J. DIFFERENTIAL GEOMETRY
86 (2010) 273-301

ON THE TOPOLOGY OF THE SPACE OF NEGATIVELY CURVED METRICS

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Abstract

We show that the space of negatively curved metrics of a closed negatively curved Riemannian n-manifold, $n \geq 10$, is highly non-connected.

0. Introduction

Let M be a closed smooth manifold. We denote by $\mathcal{MET}(M)$ the space of all smooth Riemannian metrics on M and we consider $\mathcal{MET}(M)$ with the smooth topology. Note that the space $\mathcal{MET}(M)$ is contractible. A subspace of metrics whose sectional curvatures lie in some interval (closed, open, semi-open) will be denoted by placing a superscript on $\mathcal{MET}(M)$. For example, $\mathcal{MET}^{sec<\epsilon}(M)$ denotes the subspace of $\mathcal{MET}(M)$ of all Riemannian metrics on M that have all sectional curvatures less that ϵ. Thus saying that all sectional curvatures of a Riemannian metric g lie in the interval $[a,b]$ is equivalent to saying that $g \in \mathcal{MET}^{a \leq sec \leq b}(M)$. Note that if $I \subset J$, then $\mathcal{MET}^{sec \in I}(M) \subset \mathcal{MET}^{sec \in J}(M)$. Note also that $\mathcal{MET}^{sec=-1}(M)$ is the space of hyperbolic metrics $\mathcal{H}yp\,(M)$ on M.

A natural question about a closed negatively curved manifold M is the following: Is the space $\mathcal{MET}^{sec<0}(M)$ of negatively curved metrics on M path connected? This problem has been around for some time and has been posed several times in the literature; see for instance K. Burns and A. Katok ([**2**], Question 7.1). In dimension two, Hamilton's Ricci flow [**12**] shows that $\mathcal{H}yp\,(M^2)$ is a deformation retract of $\mathcal{MET}^{sec<0}(M^2)$. But $\mathcal{H}yp\,(M^2)$ fibers over the Teichmüller space $\mathcal{T}(M^2) \cong \mathbb{R}^{6\mu-6}$ (μ is the genus of M^2), with contractible fiber $\mathcal{D} = \mathbb{R}^+ \times DIFF_0(M^2)$ [**5**], where $DIFF_0(M^2)$ denotes the group of self-diffeomorphisms of M^2 which are homotopic to the identity. Therefore $\mathcal{H}yp\,(M^2)$ and $\mathcal{MET}^{sec<0}(M^2)$ are contractible.

Received 2/15/10.

In this paper we prove that, for $n \geq 10$, $\mathcal{MET}^{sec<0}(M^n)$ is never path-connected; in fact, it has infinitely many path-components. Moreover we show that all the groups $\pi_{2p-4}(\mathcal{MET}^{sec<0}(M^n))$ are non-trivial for every prime number $p > 2$ and such that $p < \frac{n+5}{6}$. (In fact, these groups contain the infinite sum $(\mathbb{Z}_p)^\infty$ of $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$'s, and hence they are not finitely generated. Also, the restriction on $n = dim\, M$ can be improved to $p \leq \frac{n-2}{4}$. See Remark 1 below.) We also show that $\pi_1(\mathcal{MET}^{sec<0}(M^n))$ contains the infinite sum $(\mathbb{Z}_2)^\infty$ when $n \geq 14$. These results about π_k are true for each path component of $\mathcal{MET}^{sec<0}(M^n)$, i.e., relative to any base point. Before we state our Main Theorem, we need some definitions.

Denote by $DIFF(M)$ the group of all smooth self-diffeomorphisms of M. We have that $DIFF(M)$ acts on $\mathcal{MET}(M)$ pulling-back metrics: $\phi g = (\phi^{-1})^* g = \phi_* g$, for $g \in \mathcal{MET}(M)$ and $\phi \in DIFF(M)$, that is, ϕg is the metric such that $\phi : (M, g) \to (M, \phi\, g)$ is an isometry. Note that $DIFF(M)$ leaves invariant all spaces $\mathcal{MET}^{sec\in I}(M)$, for any $I \subset \mathbb{R}$. For any metric g on M, we denote by $DIFF(M)\, g$ the orbit of g by the action of $DIFF(M)$. We have a map $\Lambda_g : DIFF(M) \to \mathcal{MET}(M)$, given by $\Lambda_g(\phi) = \phi_* g$. Then the image of Λ_g is the orbit $DIFF(M)\, g$ of g. And Λ_g of course naturally factors through $\mathcal{MET}^{sec\in I}(M)$, if $g \in \mathcal{MET}^{sec\in I}(M)$. Note that if $dim\, M \geq 3$ and $g \in \mathcal{MET}^{sec=-1}(M)$, then the statement of Mostow's rigidity theorem is equivalent to saying that the map $\Lambda_g : DIFF(M) \to \mathcal{MET}^{sec=-1}(M) = \mathcal{H}yp\,(M)$ is a surjection. Here is the statement of our main result.

