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COMPUTING POSITIVE FIXED-POINTS OF DECREASING HAMMERSTEIN OPERATORS BY RELAXED ITERATIONS

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ABSTRACT. We prove global convergence of (under)relaxed Picard-like methods for fixed-point equations $u = A(u), A : C_+(\Omega) \to C_+(\Omega)$, Ω being a compact Hausdorff space. The operator A is *decreasing* and completely continuous, and possesses no pairs of distinct and comparable coupled-fixed points. Infinite- as well as finite-dimensional Hammerstein equations of this type arise in transport theory. As a numerical application, we test Picard, updated Picard, Jacobi, and Gauss-Seidel (under)relaxed iterations on the discrete "decreasing" version of Chandrasekhar H-equation. A comparison with popular Newton-like solvers is also presented.

1. Introduction. In this paper we consider as a model problem the Hammerstein equation

$$u(x) = A(u)(x) = KN(u)(x), \quad x \in \Omega, \tag{1}$$

where $K : C_+(\Omega) \to C_+(\Omega)$ is (the restriction of) a linear completely continuous operator, $C(\Omega)$ denoting the space of continuous real functions on the compact Hausdorff space Ω (endowed with $\|\cdot\|_\infty$), and $C_+(\Omega)$ its positive cone; cf. [**16, 18**]. In (1), N is the Nemytskii operator

$$N(u)(x) = f(x, u(x)), \quad x \in \Omega, \tag{2}$$

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associated with the continuous scalar function

$$f(x,u)=\frac{1}{\sigma(x,u)},\quad x\in\Omega,\quad u\geq 0, \tag{3}$$

where σ is strictly positive and continuous, with $\sigma(x,\cdot)$ *nondecreasing*, and *sublinear*, i.e.,

$$\sigma(x,\tau u)\geq\tau\sigma(x,u), \tag{4}$$

for every $\tau\in(0,1)$, $x\in\Omega$, $u>0$. Observe that the operator $A=KN:C_+(\Omega)\to C_+(\Omega)$ turns out to be completely continuous and *decreasing*, and possesses no pairs of distinct and *comparable coupled fixed-points*, cf. [**10, 15, 16, 24, 26**].

Integral equations of this type arise in nuclear physics and in the theory of radiative transfer, where some quadratic integral models can be transformed into (1), with K a linear and compact Fredholm operator with a nonnegative kernel, like

$$K(\phi)=\int_\Omega k(\cdot,t)\phi(t)\,dt,\quad \Omega\subset\mathbf{R}^d \quad \text{closed and bounded}, \tag{5}$$

such that

$$\lim_{x\to x_0}\int_\Omega |k(x,t)-k(x_0,t)|\,dt=0, \tag{6}$$

for every $x_0\in\Omega$, cf. [**16, 20, 27**]. We recall, for example, the well-known Chandrasekhar H-equation, modeling heat transfer in semi-infinite atmospheres, which can be rewritten in the form

$$h(x)=\frac{\lambda}{1-2\lambda}\int_0^1\frac{t\psi(t)}{x+t}\frac{1}{1+h(t)}\,dt,\quad x\in[0,1], \tag{7}$$

where $\lambda\in(0,1/2)$, $\psi(t)\geq 0$ and $\int_0^1\psi(t)\,dt=1$, cf. [**8, 27**]. Equation (7) does fit (1)–(6), taking $\sigma(x,u)\equiv 1+u$, cf. [**16**, Theorem 3.3.5] for the compactness of the relevant linear integral operator. The constant 2λ is a physical parameter of the model, termed the *albedo*, which measures the fraction of radiation lost due to scattering: the case $\lambda<1/2$ amounts to nonconservative instances, cf. [**7**]. It is worth

reminding that there exists also a discrete version of Chandrasekhar H-equation, which deserves to be studied and solved numerically in view of its own physical meaning [**8, 17**]. On the other hand, discretizations of (7) naturally appear within numerical solution methods, cf., e.g., [**7**] and the survey [**2**] for a general treatment of nonlinear integral equations.

