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A SINC QUADRATURE METHOD FOR THE DOUBLE-LAYER INTEGRAL EQUATION IN PLANAR DOMAINS WITH CORNERS

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ABSTRACT. A convergence and error analysis is given for a Nyström method on a graded mesh based on sinc quadrature for an integral equation of the second kind with a Mellin type singularity. An application to the double-layer integral equation for planar domains with corners is described.

1. Introduction. Sinc approximation methods have been successfully employed for problems where the solution has singularities, for example for partial differential equations and associated integral equations in domains with corners (see [**8**, Chapter 5], and [**12**, Sections 6.5, 6.6, and 7.4]). Given this success, we felt a need to explain theoretically the numerical performance by an error and convergence analysis for a particular situation. For this we have chosen the application of a sinc quadrature method for the solution of the integral equation of the second kind arising from the double-layer approach to solve the Dirichlet problem for the Laplace equation in planar domains with corners. Since the solution of the integral equation develops a singularity of the derivatives at the corner, in the discretization of this integral equation a graded mesh must be used in order to achieve a satisfactory accuracy. Quadrature or Nyström methods for the double-layer integral equation using graded meshes have been previously considered by Graham and Chandler [**4**], Atkinson and Graham [**2**], Kress [**7**], Jeon [**5**] and Elliott and Prössdorf [**3**].

Because of the Mellin type singularity of the double-layer kernel for domains with corners, the double-layer integral operator is no longer compact in the space of continuous functions. Therefore the Riesz theory cannot be immediately employed for establishing existence of a solution. Following the classical work of Radon [**11**], this difficulty can be remedied by splitting the operator into an operator with norm less than one, reflecting the singular behavior at the corner, and a compact operator (see also [**6**, p. 76]). It appears natural that for a convergence and error analysis for corresponding quadrature

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or Nyström methods in the framework of Anselone's [**1**] theory of collectively compact operators one should mimic this approach, and also split the approximating operators such that of the two resulting operator sequences one is uniformly bounded with norms uniformly less than one, and the other is collectively compact. In previous work (see [**4, 2, 7, 5, 3**]) it turned out that, due to the grading of the mesh, establishing the uniform boundedness with norm less than one posed a major difficulty in the analysis, which could be overcome only through the introduction of some cut-off procedures in the vicinity of the corner. In the present paper the grading is achieved by way of a sinc quadrature. As will become evident later in the paper, for this particular grading uniform boundedness with norm less than one can be established quite straightforwardly, and without any need for additional modifications of the approximation method. Besides the fact that the sinc quadrature can be easily implemented by a simple parameter transformation, and that it yields rapid convergence for domains which are smooth (with the exception of the corner), we consider this simplification as a major advantage for theoretical reasons. In particular, this makes us believe that eventually a similar error analysis for sinc quadrature applied to other related integral equations, such as Symm's integral equation of the first kind, should be manageable. For a comprehensive review of approximation methods for Mellin type equations we refer to [**10**].

The plan of the paper is as follows. In Section 2 we will review the basic properties of the special form of the sinc quadrature rule which we shall use in our approximation and analysis. In Section 3 we introduce a class of integral equations with Mellin type singularities which we want to solve approximately, and recall their basic properties from [**7**]. Section 4, the main part of the present paper, is devoted to establishing uniform convergence of the approximate solution. In addition, it is shown here that the fast convergence of the sinc quadrature for analytic functions carries over to the approximate solution of the integral equation. The final section, Section 5, describes the application of these results to the suitably parametrized form of the double-layer integral equation. It concludes with two numerical examples.

2. The quadrature formula. For $d > 0$, let H_d denote the Hardy space of holomorphic functions f on the strip

$$D_d := \{z = x + iy : x \in \mathbf{R}, |y| < d\} \subset \mathbf{C}$$

which are real valued for real z, and for which the integral

$$\|f\|_{H_d} := \int_{-\infty}^{\infty} |f(x+id)|\,dx$$
$$:= \lim_{a \nearrow d} \int_{-\infty}^{\infty} |f(x+ia)|\,dx$$

is finite. Then for the error in the trapezoidal rule

$$\int_{-\infty}^{\infty} f(x)\,dx \approx h \sum_{j=-\infty}^{\infty} f(hj) \tag{2.1}$$

with step width $h > 0$, i.e., for

$$R_h(f) := \int_{-\infty}^{\infty} f(x)\,dx - h \sum_{j=-\infty}^{\infty} f(hj), \tag{2.2}$$

we have the estimate, due to Martensen,

$$|R_h(f)| \leq \frac{2}{e^{2\pi d/h} - 1} \|f\|_{H_d} \tag{2.3}$$

(see [**9, 12**]). Note that the trapezoidal rule can be obtained through integrating Whittaker's cardinal series, i.e., the sinc function series. Therefore it is legitimate to consider the trapezoidal rule (2.1) as a sinc function approximation.

For $d \leq \pi/2$, the function

$$w(z) := \frac{1}{1+e^{-z}}, \quad z \in \mathbf{C}, \tag{2.4}$$

maps the strip D_d bijectively onto an eye-shaped domain $E_d := \{w(z) : z \in D_d\}$ centered around the interval $(0,1)$; the latter interval is the image of the real line under the map w. The domain E_d is symmetric with respect to the real axis and the line $\operatorname{Re} w = 1/2$. At the end points of $(0,1)$ the domain E_d has corners with interior angle $2d$. The width of E_d is $\tan(d/2)$, and E_d is contained in the cone

$$W_d := \{re^{i\theta} : r \geq 0,\ -d \leq \theta \leq d\}.$$

We proceed by employing the trapezoidal rule to derive a numerical quadrature rule for the integral

$$\int_0^1 g(t)\,dt,$$

where the real valued integrand g is continuous in $(0,1)$ but is allowed to have integrable singularities at the two endpoints $t=0$ and $t=1$. Since w maps $\mathbf{R}$ bijectively and strictly monotonically increasing onto the interval $(0,1)$, we can substitute $t=w(x)$ and consequently obtain

$$\int_0^1 g(t)\,dt = \int_{-\infty}^{\infty} f(x)\,dx,$$

where

$$f(x) := w'(x)g(w(x)), \quad -\infty < x < \infty.$$

Applying the trapezoidal rule to the transformed integral now yields the approximation

$$\int_0^1 g(t)\,dt \approx h \sum_{j=-\infty}^{\infty} w'(jh)g(w(jh)), \tag{2.5}$$

with the error given by (2.3), provided g can be extended to a holomorphic function in E_d such that the Hardy norm $\|w'g\circ w\|_{H_d}$ of the transformed function exists.