Main Theorem. *Let M be a closed smooth n-manifold and let g be a negatively curved Riemannian metric on M. Then we have the following:*

i. *The map $\pi_0(\Lambda_g) : \pi_0(\,DIFF(M)\,) \to \pi_0(\,\mathcal{MET}^{sec<0}(M)\,)$ is not constant, provided $n \geq 10$.*
ii. *The homomorphism $\pi_1(\Lambda_g) : \pi_1(\,DIFF(M)\,) \to \pi_1(\,\mathcal{MET}^{sec<0}(M)\,)$ is non-zero, provided $n \geq 14$.*
iii. *For $k = 2p - 4$, p prime integer and $1 < k \leq \frac{n-8}{3}$, the homomorphism $\pi_k(\Lambda_g) : \pi_k(\,DIFF(M)\,) \to \pi_k(\,\mathcal{MET}^{sec<0}(M)\,)$ is non-zero. (See Remark 1 below.)*

Addendum to the Main Theorem. *We have that the image of $\pi_0(\Lambda_g)$ is infinite and in cases (ii) and (iii) mentioned in the Main Theorem, the image of $\pi_k(\Lambda_g)$ is not finitely generated. In fact we have:*

i. *For $n \geq 10$, $\pi_0(\,DIFF(M)\,)$ contains $(\mathbb{Z}_2)^\infty$, and $\pi_0(\Lambda_g)|_{(\mathbb{Z}_2)^\infty}$ is one-to-one.*
ii. *For $n \geq 14$, the image of $\pi_1(\Lambda_g)$ contains $(\mathbb{Z}_2)^\infty$.*

iii. *For* $k = 2p - 4$, *p prime integer and* $1 < k \leq \frac{n-8}{3}$, *the image of* $\pi_k(\Lambda_g)$ *contains* $(\mathbb{Z}_p)^\infty$. *See Remark 1 below.*

For $a < b < 0$ the map Λ_g factors through the inclusion map $\mathcal{MET}^{\,a \leq sec \leq b}(M) \hookrightarrow \mathcal{MET}^{sec<0}(M)$ provided $g \in \mathcal{MET}^{\,a \leq sec \leq b}(M)$. Therefore we have:

Corollary 1. *Let* M *be a closed smooth* n*-manifold,* $n \geq 10$. *Let* $a < b < 0$ *and assume that* $\mathcal{MET}^{\,a \leq sec \leq b}(M)$ *is not empty. Then the inclusion map* $\mathcal{MET}^{\,a \leq sec \leq b}(M) \hookrightarrow \mathcal{MET}^{sec<0}(M)$ *is not null-homotopic. Indeed, the induced maps, at the* k*-homotopy level, are not constant for* $k = 0$, *and non-zero for the cases (ii) and (iii) mentioned in the Main Theorem. Furthermore, the image of these maps satisfy a statement analogous to the one in the addendum to the Main Theorem.*

If $a = b = -1$ we have:

Corollary 2. *Let* M *be a closed hyperbolic* n*-manifold,* $n \geq 10$. *Then the inclusion map* $\mathcal{H}yp\,(M) \hookrightarrow \mathcal{MET}^{sec<0}(M)$ *is not null-homotopic. Indeed, the induced maps, at the* k*-homotopy level, are not constant for* $k = 0$, *and non-zero for the cases (ii) and (iii) mentioned in the Main Theorem. Furthermore, the image of these maps satisfy a statement analogous to the one in the addendum to the Main Theorem.*

Hence, taking $k = 0$ (i.e., $p = 2$) in Corollary 2, we get that for any closed hyperbolic manifold (M^n, g), $n \geq 10$, there is a hyperbolic metric g' on M such that g and g' cannot be joined by a path of negatively curved metrics.

Also, taking $a = -1 - \epsilon, \quad b = -1 \ (0 \leq \epsilon)$ in Corollary 1, we have that the space $\mathcal{MET}^{\,-1-\epsilon \leq sec \leq -1}(M^n)$ of ϵ-pinched negatively curved Riemannian metrics on M has infinitely many path components, provided it is not empty and $n \geq 10$. And the homotopy groups $\pi_k(\mathcal{MET}^{\,-1-\epsilon \leq sec \leq -1}(M))$ are non-zero for the cases (ii) and (iii) mentioned in the Main Theorem. Moreover, these groups are not finitely generated.

Remark 1. The restriction on $n = dim\, M$ given in the Main Theorem, its addendum and its corollaries are certainly not optimal. In particular, in (iii) it can be improved to $1 < k < \frac{n-10}{2}$ by using Igusa's "Surjective Stability Theorem" ([**16**], p. 7).

As before, let $DIFF_0(M)$ be the subgroup of $DIFF(M)$ of all self-diffeomorphisms that are homotopic to the identity. If M is closed and negatively curved, the action of $DIFF_0(M)$ on $\mathcal{MET}^{sec<0}(M)$ is free and in [**7**] we called the quotient $\mathcal{T}^\infty(M) = \mathcal{MET}^{sec<0}(M)/DIFF_0(M)$ the Teichmüller space of negatively curved metrics on M. We have a fibration

$$DIFF_0(M) \longrightarrow \mathcal{MET}^{sec<0}(M) \longrightarrow \mathcal{T}^\infty(M).$$

In [**7**], by using diffeomorphisms that are supported on a ball, we proved that there are closed hyperbolic manifolds for which some of the connecting homomorphisms $\pi_k(\mathcal{T}^\infty(M)) \to \pi_{k-1}(DIFF_0(M))$ are non-zero. In this paper, we use diffeomorphisms supported on a tubular neighborhood of a closed geodesic to show that the homomorphism induced by the inclusion of the fiber, $\pi_k(DIFF_0(M)) \to \pi_k(\mathcal{MET}^{sec<0}(M))$, is non-zero for many values of k. For other related results, see [**8**] and [**9**].

Another interesting application of the Main Theorem shows that the answer to the following natural question is negative:

Question. Let $E \to B$ be a fiber bundle whose fibers are diffeomorphic to a closed negatively curved manifold M^n. Is it always possible to equip its fibers with negatively curved Riemannian metrics (varying continuously from fiber to fiber)?