Observe that the general formulation (1)–(4) above embodies *infinite-dimensional* instances, as well as their *finite-dimensional* versions: in the latter simply $\Omega = \{1, \dots, m\}$, $u \in \mathbf{R}^m_+ = C_+(\Omega)$ and $K = \{k_{ij}\} \in \mathbf{R}^{m\times m}_+$, i.e., (1) reads as the nonlinear system

$$u_i = A_i(u) = \sum_{j=1}^{m} k_{ij} f_j(u_j), \quad i = 1, \dots, m. \tag{8}$$

Indeed, if we discretize (1)–(6) by means of a quadrature formula with m nodes $\{x_j\}$ and *positive* weights $\{w_j\}$, then

$$k_{ij} = w_j k(x_i, x_j)/w(x_j), \qquad f_j(u_j) = N_j(u) = f(x_j, u_j), \quad 1 \le i, j \le m, \tag{9}$$

where $w(x)$ is the corresponding weight function. The same finite-dimensional structure appears after discretization of certain purely integral Boltzmann models (via reduction to a Hammerstein formulation) [**4, 5, 25**], where on the contrary the integration domain is unbounded, and infinite-dimensional compactness may fail [**14**].

Existence and uniqueness of positive continuous solutions to (7) has been studied by various authors, cf. [**6, 16, 20, 21, 27**]. More generally, in [**12**], recalling some arguments that are present also in [**20**], it has been shown that (1)–(4), in the infinite dimensional instance (5), has a unique positive and continuous solution. This result is achieved by a *constructive* theorem on decreasing operators in ordered Banach spaces, the core of its proof being *Picard method*, together with the fact that the nonlinear operator A does not possess distinct and comparable coupled fixed-points.

It is worth noting that mere existence of a solution to (8) is easily obtained by Brouwer fixed-point theorem, as the order interval $\{u \in \mathbf{R}^m : 0 \le u_i \le A_i(0, \dots, 0), 1 \le i \le m\}$ is mapped into itself by the *decreasing* and continuous operator A. The same observation applies

to the infinite-dimensional case (5)–(6), the nonlinear integral operator being completely continuous from $C_+(\Omega)$ into $C_+(\Omega)$, as Schauder fixed-point theorem can be invoked, with reference to the closed convex $\{u \in C_+(\Omega) : 0 \le u(x) \le \int_0^1 k(x,t)f(t,0)\,dt, x \in \Omega\}$.

In this paper we solve numerically (8) by *stationary underrelaxation* of *Picard* and *updated Picard* methods, and of *Jacobi* and *Gauss-Seidel* methods [**22, 23**], producing respectively the iterative processes

$$(P_\omega)\begin{cases}(u_{n+1/2})_i = \sum_{j=1}^m k_{ij} f_j((u_n)_j), & i = 1,\dots,m\\ u_{n+1} = (1-\omega)u_n + \omega u_{n+1/2},\end{cases}$$

$$(UP_\omega)\begin{cases}(u_{n+1/2})_i = \sum_{j<i} k_{ij} f_j((u_{n+1/2})_j)\\ \quad + \sum_{j\ge i} k_{ij} f_j((u_n)_j), & i = 1,\dots,m,\\ u_{n+1} = (1-\omega)u_n + \omega u_{n+1/2},\end{cases}$$

$$(J_\omega)\begin{cases}(u_{n+1/2})_i = k_{ii} f_i((u_{n+1/2})_i)\\ \quad + \sum_{j\ne i} k_{ij} f_j((u_n)_j), & i = 1,\dots,m,\\ u_{n+1} = (1-\omega)u_n + \omega u_{n+1/2},\end{cases}$$

$$(SOR)\begin{cases}(u_{n+1/2})_i = \sum_{j\le i} k_{ij} f_j((u_{n+1/2})_j)\\ \quad + \sum_{j>i} k_{ij} f_j((u_n)_j), & i = 1,\dots,m,\\ u_{n+1} = (1-\omega)u_n + \omega u_{n+1/2},\end{cases}$$

where $(u_0)_i \in [0, A_i(0,\dots,0)]$, $1 \le i \le m$, cf. (8), and $\omega \in (0,1]$.

Some questions arise quite naturally:

(i) Does the system (8) have a *unique* positive solution?

(ii) In such a case, are the relaxed methods $P_\omega, UP_\omega, J_\omega$, SOR *globally convergent* to the unique positive solution?

(iii) Are their performances comparable with other classical methods, e.g., inexact- and quasi-Newton methods, when the parameter ω is suitably chosen?

In the next sections we'll give an affirmative answer to all of these questions, embedding the problem in the general framework of fixed-point approximation for positive decreasing operators in ordered Banach spaces [**16**]. Note that global convergence of relaxed Jacobi and SOR processes cannot be recovered by classical results; in fact, the

mapping $I - KN$ is in general neither a gradient mapping nor an M-function, nor strictly diagonally dominant [**23**, Theorem 6.12]. Moreover, our result gives a sound theoretical basis to the updated iterative method effectively used in [**7**] for the Chandrasekhar H-equation and in [**5**] for the Boltzmann equation.

