JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 10, Number 2, Summer 1998

ON THE USE OF GREEN'S FUNCTION IN SAMPLING THEORY

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Dedicated to the memory of Professor A.H. Nasr

ABSTRACT. There are many papers dealing with Kramer's sampling theorem associated with self-adjoint boundary-value problems with simple eigenvalues. To the best of our knowledge, Zayed was the first to introduce a theorem that deals with Kramer's theorem associated with Green's function of not necessarily self-adjoint problems which may have multiple eigenvalues, but no examples of sampling series associated with either non-self-adjoint problems or problems with multiple eigenvalues were given. We define two classes of not necessarily self-adjoint problems for which sampling theorems can be derived and give a sampling theorem associated with Green's function of self-adjoint problems. Finally, we give some examples that illustrate our technique.

1. Introduction. Consider the boundary-value problem

$$l(y) = \sum_{k=0}^{n} p_k(x) y^{(n-k)}(x) = \lambda y, \qquad a \le x \le b, \quad \lambda \in \mathbf{C}, \tag{1.1}$$

$$U_\nu(y) = \sum_{j=1}^{n} \alpha_{j\nu} y^{(j-1)}(a) + \beta_{j\nu} y^{(j-1)}(b) = 0, \qquad \nu = 1, 2, \dots, n, \tag{1.2}$$

where $p_k(x)$ are sufficiently smooth functions [**12**, p. 6] on $[a, b]$, $p_0(x) \ne 0$ for all $x \in [a, b]$, and U_ν are n linearly independent forms of

Received by the editors on September 9, 1997, and in revised form on December 5, 1997.
1991 AMS *Mathematics Subject Classification.* 41A05, 34B05.
Key words and phrases. Boundary-value problems, Kramer's sampling theorem.
The first author, who is on leave from Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt, wishes to thank the Alexander von Humboldt foundation for the grant IV-1039259.

$y^{(j-1)}(a), y^{(j-1)}(b)$, $\alpha_{j\nu}, \beta_{j\nu} \in \mathbf{C}$, $1 \le k,\ j, \nu \le n$. Problem (1.1)–(1.2) is self-adjoint if it is equivalent to its adjoint problem

$$l^*(y) = \lambda y, \tag{1.3}$$

$$V_\nu(y) = 0, \quad \nu = 1, 2, \dots , n, \tag{1.4}$$

where $l^*(y) = \sum_{k=0}^{n} (-1)^{n-k} (\overline{p_k}(x) y(x))^{(n-k)}$ and V_ν are n linearly independent forms obtained from the equation

$$\langle l(y), z\rangle = \sum_{i=1}^{2n} U_i(y) V_{2n-i}(z) + \langle y, l^*(z)\rangle. \tag{1.5}$$

Here $U_{n+1}, \dots , U_{2n}$ are n forms of $y^{(j-1)}(a)$, $y^{(j-1)}(b)$, $1 \le j \le n$, that make $U_1, \dots , U_{2n}$ linearly independent [**12**, pp. 6–12].

Let $\{y_1(x, \lambda), \dots , y_n(x, \lambda)\}$ be a fundamental set of solutions of (1.1) determined by the initial conditions

$$y_i^{(k-1)}(a, \lambda) = \delta_{ik}, \quad 1 \le i, k \le n, \quad \forall\, \lambda \in \mathbf{C}.$$

In the following we give some definitions and relations which are needed in the sequel.

Definitions A. The number λ^* is said to be an eigenvalue of problem (1.1)–(1.2) if the boundary-value problem (1.1)–(1.2) has a nontrivial solution, $y^*(x)$, corresponding to λ^*. In this case we say that $y^*(x)$ is an *eigenfunction corresponding (belonging) to the eigenvalue* λ^*. The number of linearly independent eigenfunctions corresponding to the same eigenvalue λ^* is called the *multiplicity* of λ^*. An eigenvalue is simple if it has multiplicity one.

The eigenvalues of the problem (1.1)–(1.2) are [**12**, pp. 13–14], the zeros of the characteristic determinant

$$\Delta(\lambda) := \begin{vmatrix} U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) \\ U_2(y_1) & U_2(y_2) & \cdots & U_2(y_n) \\ \cdots & \cdots & \cdots & \cdots \\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) \end{vmatrix}. \tag{1.6}$$

The following lemma determines the relationship between the multiplicity of the zeros of $\Delta(\lambda)$ and their multiplicity as eigenvalues.

Lemma B [**12**, p. 15]. *If λ^* is a zero of $\Delta(\lambda)$ with multiplicity ν, then the multiplicity of the eigenvalue λ^* cannot be greater than ν. In particular, if λ^* is a simple zero of $\Delta(\lambda)$, then λ^* is a simple eigenvalue.*

Let us consider the problem [**15**, p. 53]

$$l(y) = -y'' = \lambda y, \quad 0 \le x \le \pi, \quad \lambda \in \mathbf{C}, \tag{1.7}$$

$$U_1(y) = y'(\pi) - y'(0) = 0, \tag{1.8}$$

$$U_2(y) = y(\pi) + 2y(0) = 0. \tag{1.9}$$

If we calculate $\Delta(\lambda)$ with respect to the fundamental set

$$\left\{ y_1(x,\lambda) = \cos\sqrt{\lambda}x, y_2(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right\}, \tag{1.10}$$

we find

$$\Delta(\lambda) = 1 - \cos\sqrt{\lambda}\pi, \quad \lambda \ne 0. \tag{1.11}$$

Hence the eigenvalues of problem (1.7)–(1.9) determined from the equation $\Delta(\lambda) = 0$, $\{\lambda_k = (2k)^2\}_{k=1}^{\infty}$, are double zeros of $\Delta(\lambda)$, while they are simple eigenvalues of the problem with corresponding sequence of eigenfunction $\{\phi_k(x) = \sin 2kx\}_{k=1}^{\infty}$. Also $\lambda = 0$ is a simple eigenvalue of this problem with eigenfunction $\phi_0(x) = 3x - \pi$. Lemma B indicates that a converse situation, i.e., to find a problem which has an eigenvalue with multiplicity higher than its multiplicity as a zero, is impossible. Lemma C below determines the relationship between the eigenvalues and the eigenfunctions of problem (1.1)–(1.2) and its adjoint (1.3)–(1.4).