The negative answer is gotten by setting $B = \mathbb{S}^{k+1}$, where k is as in the Main Theorem case (iii) (or $k = 0, 1$, cases (i) and (ii)), and the bundle $E \to \mathbb{S}^{k+1}$ is obtained by the standard clutching construction using an element $\alpha \in \pi_k(DIFF(M))$ such that $\pi_k(\Lambda_g)(\alpha) \neq 0$, for *every* negatively curved Riemannian metric g on M. Using our method for proving the Main Theorem (in particular Theorem 1 below), one sees that such elements α, which are independent of g, exist in all cases (i), (ii), (iii).

The Main Theorem follows from Theorems 1 and 2 below. Before we state these results, we need some definitions and constructions. For a manifold N let $P(N)$ be the space of topological pseudo-isotopies of N, that is, the space of all homeomorphisms $N \times I \to N \times I$, $I = [0,1]$, that are the identity on $(N \times \{0\}) \cup (\partial N \times I)$. We consider $P(N)$ with the compact-open topology. Also, $P^{diff}(N)$ is the space of all *smooth* pseudo-isotopies on N, with the smooth topology. Note that $P^{diff}(N)$ is a subset of $P(N)$. The map of spaces $P^{diff}(N) \to P(N)$ is continuous and will be denoted by ι_N, or simply by ι. The space of all self-diffeomorphisms of N will be denoted by $DIFF(N)$, considered with the smooth topology. Also $DIFF(N, \partial)$ denotes the subspace of $DIFF(N)$ of all self-diffeomorphism of N which are the identity on ∂N.

Remark 2. We will assume that the elements in $DIFF(N, \partial)$ are the identity *near* ∂N.

Note that $DIFF(N \times I, \partial)$ is the subspace of $P^{diff}(N)$ of all smooth pseudo-isotopies whose restriction to $N \times \{1\}$ is the identity. The restriction of ι_N to $DIFF(N \times I, \partial)$ will also be denoted by ι_N. The map $\iota_N : DIFF(N \times I, \partial) \to P(N)$ is one of the ingredients in the statement Theorem 1.

We will also need the following construction. Let M be a negatively curved n-manifold. Let $\alpha : \mathbb{S}^1 \to M$ be an embedding. Sometimes we will denote the image $\alpha(\mathbb{S}^1)$ just by α. We assume that the normal bundle of α is orientable, and hence trivial. Let $V : \mathbb{S}^1 \to TM \times \ldots \times TM$ be an orthonormal trivialization of this bundle: $V(z) = (v_1(z), \ldots, v_{n-1}(z))$ is an orthonormal base of the orthogonal complement of $\alpha(z)'$ in T_zM. Also, let $r > 0$ be such that $2r$ is less than the width of the normal geodesic tubular neighborhood of α. Using V and the exponential map of geodesics orthogonal to α, we identify the normal geodesic tubular neighborhood of width $2r$ minus α, with $\mathbb{S}^1 \times \mathbb{S}^{n-2} \times (0, 2r]$. Define $\Phi = \Phi^M(\alpha, V, r) : DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial) \to DIFF(M)$ in the following way. For $\varphi \in DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial)$ let $\Phi(\varphi) : M \to M$ be the identity outside $\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [r, 2r] \subset M$, and $\Phi(\varphi) = \lambda^{-1}\varphi\lambda$, where $\lambda(z, u, t) = (z, u, \frac{t-r}{r})$, for $(z, u, t) \in \mathbb{S}^1 \times \mathbb{S}^{n-2} \times [r, 2r]$. Note that the dependence of $\Phi(\alpha, V, r)$ on α and V is essential, while its dependence on r is almost irrelevant.

We denote by g the negatively curved metric on M. Hence we have the diagram

$$\begin{array}{ccccc} DIFF(\,(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times I, \partial\,) & \xrightarrow{\Phi} & DIFF(M) & \xrightarrow{\Lambda_g} & \mathcal{MET}^{sec<0}(M) \\ \iota \downarrow & & & & \\ P(\mathbb{S}^1 \times \mathbb{S}^{n-2}) & & & & \end{array}$$

where $\iota = \iota_{\mathbb{S}^1 \times \mathbb{S}^{n-2}}$ and $\Phi = \Phi^M(\alpha, V, r)$.

Theorem 1. *Let M be a closed n-manifold with a negatively curved metric g. Let α, V, r, and $\Phi = \Phi(\alpha, V, r)$ be as above, and assume that α is not null-homotopic. Then $Ker\,(\,\pi_k(\Lambda_g\Phi)\,) \subset Ker\,(\,\pi_k(\iota)\,)$, for $k < n - 5$. Here $\pi_k(\Lambda_g\Phi)$ and $\pi_k(\iota)$ are the homomorphisms at the k-homotopy group level induced by $\Lambda_g\Phi$ and $\iota = \iota_{\mathbb{S}^1 \times \mathbb{S}^{n-2}}$, respectively.*

Remark. In the statement of Theorem 1 above, by $Ker\,(\,\pi_0(\Lambda_g\Phi)\,)$ (for $k = 0$) we mean the set $(\pi_0(\,\Lambda_g\Phi\,))^{-1}\,([g])$, where $[g] \in \pi_0(\mathcal{MET}^{sec<0}(M))$ is the connected component of the metric g.

