde{H}^{1/2}(\Gamma)$

$$\|P_H v\|_{\tilde{H}^{1/2}(\Gamma)} \le C_4 \|v\|_{\tilde{H}^{1/2}(\Gamma)} \tag{5.2}$$

and

$$\|P_H v - v\|_{L^2(\Gamma)} \le C_4 H^{1/2} \|v\|_{\tilde{H}^{1/2}(\Gamma)}. \tag{5.3}$$

Consider a partition of unity on Γ composed of piecewise-linear functions θ_i which are defined as in the proof of Lemma 4.3 so that $\operatorname{supp}\theta_i = \overline{\Gamma}'_i$; see Figure 2. The step-size of the mesh on which θ_i is defined is 2δ, the size of the overlap.

For any $u \in \mathcal{S}$ we define a decomposition of u as a sum of functions in $\mathcal{S}_i$ as follows. Let $u_0 = P_H u$ and $u_i = \Pi_h(\theta_i w)$, $i = 1, \dots, J$, where $w = u - u_0$ and Π_h denotes the linear interpolation operator which interpolates a continuous function at the fine-mesh points. It is clear that $u = \sum_{i=0}^{J} u_i$. Since $\theta_i w$ is a piecewise-quadratic function on Γ, it follows from Lemma 4.1 that

$$\|u_i\|_{\tilde{H}^{1/2}(\Gamma)} \le C_5 \|\theta_i w\|_{\tilde{H}^{1/2}(\Gamma)},$$

where C_5 is independent of h. Hence, by using (5.2), (4.7) and (5.3) we obtain, noting (5.1),

$$\begin{aligned}
\sum_{i=0}^{J} \|u_i\|^2_{\tilde{H}^{1/2}(\Gamma)} &\le C_4^2 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)} + C_5^2 \sum_{i=1}^{J} \|\theta_i w\|^2_{\tilde{H}^{1/2}(\Gamma'_i)} \\
&\le C_4^2 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \\
&\quad + C_6 \left(1 + \log^2 \frac{H}{\delta}\right) \left(\frac{1}{H} \|w\|^2_{L^2(\Gamma)} + \|w\|^2_{\tilde{H}^{1/2}(\Gamma)}\right) \\
&\le C_3 \left(1 + \log^2 \frac{H}{\delta}\right) \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}.
\end{aligned}$$

The lemma is now a result of (2.2). $\square$

Combining Lemmas 3.1, 5.1 and 5.2, we obtain

Theorem 5.3. *The condition number of the additive Schwarz operator P is bounded as*

$$\kappa(P) \le C_2 C_3 \left(1 + \log^2 \frac{H}{\delta}\right).$$

Remark 5.4. If the overlap is generous enough so that $\delta \ge cH$ for some constant c, then the condition number of the additive Schwarz operator is bounded independently of h and H.

6. An overlapping method for the p version. In this section we give a sharper estimate for the condition number of the additive Schwarz operator designed in [**32**, Section 5]. We first recall the finite-dimensional space for this version and the decomposition defined in [**32**, Section 5].

Fixing a mesh of size h defined by $x_0 < x_1 < \cdots < x_{N+1}$ we define $\mathcal{S}$ to be the space of continuous functions vanishing at the endpoints of Γ whose restrictions on $\Gamma_i := (x_{i-1}, x_i)$ are polynomials of degree p. Considering overlapping subdomains

$$\Gamma_i' = \Gamma_i \cup \{x_i\} \cup \Gamma_{i+1}, \quad i = 1, \dots, N,$$

we now decompose $\mathcal{S}$ by (3.1) where $\mathcal{S}_0$ is the space of piecewise-linear functions vanishing at the endpoints of Γ, and

$$\mathcal{S}_i := \mathcal{S} \cap \tilde{H}^{1/2}(\Gamma_i'), \quad i = 1, \dots, N.$$

Let

$$\mathcal{L}_{p+1}(x) := \int_{-1}^{x} L_p(y)\, dy, \quad x \in [-1, 1],$$

where L_p is the Legendre polynomial of degree p. Note that $\mathcal{L}_{p+1}$ has $p+1$ zeros satisfying

$$-1 = z_1 < \cdots < z_{p+1} = 1.$$

Let $\mathcal{Q}[-1,1]$ be the space of polynomials of degree at most p on $[-1,1]$. We define $T_p : C[-1,1] \to \mathcal{Q}[-1,1]$ as an interpolation operator which interpolates a function $v \in C[-1,1]$ at $z_1, \dots, z_{p+1}$, i.e.,

$$T_p v \in \mathcal{Q}[-1,1] \quad \text{and} \quad T_p v(z_j) = v(z_j), \quad j = 1, \dots, p+1.$$