Of course, for actual numerical calculations the infinite series in (2.5) has to be truncated, i.e., we will finally approximate by the quadrature rule

$$\int_0^1 g(t)\,dt \approx h \sum_{j=-n}^{n} w'(jh)g(w(jh)), \tag{2.6}$$

for some $n \in \mathbf{N}$. In order to be able to estimate the error caused by this truncation, for $\alpha > 0$ we denote by $S_{\alpha,d}$ the space of all functions g which are holomorphic in E_d, real valued on $(0,1)$, and which satisfy

$$|t(1-t)|^{1-\alpha}|g(t)| \le c, \quad t \in E_d,$$

for some constant c. On $S_{\alpha,d}$ we define a norm by

$$\|g\|_{S_{\alpha,d}} := \sup_{t\in E_d} |t(1-t)|^{1-\alpha}|g(t)|.$$

The transformation w solves the differential equation

$$w' = w(1-w), \tag{2.7}$$

and from

$$w(z)[1-w(z)] = \frac{1}{1+e^{-z}}\frac{1}{1+e^{z}}$$

it can be easily seen, since $d \le \pi/2$, that

$$|w(z)[1-w(z)]| \le e^{-|\mathrm{Re}\, z|}, \quad z \in D_d. \tag{2.8}$$

Using (2.7) and (2.8) we can estimate

$$\begin{aligned}\sum_{|j|=n+1}^{\infty} &|w'(jh)g(w(jh))| \\ &\le \sum_{|j|=n+1}^{\infty} w'(jh)\{w(jh)[1-w(jh)]\}^{\alpha-1}\,\|g\|_{S_{\alpha,d}} \\ &\le 2\sum_{j=n+1}^{\infty} e^{-jh\alpha}\,\|g\|_{S_{\alpha,d}} = \frac{2e^{-(n+1)h\alpha}}{1-e^{-h\alpha}}\|g\|_{S_{\alpha,d}}\end{aligned}$$

for $\alpha > 0$. From this, with the aid of the inequality

$$\frac{he^{-h\alpha}}{1-e^{-h\alpha}} \le \frac{1}{\alpha},$$

we conclude that

$$h\sum_{|j|=n+1}^{\infty} |w'(jh)g(w(jh))| \le \frac{2}{\alpha}\,e^{-nh\alpha}\,\|g\|_{S_{\alpha,d}} \tag{2.9}$$

for $g \in S_{\alpha,d}$ and $\alpha > 0$.

Using (2.7) and (2.8) we also have that

$$|w'(z)g(w(z))| \le e^{-\alpha|\mathrm{Re}\, z|}\|g\|_{S_{\alpha,d}}, \quad z \in D_d. \tag{2.10}$$

Therefore the Hardy norm of $w'g \circ w$ can be estimated through

$$\|w'g \circ w\|_{H_d} \le \frac{2}{\alpha}\,\|g\|_{S_{\alpha,d}} \tag{2.11}$$

for $g \in S_{\alpha,d}$ and $\alpha > 0$. Now from (2.3), (2.9) and (2.11) we conclude that the error

$$E_{n,h}(g) := \int_0^1 g(t)\,dt - h \sum_{j=-n}^{n} w'(jh)g(w(jh)) \tag{2.12}$$

for the quadrature (2.6) for $h \le 2\pi d$ can be estimated through

$$|E_{n,h}(g)| \le (C_1 e^{-nh\alpha} + C_2 e^{-2\pi d/h})\|g\|_{S_{\alpha,d}} \tag{2.13}$$

for $g \in S_{\alpha,d}$ and some constants C_1 and C_2 depending only on α and d.

Provided we choose the step width h according to the relation

$$h = \frac{\lambda}{n^{1/2}} \tag{2.14}$$

for some constant $\lambda > 0$, from (2.13) we obtain the following theorem indicating faster than polynomial convergence.

Theorem 2.1. *For $g \in S_{\alpha,d}$ with $\alpha > 0$ and $0 < d < \pi/2$ the error in the quadrature* (2.6) *with step width* (2.14) *can be estimated by*

$$|E_{n,h}(g)| \le C e^{-\mu\, n^{1/2}}\|g\|_{S_{\alpha,d}} \tag{2.15}$$

for some positive constants C and μ depending on d, α and λ.

Roughly speaking, the estimate (2.15) says that increasing the number of quadrature points by a factor of four doubles the number of correct digits in the approximate value of the integral.

Obviously, the optimal choice for the constant λ is given through

$$h = \left(\frac{2\pi d}{\alpha n}\right)^{1/2} \tag{2.16}$$

in which case $\mu = (2\pi d\alpha)^{1/2}$. However, its actual numerical implementation would require the knowledge of both the parameters d and α.

In our convergence analysis for integral operators we will also need the following estimate of the error in the nontruncated quadrature

$$E_{\infty,h}(g) := \int_0^1 g(t)\,dt - h\sum_{j=-\infty}^{\infty} w'(jh)g(w(jh)) \tag{2.17}$$

for integrands g which are merely differentiable.

Theorem 2.2. *For $g \in C^1[0,1]$ the error* (2.17) *can be estimated by*

$$|E_{\infty,h}(g)| \le \frac{h}{2}\int_0^1 |t(1-t)\,g'(t) + (1-2t)\,g(t)|\,dt. \tag{2.18}$$

Proof. Define a function $B : \mathbf{R} \to \mathbf{R}$ with period h by

$$B(x) := x - \frac{h}{2}, \quad 0 \le x < h.$$

Then for $f = w'g \circ w$ by partial integration over subintervals of length h we obtain

$$\begin{aligned}\int_{-mh}^{mh} f(x)\,dx - h\sum_{j=-m}^{m} f(jh) &= -\frac{h}{2}[f(-mh) + f(mh)] \\ &\quad - \int_{-mh}^{mh} B(x)f'(x)\,dx.\end{aligned} \tag{2.19}$$