Hence to deduce the Main Theorem from Theorem 1 we need to know that $\pi_k(\iota_{\mathbb{S}^1 \times \mathbb{S}^{n-2}})$ is a non-zero homomorphism. Furthermore, to prove the addendum to the Main Theorem we have to show that $\pi_k(DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial))$ contains an infinite sum of $\mathbb{Z}_p$'s (resp. $\mathbb{Z}_2$'s) where $k = 2p - 4$, p prime (resp. $k = 1$) and $\pi_k(\iota_{\mathbb{S}^1 \times \mathbb{S}^{n-2}})$ restricted to this sum is one-to-one.

Theorem 2. *Let p be a prime integer such that max $\{9, 6p - 5\} < n$. Then for $k = 2p - 4$ we have that $\pi_k(DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial))$ contains* *a subgroup isomorphic to* $(\mathbb{Z}_p)^\infty$ *and the restriction of* $\pi_k(\iota_{\mathbb{S}^1\times\mathbb{S}^{n-2}})$ *to this subgroup is one-to-one.*

Addendum to Theorem 2. *Assume* $n \geq 14$. *Then* $\pi_1(DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2}\times I, \partial))$ *contains a subgroup isomorphic to* $(\mathbb{Z}_2)^\infty$ *and the restriction of* $\pi_1(\iota_{\mathbb{S}^1\times\mathbb{S}^{n-2}})$ *to this subgroup is one-to-one.*

The paper is structured as follows. In Section 1 we give some lemmas, including some fibered versions of the Whitney embedding Theorem. In Section 2 we give (recall) some facts about simply connected negatively curved manifolds and their natural extensions to a special class of non-simply connected ones. The results and facts in Sections 1 and 2 are used in the proof of Theorem 1, which is given in Section 3. Finally, Theorem 2 is proved in Section 4.

Before we finish this introduction, we sketch an argument that, we hope, motivates our proof of Theorem 1. To avoid complications, let's just consider the case $k = 0$. In this situation we want to show the following:
Let $\theta \in DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial) \subset P(\mathbb{S}^1 \times \mathbb{S}^{n-2})$, *and write* $\varphi = \Phi(\theta) : M \to M$. *Suppose that* θ *cannot be joined by a path to the identity in* $P(\mathbb{S}^1 \times \mathbb{S}^{n-2})$. *Then* g *cannot be joined to* $\phi_* g$ *by a path of negatively curved metrics.*

Here is an argument that we could tentatively use to prove the statement above. Suppose that there is a smooth path g_u, $u \in [0, 1]$, of negatively curved metrics on M, with $g_0 = g$ and $g_1 = \varphi_* g$. We will use g_u to show that θ can be joined to the identity in $P(\mathbb{S}^1\times\mathbb{S}^{n-2})$. We assume that α is an embedded closed geodesic in M. Let Q be the cover of M corresponding to the infinite cyclic group generated by α. Each g_u lifts to a g_u on Q (we use the same letter). Then α lifts isometrically to (Q, g) and we can identify Q with $\mathbb{S}^1\times\mathbb{R}^{n-1}$ such that α corresponds $\mathbb{S}^1 = \mathbb{S}^1 \times \{0\}$ and such that each $\{z\} \times \mathbb{R}v$, $v \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$, corresponds to a g geodesic ray emanating perpendicularly from α. For each u, the complete negatively curved manifold (Q, g_u) contains exactly one closed geodesic α_u, and α_u is freely homotopic to α. Let us assume that $\alpha_u = \alpha$, for all $u \in [0, 1]$. Moreover, let us assume that g_u coincides with g in the normal tubular neighborhood W of length one of α. Note that $Q \setminus int\, W$ can be identified with $(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [1, \infty)$. Using geodesic rays emanating perpendicularly from α, we can define a path of diffeomorphisms $f_u : (\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [1, \infty) \to (\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [1, \infty)$ by $f_u = [exp]^{-1} \circ exp^u$, where exp^u denotes the normal (to α) exponential map with respect to g_u, and $exp = exp^0$. Using "the space at infinity" $\partial_\infty Q$ of Q (see Section 2), we can extend f_u to $(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [1, \infty]$, which we identify with $(\mathbb{S}^1\times\mathbb{S}^{n-2})\times[0, 1]$. Finally, it is proved that f_1 can be joined to θ in $P(\mathbb{S}^1 \times \mathbb{S}^{n-2})$ (see Claim 6 in Section 3). This is enough because f_0 is the identity.

Along the "sketch of the proof" above we have of course made several unproven claims (that will be proven later); and we have also made a few assumptions: (1) α is an embedded closed geodesic, (2) $\alpha_u = \alpha$ for all u, (3) g_u coincides with g in a neighborhood of g. Item (1) can be obtained "after a deformation" in Q. Item (2) can also be obtained after a deformation in Q using the results of Section 2. We do not know how to obtain (3) after a deformation (and this might even be impossible to do), so we have to use some approximation methods based on Lemma 1.6 which implies that we can take a very thin normal neighborhood W of α such that all normal (to α) g_u geodesics rays will intersect ∂W transversally in one point.

Acknowledgments. The first author was partially supported by a NSF grant. The second author was supported in part by research grants from CNPq (Brazil) and NSF. We are grateful to the referee for his useful comments and suggestions.

The p-torsion Theorem that appears at the end of section 4 is crucial to the proof of Theorem 2 (when $k > 1$). This result, together with a sketch of its proof, was given to us by Tom Goodwillie in 2005 in a personal communication. We are grateful to him for this. The p-torsion theorem appeared in print in a paper by Grunewald, Klein, and Macko in 2008. We are grateful to the authors of this paper as well, in particular to John Klein for his comments and emails.