This operator was first introduced in [**25**]. It was proved in [**32**, Lemma 5.2] that

$$|T_p v|_{H^{1/2}(-1,1)} \le c|v|_{H^{1/2}(-1,1)} \quad \forall\, v \in \mathcal{Q}[-1,1]/\mathbf{R}, \tag{6.1}$$

where c is independent of p and v. An estimate of the form (3.5) with $C_1^2 = c(1+\log^2 p)$ was obtained in [**32**, Section 5] by using (6.1) and estimating the last two singular terms in the definition (2.6), resulting in a logarithmic dependence on p of the bound.

In the following we employ another approach to prove a constant bound independent of p. We first introduce from T_p another interpolation operator T_p^* and prove, analogously to (6.1), the boundedness of T_p^* in the $\tilde{H}^{1/2}$-norm. In the sequel, if v is a function defined on an interval, then we denote by $\hat{v}$ its affine image on the reference interval $[-1,1]$.

Lemma 6.1. *Let f be a continuous function on $[-1,1]$ vanishing at ± 1 such that $f^- := f|_{[-1,0]}$ and $f^+ := f|_{[0,1]}$ are polynomials of degree $p+1$. Let $T_p^*(f)$ be defined as*

$$T_p^*(f) := \begin{cases} g^- & \text{on } [-1,0), \\ g^+ & \text{on } [0,1], \end{cases}$$

where $g^\pm$ are polynomials of degree p, the affine images $\hat{g}^\pm$ of which are defined by $\hat{g}^\pm := T_p(\hat{f}^\pm)$, respectively. Then the following holds

$$\|T_p^*(f)\|_{\tilde{H}^{1/2}(-1,1)} \le C''\|f\|_{\tilde{H}^{1/2}(-1,1)}, \tag{6.2}$$

where C'' is independent of f and p.

Proof. We note that the functions $g^\pm$ so defined are polynomials of degree p and that $T_p^*(f) \in \tilde{H}^{1/2}(-1,1)$ as was shown in [**32**, Section 5]. We now prove (6.2). Let

$$\Omega_1^- := [-1,0]\times[-1,0], \quad \Omega_2^- := [-1,0]\times[0,1], \quad \Omega^- := \Omega_1^- \cup \Omega_2^-,$$
$$\Omega_1^+ := [0,1]\times[-1,0], \quad \Omega_2^+ := [0,1]\times[0,1], \quad \Omega^+ := \Omega_1^+ \cup \Omega_2^+,$$
$$\Omega := \Omega^- \cup \Omega^+ \quad \text{and} \quad I := [-1,1]\times\{0\}.$$

By Lemma 4.2 a function $F \in H_0^1(\Omega)$ exists such that

$$F|_{\Omega_i^\pm} \in \mathcal{P}_{p+1}(\Omega_i^\pm), \quad i = 1, 2, \quad F|_I = f,$$

and

$$\|F\|_{H^1(\Omega)} \le c\|f\|_{\tilde{H}^{1/2}(-1,1)}, \tag{6.3}$$

where the constant c is independent of p. By using the two-dimensional version of the interpolation operator T_p discussed in [**25**], we can define $G_i^\pm \in \mathcal{P}_p(\Omega_i^\pm)$ such that

$$|G_i^\pm|_{H^1(\Omega_i^\pm)} \le c|F|_{\Omega_i^\pm}|_{H^1(\Omega_i^\pm)}.$$

Let G denote the function on Ω such that $G|_{\Omega_i^\pm} = G_i^\pm$. Then $G \in H_0^1(\Omega)$, because $F \in H_0^1(\Omega)$, and

$$|G|_{H^1(\Omega)} \le c|F|_{H^1(\Omega)}. \tag{6.4}$$

Now let $\partial\Omega_2$ be the boundary of $\Omega_2 := \Omega_2^+ \cup \Omega_2^-$, and let $\Gamma' := \partial\Omega_2 \backslash \overline{I}$. We denote by $\tilde{g}$ the extension of $T_p^*(f)$ by 0 onto Γ'. On each of the intervals $[-1, 0]$ and $[0, 1]$, G and $T_p^*(f)$ are polynomials of degree p which coincide at $p + 1$ points, the interpolation points. Therefore, $G|_I = T_p^*(f) = \tilde{g}|_I$. On the other hand, $G|_{\Gamma'} \equiv 0 \equiv \tilde{g}|_{\Gamma'}$. Thus $G|_{\partial\Omega_2} = \tilde{g}$, which implies

$$\|T_p^*(f)\|_{\tilde{H}^{1/2}(-1,1)} = \|\tilde{g}\|_{H^{1/2}(\partial\Omega_2)} \le c\|G\|_{H^1(\Omega)} \sim |G|_{H^1(\Omega)}. \tag{6.5}$$