Lemma C [**12**, pp. 20–21]. *If λ^* is an eigenvalue of* (1.1)–(1.2) *with multiplicity ν, then $\overline{\lambda^*}$ is an eigenvalue of the adjoint problem* (1.3)–(1.4) *with multiplicity ν. Eigenfunctions of* (1.1)–(1.2) *and* (1.3)–(1.4) *corresponding to the eigenvalues λ^*, μ^*, respectively, are orthogonal if $\lambda^* \ne \overline{\mu^*}$. In particular, if the problem is self-adjoint, then all eigenvalues are real and eigenfunctions corresponding to different eigenvalues are orthogonal.*

Now we give some important relations that are needed to calculate the Green's function. Let $W(x)$ be the Wronskian of $y_1(x,\lambda),\dots,y_n(x,\lambda)$ and

$$g(x,\xi) = \frac{\pm 1}{2W(\xi)} \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1^{(n-2)}(\xi) & y_2^{(n-2)}(\xi) & \cdots & y_n^{(n-2)}(\xi) \\ \cdots & \cdots & \cdots & \cdots \\ y_1(\xi) & y_2(\xi) & \cdots & y_n(\xi) \end{vmatrix}, \tag{1.12}$$

where the positive and negative signs are taken when $x > \xi$ and $x < \xi$, respectively, $x, \xi \in [a,b]$. Hence, Green's function of problem (1.1)–(1.2) is

$$G(x,\xi,\lambda) = \frac{(-1)^n}{\Delta(\lambda)} H(x,\xi,\lambda), \tag{1.13}$$

where

$$H(x,\xi,\lambda) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) & g(x,\xi) \\ U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) & U_1(g) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) & U_n(g) \end{vmatrix}, \tag{1.14}$$

cf., [**12**, pp. 35–37]. Formula (1.13) indicates that Green's function is a meromorphic function of λ and its poles are exactly the eigenvalues of the problem. This fact does not imply that the multiplicities of the poles of Green's function are the same as the multiplicities of the zeros of $\Delta(\lambda)$. For example, according to the system (1.10), $\Delta(\lambda)$ of the problem

$$l(y) = -y'' = \lambda y, \quad 0 \le x \le \pi, \quad \lambda \in \mathbf{C}, \tag{1.15}$$

$$U_1(y) = y'(\pi) + y'(0) = 0, \tag{1.16}$$

$$U_2(y) = y(\pi) + y(0) = 0 \tag{1.17}$$

is

$$\Delta(\lambda) = -2(1 + \cos\sqrt{\lambda}\pi). \tag{1.18}$$

Thus, the zeros of $\Delta(\lambda)$ are all double, while the poles of the Green's function of this problem are all simple. See Example 3 in Section 4 below. Also the simplicity of the eigenvalues does not necessitate the simplicity of the poles of the Green's function. See the example in [**7**, p. 312].

Assume that the poles of the Green's function are all simple. Define the function

$$\Phi(x,\lambda) := p(\lambda)G(x,\xi_0,\lambda), \tag{1.19}$$

where $p(\lambda)$ is the canonical product given in [**15**, p. 229] in terms of the eigenvalues, $\{\lambda_k\}_{k=1}^{\infty}$, of problem (1.1)–(1.2), and ξ_0 is a point in $[a,b]$ such that $G(x,\xi_0,\lambda) \not\equiv 0$, $x \in [a,b]$. Zayed's theorem [**17**] reads

Theorem D. *Let $f \in L^2(a,b)$ and*

$$F(\lambda) = \int_a^b \bar{f}(x)\Phi(x,\lambda)\,dx. \tag{1.20}$$

Then $F(\lambda)$ is an entire function of λ of order not exceeding $1/n$ which admits the sampling representation

$$F(\lambda) = \sum_{k=1}^{\infty} F(\lambda_k)\frac{p(\lambda)}{(\lambda-\lambda_k)p'(\lambda_k)}. \tag{1.21}$$

Moreover, if problem (1.1)–(1.2) *is self-adjoint or if $f(x)$ satisfies the adjoint boundary conditions* (1.4), *then the series* (1.21) *converges uniformly on compact subsets of the complex plane.*

Theorem D above is obtained under the conditions that the poles of the Green's function of problem (1.1)–(1.2) are all simple and that the eigenvalues have the asymptotic behavior

$$\lambda_k = O(k^n), \quad |k| \to \infty, \tag{1.22}$$

where n is the order of the differential equation (1.1). For reasons indicated in the next section these conditions should be replaced by more restrictive ones. Under these new conditions we can obtain a sampling theorem associated with not necessarily self-adjoint problems, but the class of self-adjoint problems for which the theorem is applicable becomes smaller. So it is more practical to treat the self-adjoint problems and non-self-adjoint ones separately. Also, due to the nature of non-self-adjoint problems, no examples of sampling associated with this kind of problem are known, except for the example given by Higgins [**9**, pp. 175–176], but this example is derived using the fact that the eigenfunctions form Riesz bases.

In the next section we define two classes of not necessarily self-adjoint problems for which sampling theorems can be obtained. The relationship between these two classes is also discussed. Section 3 is devoted to the self-adjoint case. In the last section we give some illustrative examples to show how the Green's function of the presented classes can be used in sampling.