(Of course, (2.19) is just the Euler-Maclaurin summation formula of lowest order.) From (2.19) and (2.10) for the case $\alpha = 1$ (and $d = 0$), by passing to the limit $m \to \infty$, we find that

$$|E_{\infty,h}(g)| \le \frac{h}{2}\int_{-\infty}^{\infty}\left|\frac{d}{dx}\,[w'(x)g(w(x))]\right| dx,$$

provided the integral exists. Elementary calculations, based on the differential equation (2.7), yield the relation

$$[w'g \circ w]' = w'[w(1-w)g' \circ w + (1-2w)g \circ w].$$

From this, on substituting $t = w(x)$ we find that

$$\int_{-\infty}^{\infty} |[w'(x)g(w(x))]'|\,dx = \int_0^1 |t(1-t)g'(t) + (1-2t)g(t)|\,dt,$$

which completes the proof of (2.18). $\square$

For further analysis it is more convenient to renumber the quadrature points in (2.6) and write

$$\int_0^1 g(t)\,dt \approx \sum_{j=1}^{2n+1} a_j^{(n)}\, g(s_j^{(n)}) \tag{2.20}$$

with the weights and mesh points given by

$$\begin{aligned} a_j^{(n)} &= hw'(jh - nh - h), \\ s_j^{(n)} &= w(jh - nh - h), \quad j = 1, \dots, 2n+1. \end{aligned}$$

Corollary 2.3. *The quadrature rule* (2.20) *has positive weights and converges for all continuous functions* $\varphi \in C[0,1]$.

Proof. Since w is strictly monotonically increasing the weights are positive. By Theorem 2.1 the quadratures converge for all polynomials, whence convergence for all $\varphi \in C[0,1]$ follows from Steklow's theorem (see [**6**, Theorem 12.4]). $\square$

3. A Mellin type integral equation. We consider integral equations of the second kind in the form

$$\varphi(t) - \int_0^1 K(t,\tau)[\varphi(\tau) - \varphi(0)]\,d\tau + \gamma(t)\varphi(0) = f(t), \qquad 0 \le t \le 1. \tag{3.1}$$

This form and the following assumptions on the kernel are motivated by the double-layer integral equation of the second kind for the Dirichlet problem of potential theory in planar domains with corners. We look for 1-periodic continuous solutions, i.e., $\varphi(0) = \varphi(1)$, with possible singularities of the derivatives at the endpoints $t = 0$ and $t = 1$. The functions γ and the righthand side f are both assumed to be 1-periodic and continuous. The kernel K is assumed to have period one with respect to t and to be continuous for $0 \leq t,\, \tau \leq 1$, with the exception of the four corners of the square $[0,1] \times [0,1]$. In these corners K has Mellin type singularities which, for notational brevity, we describe in detail for the case of singularities at only the corners $t = \tau = 0$ and $t = \tau = 1$. However, we note that our analysis can be readily carried over to the case of singularities at all four corners of $[0,1] \times [0,1]$.

For convenience we set

$$Q := \{(t,\tau) \in [0,1] \times [0,1] : 0 < t + \tau < 2\},$$

i.e., Q coincides with the square $[0,1] \times [0,1]$ except that the two corners $t = \tau = 0$ and $t = \tau = 1$ are excluded. We assume K to be a sum of two functions

$$K(t,\tau) = L(t,\tau) + M(t,\tau),$$

which both are continuous on Q. In addition we require L to be either nonnegative or nonpositive and M to be bounded. Furthermore we assume that there exists a bounded differentiable function $k : [0,\infty) \to [0,\infty)$ with bounded derivative k' and the properties

$$k(0) = 0 \tag{3.2}$$

and

$$\int_0^\infty \frac{k(s)}{s}\, ds < 1 \tag{3.3}$$

such that

$$\begin{aligned} |L(t,\tau)| &\leq \frac{1}{\tau} k\left(\frac{t}{\tau}\right), \quad (t,\tau) \in Q, \quad t \leq 1/2, \\ |L(t,\tau)| &\leq \frac{1}{1-\tau} k\left(\frac{1-t}{1-\tau}\right), \quad (t,\tau) \in Q, \quad t \geq 1/2. \end{aligned} \tag{3.4}$$

Using (3.4), for $0 < t \le 1/2$ we can estimate

$$\int_0^1 |L(t,\tau)|\,d\tau \le \int_0^1 \frac{1}{\tau} k\left(\frac{t}{\tau}\right) d\tau \le \int_t^\infty \frac{k(s)}{s}\,ds.$$

Analogously, for $1/2 \le t < 1$ we have

$$\int_0^1 |L(t,\tau)|\,d\tau \le \int_0^1 \frac{1}{1-\tau}\, k\left(\frac{1-t}{1-\tau}\right) d\tau \le \int_{1-t}^\infty \frac{k(s)}{s}\,ds.$$

Therefore the assumption (3.4) implies that

$$\int_0^1 |L(t,\tau)|\,d\tau \le \int_0^\infty \frac{k(s)}{s}\,ds < 1, \quad 0 < t < 1. \tag{3.5}$$

We write the integral equation (3.1) in operator notation as

$$\varphi - A\varphi - B\varphi = f$$

with the two integral operators defined by

$$(A\varphi)(t) := \int_0^1 L(t,\tau)[\varphi(\tau) - \varphi(0)]\,d\tau, \quad t \in [0,1],$$

and

$$(B\varphi)(t) := \int_0^1 M(t,\tau)[\varphi(\tau) - \varphi(0)]\,d\tau - \gamma(t)\varphi(0), \quad t \in [0,1].$$

Note that the conditions (3.2) and (3.4) imply that

$$L(0,\cdot) = L(1,\cdot) = 0, \tag{3.6}$$

and consequently

$$(A\varphi)(0) = (A\varphi)(1) = 0. \tag{3.7}$$

We introduce an additional norm

$$\|\varphi\|_{\infty,0} := \max_{0\le t\le 1} |\varphi(t) - \varphi(0)| + |\varphi(0)|$$

on $C[0,1]$ which is equivalent to the usual maximum norm on $C[0,1]$. Recall that the corresponding operator norm is given by $\|A\|_{\infty,0} := \sup_{\|\varphi\|_{\infty,0} \leqq 1} \|A\varphi\|_{\infty,0}$. Then the following theorem, from [**7**], is easily proved.