The estimate (6.2) hence follows from (6.3), (6.4) and (6.5). □

Using the projection defined in the proof of Lemma 5.2, the interpolation operator T_p^*, and the transformation A, introduced in Lemma 4.4, we can define for each function $u \in \mathcal{S}$ a representation of the form $u = \sum_{i=1}^N u_i$ with $u_i \in \mathcal{S}_i$ as follows.

Let $\{\theta_i : i = 1, \ldots, N\}$ be a partition of unity composed of piecewise-linear functions θ_i satisfying

$$\operatorname{supp}\theta_i \subset \overline{\Gamma}_i', \quad 0 \le \theta_i \le 1, \quad \sum_{i=1}^N \theta_i = 1, \quad \left|\frac{d\theta_i}{dx}\right| \le \frac{1}{h}. \tag{6.6}$$

Let u_0 be the projection of u onto $\mathcal{S}_0$ defined as in the proof of Lemma 5.2 so that

$$\|u_0\|_{\tilde{H}^{1/2}(\Gamma)} \le C_8 \|u\|_{\tilde{H}^{1/2}(\Gamma)} \tag{6.7}$$

and

$$\|u - u_0\|_{L^2(\Gamma)} \le C_8 h^{1/2} \|u\|_{\tilde{H}^{1/2}(\Gamma)}. \tag{6.8}$$

Let $w := u - u_0$. Then $\theta_i w$ having support in $\overline{\Gamma}_i'$ comprises two parts, $\theta_i w|_{\Gamma_i}$ and $\theta_i w|_{\Gamma_{i+i}}$, each being a polynomial of degree at most $p+1$. Defining

$$u_i = A^{-1} T_p^* A(\theta_i w), \quad i = 1, \ldots, N,$$

we can prove that, see [**32**], $u_i \in \mathcal{S}_i$ and $u = \sum_{i=0}^N u_i$.

Lemma 6.2. *There exists a constant $C_9 > 0$ independent of p and N such that with u_i defined as above, the following holds*

$$\sum_{i=0}^N a(u_i, u_i) \le C_9 a(u, u).$$

Proof. It follows from Lemmas 4.4 and 6.1 that

$$\begin{aligned} \|u_i\|^2_{\tilde{H}^{1/2}(\Gamma_i')} &\le \mu \|T_p^* A(\theta_i w)\|^2_{\tilde{H}^{1/2}(-1,1)} \\ &\le \mu C'' \|A(\theta_i w)\|^2_{\tilde{H}^{1/2}(-1,1)} \\ &\le \mu^2 C'' \|\theta_i w\|^2_{\tilde{H}^{1/2}(\Gamma_i')}, \end{aligned}$$

where $\mu = h_{\max}/h_{\min}$. By using (4.6), (6.7) and (6.8), noting that $\delta \sim h$ and $|\Gamma_i'| \sim 2h$, we obtain

$$\begin{aligned} \sum_{i=0}^N \|u_i\|^2_{\tilde{H}^{1/2}(\Gamma_i')} &\le C_8 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)} + \mu^2 C'' \left(\frac{16}{h} \|w\|^2_{L^2(\Gamma)} + 4 \|w\|^2_{\tilde{H}^{1/2}(\Gamma)} \right) \\ &\le C_9 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}, \end{aligned}$$