1. Preliminaries

For smooth manifolds A, B, with A compact, $C^\infty(A,B)$, $DIFF(A)$, $Emb\,(A,B)$, denote the space of smooth maps, smooth self-diffeomorphisms, and smooth embeddings of A into B, respectively. We consider these spaces with the smooth topology. The l-disc will be denoted by $\mathbb{D}^l$. We choose $u_0 = (1,0,\dots,0)$ as the base point of $\mathbb{S}^l \subset \mathbb{D}^{l+1}$. For a map $f : A \times B \to C$, we denote by f_a the map given by $f_a(b) = f(a,b)$. A map $f : \mathbb{D}^l \times A \to B$ is *radial near* ∂ if $f_u = f_{tu}$ for all $u \in \partial\mathbb{D}^l = \mathbb{S}^{l-1}$ and $t \in [1/2, 1]$. Note that any map $f : \mathbb{D}^l \times A \to B$ is homotopic rel $\partial\mathbb{D}^l \times A$ to a map that is radial near ∂. The next lemma is a special case of a parametrized version of Whitney's embedding theorem.

Lemma 1.1. *Let P^m and D^{k+1} be compact smooth manifolds and let T be a closed smooth submanifold of P. Let Q be an open subset of $\mathbb{R}^n$ and let $H' : D \times P \to Q$ be a smooth map such that (1) $H'_u|_T : T \to Q$ is an embedding for all $u \in D$ and (2) H'_u is an embedding for all $u \in \partial D$. Assume that that $k + 2m + 1 < n$. Then H' is homotopy equivalent to a smooth map $\bar{H} : D \times P \to Q$ such that:*

1. *$\bar{H}_u : P \to Q$ is an embedding, for all $u \in D$.*
2. *$\bar{H}|_{D\times T} = H'|_{D\times T}$.*
3. *$\bar{H}|_{\partial D\times P} = H'|_{\partial D\times P}$.*

Proof. It is not difficult to construct a smooth map $g : P \to \mathbb{R}^q$, for some q, such that (i) $g : P \setminus T \to \mathbb{R}^q \setminus \{0\}$ is a smooth embedding, (ii) $g(T) = \{0\} \in \mathbb{R}^q$, and (iii) $D_p\, g\,(v) \neq 0$, for every $p \in T$ and $v \in T_pP \setminus T_pT$. Let $\varpi : D \to [0,1]$ be a smooth map such that $\varpi^{-1}(0) = \partial D$. Define $G = H' \times g : D \times P \to Q \times \mathbb{R}^q$, $G(u,p) = (H'(u,p), \varpi(u)g(p))$. Then, for each $u \in D$, $G_u : P \to Q \times \mathbb{R}^q$ is an embedding. Moreover, $G|_{D\times T} = H'_{D\times T}$, where we consider $Q = Q \times \{0\} \subset Q \times \mathbb{R}^q$. Also, $G|_{\partial D\times P} = H'|_{\partial D\times P}$. Note that G is homotopic to H' because g is homotopically trivial. Now, as in the proof of Whitney's theorem, we want to reduce the dimension q to $q - 1$. So assume $q > 0$. Given $w \in \mathbb{S}^{n+q-1} \subset \mathbb{R}^{n+q} = \mathbb{R}^n \times \mathbb{R}^q$, $w \notin \mathbb{R}^n \times \mathbb{R}^{q-1} = \mathbb{R}^{n+q-1}$, denote by $L_w : \mathbb{R}^{n+q} \to \mathbb{R}^{n+q-1}$ the linear projection "in the w-direction." As in the proof of Whitney's theorem, using the dimension restriction and Sard's theorem, we can find a "good" w:

Claim. There is a w such that $L_w|_{G_u(P)} : G_u(P) \to \mathbb{R}^{n+q-1}$ is an embedding, for all $u \in D$.

For this consider the following:

$$r : D \times (\,(P \times P) \setminus \Delta(P)\,) \to \mathbb{R}^{n+q}, \quad r(u,p,q) = \tfrac{G_u(p)-G_u(q)}{|G_u(p)-G_u(q)|}$$

$$s : D \times SP \to \mathbb{R}^{n+q}, \qquad s(u,v) = \tfrac{D_p(G_u)(v)}{|D_p(G_u)(v)|},\ \ v \in T_pP.$$

Here $\Delta(P) = \{(p,p) : p \in P\}$ and SP is the sphere bundle of P (with respect to any metric). Since $(k+1) + 2m < n$ and $q > 0$, by Sard's theorem the images of r and s have measure zero in $\mathbb{S}^{n+q-1}$. This proves the claim.

Also, since D and P are compact, we can choose w close enough to $(0,\ldots,0,1)$ such that $L_w(\,G(D \times P)\,) \subset Q \times \mathbb{R}^{q-1}$. Define $G_1 = L_wG$. In the same way, we define $G_2 : D \times P \to Q \times \mathbb{R}^{q-2}$, and so on. Our desired map $\bar{H}$ is $\bar{H} = G_q$. This proves the lemma. q.e.d.

In what follows of this section we consider $Q = \mathbb{S}^1 \times \mathbb{R}^{n-1} = (\mathbb{S}^1 \times \mathbb{R}) \times \mathbb{R}^{n-2} \subset \mathbb{R}^2 \times \mathbb{R}^{n-2}$, where the inclusion $\mathbb{S}^1 \times \mathbb{R} \hookrightarrow \mathbb{R}^2$ is given by $(z,s) \mapsto e^s z$. That is, we identify $\mathbb{S}^1 \times \mathbb{R}$ with the open set $\mathbb{R}^2 \setminus \{0\}$, and hence we identify $Q = \mathbb{S}^1 \times \mathbb{R}^{n-1}$ with $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{n-2} = \mathbb{R}^n \setminus (\,\{0\} \times \mathbb{R}^{n-2}\,)$. Also, identify $\mathbb{S}^1$ with $\mathbb{S}^1 \times \{0\} \subset Q$ and denote by $h_0 : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}^{n-1} = Q$ the inclusion. For $t > 0$ denote by $\kappa_t : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^2 \times \mathbb{R}^{n-2}$ the map given by $\kappa_t(a,b) = (ta,b)$. Note that κ_t restricts to $Q = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{n-2}$.