Theorem 3.1. *Under the assumptions* (3.2)–(3.4) *the integral operator A is bounded from $C[0,1]$ into $C[0,1]$ with*

$$\|A\|_{\infty,0} \leq \int_0^\infty \frac{k(s)}{s}\, ds < 1.$$

Foreshadowing the subsequent error analysis, let $\psi \in C[0,1]$ be a nonnegative function such that $1-\psi$ vanishes in a neighborhood of $t=0$ and $t=1$, and consider the operator $\tilde{B}$ defined by

$$(\tilde{B}\varphi)(t) := \int_0^1 M(t,\tau)\psi(\tau)[\varphi(\tau)-\varphi(0)]\, d\tau, \quad t \in [0,1]. \tag{3.8}$$

Since the kernel M is assumed to be bounded, we can choose ψ such that the norm of $\tilde{B}$ is small enough to ensure that $\|A\|_{\infty,0}+\|\tilde{B}\|_{\infty,0} < 1$. Then $I-A-\tilde{B}$ has a bounded inverse on $C[0,1]$. The integral term of $B-\tilde{B}$ is continuous and therefore $B-\tilde{B} : C[0,1] \to C[0,1]$ is a compact operator. Hence the Riesz theory (see [**6**]) can be applied to the operator $I-A-B = (I-A-\tilde{B})[I-(I-A-\tilde{B})^{-1}(B-\tilde{B})]$. In particular, this implies that if $I-A-B$ is injective then the inverse operator $(I-A-B)^{-1} : C[0,1] \to C[0,1]$ exists and is bounded. Hence we can state the following theorem.

Theorem 3.2. *Provided the homogeneous equation*

$$\varphi - A\varphi - B\varphi = 0$$

has only the trivial solution $\varphi = 0$, then for each continuous 1*-periodic function f the inhomogeneous equation*

$$\varphi - A\varphi - B\varphi = f$$

has a unique continuous 1*-periodic solution φ.*

4. A Nyström method. In the Nyström or quadrature method we approximate the integral in (3.1) by the quadrature formula (2.20), i.e., we approximate the integral equation (3.1) by

$$\varphi_n(t) - \sum_{j=1}^{2n+1} a_j^{(n)} K(t, s_j^{(n)})[\varphi_n(s_j^{(n)}) - \varphi_n(0)] + \gamma(t)\varphi_n(0) = f(t), \quad 0 \le t \le 1. \tag{4.1}$$

Solving (4.1) reduces to solving a finite dimensional linear system. For any solution of (4.1) the values $\varphi_{n,i} = \varphi_n(s_i^{(n)})$, $i = 1, \dots, 2n+1$, at the quadrature points and $\varphi_{n,0} = \varphi_n(0)$ associated with $s_0^{(n)} = 0$ trivially satisfy the linear system

$$\varphi_{n,i} - \sum_{j=1}^{2n+1} a_j^{(n)} K(s_i^{(n)}, s_j^{(n)})[\varphi_{n,j} - \varphi_{n,0}] + \gamma(s_i^{(n)})\varphi_{n,0} = f(s_i^{(n)}), \quad i = 0, \dots, 2n+1. \tag{4.2}$$

And, conversely, given a solution $\varphi_{n,i}$, $i = 0, \dots, 2n+1$, of the system (4.2), then the function φ_n defined by

$$\varphi_n(t) := \sum_{j=1}^{2n+1} a_j^{(n)} K(t, s_j^{(n)})[\varphi_{n,j} - \varphi_{n,0}] - \gamma(t)\varphi_{n,0} + f(t), \quad 0 \le t \le 1,$$

is readily seen to satisfy (4.1).

We write the approximating equation (4.1) in operator notation as

$$\varphi_n - A_n\varphi_n - B_n\varphi_n = f$$

with the approximating operators A_n and B_n defined by

$$(A_n\varphi)(t) := \sum_{j=1}^{2n+1} a_j^{(n)} L(t, s_j^{(n)})[\varphi(s_j^{(n)}) - \varphi(0)], \quad 0 \le t \le 1,$$

and

$$(B_n\varphi)(t) := \sum_{j=1}^{2n+1} a_j^{(n)} M(t, s_j^{(n)})[\varphi(s_j^{(n)}) - \varphi(0)] - \gamma(t)\varphi(0),$$
$$0 \le t \le 1.$$

Note that

$$(A_n\varphi)(0) = (A_n\varphi)(1) = 0 \tag{4.3}$$

as a consequence of (3.6).

In accordance with the existence analysis of the previous section, we will base our error analysis on showing that under appropriate assumptions the operators A_n are uniformly bounded with norm less than one for sufficiently large n. To accomplish this, we want to apply the error analysis of Section 2 for the quadrature rule. For this we need additional assumptions on the regularity of the kernels. We assume that $L(t, \tau)$ is continuously differentiable with respect to the variable τ for each fixed $t \in [0, 1]$. In addition, analogously to (3.4), we require that there exists a continuous nonnegative function $k_1 : [0, \infty) \to [0, \infty)$ with the property

$$\int_0^\infty \frac{k_1(s)}{s}\, ds < \infty \tag{4.4}$$

such that

$$\begin{aligned} \left|\frac{\partial}{\partial\tau} L(t,\tau)\right| &\le \frac{1}{\tau^2}\, k_1\!\left(\frac{t}{\tau}\right), \quad (t,\tau) \in Q, \quad t \le 1/2, \\ \left|\frac{\partial}{\partial\tau} L(t,\tau)\right| &\le \frac{1}{(1-\tau)^2}\, k_1\!\left(\frac{1-t}{1-\tau}\right), \quad (t,\tau) \in Q, \quad t \ge 1/2. \end{aligned} \tag{4.5}$$

Proceeding as in the proof of the estimate (3.5) it can be seen that the assumption (4.5) implies that

$$\int_0^1 \tau(1-\tau)\left|\frac{\partial}{\partial\tau} L(t,\tau)\right| d\tau \le \int_0^\infty \frac{k_1(s)}{s}\, ds, \quad 0 < t < 1. \tag{4.6}$$

Lemma 4.1. *Under the assumptions* (3.2)–(3.4), (4.4) *and* (4.5), *for* $\psi \in C^1[0,1]$ *we have*

$$h \sum_{j=-\infty}^{\infty} w'(jh) L(t, w(jh)) \psi(w(jh)) \longrightarrow \int_0^1 L(t,\tau)\psi(\tau)\,d\tau, \quad h \to 0,$$

with the convergence uniform for all $t \in (0,1)$.