The paper is organized as follows. In Section 2 we analyze global convergence of relaxed Picard-like iterations for compact decreasing operators: the main convergence theorem is stated and proved in Section 2.1, in the general setting of abstract cones, and in Section 2.2 such a result is applied to the Hammerstein equation (1)–(6), with special attention to the finite-dimensional case (8)–(9). In Section 3 some numerical examples are presented concerning the discrete version of the Chandrasekhar H-equation in the "decreasing" form (7). The convergence behavior varying the relaxation parameter is exhibited, and the remarkable speedup given by "optimal" relaxation is pointed out. The key property of mesh-independence is also experimentally shown and discussed. Finally, the performance of optimally relaxed Picard-like methods is compared to that of popular Newton-like methods.

2. Convergence of relaxed Picard-like iterations.

2.1 The abstract setting. Before proving our main theorem, we introduce some basic definitions and a technical lemma. For the theory of monotone operators in partially ordered Banach spaces, we refer the reader to [**16**] and to the classics [**18, 19**].

Let X be a real Banach space, $P \subset X$ a cone and $\preceq$ the partial ordering defined by P. Hereafter θ denotes the *zero* element of X; moreover, following [**16**], the "*order interval*" $\{z \in X : u \preceq z \preceq v\}$ will be denoted as $[u, v]$.

(i) An operator $A : D \subseteq X \to X$ is said to be *increasing*, or *isotone*, when

$$s_1, s_2 \in D, s_1 \preceq s_2 \Longrightarrow A(s_1) \preceq A(s_2),$$

and *decreasing*, or *antitone*, if

$$s_1, s_2 \in D, s_1 \preceq s_2 \Longrightarrow A(s_2) \preceq A(s_1);$$

(ii) two points $s^*, t^* \in D$ are called *coupled fixed-points of* A if $t^* = A(s^*)$ and $s^* = A(t^*)$;

(iii) P is *normal* if and only if $x_n \preceq z_n \preceq y_n$, $\|x_n - x\| \to 0$, $\|y_n - x\| \to 0$, imply $\|z_n - x\| \to 0$, i.e., the *two militia men rule* holds;

(iv) P is *regular* if and only if every increasing (decreasing) and bounded in order from above (below) sequence in X has a limit.

We recall that every regular cone is normal, the converse being true in reflexive ordered spaces, e.g., in finite-dimensional ordered spaces. As a classical example of a normal but nonregular cone, which is of interest in the present paper, we quote $P = C_+[0,1] = \{u \in C[0,1] : u(x) \geq 0, \forall x \in [0,1]\}$ in $X = C[0,1]$, endowed with $\|\cdot\|_\infty$.

Lemma 2.1. *Let X be a Banach space partially ordered by a normal cone P and $A : P \to P$ a completely continuous and decreasing operator. Let $\{s_n\}, \{t_n\}, \{s_{n+1/2}\}, \{t_{n+1/2}\}$ be defined recursively by*

$$\begin{cases} t_0 = A(\theta) \\ t_{n+1/2} = A(t_n) \\ s_{n+1} = (1-\omega)s_n + \omega t_{n+1/2}, \end{cases} \tag{10}$$

$$\begin{cases} s_0 = \theta \\ s_{n+1/2} = A(s_n) \\ t_{n+1} = (1-\omega)t_n + \omega s_{n+1/2}, \end{cases} \tag{11}$$

where $\omega \in (0,1]$. Then there exist $\lim_n s_n = s^* \in P$, $\lim_n t_n = t^* \in P$ and the following chain of inequalities holds

$$\theta = s_0 \preceq s_1 \preceq \cdots \preceq s_n \preceq s^* \preceq t^* \preceq t_n \preceq \cdots \preceq t_1 = t_0. \tag{12}$$

Proof. First we observe that the operator $(u,v) \mapsto (1-\omega)u + \omega A(v)$ is *mixed-monotone* on $P \times P$, i.e., it is increasing in u and decreasing in v. Following the reasoning developed in [**13**], cf. also [**16**, Theorem 2.1.7], it is not difficult to show that

$$\theta \preceq s_0 \preceq s_1 \preceq \cdots \preceq s_n \preceq t_n \preceq \cdots \preceq t_1 = t_0 = A(\theta).$$