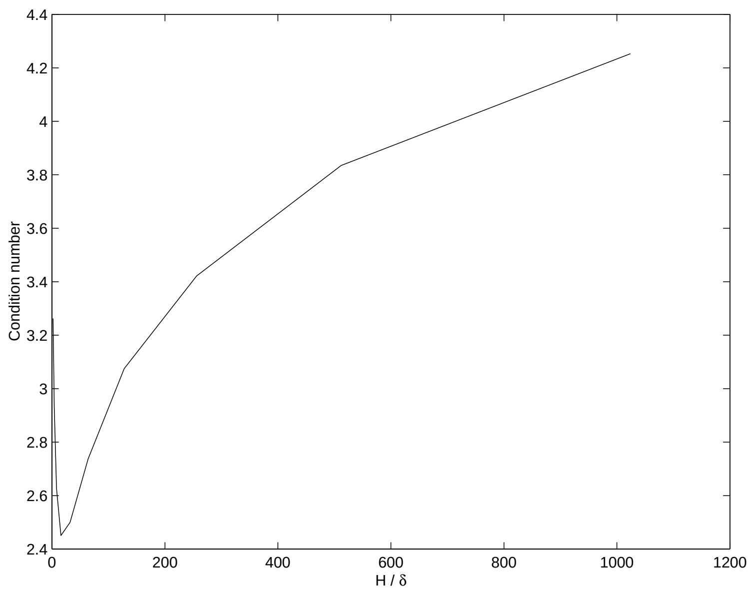


FIGURE 3. Condition number of OP as a function of H/δ when DoF $= 2047$.

completing the proof of the lemma. The constant C_9 is independent of p and N, but may depend on μ which can be chosen to be constant. $\square$

The following lemma was proved in [**32**].

Lemma 6.3. *There exists a constant $C_{10} > 0$ independent of p and N such that, for any $u \in \mathcal{S}$, if $u = \sum_{i=0}^{N} u_i$ for some $u_i \in \mathcal{S}_i$, then*

$$a(u,u) \leq C_{10} \sum_{i=0}^{N} a(u_i, u_i).$$

Lemmas 3.1, 6.2 and 6.3 yield

Theorem 6.4. *The condition number of the additive Schwarz operator P is bounded independently of p and N.*

TABLE 1. Condition numbers with overlapping preconditioner: $h = 2/\text{DoF}$.

DoF	$\delta = h$	$\delta = 2h$	$\delta = 4h$	$\delta = h$	$\delta = 2h$	$\delta = 4h$
	$H/\delta = 2$			$H/\delta = 4$		
7	3.026	2.838		2.515		
15	3.035	3.262	2.876	2.839	2.554	
31	3.087	3.266	3.328	2.930	2.900	2.577
63	3.103	3.294	3.342	2.896	2.986	2.932
127	3.117	3.302	3.367	2.900	2.955	3.015
255	3.123	3.303	3.371	2.895	2.957	2.985
511	3.125	3.303	3.371	2.890	2.952	2.987
1023	3.124	3.299	3.369	2.885	2.947	2.982
2047	3.122	3.298	3.367	2.882	2.943	2.976
	$H/\delta = 8$			$H/\delta = 16$		
15	2.298					
31	2.561	2.320		2.238		
63	2.597	2.594	2.333	2.468	2.219	
127	2.621	2.630	2.613	2.480	2.472	2.208
255	2.614	2.654	2.648	2.482	2.495	2.478
511	2.605	2.648	2.673	2.460	2.490	2.501
1023	2.597	2.639	2.666	2.442	2.467	2.496
2047	2.592	2.632	2.657	2.429	2.449	2.474
	$H/\delta = 32$			$H/\delta = 64$		
63	2.388					
127	2.550	2.272		2.631		
255	2.575	2.567	2.259	2.786	2.609	
511	2.534	2.568	2.564	2.797	2.764	2.600
1023	2.499	2.529	2.566	2.752	2.775	2.755
2047	2.475	2.495	2.529	2.716	2.731	2.766

TABLE 1. Continued.

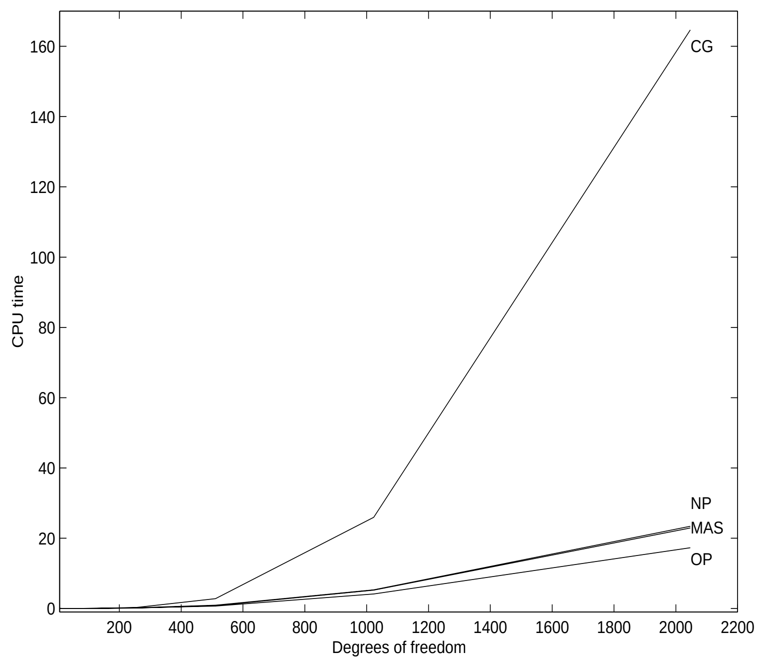
	$H/\delta = 128$			$H/\delta = 256$		
255	2.954					
511	3.120	2.927		3.338		
1023	3.116	3.092	2.915	3.505	3.306	
2047	3.056	3.090	3.079	3.499	3.473	3.292
	$H/\delta = 512$			$H/\delta = 1024$		
1023	3.773					
2047	3.933	3.737		4.253		