Lemma 1.2. *Let $h, h' : \mathbb{D}^{k+1} \times \mathbb{S}^1 \to Q$ be continuous maps such that h_u, h'_u are homotopic, for all $u \in \mathbb{S}^k$. That is, there is $H' : \mathbb{S}^k \times \mathbb{S}^1 \times I \to Q$ such that $H'(u,z,0) = h(u,z)$, $H'(u,z,1) = h'(u,z)$, for all $(u,z) \in \mathbb{S}^k \times \mathbb{S}^1$. For $k = 0$ also assume that the loop $h(t,1) * H'(1,1,t) * [h'(t,1)]^{-1} * [H'(-1,1,t)]^{-1}$ is null-homotopic. Then H' extends to $H' : \mathbb{D}^{k+1} \times \mathbb{S}^1 \times I \to Q$ such that H'_u is a homotopy from h_u to h'_u, that is,*

1. $H'_u|_{\mathbb{S}^1\times\{0\}} = h_u$, *for* $u \in \mathbb{D}^{k+1}$,
2. $H'_u|_{\mathbb{S}^1\times\{1\}} = h'_u$, *for* $u \in \mathbb{D}^{k+1}$.

Proof. First define $H' = h$ on $\mathbb{D}^{k+1} \times \mathbb{S}^1 \times \{0\}$ and $H' = h'$ on $\mathbb{D}^{k+1} \times \mathbb{S}^1 \times \{1\}$. Note that H' is defined on $\partial(\mathbb{D}^{k+1} \times \{1\} \times I)$. Since Q is aspherical, we can extend H' to $\mathbb{D}^{k+1} \times \{1\} \times I$ (for $k = 0$ use the assumption given in the statement of the lemma). H' is now defined on $A = \partial(\mathbb{D}^{k+4} \times \mathbb{S}^1 \times I) \cup \mathbb{D}^{k+1} \times \{1\} \times I$. Since $\mathbb{D}^{k+1} \times \mathbb{S}^1 \times I$ is obtained from A by attaching a $(k+3)$-cell and Q is aspherical, we can extend H' to $\mathbb{D}^{k+1} \times \mathbb{S}^1 \times I$. This proves the lemma. q.e.d.

Lemma 1.3. *Let* $h : \mathbb{D}^{k+1} \times \mathbb{S}^1 \to Q$ *be a smooth map which is radial near* ∂. *Assume that* $h_u \in Emb(\mathbb{S}^1, Q)$ *for all* $u \in \mathbb{D}^{k+1}$ *and* $h_u = h_0$, *for all* $u \in \mathbb{S}^k$. *(Here* $h_0 = h_{u_0}$.*) For* $k = 0$ *assume that the loop* $h(u, 1)$ *is homotopically trivial. If* $k+5 < n$ *then there is a smooth map* $\hat{H} : \mathbb{D}^{k+1} \times \mathbb{S}^1 \times I \to Q$ *such that:*

1. $\hat{H}_u|_{\mathbb{S}^1\times\{0\}} = h_u$, *for* $u \in \mathbb{D}^{k+1}$.
2. $\hat{H}_u|_{\mathbb{S}^1\times\{1\}} = h_0$, *for* $u \in \mathbb{D}^{k+1}$.
3. $\hat{H}_u$ *is a smooth isotopy from* h_u *to* h_0.
4. $(\hat{H}_u)_t = h_0$, *for all* $u \in \mathbb{S}^k$ *and* $t \in I$. *Here* $(\hat{H}_u)_t(z) = \hat{H}(u, z, t)$.

Proof. During this proof some isotopies and functions have to be smoothed near endpoints and boundaries. We do not do this to avoid unnecessary technicalities.

Let $D = \mathbb{D}^{k+1}_{1/2}$ be the closed $(k+1)$-disc of radius $1/2$. Since $h(\mathbb{D}^{k+1} \times \mathbb{S}^1) \subset Q = \mathbb{R}^n \setminus (\{0\} \times \mathbb{R}^{n-2})$, we have that $h(\mathbb{D}^{k+1} \times \mathbb{S}^1)$ does not intersect $\{0\} \times \mathbb{R}^{n-2}$. Therefore the distance d from $h(\mathbb{D}^{k+1} \times \mathbb{S}^1)$ to $\{0\} \times \mathbb{R}^{n-2}$ is positive. Let $c < 1$ be such that $c < d$.

Definition of $(\hat{H}_u)_t$ **for** $t \in [1/2, 1]$**.** In this case define for $u \in \mathbb{S}^k$, $(\hat{H}_{su})_t = \kappa_\lambda h_0$, where **(1)** $\lambda = 1 - 4(1-t)(1-s) + 4(1-t)(1-s)c$ if $s \in [1/2, 1]$ and **(2)** $\lambda = (2t-1) + (2-2t)c$ $s \in [0, 1/2]$.

Definition of $(\hat{H}_{su})_t$ **for** $t \in [0, 1/2]$ **and** $s \in [1/2, 1]$**.** Define for $u \in \mathbb{S}^k$, $s \in [1/2, 1]$: $(\hat{H}_{su})_t = \kappa_\lambda$, where $\lambda = 1 - 4t(1-s) + 4t(1-s)c$, for $t \in [0, 1/2]$.