Now the operator from $P \times P$ into $P \times P$ defined by

$$\begin{aligned} (u,v) &\longmapsto ((1-\omega)u + \omega A(v), (1-\omega)v + \omega A(u)) \\ &= (1-\omega)(u,v) + \omega(A(v), A(u)) \end{aligned}$$

being the sum of a contractive operator, with constant $1-\omega$, and a completely continuous operator on the product space, is *condensing*, cf. [**28**, Example 11.7]. By a standard argument [**16**, Theorem 2.1.1], we obtain that the Kuratowski measure of noncompactness of the bounded sequence (s_n, t_n) is zero, i.e., that $\{(s_n, t_n)\}$ is relatively compact, which implies existence of a convergent subsequence $(s_{n_k}, t_{n_k}) \to (s^*, t^*)$, $k \to \infty$. By normality of P and monotonicity of s_n and t_n, it follows that $s_n \uparrow s^*$, $t_n \downarrow t^*$, $n \to \infty$. □

We are now ready to state and prove the following:

Theorem 2.2. *Let X be a Banach space with a normal cone P and A, E, F: $P \to P$ decreasing operators, with $A = E + F$ completely continuous. Assume that the fixed-point equation $u = E(u) + b$ has a unique solution in P for every $b \in [\theta, F(\theta)] \subseteq [\theta, A(\theta)]$. Then the sequences $\{s_n\}, \{t_n\}, \{u_n\}$, defined recursively by*

$$\begin{cases} t_0 = A(\theta) \\ t_{n+1/2} = E(t_n) + F(t_n) = A(t_n) \\ s_{n+1} = (1-\omega)s_n + \omega t_{n+1/2}, \end{cases} \tag{13}$$

$$\begin{cases} s_0 = \theta \\ s_{n+1/2} = E(s_n) + F(s_n) = A(s_n) \\ t_{n+1} = (1-\omega)t_n + \omega s_{n+1/2}, \end{cases} \tag{14}$$

$$\begin{cases} u_0 \in [\theta, A(\theta)] \\ u_{n+1/2} = E(u_{n+1/2}) + F(u_n) \\ u_{n+1} = (1-\omega)u_n + \omega u_{n+1/2}, \end{cases} \tag{15}$$

where $\omega \in (0, 1]$ satisfy

$$\theta = s_0 \preceq s_1 \preceq \cdots \preceq s_n \preceq u_n \preceq t_n \preceq \cdots \preceq t_1 \preceq t_0, \quad n = 0, 1, 2, \ldots . \tag{16}$$

If, in addition, A has no distinct comparable coupled fixed-points in $[\theta, A(\theta)] \subset P$, then A has a unique fixed-point in P, say u^*, and the sequences $\{s_n\}$, $\{t_n\}$, $\{u_n\}$ converge to u^*.

Proof. Note first that the sequence $\{u_n\}$ is well-defined, since $u = E(u) + b$ has a unique solution in P, for each $b \in [\theta, F(\theta)]$.

We prove now by induction that

$$s_n \preceq u_n \preceq t_n, \quad n = 0, 1, 2, \dots, \tag{17}$$

which is trivially verified for $n = 0$. Assume that (17) holds for a fixed $n \geq 0$. Our basic step consists in proving

$$t_{m+1/2} \preceq u_{n+1/2} \preceq s_{m+1/2}, \tag{18}$$

for $m = 0, 1, \dots, n$. Again, we proceed by induction (on m). For $m = 0$, by (13)–(15) we have $u_{n+1/2} = E(u_{n+1/2}) + F(u_n) \preceq E(\theta) + F(\theta) = s_{1/2}$ and $t_{1/2} = E(A(\theta)) + F(A(\theta)) \preceq E(u_{n+1/2}) + F(u_n) = u_{n+1/2}$, i.e., the basis of the induction is verified. Assume that (18) holds for a fixed $m < n$. By the main inductive assumption (17) and (12), we get $s_m \preceq u_n \preceq t_m$ and thus, using (13)–(15), the inequalities $s_{m+1} \preceq u_{n+1/2} \preceq t_{m+1}$ are derived. Moreover, (17) and (12) also give immediately $s_m \preceq s_{m+1} \preceq u_n \preceq t_{m+1} \preceq t_m$. Finally, by monotonicity of E, F and (13)–(15), we have

$$\begin{aligned} t_{m+3/2} &= E(t_{m+1}) + F(t_{m+1}) \preceq E(u_{n+1/2}) + F(u_n) = u_{n+1/2}, \\ u_{n+1/2} &= E(u_{n+1/2}) + F(u_n) \preceq E(s_{m+1}) + F(s_{m+1}) = s_{m+3/2}, \end{aligned} \tag{19}$$

i.e., the inner induction is completed. In particular, for $m = n$ we have $t_{n+1/2} \preceq u_{n+1/2} \preceq s_{n+1/2}$. This latter, together with the main inductive assumption $s_n \preceq u_n \preceq t_n$ and the definition of the iterative schemes (13)–(15), finally provides the inequalities $s_{n+1} \preceq u_{n+1} \preceq t_{n+1}$, i.e., also the outer induction is completed.