Proof. By Theorem 2.2 it suffices to show that the integral

$$I(t) := \int_0^1 \left| \tau(1-\tau)\, \frac{\partial L(t,\tau)\psi(\tau)}{\partial \tau} + (1-2\tau) L(t,\tau)\psi(\tau) \right| d\tau$$

is uniformly bounded for all $t \in (0,1)$. Using (3.5) and (4.6) we can estimate

$$\begin{aligned} I(t) &\leq \|\psi\|_\infty \int_0^1 \tau(1-\tau) \left| \frac{\partial L(t,\tau)}{\partial \tau} \right| d\tau \\ &\quad + \{\|\psi\|_\infty + \|\psi'\|_\infty\} \int_0^1 |L(t,\tau)|\,d\tau \\ &\leq \|\psi\|_\infty \int_0^\infty \frac{k_1(s)}{s}\,ds + \{\|\psi\|_\infty + \|\psi'\|_\infty\} \int_0^\infty \frac{k(s)}{s}\,ds \end{aligned}$$

for all $t \in (0,1)$. In view of (3.3) and (4.4) the proof is finished. □

Lemma 4.2. *Under the assumptions* (3.2)–(3.4), (4.4) *and* (4.5) *the operators* A_n *are uniformly bounded. Moreover, for each* $\varepsilon > 0$ *we have that*

$$\|A_n\|_{\infty,0} \leq \int_0^\infty \frac{k(s)}{s}\,ds + \varepsilon \tag{4.7}$$

for all sufficiently large n.

Proof. Since $(A_n\varphi)(0) = 0$, the norm of the operator A_n is easily seen to be given by

$$\|A_n\|_{\infty,0} = \max_{0 \leq t \leq 1} \sum_{j=1}^{2n+1} a_j^{(n)} |L(t, s_j^{(n)})|.$$

Substituting for the quadrature points and weights in terms of w, we can estimate

$$\sum_{j=1}^{2n+1} a_j^{(n)} |L(t, s_j^{(n)})| = h \sum_{j=-n}^{n} w'(jh) |L(t, w(jh))|$$
$$\leq h \sum_{j=-\infty}^{\infty} w'(jh) |L(t, w(jh))|.$$

Applying the previous Lemma 4.1 with $\psi = 1$ and using the fact that L is either nonnegative or nonpositive it follows that for each ε and all $t \in (0, 1)$ we have

$$h \sum_{j=-\infty}^{\infty} w'(jh) |L(t, w(jh))| \leq \int_0^1 |L(t, \tau)| \, d\tau + \varepsilon$$

for all sufficiently large n. In view of (3.5), the proof is complete. ◻

Lemma 4.3. *Under the assumptions* (3.2)–(3.4), (4.4) *and* (4.5) *the approximate operators* A_n *are pointwise convergent,*

$$\|A_n \varphi - A\varphi\|_{\infty,0} \longrightarrow 0, \quad n \to \infty,$$

for all functions $\varphi \in C[0, 1]$ *satisfying* $\varphi(0) = \varphi(1)$.

Proof. Because $C^1[0, 1]$ is dense in $C[0, 1]$, it follows from the uniform boundedness of the operators A_n (Lemma 4.2) that it suffices to show the pointwise convergence for $\varphi \in C^1[0, 1]$. For this, by Lemma 4.1 we only need to show that the truncation error converges

$$h \sum_{|j|=n+1}^{\infty} w'(jh) L(t, w(jh)) [\varphi(w(jh)) - \varphi(0)] \longrightarrow 0, \quad n \to \infty,$$

uniformly for all $t \in (0, 1)$. As a consequence of (3.4), by using the mean value theorem, we can estimate

$$|L(t, \tau)[\varphi(\tau) - \varphi(0)]| = |L(t, \tau)[\varphi(\tau) - \varphi(1)]|$$
$$\leq \|k\|_\infty \|\varphi'\|_\infty, \quad 0 < t, \tau < 1.$$

This implies that

$$h \sum_{|j|=n+1}^{\infty} |w'(jh)L(t,w(jh))[\varphi(w(jh))-\varphi(0)]| \\ \leq h\|k\|_\infty \|\varphi'\|_\infty \sum_{|j|=n+1}^{\infty} w'(jh),$$

and from (2.9) with $\alpha = 1$ we have the estimate

$$h \sum_{|j|=n+1}^{\infty} w'(jh) \leq 2e^{-hn} = 2e^{-\lambda\, n^{1/2}} \longrightarrow 0, \quad n \to \infty,$$

which finishes the proof. $\square$

Now recall the definition (3.8) of the operator $\tilde{B}$ with the help of a nonnegative function $\psi \in C[0,1]$ for which $1-\psi$ vanishes in a neighborhood of $t = 0$ and $t = 1$. Define the corresponding approximation operator $\tilde{B}_n$ by

$$(\tilde{B}_n\varphi)(t) := \sum_{j=1}^{2n+1} a_j^{(n)} M(t, s_j^{(n)})\psi(s_j^{(n)})[\varphi(s_j^{(n)}) - \varphi(0)], \\ 0 \leq t \leq 1. \tag{4.8}$$

Then we can state the following lemma on the properties of the approximate operators B_n.

Lemma 4.4. *Assume that M is continuous and bounded on Q. Then the sequence $(B_n - \tilde{B}_n)$ is pointwise convergent to $B - \tilde{B}$ and collectively compact. The sequence $(\tilde{B}_n)$ is pointwise convergent to $\tilde{B}$ and uniformly bounded, with*

$$\|\tilde{B}_n\|_{\infty,0} \leq 4 \sup_{(t,\tau)\in Q} |M(t,\tau)| \int_0^1 \psi(\tau)\, d\tau$$

for all sufficiently large n.

Proof. That the sequence $(B_n - \tilde{B}_n)$ is pointwise convergent to $B-\tilde{B}$ and collectively compact follows from the continuity of the kernel $M(t,\tau)[1-\psi(\tau)]$ on $[0,1]\times[0,1]$ (see [**1**] or [**6**, Theorem 12.8]). The norm of the operator $\tilde{B}_n$ can be estimated by

$$\|\tilde{B}_n\|_{\infty,0} \le 3 \sup_{(t,\tau)\in Q} |M(t,\tau)| \sum_{j=1}^{2n+1} a_j^{(n)} \psi(s_j^{(n)}).$$

This implies the statement on the uniform boundedness, since from Corollary 2.3 we have

$$\sum_{j=1}^{2n+1} a_j^{(n)}\, \psi(s_j^{(n)}) \longrightarrow \int_0^1 \psi(\tau)\, d\tau, \quad n\to\infty.$$