2. Problems that are not necessarily self-adjoint. Under the conditions that the poles of Green's function are all simple and that the eigenvalues have the asymptotic behavior (1.22), the sampling theorem in [**17**, pp. 230–233] is obtained. The sampling series (1.21) is obtained by applying Parseval's identity on the eigenfunction expansions of both $f(x)$ and the kernel, $\Phi(x, \lambda)$, of the integral transform (1.20), cf. [**17**]. But the class of functions that have such expansions [**7**, p. 311] seems to be more restrictive than $L^2(a, b)$. This fact has been brought to our attention by Professor J.R. Higgins, to whom we are very grateful. In [**7**, pp. 300–308] the eigenfunction expansion of a summable function is shown to be uniformly equiconvergent with the Fourier expansion of this function, provided that the boundary conditions are regular. This result is due to Stone [**14**], see also [**2, 3**]. Then [**7**, p. 311] an eigenfunction expansion theorem is given under the condition that the poles of Green's function are all simple. The class of functions that have such expansions is described in [**7**, p. 299] to be more restrictive than $L^2(a, b)$. To illustrate the nature of classes of functions that have eigenfunction expansions associated with not necessarily self-adjoint problems, we refer the reader, in addition to the above-mentioned references, to Birkhoff [**4**] for problems with regular boundary conditions; Hopkins-Ward [**10, 16**]; and a more general theorem for the problem with separate type boundary conditions in [**12**, pp. 91–103].

To remove this difficulty we should add further conditions to problem (1.1)–(1.2). First we assume that the boundary conditions (1.2) are normal [**12**, p. 56], the adjoint problem (1.3)–(1.4) always exists, and that the differential expression (1.1) is of the type described in [**12**, p. 43]. In the following we define two classes of not necessarily self-adjoint problems for which L^2-eigenfunction expansion theorems, and hence sampling theorems, hold.

Definition 1.2. The boundary-value problem (1.1)–(1.2) is said to be of *class* 1 if:

(I) the boundary conditions (1.2) are regular;

(II) the zeros of the characteristic determinant $\Delta(\lambda)$ are all simple.

The definition of regular boundary conditions is very technical, so it is omitted here, but interested readers are referred to [**4**, pp. 382–383]. Also, in [**12**, pp. 56–60], regular boundary conditions are defined in detail in terms of numbers θ_0, $\theta_{\pm 1}$. Moreover, in [**12**], many examples of regular boundary conditions are given. These examples include conditions of Sturm type when n, the order of $l(.)$, is even [**12**, pp. 60–61] provided that the orders of the derivatives which appear in the boundary conditions satisfy some conditions; conditions of periodic type [**12**, pp. 61–62], and all regular boundary conditions when $n = 2$ are listed in three classes in [**12**, pp. 62–63].

Condition (II) necessitates the simplicity of the poles of $G(x, \xi, \lambda)$ and the simplicity of the eigenvalues, but the converse is not necessarily true as we have seen in Section 1 above. Also [**12**, p. 64] condition (I) implies (1.22), but the converse is not always true. For example [**12**, p. 94], problems associated with the differential equation $l(y) = y^{(n)} = \lambda y$ and separate type boundary conditions have eigenvalues with the asymptotic behavior (1.22), but separate-type boundary conditions are not necessarily regular, cf. [**10, 16**].

Under these two conditions we can extend the eigenfunction expansion theorem in [**12**, p. 89] to be a global L^2-expansion theorem since the class of functions that have uniform convergence eigenfunction expansions is dense in $L^2(a, b)$ [**12**, p. 90]. The convergence of the resulting theorem is in L^2-norm. From now on, $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\bar{\lambda}_k\}_{k=1}^{\infty}$ will denote the eigenvalues of problem (1.1)–(1.2) and its adjoint (1.3)–(1.4) respectively with $\lambda_k \neq 0$ for all k; $\{\phi_k(x)\}_{k=1}^{\infty}$ and $\{\psi_k(x)\}_{k=1}^{\infty}$ are respectively the corresponding sequences of eigenfunctions which satisfy the biorthonormality relation

$$\langle \phi_k, \psi_m \rangle = \int_a^b \phi_k(x)\bar{\psi}_m(x)\, dx = \delta_{km}. \tag{2.1}$$

In the following we state and prove two important lemmas.

Lemma 2.2. *Let problem* (1.1)–(1.2) *be of* class 1. *Then, for* $\lambda \neq \lambda_k$, $k = 1, 2, \dots$, *the Green's function of problem* (1.1)–(1.2) *has the uniformly convergent expansion*

$$G(x, \xi, \lambda) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\bar{\psi}_k(\xi)}{\lambda_k - \lambda}, \quad x, \xi \in [a, b]. \tag{2.2}$$

Proof. Under the conditions of the lemma, the Green's function of the problem $l(y) = 0$, $U_\nu(y) = 0$, $\nu = 1, \dots, n$, $G(x, \xi)$, has the uniformly convergent expansion [**12**, p. 89]

$$G(x, \xi) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\bar{\psi}_k(\xi)}{\lambda_k}. \tag{2.3}$$

Let $\lambda \in \mathbf{C}$, $\lambda \neq \lambda_k$ for all k. Obviously, $\{\lambda_k - \lambda\}_{k=1}^{\infty}$ are the eigenvalues of the problem $(l - \lambda)y = 0$, $U_\nu(y) = 0$, $\nu = 1, \dots, n$, with the eigenfunctions $\{\phi_k(x)\}_{k=1}^{\infty}$. Also $\{\overline{\lambda_k - \lambda}\}_{k=1}^{\infty}$ are the eigenvalues of the adjoint problem with the eigenfunctions $\{\psi_k(x)\}_{k=1}^{\infty}$. Substituting $\lambda_k - \lambda$ for λ_k in (2.3), we obtain (2.2). □

Lemma 2.3. *Green's function of the boundary-value problem* (1.1)–(1.2)*, which is assumed to be of* class 1*, equals the resolvent kernel of the Fredhholm integral equation*

$$y(x) = \lambda \int_a^b G(x, \xi) y(\xi)\, d\xi, \tag{2.4}$$

where $G(x, \xi)$ *is given by* (2.3)*.*

Proof. We prove the lemma by showing that the resolvent kernel of (2.4) has the expansion (2.2). To achieve this aim we use the Neumann series expansion for the resolvent kernel. The integral equation (2.4) is equivalent to the boundary-value problem (1.1)–(1.2) [**12**, p. 35]. Thus the eigenvalues and the eigenfunctions of problem (1.1)–(1.2) and the integral operator (2.4) are exactly the same. Now following [**6**, pp. 24–25], when λ is not an eigenvalue, the resolvent kernel has the expansion

$$R_G(x,\xi,\lambda) = \sum_{\nu=1}^{\infty} \lambda^{\nu-1} G^{\nu}(x,\xi), \tag{2.5}$$

where the sequence $G^{\nu}(x,\xi)$ is given by

$$\begin{aligned} G^1(x,\xi) &= G(x,\xi); \\ G^2(x,\xi) &= GG = \int_a^b G(x,t)G(t,\xi)\,dt; \\ G^{\nu}(x,\xi) &= GG^{\nu-1} = \int_a^b G(x,t)G^{(\nu-1)}(t,\xi)\,dt, \quad \nu = 3,\dots . \end{aligned}$$

