JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 12, Number 1, Spring 2000

A NOTE ON THE SOLUTION SET OF INTEGRAL INCLUSIONS

R. KANNAN AND DONAL O'REGAN

ABSTRACT. In this note we discuss the topological structure of the set of solutions of integral and differential inclusions.

1. Introduction. This paper discusses the structure of the solution set of the Volterra integral inclusion

$$y(t) \in h(t) + \int_0^t k(t,s)F(s,y(s))\,ds \quad \text{for } t \in [0,T]. \tag{1.1}$$

Throughout $k : [0,T] \times [0,t] \to \mathbf{R}$ and $F : [0,T] \times \mathbf{R}^n \to CK(\mathbf{R}^n)$; here $CK(\mathbf{R}^n)$ denotes the family of all nonempty, compact, convex subsets of $\mathbf{R}^n$. In the literature only a few results have appeared on the structure of the solution set of (1.1); we refer the reader to [**1**, p. 219] and the references therein. For completeness we state here the main result available in the literature [**1**]. Let $S(h;\mathbf{R}^n)$ denote the solution set of (1.1).

Theorem 1.1. *Let* $k : [0,T]\times[0,t] \to \mathbf{R}$, $F : [0,T]\times\mathbf{R}^n \to CK(\mathbf{R}^n)$ *and suppose the following conditions hold*:

$$t \longmapsto F(t,x) \quad \textit{is measurable for every } x \in \mathbf{R}^n \tag{1.2}$$

$$\begin{cases} x \longmapsto F(t,x) \quad \textit{is upper semicontinuous } (u.s.c.) \\ \textit{for a.e. } t \in [0,T] \end{cases} \tag{1.3}$$

$$\begin{cases} \textit{there exists } h \in L^1[0,T] \textit{ with } \|F(t,x)\| \le h(t) \\ \textit{for a.e. } t \in [0,T] \textit{ and every } x \in R^n \end{cases} \tag{1.4}$$

Accepted for publication on November 18, 1998.

$$h \in C[0,T] \tag{1.5}$$

$$\begin{cases} \textit{for each } t \in [0,T],\ k(t,s) \textit{ is measurable on } [0,t] \textit{ and} \\ k(t) = \operatorname{ess\,sup}|k(t,s)|, 0 \le s \le t, \text{ is bounded on } [0,T] \end{cases} \tag{1.6}$$

and

$$\begin{cases} \textit{the map } t \mapsto k_t \textit{ is continuous from } [0,T] \textit{ to } L^\infty[0,T]; \\ \textit{here } k_t(s) = k(t,s). \end{cases} \tag{1.7}$$

Then $S(h;\mathbf{R}^n)$ *is nonempty, connected and compact.*

Remark 1.1. In [**1**]

$$F(\cdot,x) \text{ possesses a measurable selection} \tag{1.8}$$

was assumed instead of (1.2). Notice [**3**, p. 22] implies if (1.2) is true then automatically (1.8) is true.

One of the main goals of this paper is to remove the "global" integrably boundedness assumption, see (1.4), on F. By using Theorem 1.1 and a trick involving the Urysohn function we are able to accomplish this if we assume a "local" integrably boundedness assumption on F. This is exactly what we need from an application viewpoint.

2. Solution set. First we establish a general existence principle for (1.1). We assume (1.2), (1.3), (1.5), (1.6) and (1.7) hold. In addition suppose the following conditions are also satisfied:

$$\begin{cases} \text{for each } r > 0 \text{ there exists } h_r \in L^1[0,T] \text{ with} \\ \|F(t,x)\| \le h_r(t) \text{ for a.e. } t \in [0,T] \text{ and every} \\ x \in \mathbf{R}^n \text{ with } \|x\| \le r \end{cases} \tag{2.1}$$

and

$$\begin{cases} \text{there exists a constant } M > \|h\|_0 = \sup_{t\in[0,T]} \|h(t)\| \text{ with} \\ \|y\|_0 < M \text{ for any possible solution to (1.1).} \end{cases} \tag{2.2}$$

Let $\varepsilon > 0$ be given, and let $\tau_\varepsilon : \mathbf{R}^n \to [0,1]$ be the Urysohn function for

$$(\overline{B}(0,M), \mathbf{R}^n \backslash B(0,M+\varepsilon))$$

such that $\tau_\varepsilon(x) = 1$ if $\|x\| \le M$ and $\tau_\varepsilon(x) = 0$ if $\|x\| \ge M + \varepsilon$. Let $\tilde{F}(t, x) = \tau_\varepsilon(x)F(t, x)$ and consider the problem

$$y(t) \in h(t) + \int_0^t k(t, s)\tilde{F}(s, y(s))\, ds \quad \text{for } t \in [0, T]. \tag{2.3}$$

Let $S_\varepsilon(h; \mathbf{R}^n)$ denote the solution set of (2.3).