The pointwise convergence can be seen analogously. $\square$

In order to establish a convergence order analogous to Theorem 2.1 we need to assume that the kernels L and M are analytic with respect to τ. More precisely we assume that for each $0<t<1$ there exist holomorphic extensions of $L(t,\cdot)$ and $M(t,\cdot)$ onto the eye-shaped domain E_d such that L and M are continuous on $[0,1]\times\overline{E}_d$ with the exception of $t=\tau=0$ and $t=\tau=1$. We also require that the function k of the conditions (3.2)–(3.4) can be extended as a bounded function on the cone W_d, such that

$$\begin{aligned} &|L(t,\tau)| \le \frac{1}{|\tau|}\left|k\left(\frac{t}{\tau}\right)\right|, \quad 0<t\le 1/2, \quad \tau\in E_d,\\ &|L(t,\tau)| \le \frac{1}{|1-\tau|}\left|k\left(\frac{1-t}{1-\tau}\right)\right|, \quad 1/2\le t<1, \quad \tau\in E_d. \end{aligned} \tag{4.9}$$

Lemma 4.5. *Under the assumption* (4.9) *let* $\varphi-\varphi(0)\in S_{\alpha+1,d}$ *for* $\alpha>0$. *Then*

$$\|(A-A_n)\varphi\|_\infty \le C\, e^{-\mu n^{1/2}} \|\varphi-\varphi(0)\|_{S_{\alpha+1,d}}$$

for some positive constants C *and* μ.

Proof. Using (4.9) and denoting by b a bound for the function k on W_d, we can estimate

$$\begin{aligned}|\tau^{1-\alpha}L(t,\tau)[\varphi(\tau)-\varphi(0)]| &\le |\tau^{-\alpha}|\left|k\left(\frac{t}{\tau}\right)\right||\varphi(\tau)-\varphi(0)|\\ &\le b|\tau^{-\alpha}[\varphi(\tau)-\varphi(0)]|\end{aligned}$$

for all $0 < t \le 1/2$. Here, we have used the fact that $t/\tau \in W_d$ for $t > 0$ and $\tau \in E_d$. Since $|1-\tau| \le 1$ on E_d (in the limiting case $d = \pi/2$ the domain E_d is a disk of radius $1/2$ centered at $\tau = 1/2$) the latter inequality implies that

$$|\tau(1-\tau)|^{1-\alpha}|L(t,\tau)[\varphi(\tau)-\varphi(0)]| \le b|\tau(1-\tau)^{-\alpha}[\varphi(\tau)-\varphi(0)]|$$

for all $0 < t \le 1/2$. Analogously it can be shown that this inequality also holds for all $1/2 \le t < 1$. Therefore

$$\|L(t,\cdot)[\varphi-\varphi(0)]\|_{S_{\alpha,d}} \le b\|\varphi-\varphi(0)\|_{S_{\alpha+1,d}}$$

for all $0 < t < 1$. Now the statement of the lemma is a consequence of Theorem 2.1, applied to $g = L(t,\cdot)[\varphi-\varphi(0)]$. □

Now we are ready to formulate our main result.

Theorem 4.6. *Let L be continuous on $\{(t,\tau) \in [0,1]\times[0,1] : 0 < t+\tau < 2\}$ such that $L(t,\cdot)$ is continuously differentiable with respect to τ for each $t \in [0,1]$ and that the conditions* (3.2)–(3.4), (4.4) *and* (4.5) *are satisfied, and let M be continuous and bounded on $\{(t,\tau) \in [0,1]\times[0,1] : 0 < t+\tau < 2\}$. Then, if $I-A-B$ is injective, for sufficiently large n the operators $I-A_n-B_n$ are invertible and their inverses are uniformly bounded. For the solutions of*

$$\varphi - A\varphi - B\varphi = f$$

and

$$\varphi_n - A_n\varphi_n - B_n\varphi_n = f$$

we have that

$$\|\varphi_n - \varphi\|_\infty \to 0, \quad n \to \infty.$$

If, in addition, $L(t,\cdot)$ and $M(t,\cdot)$ for each $t \in [0,1]$ can be extended to holomorphic functions on E_d such that L and M are continuous on $[0,1] \times \bar{E}_d$ and (4.9) is satisfied, and if $\varphi - \varphi(0) \in S_{1+\alpha,d}$ with $0 < d \le \pi/2$ and $\alpha > 0$, the error estimate

$$\|\varphi_n - \varphi\|_\infty \le C\, e^{-\mu n^{1/2}} \|\varphi - \varphi(0)\|_{S_{1+\alpha,d}} \tag{4.10}$$

holds for some constants C and μ and all sufficiently large n.

Proof. By assumption (3.3) we can select a constant q such that

$$\int_0^\infty \frac{k(s)}{s}\, ds < q < 1.$$

Then, from Lemmas 4.2 and 4.4 it follows that we can choose the support of the function ψ in the definitions (3.8) and (4.8) of $\tilde{B}$ and $\tilde{B}_n$ small enough such that

$$\|A_n + \tilde{B}_n\|_{\infty,0} \le q$$

for all sufficiently large n. Hence, by the Neumann series, the inverse operators $(I - A_n - \tilde{B}_n)^{-1}$ exist and are uniformly bounded. By Lemmas 4.3 and 4.4 the operators $A_n + \tilde{B}_n$ are pointwise convergent and by Lemma 4.4 the operators $B_n - \tilde{B}_n$ are collectively compact and pointwise convergent. These properties ensure the existence and the uniform boundedness of the inverse $(I - A_n - B_n)^{-1}$ (see [**1**] or [**6**, Problem 10.3]).

Now, writing

$$\varphi_n - \varphi = (I - A_n - B_n)^{-1}(A_n + B_n - A - B)\varphi,$$

we obtain an estimate

$$\|\varphi_n - \varphi\|_{\infty,0} \le C_1 \|(A_n + B_n - A - B)\varphi\|_{\infty,0} \tag{4.11}$$

with some constant C_1. This implies convergence since the sequence $(A_n + B_n)$ is pointwise convergent.

For $\varphi - \varphi(0) \in S_{1+\alpha,d}$, the error estimate (4.10) follows from Lemma 4.5, together with the corresponding estimate for $B_n - B$, which is also valid since the kernel of B satisfies the assumption (4.9) with k a constant function. □

5. The double-layer integral equation. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with a connected boundary $\Gamma = \partial\Omega$. By ν we denote the unit normal to Γ directed into the exterior of Ω. We consider the Dirichlet problem for the Laplace equation

$$\triangle u = 0 \quad \text{in } \Omega$$

with boundary condition

$$u = g \quad \text{on } \Gamma.$$

We assume Γ to be analytic with the exception of a corner at a point x_0 with interior angle $0 < \beta < 2\pi$.