Using mathematical induction and the uniform convergence expansion of $G(x,\xi)$, we get

$$G^{\nu}(x,\xi) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\bar{\psi}_k(\xi)}{\lambda_k^{\nu}}.$$

Hence

$$R_G(x,\xi,\lambda) = \sum_{k=1}^{\infty}\sum_{\nu=1}^{\infty} \phi_k(x)\bar{\psi}_k(\xi)\frac{1}{\lambda_k}\left(\frac{\lambda}{\lambda_k}\right)^{\nu-1}.$$

Since $|\lambda_k| \to \infty$ as $|k| \to \infty$ [**12**, p. 14], we can find ν_0 such that $|\lambda/\lambda_\nu| < 1$ for $\nu > \nu_0$. Hence, dividing the above summation into two summations according to ν_0, one for $\nu \le \nu_0$ and the other for $\nu > \nu_0$, and using the formulae of the partial and infinite sums of the geometrical series, we get expansion (2.2), which implies the equality of the Green's function and the resolvent kernel. □

Now we define a second class of problems for which an eigenfunction expansion theorem is guaranteed.

Definition 2.4. The boundary-value problem (1.1)–(1.2) is said to be of *class* 2 if:

(i) the boundary conditions (1.2) are strongly regular, i.e., [**11**, p. 852], the numbers θ_0, $\theta_{\pm 1}$ mentioned above satisfy the regularity conditions and in the case when n is even they satisfy $\theta_0^2 \neq 4\theta_{-1}\theta_1$;

(ii) the poles of Green's function are all simple.

Condition (i) is sufficient to guarantee an L^2-eigenfunction expansion theorem associated with *class* 2, [**11**, p. 853], see also [**3, 12**, pp. 90, 15]. But we added condition (ii) for technical reasons related to sampling theory. There are many examples of problems with strongly regular boundary conditions. For instance, regular boundary conditions of Sturm type, when the order of the differential equation is even, are strongly regular [**12**, pp. 60–61]; periodic boundary conditions [**12**, pp. 61–62] are strongly regular when and only when the order of the differential equation is odd; two classes of problems of the second order listed in [**12**, pp. 62–63] are strongly regular while the third one must satisfy another condition to be strongly regular.

Obviously, condition (i) in Definition 2.4 is more restrictive than condition (I) in Definition 2.1, and condition (II) in Definition 2.1 is more restrictive than condition (ii) in Definition 2.4. Thus, it may be reasonable to discuss the relationship between the two classes. Let A, B denote the set of all problems of *class* 1 and *class* 2, respectively. Then we have the following lemma.

Lemma 2.5. (a) $A \cap B \neq \phi$,

(b) $A - B \neq \phi$.

Proof. (a) See Examples 1 and 2 in Section 4 below.

(b) We show that $A - B \neq \phi$. We give a boundary-value problem which belongs to A and does not belong to B. It is the problem

$$l(y) = -y'' = \lambda y, \quad 0 \leq x \leq \pi, \quad \lambda \in \mathbf{C}, \tag{2.6}$$

$$U_1(y) = y'(0) + y'(\pi) + y(\pi) = 0, \tag{2.7}$$

$$U_2(y) = y(0) = 0. \tag{2.8}$$

In this case, see [**12**, p. 63],

$$\theta_1 = \theta_{-1} = \omega_1, \quad \theta_0 = 2\omega_1,$$

where ω_1 is one of the two complex roots of -1, chosen as indicated in [**12**, pp 43–45]. Hence, the problem is regular since $\theta_{\pm 1} \neq 0$ but not strongly regular since $\theta_0^2 = 4\theta_1\theta_{-1} = -4$. According to the fundamental set (1.10), we have

$$\Delta(\lambda) = -\left(1 + \cos\sqrt{\lambda}\pi + \frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\right), \quad \lambda \neq 0.$$

If we set $\Delta(\lambda) = 0$, then we obtain

$$\cos\left(\frac{\sqrt{\lambda}\pi}{2}\right)\left\{\sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}\pi}{2}\right) + \sin\left(\frac{\sqrt{\lambda}\pi}{2}\right)\right\} = 0.$$

Thus the zeros of $\Delta(\lambda)$ are either the solutions of $\cos\sqrt{\lambda}(\pi/2) = 0$ which are all real and simple, or the solutions of $\tan\sqrt{\lambda}(\pi/2) + \sqrt{\lambda} = 0$, $\cos\sqrt{\lambda}(\pi/2) \neq 0$ which are also real and simple. The reason is that the eigenvalues of the self-adjoint problem

$$l(y) = -y'' = \lambda y, \qquad y(0) = 0, \qquad y'(\pi) + y(\pi) = 0$$

are also the zeros of the same equation, $\tan\sqrt{\lambda}\pi = -\sqrt{\lambda}$, which are known to be real and simple. Also it is easy to see that zero is not an eigenvalue. Hence, problem (2.6)–(2.8) is of *class* 1, but not of *class* 2.

We have no answer to the question of whether $B - A \neq \phi$, or $B \subset A$. From a theoretical point of view, it is expected that $B - A \neq \phi$ since the eigenvalues of any problem of *class* 1 are all simple, while the eigenvalues of problems of *class* 2 are not necessarily simple, but asymptotically simple, i.e., the number of multiple eigenvalues is finite [**3, 11, 12**, p. 65]. An example of a boundary-value problem which is neither of *class* 1 nor of *class* 2 is the example in [**7**, p. 312]. In this example the boundary conditions are regular but not strongly regular. Moreover, the zeros of the characteristic determinant as well as the poles of Green's function are double while all eigenvalues are simple. It is also known [**15**, p. 31] that the Green's function of a problem of *class* 2 has a uniformly convergent expansion (2.2), and this Green's function equals the resolvent kernel of the corresponding integral equation.