Theorem 2.1. *Suppose* (1.2), (1.3), (1.5), (1.6), (1.7), (2.1) *and* (2.2) *hold. Let* $\varepsilon > 0$ *be given and assume*

$$\|w\|_0 < M \quad \textit{for any possible solution } w \in C[0, T] \textit{ to } (2.3). \tag{2.4}$$

Then $S(h; \mathbf{R}^n)$ *is nonempty, connected and compact.*

Proof. Notice (2.2) and (2.4) imply $S(h; \mathbf{R}^n) = S_\varepsilon(h; \mathbf{R}^n)$. It is easy to see that $\tilde{F} : [0, T] \times \mathbf{R}^n \to CK(\mathbf{R}^n)$ satisfies (1.2) and (1.3), with F replaced by $\tilde{F}$. Also (2.1) and the definition of τ_ε imply that $\tilde{F}$ satisfies (1.4), with F replaced by $\tilde{F}$. Thus, Theorem 1.1 implies $S_\varepsilon(h; \mathbf{R}^n)$ is nonempty, connected and compact. □

The existence principle, Theorem 2.1, can now be used to establish some applicable results. We illustrate the ideas involved with the following theorem.

Theorem 2.2. *Let* $k : [0, T] \times [0, t] \to \mathbf{R}$, $F : [0, T] \times \mathbf{R}^n \to CK(\mathbf{R}^n)$ *and assume* (1.2), (1.3), (1.5), (1.6), (1.7) *and* (2.1) *hold. In addition, suppose the following conditions are satisfied*:

$$\begin{cases} \text{there exists } \alpha \in L^1[0, T] \text{ and } g : [0, \infty) \to (0, \infty) \text{ a} \\ \text{nondecreasing continuous function such that} \\ \|k(t, s)F(s, u)\| \le \alpha(s)g(\|u\|) \text{ for a.e. } s \in [0, t], \\ \text{a.e. } t \in [0, T] \text{ and all } u \in R^n \end{cases} \tag{2.5}$$

and

$$\int_0^T \alpha(s)\, ds < \int_{\|h\|_0}^\infty \frac{dx}{g(x)}. \tag{2.6}$$

Then $S(h;\mathbf{R}^n)$ *is nonempty, connected and compact.*

Proof. Let $\varepsilon > 0$ be given,

$$M_0 = I^{-1}\biggl(\int_0^T \alpha(s)\,ds\biggr),$$

where

$$I(z) = \int_{\|h\|_0}^{z} \frac{dx}{g(x)}, \quad \text{and} \quad M = M_0 + 1.$$

We will show any possible solution u of (1.1) satisfies $\|u\|_0 \leq M_0$ and any possible solution y of (2.3) satisfies $\|y\|_0 \leq M_0$. If this is true, then Theorem 2.1 guarantees the result.

Suppose u is a possible solution of (2.1). Then

$$\|u(t)\| \leq \|h\|_0 + \int_0^t \alpha(s) g(\|u(s)\|)\,ds \equiv w(t) \quad \text{for } t \in [0,T].$$

Now $w'(t) = \alpha(t) g(\|u(t)\|) \leq \alpha(t) g(w(t))$ almost everywhere and so

$$\int_{\|h\|_0}^{w(x)} \frac{ds}{g(s)} = \int_0^x \frac{w'(s)}{g(w(s))}\,ds \leq \int_0^x \alpha(s)\,ds \leq \int_0^T \alpha(s)\,ds$$

for $x \in [0,T]$. Thus $w(x) \leq M_0$ for any $x \in [0,T]$ and so $\|u(x)\| \leq M_0$ for all $x \in [0,T]$.