The classical approach to solve this boundary value problem is to seek the solution in the form of a double-layer potential which we modify into the form

(5.1)
$$u(x) = \frac{1}{2\pi}\int_\Gamma [\psi(y) - \psi(x_0)]\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,ds(y) - \psi(x_0), \quad x \in \Omega,$$

which is more convenient for dealing with the corner singularity. By the potential theoretic jump relations, this double-layer potential solves the Dirichlet problem provided the density $\psi \in C(\Gamma)$ satisfies the integral equation

(5.2)
$$\psi(x) - \frac{1}{\pi}\int_\Gamma [\psi(y)-\psi(x_0)]\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,ds(y) + \psi(x_0) = -2g(x),$$
$$x \in \Gamma.$$

Note that there is no change in the residual term in the jump relation at the corner since the density $\psi - \psi(x_0)$ vanishes in the corner.

Following the uniqueness proof for the classical double-layer integral equation of the second kind (see [**6**, Theorem 6.16]), it can be seen that (5.2) has at most one continuous solution. A classical existence proof for the double-layer integral equation for a planar domain with corners is due to Radon (see [**11**] and [**6**, p. 76]). Its main idea is to decompose the integral operator into a compact operator and a bounded operator with norm less than one reflecting the behavior at the corner. (This classical proof motivated our analysis in Section 3.) Consequently, existence can be concluded with the aid of Theorem 3.1 from the following parametrized version of (5.2).

We assume that the boundary curve Γ is given in parametric form

$$x(t) = (x_1(t), x_2(t)), \quad 0 \le t \le 1,$$

in counter-clockwise orientation satisfying $x'(t) \neq 0$ for all $0 < t < 1$. We assume that the parameter t is equivalent to arc length s on the boundary in the sense that there exist positive constants c_1 and c_2 such that $c_1 t \le s \le c_2 t$. The corner x_0 of Γ corresponds to the parameter $t = 0$. Then, by straightforward calculations, the boundary integral equation (5.2) can be transformed into the parametric form

$$\varphi(t) - \int_0^1 K(t,\tau)[\varphi(\tau) - \varphi(0)]\, d\tau + \varphi(0) = f(t), \quad 0 \le t \le 1, \tag{5.3}$$

where we have set $\varphi(t) := \psi(x(t))$ and $f(t) := -2g(x(t))$, and where the kernel is given by

(5.4)
$$K(t,\tau) = \begin{cases} (1/\pi)([x'(\tau)]^{\perp} \cdot [x(t) - x(\tau)])/|x(t) - x(\tau)|^2, & t \neq \tau, \\ (1/(2\pi))([x'(t)]^{\perp} x''(t))/|x'(t)|^2, & t = \tau. \end{cases}$$

Here, we have set $a^{\perp} := (a_2, -a_1)$ for vectors $a = (a_1, a_2) \in \mathbf{R}^2$. It can be seen that this kernel is analytic for $(t,\tau) \in Q$ since $\Gamma \setminus x_0$ is analytic. Singularities occur when $t \to 0$ and $\tau \to 1$ and when $t \to 1$ and $\tau \to 0$, i.e., when t and τ approach the corner of Γ on the two different arcs intersecting at the corner.

We want to approximately solve the integral equation (5.3) using the method (4.1). Hence, we need to establish that the kernel (5.4) satisfies the assumptions of the previous two sections.

Consider the two tangent lines at the corner. Without loss of generality we may assume that they are given in parametric form by

$$\xi_0(t) = (t, 0), \quad 0 \le t \le 1,$$

and

$$\xi_1(t) = ((1-t)\cos\beta, (1-t)\sin\beta), \quad 0 \le t \le 1.$$

Denoting the kernel (5.4) for the case of these two straight lines by K_0, we obtain that

$$K_0(t,\tau) = H(t, 1-\tau)$$

in a neighborhood of $t = 0$ and $\tau = 1$ and

$$K_0(t,\tau) = H(1-t, \tau)$$

in a neighborhood of $t = 1$ and $\tau = 0$. Here we have set

$$H(t,\tau) = \frac{1}{\pi} \frac{t\sin\beta}{t^2 - 2t\tau\cos\beta + \tau^2}. \tag{5.5}$$

This kernel is of the form

$$H(t,\tau) = \frac{1}{\tau} k\left(\frac{t}{\tau}\right)$$

with

$$k(s) = \frac{1}{\pi} \, \frac{s\sin\beta}{1 - 2s\cos\beta + s^2}$$

and it can be checked that (5.5) has the properties (3.2)–(3.4), (4.4), (4.5) and (4.9) required in the analysis of the two previous sections above, with the integral of $k(s)/s$ bounded by

$$\int_0^\infty \frac{k(s)}{s} \, ds = 1 - \frac{\beta}{\pi} < 1.$$

Moreover,

$$\frac{\partial}{\partial\tau} H(t,\tau) = \frac{1}{\tau^2} \, k_1\left(\frac{t}{\tau}\right)$$

where

$$k_1(s) := -k(s) - sk'(s).$$

The function k has poles in the complex plane. However, the parameter d obviously can be chosen such that k is bounded on the cone W_d.

To investigate the kernel for the curve Γ itself we note that

$$|x(t) - \xi_0(t)| \le ct^2, \qquad |x'(t) - \xi_0'(t)| \le ct \tag{5.6}$$

and

$$(5.7) \qquad |x(t) - \xi_1(t)| \le c(1-t)^2, \qquad |x'(t) - \xi_1'(t)| \le c(1-t)$$

for some constant c. The kernel K can be expressed in the form

$$K(t,\tau) = \frac{1}{\pi} \frac{t \sin\beta + a(t,\tau)}{t^2 - 2t(1-\tau)\cos\beta + (1-\tau)^2 + b(t,\tau)}$$

in the neighborhood of $t = 0$ and $\tau = 1$. Here we have set

$$\begin{aligned} a(t,\tau) := {}& [\xi_1'(\tau)]^\perp \cdot [x(t) - \xi_0(t) - x(\tau) + \xi_1(\tau)] \\ & + [x'(\tau) - \xi_1'(\tau)]^\perp \cdot [x(t) - x(\tau)] \end{aligned}$$

and

$$\begin{aligned} b(t,\tau) := {}& 2[\xi_0(t) - \xi_1(\tau)] \cdot [x(t) - \xi_0(t) - x(\tau) + \xi_1(\tau)] \\ & + |x(t) - \xi_0(t) - x(\tau) + \xi_1(\tau)|^2. \end{aligned}$$