Now, from (13), (14) and (12), taking the limit as $n \to \infty$, we get $s^* = (1 - \omega)s^* + \omega A(t^*)$ and $t^* = (1 - \omega)t^* + \omega A(s^*)$, i.e., $s^* = A(t^*)$, $t^* = A(s^*)$. But A has no distinct comparable fixed-points, hence $s^* = t^* = u^*$. Being the cone P normal, we can apply the "two militia men rule", obtaining $u_n \to u^*$ as $n \to \infty$.

Concerning *uniqueness* of the fixed-points of A in P, it is easy to show that if there exists $\hat{u} \in P$ such that $\hat{u} = A(\hat{u})$, then $A^{2n}(\theta) \preceq \hat{u} \preceq A^{2n+1}(\theta)$ for every $n \in \mathbf{N}$. Taking $E \equiv 0$ and $\omega = 1$ in (15), i.e., the standard Picard method, and starting from $u_0 = \theta$ we obtain

$u_n = A^n(\theta) \to u^*$, $n \to \infty$; the two militia men rule allows us finally to conclude that $\hat{u} = u^*$. □

Remark 2.3. The proof of Theorem 2.2 does not seem to provide a way to estimate the effective convergence rate of the relaxed schemes, being essentially based on the *qualitative* property of decreasing monotonicity of all the operators involved. On the other hand, in Section 3, the convergence behavior varying the relaxation parameter, and the optimal choice of such a parameter, which are commonly recognized as difficult theoretical topics even in the linear case, will be analyzed "experimentally" on the decreasing version (7) of the Chandrasekhar H-equation.

Remark 2.4. The thesis of Theorem 2.2 can be proved also assuming that the operator A is merely *demicontinuous*, i.e., $x_n \to x$ strongly implies $A(x_n) \to A(x)$ weakly, provided that the cone P is *regular*. In fact, existence of the limits s^* and t^* in point (a) of Lemma 2.1 becomes trivial, while the fact that $s^* = A(t^*)$ and $t^* = A(s^*)$ can be recovered even when A is demicontinuous, cf. [**16**, Theorem 2.1.7].

Remark 2.5. A *sufficient* condition ensuring nonexistence of distinct and comparable coupled fixed-points is given by *uniqueness* of (positive) fixed-points for $A^2 = A \circ A$. In fact, *coupled* fixed-points of A are *separately* fixed-points of A^2.

2.2. Application to Hammerstein integral equations. In the present section we focus our attention on the discrete Hammerstein equation (8)–(9), and on its approximate solution by the stationary (under)relaxed iterative processes P_ω, UP_ω, J_ω and SOR described in the introduction. Writing the matrix K in (8) as

$$K = D + L + U, \tag{20}$$

D, L and U being its diagonal, lower and upper triangular parts, respectively, such processes correspond to the following splittings $A =$

$KN = E + F$ of the nonlinear operator in (8):

$$
\begin{aligned}
(P_\omega) \quad & E = 0, \quad F = KN; \\
(UP_\omega) \quad & E = LN, \quad F = (U + D)N; \\
(J_\omega) \quad & E = DN, \quad F = (L + U)N; \\
(SOR) \quad & E = (L + D)N, \quad F = UN.
\end{aligned} \tag{21}
$$

The operators A, E and F are *decreasing* and (completely) *continuous* from $\mathbf{R}_+^m$ into $\mathbf{R}_+^m$ as $k_{ij} \geq 0$ for all i, j and the $f_j : \mathbf{R}_+ \to \mathbf{R}_+$ are decreasing and continuous scalar functions. Moreover, the system $u = E(u) + b$ has a unique solution in $\mathbf{R}_+^m$ for any $b \in \mathbf{R}_+^m$; this is trivial for P_ω and UP_ω, such methods being indeed *explicit*, it is immediate for J_ω, since the mapping E is decreasing and *diagonal*, while it can be easily proved via backward substitution for SOR. In both the latter cases we use the fact that a positive decreasing and continuous scalar function defined on $\mathbf{R}_+$ has a unique fixed-point. Thus, taking $X = (\mathbf{R}^m, \|\cdot\|_\infty)$, $P = \mathbf{R}_+^m = [0, +\infty)^m$, the general result given by Theorem 2.2 ensures global convergence of all of the four relaxed methods above to the unique positive solution of (8), provided that the operator $A = KN$ does not possess distinct and comparable positive coupled fixed-points.