Next let y be a possible solution of (2.3). For $t \in [0,T]$, since $\tau_\varepsilon : \mathbf{R}^n \to [0,1]$, we have

$$\|y(t)\| \leq \|h\|_0 + \int_0^t \alpha(s) g(\|y(s)\|)\,ds$$

and again we have $\|y(x)\| \leq M_0$ for all $x \in [0,T]$. □

Next we discuss the differential inclusion

$$\begin{cases} y' \in F(t,y) & \text{a.e. } t \in [0,T] \\ y(0) = y_0. \end{cases} \tag{2.7}$$

We discuss a more general situation than before, namely when $F : [0,T] \times E \to CK(E)$; here E is a real Banach space. By a solution to (2.7) we mean a function $y \in W^{1,1}([0,T],E)$, see [**7**], which satisfies the differential inclusion almost everywhere on $[0,T]$ and the stated initial data. Let $S(y_0;E)$ denote the solution set of (2.7). The analogue of Theorem 1.1, in this situation, may be found in [**3**, p. 118]; we state it here.

Theorem 2.3. *Let E be a separable Banach space and $F : [0,T] \times E \to CK(E)$. Suppose the following conditions are satisfied*:

$$t \longmapsto F(t,x) \quad \text{is measurable for every } x \in E \tag{2.8}$$

$$x \longmapsto F(t,x) \quad \text{is u.s.c. for a.e. } t \in [0,T] \tag{2.9}$$

$$\begin{cases} \textit{there exists } h \in L^1[0,T] \textit{ such that } \|F(t,x)\| \le h(t) \\ \textit{for a.e. } t \in [0,T] \textit{ and all } x \in E \end{cases} \tag{2.10}$$

and

$$\textit{for any bounded set } A \subseteq E \textit{ we have } \alpha(F([0,T] \times A)) = 0. \tag{2.11}$$

Then $S(y_0;E)$ is nonempty, connected and compact.

Remark 2.1. In fact [**3**], $S(y_0;E)$ is an R_δ set.

Remark 2.2. α denotes the Kuratowskii measure of noncompactness.

Remark 2.3. Instead of Theorem 2.3, we could state a result in [**10**, p. 1093].

Now we establish a general existence principle for (2.7). We assume (2.8) and (2.9) hold. In addition, suppose the following conditions are satisfied:

$$\begin{cases} \text{for each } r > 0 \text{ there exists } h_r \in L^1[0,T] \text{ s.t.} \\ \|F(t,x)\| \le h_r(t) \text{ for a.e. } t \in [0,T] \text{ and all} \\ x \in E \text{ with } \|x\| \le r \end{cases} \tag{2.12}$$

and

$$\left\{\begin{array}{l}\text{there exists a constant } M > \|y_0\| \text{ with } \|y\|_0 < M \\ \text{for any possible solution to (2.7).}\end{array}\right. \tag{2.13}$$

Let $\varepsilon > 0$ be given, and let $\tau_\varepsilon : E \to [0,1]$ be the Urysohn function for

$$(\overline{B}(0,M), E\backslash B(0,M+\varepsilon))$$

such that $\tau_\varepsilon(x) = 1$ if $\|x\| \le M$ and $\tau_\varepsilon(x) = 0$ if $\|x\| \ge M + \varepsilon$. Let $\tilde{F}(t,x) = \tau_\varepsilon(x) F(t,x)$ and consider

$$\left\{\begin{array}{l} y'(t) \in \tilde{F}(t,y(t)) \quad \text{a.e. } t \in [0,T] \\ y(0) = y_0. \end{array}\right. \tag{2.14}$$

Let $S_\varepsilon(y_0;E)$ denote the solution set of (2.14).

Theorem 2.4. *Let E be a separable Banach space and $F : [0,T] \times E \to CK(E)$. Suppose* (2.8), (2.9), (2.11), (2.12) *and* (2.13) *hold. Let $\varepsilon > 0$ be given and assume*

$$\left\{\begin{array}{l} \|w\|_0 < M \textit{ for any possible solution} \\ w \in W^{1,1}([0,T],E) \textit{ to } (2.14). \end{array}\right. \tag{2.15}$$

Then $S(y_0;E)$ is nonempty, connected and compact.