From (5.6) and (5.7) it follows that

$$|a(t,\tau)| \le C[t^2 + (1-\tau)^2]$$

and

$$|b(t,\tau)| \le C[t^3 + (1-\tau)^3]$$

for some constant C and all sufficiently small t and $1-\tau$. From the arithmetic-geometric mean inequality we have that

$$t^2 - 2t(1-\tau)\cos\beta + (1-\tau)^2 \ge (1-\cos\beta)[t^2 + (1-\tau)^2].$$

These estimates now can be used to show that the difference $K - K_0$ is bounded for $t \to 0$ and $\tau \to 1$. Analogously we also have boundedness of this difference for $t \to 1$ and $\tau \to 0$.

Now we choose a nonnegative function $\chi \in C[0,1]$ such that $\chi(t) = 1$ for $t \le 1/4$ and $\chi(t) = 0$ for $t \ge 1/3$. Then we define

$$L(t,\tau) := \chi(t)H(t, 1-\tau) + \chi(1-t)H(1-t, \tau)$$

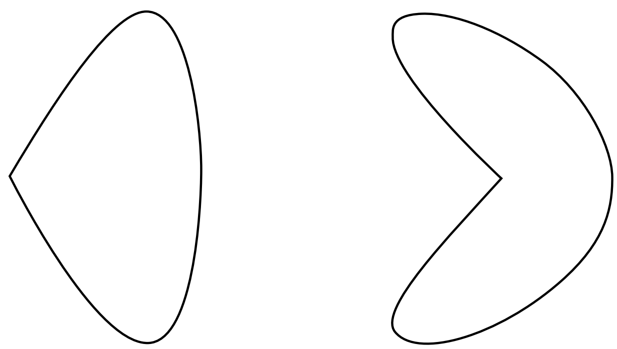

FIGURE 1. Shape of domains for examples (5.8) and (5.9).

and

$$M(t,\tau) := K(t,\tau) - L(t,\tau)$$

and can show that for this decomposition all assumptions of the previous two sections are satisfied (with the singularities at the two corners $t = 0$, $\tau = 1$ and $t = 1$, $\tau = 0$ of $[0,1] \times [0,1]$).

We conclude the paper with two numerical examples. The first example is a drop-shaped domain with the boundary curve given by the parametric representation

$$x(t) = ((2/\sqrt{3}) \sin \pi t, - \sin 2\pi t), \quad 0 \le t \le 1. \tag{5.8}$$

It is illustrated on the lefthand side of Figure 1 and has a corner at $t = 0$ with interior angle $2\pi/3$. The boundary data are given through the harmonic function

$$u(r,\theta) = r^{3/2} \cos(3\theta/2)$$

in polar coordinates r, θ.

Table 1 gives the values of the approximate solution φ_n at the parameter points $t = j/8$, $j = 0, \dots, 4$, for the coupling parameter in (2.14) chosen to be $\lambda = 1$ and for various n.

TABLE 1. Numerical results for integral equation for the domain (5.8).

n	$t=0$	$t=1/8$	$t=1/4$	$t=3/8$	$t=1/2$
2	0.85758726	0.85579494	-0.21431687	-1.93338248	-3.59701072
8	0.84586785	0.91699448	-0.23866274	-1.95721622	-3.62355053
32	0.84910651	0.91501436	-0.24106009	-1.96396598	-3.63804282
128	0.84911047	0.91500668	-0.24106446	-1.96397292	-3.63805727
4	0.83877322	0.91531915	-0.23505716	-1.93699652	-3.58397848
16	0.84885770	0.91530087	-0.24083257	-1.96349324	-3.63703350
64	0.84911046	0.91500670	-0.24106445	-1.96397291	-3.63805725
256	0.84911047	0.91500668	-0.24106446	-1.96397292	-3.63805727

Table 2 illustrates the convergence behavior through the difference between the exact solution u and the approximate solution u_n at four interior points. Of course, the approximation u_n is obtained from φ_n via (5.1) by the corresponding quadrature (2.20). Through the estimate (2.15) the error order (4.10) from Theorem 4.6 carries over to the approximation u_n for the solution u to the boundary value problem. Clearly, both tables exhibit the predicted fast convergence: the number of correct digits at least doubles when the number of quadrature points is increased by a factor of four.

TABLE 2. Errors in computed results for the potential for the domain (5.8).

n	$x=(0.2,0)$	$x=(0.4,0)$	$x=(0.6,0)$	$x=(0.2,0.2)$
2	0.06148882	0.10384493	0.20515512	0.06784151
8	0.02730672	0.01876371	0.01692735	0.03812156
32	0.00005193	0.00001788	0.00002043	0.00000473
4	0.04503861	0.05334105	0.07593531	0.05904907
16	0.00382860	0.00142027	0.00118257	0.00098782
64	0.00000008	0.00000003	0.00000007	0.00000014

Inspired by the environment in which the present work was done, the second example is a boomerang shaped domain, which is illustrated on the righthand side of Figure 1 and for which the boundary curve is given by

$$x(t) = \left(-\frac{2}{3}\sin 3\pi t, -\sin 2\pi t\right), \quad 0 \le t \le 1. \tag{5.9}$$

It has a re-entrant corner at $t = 0$ with interior angle $3\pi/2$. Table 3 gives the corresponding numerical values for the difference between u and u_n with the exact solution given by

$$u(r,\theta) = r^{2/3}\cos\left(\frac{2}{3}\theta\right).$$

TABLE 3. Errors in computed results for the potential for the domain (5.9).

n	$x=(0.15,0)$	$x=(0.3,0)$	$x=(0.45,0)$	$x=(0.2,0.2)$
4	-0.10169820	-0.02574033	0.67211123	-0.18349193
16	-0.02857248	-0.01668120	0.05518239	-0.01892663
64	-0.00023420	-0.00011317	0.00079967	-0.00010927
8	-0.06005745	-0.04048135	0.22146399	-0.03776889
32	-0.00482370	-0.00240384	0.00935449	-0.00233919
128	-0.00000290	-0.00000140	0.00002398	-0.00000133

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