In the following we state and prove the first sampling theorem of this paper. We assume that all eigenvalues of problem (1.1)–(1.2) are simple. Let $\xi_0 \in [a,b]$ such that $G(x,\xi_0,\lambda) \not\equiv 0$ on $[a,b]$. Define the entire function

$$\Phi(x,\lambda) := p(\lambda)G(x,\xi_0,\lambda), \tag{2.9}$$

where $G(x,\xi,\lambda)$ is the Green's function of problem (1.1)–(1.2) which is assumed to be of *class* 1 or *class* 2. Here $p(\lambda)$ is the canonical product mentioned above, which has the form

$$p(\lambda) = \begin{cases} \prod_{k=1}^{\infty}(1-\lambda/\lambda_k) & \text{if } n>1, \\ \prod_{k=1}^{\infty}(1-\lambda/\lambda_k)e^{(\lambda/\lambda_k)} & \text{if } n=1, \end{cases}$$

where n is the order of the differential equation (1.1); see also Equation (1.22). This infinite product converges since the eigenvalues satisfy (1.22). Let γ be the positive number

$$\gamma := \frac{1}{2}\|G(x,\xi)\|^2,$$

where $G(x,\xi)$ is given in (2.3) and the norm is the norm of $L^2([a,b]\times[a,b])$.

Lemma 2.6. *There is a positive constant C such that, for each $\lambda \in \mathbf{C}$,*

$$|\Phi(x,\lambda)| \le Ce^{\gamma|\lambda|^2}\{1+\sqrt{\lambda}\} \tag{2.10}$$

uniformly on $[a,b]$. Thus $\Phi(x,\lambda)$ is an entire function of λ of order not exceeding 2 and type not exceeding γ.

Proof. Since $G(x,\xi,\lambda)$ is the resolvent kernel of the integral operator (2.4), which is a continuous function of $x,\xi \in [a,b]$, then [**6**, p. 50], for $\lambda \in \mathbf{C}$,

$$|p(\lambda)G(x,\xi,\lambda)| \le e^{\lambda|\gamma|^2}\{|G(x,\xi)| + |\lambda|\sqrt{e}[G_1(x)G_2(\xi)]^{1/2}\}, \tag{2.11}$$

for all $x,\xi \in [a,b]$, where

$$G_1(x) = \int_a^b |G(x,z)|^2\,dz, \qquad G_2(\xi) = \int_a^b |G(z,\xi)|^2\,dz.$$

Obviously, using the continuity of $G(x,\xi,\lambda)$, $G(x,\xi)$, for any ξ_0 we can find $C>0$ such that (2.10) holds. $\square$

Notice that Lemma 2.6 holds if problem (2.1)–(2.2) is either of *class* 1 or of *class* 2.

Theorem 2.7. *Let $f(x) \in L^2(a,b)$ and $\Phi(x,\lambda)$ be the function defined in* (2.9). *Let $F(\lambda)$ be the integral transform*

$$F(\lambda) = \int_a^b \bar{f}(x)\Phi(x,\lambda)\,dx. \tag{2.12}$$

Then $F(\lambda)$ is an entire function of order not exceeding 2 *and type not exceeding γ which has the sampling expansion*

$$F(\lambda) = \sum_{k=1}^{\infty} F(\lambda_k)\frac{p(\lambda)}{(\lambda-\lambda_k)p'(\lambda_k)}, \tag{2.13}$$

where $p(\lambda)$, λ_k and γ are given in the paragraph preceding Lemma 2.6. *The series* (2.13) *converges uniformly on compact subsets of* $\mathbf{C}$.

Proof. Since series (2.2) converges uniformly, then

$$\begin{aligned} F(\lambda) &= \int_a^b \sum_{k=1}^{\infty} \frac{p(\lambda)}{\lambda_k-\lambda}\bar{\psi}_k(\xi_0)\bar{f}(x)\phi_k(x)\,dx \\ &= \sum_{k=1}^{\infty} \frac{p(\lambda)}{\lambda_k-\lambda}\bar{\psi}_k(\xi_0)\int_a^b \bar{f}(x)\phi_k(x)\,dx. \end{aligned} \tag{2.14}$$

A simple calculation yields

$$F(\lambda_k) = -p'(\lambda_k)\bar{\psi}_k(\xi_0)\int_a^b \bar{f}(x)\phi_k(x)\,dx. \tag{2.15}$$

Combining (2.14) and (2.15), one gets (2.13). For the proof of the uniform convergence of (2.13), let $M \subset \mathbf{C}$ be compact. For $\lambda \in M$,

$N > 0$, we have

$$\begin{aligned}
(2.16)\quad \left| F(\lambda) - \sum_{k=1}^{N-1} F(\lambda_k) \frac{p(\lambda)}{(\lambda-\lambda_k)p'(\lambda_k)} \right| \\
&= \left| F(\lambda) - \sum_{k=1}^{N-1} \overline{\langle f, \phi_k \rangle} \langle \Phi, \psi_k \rangle \right| \\
&\le \left(\sum_{k=N}^{\infty} |\langle f, \phi_k \rangle|^2 \right)^{1/2} \left(\sum_{k=N}^{\infty} |\langle \Phi, \psi_k \rangle|^2 \right)^{1/2}.
\end{aligned}$$