Definition of $(\hat{H}_{su})_t$ **for** $t \in [0, 1/2]$ **and** $s \in [0, 1/2]$**.** Note that $D = \{su : u \in \mathbb{S}^k,\ s \in [0, 1/2]\}$. We now want to define $\hat{H}$ on $D \times \mathbb{S}^1 \times [0, 1/2]$. To do this first apply Lemma 1.2 to h and $\mathbb{D} \times \mathbb{S}^1 \times I$, with $h'_u = \kappa_c h_0$ for all $u \in D$, $H'(u, z, t) = \hat{H}(u, z, t/2)$, for $(u, z, t) \in \partial D \times \mathbb{S}^1 \times I$. Hence H' extends to $D \times \mathbb{S}^1 \times I$. Now apply Lemma 1.1, taking $P = \mathbb{S}^1 \times I$, $T = \mathbb{S}^1 \times \{0, 1\}$. To apply this lemma, note that $H'_u|_{\mathbb{S}^1\times\{0,1\}}$ is an embedding, for all $u \in D$, because $H'_u|_{\mathbb{S}^1\times\{0\}} = h_u$, $H'_u|_{\mathbb{S}^1\times\{1\}} = \kappa_c h_0$ are embeddings and the images of h_u and $\kappa_c h_0$ are disjoint (by the choice of c). Let then $\bar{H}$ be the map given by Lemma 1.1. Finally, define $\hat{H}(u,z,t) = \bar{H}(u,z,2t)$. This proves the lemma. q.e.d.

Extending the isotopies $\hat{H}_u$ between h_u and h'_u given in the lemma above, to compactly supported ambient isotopies we obtain as a corollary the following lemma.

Lemma 1.4. *Let $h : \mathbb{D}^{k+1} \times \mathbb{S}^1 \to Q$ be a smooth map which is radial near ∂. Assume $h_u \in Emb(\mathbb{S}^1, Q)$ for all $u \in \mathbb{D}^{k+1}$ and that $h_u = h_0 \in Emb\,(\mathbb{S}^1, Q)$ for all $u \in \mathbb{S}^k$, and $k+5 < n$. (Here $h_0 = h_{u_0}$.) Identify $\mathbb{S}^1$ with $\mathbb{S}^1 \times \{0\} \subset Q$. For $k = 0$ assume that the loop $h(u,1)$ is null-homotopic. Then there is a smooth map $H : \mathbb{D}^{k+1} \times Q \times I \to Q$ such that:*

1. *$H_u|_{\mathbb{S}^1 \times \{0\}} = h_u$, for $u \in \mathbb{D}^{k+1}$.*
2. *$H_u|_{\mathbb{S}^1 \times \{1\}} = h_0$, for $u \in \mathbb{D}^{k+1}$.*
3. *H_u is an ambient isotopy from h_u to h_0, that is $(H_u)_t : Q \to Q$ is a diffeomorphism for all $u \in \mathbb{D}^{k+1}$, $t \in I$ and $(H_u)_1 = 1_Q$. Also, H_u is supported on a compact subset $K \subset Q$, where K is independent of $u \in \mathbb{D}^{k+1}$.*
4. *$(H_u)_t = 1_Q$, for all $u \in \mathbb{S}^k$ and $t \in I$.*

We will also need the result stated in Lemma 1.6 below. First we prove a simplified version of it. The k-sphere of radius δ, $\{v \in \mathbb{R}^{k+1} : |v| = \delta\}$, will be denoted by $\mathbb{S}^k(\delta)$.

Lemma 1.5. *Let X be a compact space and $f : X \to DIFF(\mathbb{R}^l)$ be continuous and write $f_x : \mathbb{R}^l \to \mathbb{R}^l$ for the image of x in $DIFF(\mathbb{R}^l)$. Assume $f_x(0) = 0 \in \mathbb{R}^l$, for all $x \in X$. Then there is a $\delta_0 > 0$ such that, for every $x \in X$ and $\delta \leq \delta_0$, the map $\mathbb{S}^{l-1}(\delta) \to \mathbb{S}^{l-1}$ given by $v \mapsto \frac{f_x(v)}{|f_x(v)|}$ is a diffeomorphism. Moreover, the map $X \to DIFF(\mathbb{S}^{l-1}(\delta), \mathbb{S}^{l-1})$, given by $x \mapsto (v \mapsto \frac{f_x(v)}{|f_x(v)|})$, is continuous.*

Proof. First note that for all $x \in X$ and $\delta > 0$, the maps in $DIFF$ $(\mathbb{S}^{l-1}(\delta), \mathbb{S}^{l-1})$ given by $(v \mapsto \frac{f_x(v)}{|f_x(v)|})$ all have degree 1 or -1. For $v \in \mathbb{R}^l \backslash \{0\}$, denote by $L_x(v)$ the image of the tangent space $T_v(\mathbb{S}^{l-1}(|v|))$ by the derivative of $f_x : \mathbb{R}^l \to \mathbb{R}^l$. It is enough to prove that there is $\delta_0 > 0$ such that $f_x(v) \notin L_x(v)$, for all $x \in X$ and $v \in \mathbb{R}^l$ satisfying $0 < |v| \leq \delta_0$ (because then the maps $(v \mapsto \frac{f_x(v)}{|f_x(v)|})$ would be immersions of degree 1 (or -1), and hence diffeomorphisms).