Note that this is equivalent to nonexistence of such a pair in $[\theta, A(\theta)]$, as required in the statement of the theorem, since positive coupled fixed-points belong necessarily to $[\theta, A(\theta)]$, A being decreasing. Such a requirement is trivially satisfied, for example, when A is a *contraction* on $[\theta, A(\theta)]$. In the case of the (discrete) Chandrasekhar H-equation (7), (23), this amounts to the constraint $\lambda < 1/3$, while instances of the equation with λ close to $1/2$ are physically meaningful.

On the other hand, as already pointed out in the introduction, nonexistence of coupled fixed-points in $C_+(\Omega)$ can be proved directly for the general formulation (1)–(4), applying essentially the same technique of [**10**], cf. also [**24, 26**]. As for Chandrasekhar H-equation, we are then entitled to apply the relaxed Picard-like methods for any value of $\lambda \in (0, 1/2)$.

Remark 2.6. It is worth pointing out that, besides Picard, also updated Picard and Gauss-Seidel iterations have a plain *infinite-dimensional counterpart*, in the case of one-dimensional Hammerstein

integral equations. Indeed, consider equation (1)–(6) on a real interval $\Omega = [\alpha, \beta]$. Taking $X = (C[\alpha, \beta], \|\cdot\|_\infty)$, $P = C^+[\alpha, \beta]$, updated Picared and SOR both correspond to the infinite-dimensional splitting

$$\begin{aligned} E &= \int_\alpha^x k(x,t) f(t, u(t))\, dt, \\ F &= \int_x^\beta k(x,t) f(t, u(t))\, dt, \end{aligned} \tag{22}$$

in Theorem 2.2, which is applicable in any dimension. The only delicate point concerns existence and uniqueness of a positive continuous solution to the Volterra-Hammerstein equation $u = E(u) + b$ for every $b \in [\theta, F(\theta)]$. Observing that, by (6) and the Ascoli-Arzelà theorem [**11**], E is itself completely continuous, the required existence and uniqueness become again a consequence of the basic fixed-point theorem for decreasing operators in [**16**, Theorem 2.1.5].

3. Numerical examples. In this section we test the relaxed iterations P_ω, UP_ω, J_ω and SOR, see (20)–(21), on the following discretization of the Chandrasekhar H-equation in its decreasing form (7)

$$u_i = \frac{\lambda}{1 - 2\lambda} \sum_{j=1}^{m} w_j \frac{k(x_i, x_j)}{1 + u_j}, \quad 1 \le i \le m, \tag{23}$$

for some values of the parameter $\lambda \in (0, 1/2)$, where the $\{w_j\}$ are the weights of the trapezoidal quadrature formula with constant step $h = 1/(m-1)$, and $x_j = (j-1)h$ for $j = 1, \ldots, m$; moreover, $k(x,t) = t/(x+t)$ for $0 \le t \le 1$, $0 < x \le 1$, and $k(0,0) = 1$, which corresponds physically to the case of *isotropic* scattering, cf. [**7**]. The four relaxed methods have been implemented in the MATLAB environment, choosing the null vector as starting approximation, and Newton method as scalar solver for J_ω and SOR. As termination criterion we used the simple *relative step in the* ∞*-norm*, i.e., iterations are stopped as soon as

$$\|u_n - u_{n-1}\|_\infty \le \|u_n\|_\infty \cdot r\mathrm{tol}, \quad n = 1, 2, \ldots, \tag{24}$$

where $r$tol is a relative error tolerance. Such criterion is indeed reliable in the present applications, since all of the relaxed methods above

turn out to be *linearly convergent*, with asymptotic error constants well bounded away from 1, cf. [**17**, Section 5.2].

In Tables 1–3 below, we illustrate the performance of the relaxed methods on system (23) with $m = 100$ and three values of λ in the neighborhood of $1/2$, for the set of discrete values $\omega = 0.1, 0.2, \ldots, 1$. For each value of ω and each method, we reported the number of iterations and of Kflops required to satisfy the termination test (24) with $r\text{tol} = 10^{-6}$. Such quantities exhibit an apparently convex dependence on ω with internal minimum for P_ω and J_ω, and with a decreasing behavior for UP_ω and SOR (with minimum at $\omega = 1$, i.e., for the unrelaxed versions UP and GS). Relatively small oscillations appear on finer ω-discretizations, in particular for Jacobi method, but still a quasi-optimal parameter can be recovered by a simple bisection procedure. Comparison with the unrelaxed Picard and Jacobi iterations shows a remarkable speedup even for rough approximations of the optimal parameters. For example, in the case $\lambda = 0.4999995$, the relaxed Picard method for $\omega = 0.6$ is more than 50 times faster than the unrelaxed one, see Table 3.