From Bessel's inequality and following Lemma 2.6, we can find a positive constant C_M such that

$$(2.17)\qquad \left(\sum_{k=N}^{\infty} |\langle \Phi, \psi_k \rangle|^2 \right)^{1/2} \le \|\Phi(.,\lambda)\| \le C_M, \quad \forall\, \lambda \in M$$

since the righthand side of (2.10) is bounded on compact sets. From the relation $\sum_{k=1}^{\infty} |\langle f, \phi_k \rangle|^2 \le c\|f\|^2 < \infty$ for some constant c and (2.17) we obtain the uniform convergence of series (2.13), since the righthand side of (2.16) approaches zero when N approaches ∞ independent of λ. From the uniform convergence of the sampling expansion of $F(\lambda)$, we get the analyticity of $F(\lambda)$ on compact subsets of $\mathbf{C}$. Thus F is entire. To prove the growth properties of $F(\lambda)$, we use Lemma 2.6. Indeed, there is a positive constant C such that, for $\lambda \in \mathbf{C}$,

$$|F(\lambda)| \le C e^{\gamma|\lambda|^2} \{1 + \sqrt{\lambda}\} \int_a^b |f(x)|\, dx.$$

Hence,

$$|F(\lambda)| \le C\sqrt{b-a}\|f\| e^{\lambda|\gamma|^2} \{1 + \sqrt{\lambda}\}.$$

Then the order of $F(\lambda)$ does not exceed 2 and its type does not exceed γ. $\square$

Remark 1. In the above theorem the definition of the sampled integral transform depends on the choice of ξ_0. Such a ξ_0 can be chosen arbitrarily in $[a, b]$. If we choose $\xi_0 \in [a, b]$ such that $G(x, \xi_0, \lambda) \equiv 0$

on $[a, b]$, then we will get the trivial case $F(\lambda) \equiv 0$. Thus we have a sampling theorem for a family of integral transforms.

3. The self-adjoint case. Although Theorem 2.7 holds for self-/non-self-adjoint problems, provided that the problems are assumed to be of *class* 1 or *class* 2, the class of self-adjoint problems for which the theorem is applicable is very restrictive. For instance, problems (1.15)–(1.17) is not included since the zeros of $\Delta(\lambda)$ are not simple, and the boundary conditions are not strongly regular. Also if (1.1)–(1.2) is self-adjoint, then, without adding any conditions on the problem, the set of eigenfunctions is a complete orthonormal set in $L^2(a, b)$, [**7**, p. 199, **12**, p. 82]. During the rest of this section, we assume that problem (1.1)–(1.2) is self-adjoint and $\{\lambda_k\}_{k=1}^{\infty}$, $\lambda_k \neq 0$ for all k, $\{\phi_k(x)\}_{k=1}^{\infty}$ are the eigenvalues and eigenfunctions of the problem. Since the eigenvalues are not necessarily simple, an eigenvalue in this sequence is repeated as many times as its multiplicity. Following [**6**, p. 194] and [**7**, p. 202], we can get the following lemma.

Lemma E. *Let problem* (1.1)–(1.2) *be self adjoint. Then the Green's function of the problem, which has only simple poles at the eigenvalues, has the L^2-convergent expansion*

$$G(x, \xi, \lambda) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\bar{\phi}_k(\xi)}{\lambda_k - \lambda}. \tag{3.1}$$

Moreover, Green's function $G(x, \xi, \lambda)$ equals the resolvent kernel of the integral equation corresponding to problem (1.1)–(1.2).

Now we give a sampling theorem associated with problems (1.1)–(1.2). We assume (1.22) and that $p(\lambda)$, ξ_0, γ, $\Phi(x, \lambda)$ are defined in a similar way to that in the above section.

Theorem 3.1. *Let $f \in L^2(a, b)$, and*

$$F(\lambda) = \int_a^b \bar{f}(x)\Phi(x, \lambda)\, dx. \tag{3.2}$$

Then $F(\lambda)$ is an entire function of order ≤ 2 and type $\leq \gamma$ which has the sampling expansion

$$F(\lambda)=\sum_{k=1}^{\infty} F(\lambda_k)\frac{p(\lambda)}{(\lambda-\lambda_k)p'(\lambda_k)}. \tag{3.3}$$

The series (3.3) *converges uniformly on compact subsets of* $\mathbf{C}$.

Proof. Let ν_k be the multiplicity of the eigenvalue λ_k. Then

$$\Phi(x,\lambda)=\sum_{k=1}^{\infty}\sum_{\nu=1}^{\nu_k}\left(\frac{p(\lambda)}{(\lambda_k-\lambda)}\bar{\phi}_{\nu,k}(\xi_0)\right)\phi_{\nu,k}(x), \tag{3.4}$$

where $\{\phi_{\nu,k}(x)\}_{\nu=1}^{\nu_k}$ are eigenfunctions corresponding to λ_k. Expansion (3.4) is the Fourier expansion of $\Phi(x,\lambda)$ with respect to the complete orthornomal system of eigenfunctions. Applying Parseval's equality on (3.2), one gets

$$F(\lambda)=\sum_{k=1}^{\infty}\frac{p(\lambda)}{(\lambda_k-\lambda)}\sum_{\nu=1}^{\nu_k}\bar{\phi}_{\nu,k}(\xi_0)\overline{\langle f,\phi_{\nu,k}\rangle}. \tag{3.5}$$

Taking the limit when $\lambda\to\lambda_k$ in (3.2) and using (3.4) we get

$$F(\lambda_k)=-p'(\lambda_k)\sum_{\nu=1}^{\nu_k}\bar{\phi}_{\nu,k}(\xi_0)\overline{\langle f,\phi_{\nu,k}\rangle}. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain (3.3).

The proof of the uniform convergence as well as the analytic and growth properties of $F(\lambda)$ can be established as in Theorem 2.7 above since Lemma 2.6 holds in this case because of the continuity of Green's function. □

Remark 2. In [**8, 5, 18**], sampling expansions are derived when the kernels of the sampled integral transforms are solutions of second order or nth order self-adjoint boundary value problems provided that the eigenvalues are all simple. Sampling expansions associated with general self-adjoint Fredholm integral operators of the second kind are treated in [**1**].

4. Examples. In this section we give three detailed examples illustrating the obtained results. Examples 1 and 2 are devoted to the non-self-adjoint case. In Example 1 the boundary conditions are taken to be self-adjoint while $l(.)$ is not self-adjoint. The converse situation is discussed in the second example. Moreover, in Example 2, we compare the sampling series obtained and that established in [**9**, p. 176]. In the third example we give a sampling expansion associated with a problem whose eigenvalues are all double.