TABLE 2. Condition number and number of iterations of different methods. CG: without preconditioner (number in brackets: order of increase); MAS: multilevel additive Schwarz method; NP: nonoverlapping preconditioner with $H = 1/2$; OP: overlapping preconditioner with $H = 1/2$ and $\delta = h = 2/\text{DoF}$.

DoF	Condition number					Iterations			
	CG		MAS	NP	OP	CG	MAS	NP	OP
3	0.2012E+01		1.644			2	2		
7	0.3865E+01	(0.77)	2.407	2.608	3.026	4	4	4	4
15	0.7578E+01	(0.88)	3.035	3.597	2.839	5	6	6	6
31	0.1535E+02	(0.97)	3.461	4.667	2.561	9	8	7	7
63	0.3089E+02	(0.99)	3.755	5.826	2.468	13	9	8	7
127	0.6220E+02	(1.00)	3.971	7.115	2.550	19	10	9	7
255	0.1249E+03	(1.00)	4.131	8.564	2.786	27	10	9	8
511	0.2503E+03	(1.00)	4.255	10.190	3.120	39	11	10	8
1023	0.5010E+03	(1.00)	4.350	12.010	3.505	55	11	11	8
2047	0.1003E+04	(1.00)	4.425	14.010	3.933	79	11	11	8

7. Numerical experiments. The numerical results for the p version were presented in [**32**]. In this section we present results from our numerical experiments for the overlapping method for the h version. The experiments were carried out on the machine SUN Ultra-I.

We solved the equation (2.1) with $\Gamma = [-1, 1]$ and $f(x) = 2$ which



DoF	NP	OP
7	$1/2$	$1/2$
15	$1/2$	$1/2$
31	$1/2$	$1/2$
63	$1/2$	$1/2$
127	$1/4$	$1/4$
255	$1/8$	$1/8$
511	$1/32$	$1/16$
1023	$1/64$	$1/32$
2047	$1/128$	$1/64$

FIGURE 4. CPU time of different methods with H chosen as in Table and $\delta = h = 2/\text{DoF}$ for OP.

has as exact solution $u(x) = -2(1-x^2)^{1/2}$. We tested the overlapping method as a preconditioner for the conjugate gradient method with different values of H and δ, the sizes of the coarse mesh and overlap, respectively. The stopping criterion for the iteration is

$$\frac{\|\mathcal{A}u^{(m)} - b\|_2}{\|b\|_2} \le 10^{-4},$$

where $\mathcal{A}$ and b are the stiffness matrix and righthand side, respectively, obtained from (2.4) with $\mathcal{S}$ defined as in Section 5, and $\|\cdot\|_2$ denotes the l^2-norm.

In Table 1 we present the condition numbers of the preconditioned matrix with different values of H/δ. It is clear that when H/δ is fixed the condition number is bounded even though the degree of freedom increases. When H/δ changes, a slight change in the condition number indicates a logarithmic behavior of the bound, as can be clearly seen in Figure 3.

In Table 2 we compare the condition numbers and numbers of iterations of four methods, namely, the conjugate gradient method (CG) without preconditioners, the multilevel additive Schwarz preconditioner (MAS) developed in [**31**], the nonoverlapping preconditioner (NP) discussed in [**30**], and the present overlappping preconditioner (OP). For both the nonoverlapping and overlapping methods we took the mesh with size $H = 1/2$ to be the coarse mesh, and for the overlapping method we chose $\delta = h$, which means only a small overlap was used. In theory, the condition numbers behave like $O(h^{-1}), O(1)$ and $O(H/h)$ for CG, MAS and NP, respectively. The numbers seem to suggest the best performance of the overlapping method.

In terms of the CPU time, if the coarse mesh is chosen properly, the overlapping method performs better than the simple MAS method and, of course, the other two methods; see Figure 4. This empirical conclusion might inspire a further study.

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