Proof. Notice $S(y_0;E) = S_\varepsilon(y_0;E)$. Also (2.12) and the definition of τ_ε implies that $\tilde{F}$ satisfies (2.10), with F replaced by $\tilde{F}$. Notice also if A is a bounded subset of E, then

$$\tilde{F}([0,T] \times A) \subseteq \overline{\text{co}}(F([0,T] \times A) \cup \{0\}).$$

This together with (2.11) and the properties of the measure of noncompactness yields $\alpha(\tilde{F}([0,T] \times A)) = 0$. Thus, Theorem 2.3 implies $S_\varepsilon(y_0;E)$ is nonempty, connected and compact. $\square$

At this stage we could easily establish an existence result of the type in Theorem 2.2 for the differential inclusion (2.7); we leave this to the reader. Instead, to illustrate the generality of the method described above, we establish a new result for differential equations when $E = \mathbf{R}$. In particular, we discuss

$$\begin{cases} y'(t) = \alpha(t)g(y(t)) & \text{for } t \in (0,T) \\ y(0) = 0. \end{cases} \tag{2.16}$$

Let $S(0;\mathbf{R})$ denote the solution set of (2.16).

Theorem 2.5. *Let* $g : \mathbf{R} \to \mathbf{R}$ *be continuous and* $\alpha : (0,T) \to [0,\infty)$ *be such that* $\alpha \in C(0,T]$*. In addition, suppose the following conditions hold*:

$$\alpha > 0 \quad \text{on } (0,T] \quad \text{with } \alpha \in L^1[0,T] \tag{2.17}$$

$$g(0) > 0 \tag{2.18}$$

$$\begin{cases} g \text{ has a positive zero (let } r_1 \text{ be} \\ \text{the smallest positive zero of } g) \end{cases} \tag{2.19}$$

and

$$\int_0^T \alpha(x)\,dx \le \int_0^{r_1} \frac{dx}{g(x)}. \tag{2.20}$$

Then $S(0;\mathbf{R})$ *is nonempty, connected and compact.*

Remark 2.4. Solutions to (2.16) will lie in $W^{1,1}([0,T],\mathbf{R}) \cap C^1(0,T]$.

Proof. Let $\varepsilon > 0$ be given, and let $M = r_1 + 1$. We will show any possible solution u of (2.16) satisfies $\|u\|_0 < M$ and any possible solution y of

$$\begin{cases} y'(t) = \tau_\varepsilon(y(t))\alpha(t)g(y(t)) & \text{for } t \in (0,T) \\ y(0) = 0 \end{cases} \tag{2.21}$$

satisfies $\|y\|_0 < M$. If this is true, then Theorem 2.4 guarantees the result.

Suppose u is a possible solution of (2.16). Now (2.17) and (2.18) imply $u' > 0$ in a neighborhood of zero. Suppose $u' > 0$ on $(0, \delta)$ and $u'(\delta) = 0$. Then $u(\delta) = r_1$. If $\delta < T$, then we have

$$\int_0^{r_1} \frac{dx}{g(x)} = \int_0^{\delta} \alpha(x)\, dx < \int_0^T \alpha(x)\, dx,$$

a contradiction. Thus $u' > 0$ on $(0, T)$ so $0 < u(t) < r_1$ for $t \in (0, T)$. Thus $0 \le u(t) \le r_1$ for $t \in [0, T]$.

Now let y be a possible solution of (2.21). Since $y(0) = 0$ and $M = r_1 + 1$ there exists an interval $(0, \delta_1)$ with $\tau_\varepsilon(y(t)) = 1$ for $t \in [0, \delta_1)$. Now (2.17) and (2.18) together with $\tau_\varepsilon(y(t)) = 1$ on $[0, \delta_1)$ implies $y' > 0$ in a neighborhood of zero, say on $(0, \delta_2)$. Note $\tau_\varepsilon(y(t)) = 1$ on $[0, \delta_2)$. To see this, notice if not, then there exists a $\delta_3 \in (0, \delta_2)$ with $\tau_\varepsilon(y(\delta_3)) \in [0, 1)$. Thus $y(\delta_3) > M = r_1 + 1$ so there exists $\delta_4 \in (0, \delta_3)$, since $y(0) = 0$, with $y(\delta_4) = r_1$. Hence $y'(\delta_4) = 0$, a contradiction. Thus $\tau_\varepsilon(y(t)) = 1$ on $[0, \delta_2)$. Suppose $y'(\delta_2) = 0$. Then $y' > 0$ on $(0, \delta_2)$ with $y'(\delta_2) = 0$ and $y'(t) = \alpha(t) g(y(t))$ for $t \in (0, \delta_2)$. If $\delta_2 < T$ we obtain, as above, a contradiction. Thus $y' > 0$ on $(0, T)$ and so we have $0 \le y(t) \le r_1$ for $t \in [0, T]$. □