Example 1. Consider the boundary-value problem

$$l(y) = y'' + 2y' + (\lambda+1)y = 0, \quad x \in [0,\pi], \quad \lambda \in \mathbf{C}, \tag{4.1}$$

$$U_1(y) = y(0) = 0, \qquad U_2(y) = y(\pi) = 0. \tag{4.2}$$

The adjoint problem is

$$l^*(y) = y'' - 2y' + (\lambda+1)y = 0, \tag{4.3}$$

$$V_1(y) = U_1(y) = y(0) = 0, \qquad V_2(y) = U_2(y) = y(\pi) = 0. \tag{4.4}$$

Hence the problem is not self-adjoint since $l \neq l^*$. The fundamental set of solutions defined by $y_i^{(j-1)}(0,\lambda) = \delta_{ij}$, $1 \leq i,\ j \leq 2$, is

$$\left\{ y_1(x,\lambda) = e^{-x}\left(\cos\sqrt{\lambda}x + \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right), y_2(x,\lambda) = e^{-x}\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right\}. \tag{4.5}$$

According to system (4.5), we have $\Delta(\lambda) = (\sin\sqrt{\lambda}\pi)/\sqrt{\lambda}$, $\lambda \neq 0$. Observing that zero is not an eigenvalue, the eigenvalues of this problem are $\{\lambda_k = k^2\}_{k=1}^{\infty}$. These eigenvalues are all simple, and the corresponding sequence of eigenfunctions is $\{\phi_k(x) = \sqrt{(2/\pi)}e^{-x}\sin kx\}_{k=1}^{\infty}$. Also the adjoint problem has the same set of simple eigenvalues with

$\{\psi_k(x) = \sqrt{(2/\pi)}e^x \sin kx\}_{k=1}^{\infty}$ the corresponding sequence of eigenfunctions. The appearance of the constant $\sqrt{2/\pi}$ is necessary for the biorthonormality relation $\langle \phi_k, \psi_m \rangle = \delta_{km}$. Since the zeros of $\Delta(\lambda)$ are all simple and the boundary conditions are strongly regular [**12**, p. 63], then problem (4.1)–(4.2) is of *class* 1 and of *class* 2. Hence Theorem 2.7 above is applicable. Using Equations (1.6) and (1.12)–(1.14), the Green's function of this problem for $\lambda \neq k^2$, $k = 1, 2, \dots$, is

$$\begin{aligned} G(x,\xi,\lambda) &= \frac{2e^{\xi-x}}{\pi}\sum_{k=1}^{\infty}\frac{\sin kx \sin k\xi}{k^2-\lambda} \\ &= \frac{e^{\xi-x-\pi}}{\sqrt{\lambda}\sin\sqrt{\lambda}\pi}\begin{cases}\sin\sqrt{\lambda}\xi\sin\sqrt{\lambda}(\pi-x) & x\geq\xi, \\ \sin\sqrt{\lambda}x\sin\sqrt{\lambda}(\pi-\xi) & x\leq\xi.\end{cases}\end{aligned} \tag{4.6}$$

Also, we have

$$p(\lambda) = \prod_{k=1}^{\infty}\left(1-\frac{\lambda}{k^2}\right) = \frac{\sin\sqrt{\lambda}\pi}{\pi\sqrt{\lambda}};$$

$$p'(k^2) = \frac{(-1)^k}{2k^2}.$$

Let $\xi_0 \in [0,\pi]$ be as described above and $\Phi(x,\lambda) = p(\lambda)G(x,\xi_0,\lambda)$. Then, for $f(x) \in L^2(0,\pi)$, the integral transform

$$F(\lambda) = \int_0^{\pi} \bar{f}(x)\Phi(x,\lambda)\,dx$$

is an entire function of order 1/2 which can be recovered in the sampling form

$$F(\lambda) = \sum_{k=1}^{\infty}(-1)^k F(k^2)\frac{2k^2\sin\sqrt{\lambda}\pi}{\pi\sqrt{\lambda}(\lambda-k^2)}. \tag{4.7}$$

Example 2. Consider the boundary-value problem

$$l(y) = -y'' = \lambda y, \quad 0 \leq x \leq \pi, \quad \lambda \in \mathbf{C}, \tag{4.8}$$

$$U_1(y) = y'(0) = 0; \qquad U_2(y) = y(0) + y'(\pi) = 0. \tag{4.9}$$

Its adjoint problem [**9**, p. 175], is

$$l^*(y) = l(y) = -y'' = \lambda y, \tag{4.10}$$

$$V_1(y) = y'(\pi) = 0; \qquad V_2(y) = y'(0) - y(\pi) = 0. \tag{4.11}$$

Conditions (4.9) and (4.11) are strongly regular. According to system (1.10), we have $\Delta(\lambda) = 1 - \sqrt{\lambda} \sin \sqrt{\lambda}\pi$. The eigenvalues of the problem, $\{\lambda_k\}_{k=1}^{\infty}$, are the positive zeros of $\Delta(\lambda)$, [**9**, p. 175]. Moreover, we have the following lemma which proves that problem (4.8)–(4.9) is of *class* 1 and of *class* 2.

Lemma 4.1. *The zeros of the function $f(t) = 1/t - \sin t\pi$, $0 < t < \infty$, are simple.*

Proof. We prove the simplicity of the zeros of $f(t)$ by discussing the distribution of the zeros of $f(t)$ and its derivative $f'(t) = -1/t^2 - \pi \cos t\pi$ in the intervals $[2n, 2n+2]$, $n > 1$. We divide the interval $[2n, 2n+2]$ into eight equal subintervals with the length $1/4$. Noting the sign of the continuous functions f, f' on the boundary and along each subinterval, we find that the zeros of f lie in $]2n, 2n+1/4[\cup]2n+3/4, 2n+1[$, while the zeros of f' lie in $]2n+1/2, 2n+3/4[\cup]2n+5/4, 2n+6/4[$. A similar situation holds for the interval $]0, 4]$. Hence, f, f' cannot have common zeros.