In Tables 4–6 we compare the four relaxed methods with Newton-GMRES and Broyden solvers on system (23), for the three values of λ above, and $m = 10$, 50, 100, 200, 400. All methods start from $u_0 = \theta = (0, \ldots, 0)$ and stop as soon as (24) is satisfied with $r\text{tol} = 10^{-6}$; we recall that such a termination criterion is reliable

TABLE 1. Number of iterations and Kflops for the relaxed methods, $\lambda = 0.45$.

ω	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
it. P_ω	78	42	28	21	16	13	11	10	15	22
Kfl. P_ω	1706	928	625	474	366	301	258	236	344	479
it. UP_ω	104	53	235	25	20	16	13	10	8	8
Kfl. UP_ω	2269	1166	777	560	452	366	301	236	193	187
it. J_ω	80	43	29	21	17	13	11	10	15	21
Kfl. J_ω	2109	1136	769	560	455	354	304	282	412	562
it. SOR	104	53	35	25	20	16	13	10	8	8
Kfl. SOR	2773	1420	944	680	547	441	362	284	232	225

TABLE 2. Number of iterations and Kflops for the relaxed methods, $\lambda = 0.49995$.

ω	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
it. P_ω	100	54	37	27	21	17	18	29	59	660
Kfl. P_ω	2182	1187	820	604	474	387	409	647	1295	13821
it. UP_ω	108	60	43	32	25	20	17	14	13	12
Kfl. UP_ω	2355	1317	950	712	560	452	387	323	301	271
it. J_ω	99	54	37	28	22	17	18	27	53	298
Kfl. J_ω	2631	1453	1006	767	608	479	506	740	1417	7779
it. SOR	109	60	43	32	25	20	17	14	13	12
Kfl. SOR	2937	1632	1174	881	693	559	478	398	369	331

TABLE 3. Number of iterations and Kflops for the relaxed methods, $\lambda = 0.4999995$.

ω	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
it. P_ω	122	65	43	32	25	20	20	31	66	> 1000
Kfl. P_ω	2657	1425	949	712	560	452	452	690	1447	> 20000
it. UP_ω	132	74	50	37	28	22	19	16	14	12
Kfl. UP_ω	2874	1620	1101	820	625	496	431	366	323	271
it. J_ω	121	64	43	32	25	20	20	29	58	497
Kfl. J_ω	3350	1801	1225	920	725	586	582	809	1564	12951
it. SOR	132	74	50	37	29	23	19	16	14	12
Kfl. SOR	3613	2029	1378	1026	809	646	537	456	399	332

also for Newton-like solvers, cf. [**17**, Section 5.2]. The basic MATLAB implementations of Newton-GMRES and Broyden methods are taken from the electronically available packages accompanying [**17**]. Very similar numerical results are obtained correspondingly to the other natural choices, $u_0 = A(\theta) = KN(\theta) = Ke$, $e = (1, \dots, 1)$, cf. (1), (8) and (23), or $u_0 = A(\theta)/2$, recall that necessarily $[\theta, A(\theta)] \ni u^*$.

First we stress the fact that the quasi-optimal relaxation parameters, computed by a bisection algorithm, are essentially mesh-independent. In particular, it is experimentally confirmed that the optimal parameter is $\omega = 1$ for updated Picard and SOR methods. As for Picard and Jacobi methods, this suggests that a quasi-optimal parameter can be computed cheaply on a rough discretization and conveniently used on a finer one.

This (asymptotic) *mesh-independence* is also evident concerning the number of iterations, for all the six methods. Mesh-independence principles for Newton-like solvers are well-known and have been deeply investigated, cf., e.g., [**1**]. It is noteworthy that such principles arise also in the context of relaxed Picard-like methods: as a possible *qualitative* interpretation, we can reasonably expect that the convergence properties of (optimally) relaxed Picard, updated Picard and SOR methods on the discretizations become close to those of their infinite-dimensional counterparts, cf. Remark 2.6, for m sufficiently large. This is confirmed by the fact that in Tables 4–6, the methods UP and GS exhibit the same number of iterations on large m.