More generally we may consider

$$\text{(2.22)} \qquad \begin{cases} y'(t) = \alpha(t) f(t, y(t)) & \text{for } t \in (0, T) \\ y(0) = 0. \end{cases}$$

Let $S_f(0; \mathbf{R})$ denote the solution set of (2.22). Essentially the same reasoning as in Theorem 2.5 establishes the following result.

Theorem 2.6. *Let* $f : [0, T] \times \mathbf{R} \to \mathbf{R}$ *be continuous and* $\alpha : (0, T) \to [0, \infty)$ *be such that* $\alpha \in C(0, T]$. *In addition, suppose* (2.17) *holds and also assume the following conditions are satisfied*:

$$\text{(2.23)} \qquad \begin{cases} \textit{there exists a continuous function } g : \mathbf{R} \to \mathbf{R} \\ \textit{with } g(0) > 0 \textit{ and } |f(t, y)| \le |g(y)| \textit{ for} \\ t \in [0, T] \textit{ and } y \in \mathbf{R} \end{cases}$$

$$\text{(2.24)} \qquad f(0, 0) > 0$$

$$\text{if } r \neq 0 \text{ and } f(t,r) = 0 \text{ for some } t \in (0,T), \text{ then } g(r) = 0 \tag{2.25}$$

$$\begin{cases} g \text{ has a positive zero (let } r_1 \text{ be} \\ \text{the smallest positive zero of } g) \end{cases} \tag{2.26}$$

and

$$\int_0^T \alpha(x)\,dx \leq \int_0^{r_1} \frac{dx}{g(x)}. \tag{2.27}$$

Then $S_f(0;\mathbf{R})$ *is nonempty, connected and compact.*

Remark 2.5. The argument in the proof of Theorem 2.5 based on approaching the "barriers" $y = 0$ and $y = r_1$ from the inside was introduced [**6**] in 1990 (it has been extended, using a similar type of argument, in [**5**]). In some sense the argument given in Theorem 2.5 is the opposite to the "upper and lower solution" type approach, see [**7**]. As was seen above the approach in Theorem 2.5 guarantees that *all* solutions are a priori bounded. However, notice that the upper and lower solution type approach only guarantees that there exists at least one solution that is bounded by the barriers.

REFERENCES

1. C. Corduneanu, *Integral equations and applications*, Cambridge Univ. Press, New York, 1991.

2. ———, *Kneser property for abstract functional differential equations of Volterra type*, World Scientific Series in Appl. Math. **I**, World Scientific, Singapore, 1991.

3. K. Deimling, *Multivalued differential equations*, Walter de Gruyter, Berlin, 1992.

4. L. Gorniewicz, *Topological approach to differential inclusions*, in *Topological methods in differential equations and inclusions* (A. Granas and M. Frigon, eds.), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **472**, Kluwer Acad. Publishers, Dordrecht, 1995.

5. P. Kelevedijev, *Existence of solutions for two point boundary value problems*, Nonlinear Anal. **22** (1994), 217–224.

6. A.G. O'Farrell and D. O'Regan, *Existence results for some initial and boundary value problems*, Proc. Amer. Math. Soc. **110** (1990), 661–673.

7. D. O'Regan, *Existence theory for nonlinear ordinary differential equations*, Kluwer Acad. Publ., Dordrecht, 1997.

8. ———, *A note on the topological structure of the solution set of abstract Volterra equations*, Proc. Roy. Irish Acad. Sect. A **99** (1999), 67–74.

9. S. Szufla, *Sets of fixed points of nonlinear mappings in function spaces*, Funkcial. Ekvac. **22** (1979), 121–126.

10. Y.I. Umanskii, *One property of the set of solutions of differential inclusions in a Banach space*, Differential Equations **28** (1992), 1091–1096.

Department of Mathematics, The University of Texas, Arlington, TX 76019
E-mail address: `kannan@math.uta.edu`

Department of Mathematics, National University of Ireland Galway, Galway, Ireland
E-mail address: Donal.ORegan@NUIGALWAY.IE