Since the boundary conditions are regular, then $\lambda_k = O(k^2)$ as $k \to \infty$. One can see easily that zero is not an eigenvalue. The sequence of corresponding eigenfunctions is $\{\phi_k(x) = \cos \sqrt{\lambda_k} x\}_{k=1}^{\infty}$. Also $\{\lambda_k\}_{k=1}^{\infty}$ is the sequence of eigenvalues of the adjoint problem with the corresponding sequence of eigenfunctions $\{\psi_k(x) = c_k \cos \sqrt{\lambda_k}(x - \pi)\}_{k=1}^{\infty}$. The sequence of constants c_k is taken to guarantee the biorthonormality relation. Green's function of this problem, $\lambda \neq \lambda_k$, is

(4.12)

$$G(x,\xi,\lambda) = \sum_{k=1}^{\infty} c_k \frac{\cos\sqrt{\lambda}_k x \cos\sqrt{\lambda}_k(\xi-\pi)}{\lambda_k - \lambda}$$

$$= \begin{cases} \dfrac{-\cos\sqrt{\lambda}\xi\cos\sqrt{\lambda}(x+\pi) + (1/\sqrt{\lambda})\sin\sqrt{\lambda}(\xi-x)}{1-\sqrt{\lambda}\sin\sqrt{\lambda}\pi} & x \geq \xi, \\ \dfrac{-\cos\sqrt{\lambda}x\cos\sqrt{\lambda}(\xi+\pi)}{1-\sqrt{\lambda}\sin\sqrt{\lambda}\pi} & x \leq \xi. \end{cases}$$

Letting $p(\lambda) = \prod_{k=1}^{\infty}(1-\lambda/\lambda_k)$, ξ_0, $\Phi(x,\lambda)$, be as defined above, every integral transform

$$F(\lambda) = \int_0^{\pi} \bar{f}(x)\Phi(x,\lambda)\,dx, \quad f \in L^2(0,\pi)$$

is an entire function of λ of order $1/2$ which has the interpolation expansion

$$F(\lambda) = \sum_{k=1}^{\infty} F(\lambda_k)\frac{p(\lambda)}{(\lambda-\lambda_k)p'(\lambda_k)}. \tag{4.13}$$

This series converges uniformly on compact subsets of $\mathbf{C}$. $\square$

Remark 4. In the above example, if the function $p(\lambda)$ is replaced by $\Delta(\lambda)$, then the interpolating functions $\Delta(\lambda)/((\lambda-\lambda_k)\Delta'(\lambda_k))$ become the same as those in [**9**, p. 176] with the constants c_k instead of ν_k used there.

Example 3. Consider the boundary-value problem (1.15)–(1.17). This problem is a self-adjoint problem with anti-periodic boundary conditions (1.16)–(1.17). This problem is not of *class* 1 or *class* 2. In this problem [**12**, p. 63], the boundary conditions are not strongly regular. Also, according to system (1.10), $\Delta(\lambda) = -2(1+\cos\sqrt{\lambda}\pi)$, i.e., the zeros of $\Delta(\lambda)$, which are exactly the eigenvalues, are all double. The eigenvalues are $\{\lambda_k = (2k-1)^2\}_{k=1}^{\infty}$ and the corresponding sequence of eigenfunctions is the sequence

$$\left\{\phi_k^1(x) = \sqrt{\frac{2}{\pi}}\sin(2k-1)x, \phi_k^2(x) = \sqrt{\frac{2}{\pi}}\cos(2k-1)x\right\}.$$

That is, the eigenvalues of the problem are all double. Green's function of this problem, when λ is not an eigenvalue, has the form

(4.14)

$$\begin{aligned} G(x,\xi,\lambda) &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)(\xi-x)}{(2k-1)^2-\lambda} \\ &= \begin{cases} \dfrac{\Delta(\lambda)\sin\sqrt{\lambda}(\xi-x) - 2\sin\sqrt{\lambda}\pi\cos\sqrt{\lambda}(\xi-x)}{2\sqrt{\lambda}\Delta(\lambda)} & x \geq \xi, \\ \dfrac{-\Delta(\lambda)\sin\sqrt{\lambda}(\xi-x) - 2\sin\sqrt{\lambda}\cos\sqrt{\lambda}(\xi-x)}{2\sqrt{\lambda}\Delta(x)} & x \leq \xi. \end{cases} \end{aligned}$$

Observe that the poles of Green's function are simple. We also have

$$p(\lambda) = \cos\sqrt{\lambda}\frac{\pi}{2}, \qquad p'((2k-1)^2) = (-1)^k\frac{\pi}{4(2k-1)}.$$

Hence the integral transforms described in Theorem 3.1 above associated with problem (1.15)–(1.17) has the sampling representation

$$F(\lambda) = \sum_{k=1}^{\infty}(-1)^k F((2k-1)^2)\frac{4(2k-1)\cos\sqrt{\lambda}(\pi/2)}{\pi(\lambda-(2k-1)^2)}. \tag{4.15}$$

Remark 5. A similar situation to the above example holds when the anti-periodic boundary conditions (1.16)–(1.17) replaced by the periodic boundary conditions

$$U_1(y) = y(0) - y(\pi) = 0, \quad U_2(y) = y'(0) - y'(\pi) = 0. \tag{4.16}$$

In this problem all eigenvalues, except the eigenvalue $\lambda_0 = 0$, are double. The poles of Green's function are simple [**13**, pp. 427–429]. But in this case zero is an eigenvalue, so we may need to replace the eigenvalue parameter λ by $\lambda - c$, where c is a constant to avoid this difficulty. This is always possible since the sequence of eigenvalues has no finite limit point.

Acknowledgment. The authors wish to express their gratitude to Professor J.R. Higgins for fruitful discussions about eigenfunction expansions of non-self-adjoint boundary value problems. Also they would like to thank Professor P.L. Butzer who read the original form of the manuscript and gave constructive comments which improved the paper.

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