It should also be noticed that the number of iterations of the relaxed methods stabilizes as λ approaches $1/2$, while that of Newton-GMRES and Broyden methods increases. Such a worsening of Newton-like solvers near $\lambda = 1/2$ has already been observed in the applications to the standard, increasing, form of the discrete Chandrasekhar H-equation, cf. [**17**], and is there ascribed, essentially, to the fact that the Jacobian matrix of the system becomes nearly singular. The singularity disappears in the present "decreasing" Hammerstein formulation, and the convergence slowing down could be overcome by suitably improving the initial approximation. A *hybrid* Picard-Newton solver could then represent an appealing choice in order to exploit *local superlinearity* of Newton-like methods together with *global convergence* of optimally relaxed Picard-like iterations, in particular for λ close to $1/2$. Numerical experiments, however, have shown that such hybrid methods do not exhibit a substantial improving with respect to the corresponding Picard-like iterations, in the precision range considered.

We stress that, indeed, UP and GS work satisfactorily for any value of $\lambda \in (0, 1/2)$, and for any nonnegative starting vector. There is numerical evidence of the good performance of UP, GS and optimally relaxed Picard, especially for λ close to $1/2$, where they overcome even Broyden method (Newton-GMRES turns out to be the worst method in the present context). In particular, UP appears the best choice with the present implementations and in the (relative) precision range considered.

It should be recalled that the good performance of updated Picard iterations UP, has already been experimentally recognized in the physical literature, cf. [**5, 7**], where the unrelaxed method is applied directly

to the original quadratic integral models. Indeed, [**7**] considers accurate numerical computation of H-functions for various types of scattering, resorting to the Chandrasekhar H-equation in its "decreasing" form (7). On the other hand, [**5**] concerns the numerical solution of certain purely integral instances of the Boltzmann equation by means of Picard and updated Picard iterations. Performance of UP, however, is there only compared with that of the basic Picard method, and convergence is proved only for the latter. Our Theorem 2.2, via reduction to Hammerstein form, now gives a theoretical basis to such a method, inserting it in the more general context of relaxed Picard-like solvers for "decreasing" fixed-point problems. We omit, for brevity, the details concerning equivalence of the updated iterations in the quadratic and in the Hammerstein setting, cf., e.g., [**16**, Theorem 3.3.5] and [**25**] for the basic transformation procedure.

TABLE 4. Number of iterations and Kflops for "optimally" relaxed and N-type methods, $\lambda = 0.45$.

m	$P_{0.760}$		UP		$J_{0.760}$		GS		N_{gmres}		Bro	
10	9	4	9	3	10	9	8	7	6	12	6	3
50	9	57	9	54	9	77	8	68	6	122	7	48
100	9	214	8	187	9	253	8	225	6	423	7	177
200	9	828	8	735	9	906	8	810	6	1564	7	673
400	9	3257	8	2910	9	3405	8	3050	6	6007	7	2627

TABLE 5. Number of iterations and Kflops for "optimally" relaxed and N-type methods, $\lambda = 0.49995$.

m	$P_{0.647}$		UP		$J_{0.635}$		GS		N_{gmres}		Bro	
10	14	5	13	4	17	17	11	10	10	17	14	9
50	14	86	12	70	16	141	12	101	10	183	14	102
100	14	322	12	271	16	448	12	331	10	635	14	354
200	14	1244	12	1062	16	1571	12	1180	10	2348	14	1307
400	14	4889	12	4204	16	5857	12	4436	10	8673	14	5013

TABLE 6. Number of iterations and Kflops for "optimally" relaxed and N-type methods, $\lambda = 0.4999995$.

m	$P_{0.708}$		UP		$J_{0.650}$		GS		N_{gmres}		Bro	
10	17	6	14	4	19	22	11	10	13	19	18	13
50	15	92	12	70	18	171	12	102	13	209	18	137
100	15	344	12	271	18	527	12	332	13	725	18	464
200	15	1328	12	1062	18	1805	12	1181	13	2688	18	1686
400	15	5215	12	4204	18	6633	12	4438	13	10334	18	6412

As a final remark, concerning *storage* allocation, we recall that the Broyden method requires at least $n + 3$ vectors at the nth iteration, so that it has to be restarted in the presence of large dimension and/or large number of iterations [**17**]. We note that Picard and Jacobi methods use essentially two vectors while UP and GS can be implemented even with one vector (there is no need to precompute the kernel matrix when the analytic expression of the kernel is at hand). This feature makes the updated Picard method even more attractive at large dimension, for example, in the numerical treatment of the 3D purely integral Boltzmann models described in [**4**] (stationary, space homogeneous, forceless case of the "scattering-kernel" formulation of the Boltzmann equation).

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