Suppose this does not happen. Then there is a sequence of points $(x_m, v_m) \in X \times \mathbb{R}^l \setminus \{0\}$ with

a. $v_m \to 0$,
b. $f_{x_m}(v_m) \in L_{x_m}(v_m)$.

Write $w_m = \frac{v_m}{|v_m|} \in \mathbb{S}^{l-1}$, $r_m = |v_m|$, $f_m = f_{x_m}$, and $D_m = D_{v_m} f_m$. We can assume that $x_m \to x \in X$, and that $w_m \to w \in \mathbb{S}^{l-1}$. It follows that there is an $u_m \in T_{v_m}(\mathbb{S}^{l-1}(r_m))$, $|u_m| = 1$, such that $D_m.\, u_m$ is parallel to $f_m(v_m)$. Note that $\langle u_m, v_m\rangle = 0$ and $D_m(u_m) \neq 0$. By changing the sign of u_m, we can assume that $\frac{D_m(u_m)}{|D_m(u_m)|} = \frac{f_m(v_m)}{|f_m(v_m)|}$. Also, we can suppose that $u_m \to u \in \mathbb{S}^{l-1}$.

Claim. We have that $\frac{f_m(v_m)}{|f_m(v_m)|} \to \frac{D_0 f_x(w)}{|D_0 f_x(w)|}$, as $m \to \infty$.

Proof of the claim. Since f is continuous, all second-order partial derivatives of the coordinate functions of the f_x at v, with, say, $|v| \leq 1$, are bounded by some constant. Hence there is a constant $C > 0$ such that $|f_m(v_m) - D_0 f_m(v_m)| = |f_m(v_m) - f_m(0) - D_0 f_m(v_m)| \leq C\,|v_m|^2$, for sufficiently large m. It follows that $\frac{f_m(v_m)}{|v_m|} \to lim_{m\to\infty} \frac{D_0 f_m(v_m)}{|v_m|} = D_0 f_x(w) \neq 0$. This implies that $\frac{|f_m(v_m)|}{|v_m|} \to |D_0 f_x(w)| \neq 0$, and thus $\frac{|v_m|}{|f_m(v_m)|} \to \frac{1}{|D_0 f_x(w)|}$. Therefore $lim_{m\to\infty} \frac{f_m(v_m)}{|f_m(v_m)|)} = lim_{m\to\infty} \frac{f_m(v_m)}{|v_m|}\frac{|v_m|}{|f_m(v_m)|} = D_0 f_x(w)\,\frac{1}{|D_0 f_x(w)|}$. This proves the claim.

But $\frac{D_m(u_m)}{|D_m(u_m)|} \to \frac{D_0 f_x(u)}{|D_0 f_x(u)|}$; therefore $\frac{D_0 f_x(u)}{|D_0 f_x(u)|} = \frac{D_0 f_x(w)}{|D_0 f_x(w)|}$. This is a contradiction since $D_0 f_x$ is an isomorphism and $u, w \in \mathbb{S}^{l-1}$ are linearly independent (because $\langle u, w\rangle = lim_m \langle u_m, \frac{v_m}{|v_m|}\rangle = 0$). This proves the lemma. q.e.d.

Lemma 1.6. *Let X be a compact space, N a closed smooth manifold, and $f : X \to DIFF(N \times \mathbb{R}^l)$ be continuous and write $f_x = (f_x^1, f_x^2) : N \times \mathbb{R}^l \to N \times \mathbb{R}^l$ for the image of x in $DIFF(N \times \mathbb{R}^l)$. Assume $f_x(z, 0) = (z, 0)$, for all $x \in X$ and $z \in N$, that is, $f_x|_N = 1_N$, where we identify N with $N \times \{0\}$. Then there is a $\delta_0 > 0$ such that, for every $x \in X$, the map $N \times \mathbb{S}^{l-1}(\delta) \to N \times \mathbb{S}^{l-1}$ given by $(z, v) \mapsto (f_x^1(z, v), \frac{f_x^2(z,v)}{|f_x^2(z,v)|})$ is a diffeomorphism for all $\delta \leq \delta_0$. Moreover, the map $X \to DIFF(N \times \mathbb{S}^{l-1}(\delta),\, N \times \mathbb{S}^{l-1})$, given by $x \mapsto (\,(z, v) \mapsto (f_x^1(z, v), \frac{f_x^2(z,v)}{|f_x^2(z,v)|})\,)$, is continuous.*

Proof. The proof is similar to the proof of the lemma above. Here are the details. Let $d = dim\, N$ and consider N with some Riemannian metric. For $(z, v) \in N \times \mathbb{R}^l \setminus \{0\}$, denote by $L_x(z, v)$ the image of the tangent space $T_{(z,v)}(N \times \mathbb{S}^{l-1}(|v|))$ by the derivative of f_x. As before it is enough to prove that there is $\delta_0 > 0$ such that $(0, f_x^2(z, v)) \notin L_x(z, v) \subset (T_z N) \times \mathbb{R}^l = T_{(z,v)}(N \times \mathbb{R}^l)$, for all $x \in X$ and $(z, v) \in N \times \mathbb{R}^l$ satisfying $0 < |v| \leq \delta_0$. Before we prove this we have a claim.

Claim 1. *We have:*

1. $D_{(z,0)} f_x^1(y, 0) = y$, *for all $z \in N$ and $y \in T_z N$.*
2. $D_{(z,0)} f_x^2(y, u) = 0$ *implies that $u = 0$.*

Proof of Claim 1. Since $f_x|_N = 1_N$ we have that $D_{(z,0)} f_x(y, 0) = (y, 0)$, for all $y \in T_z N$. Hence (1) holds. Suppose $D_{(z,0)} f_x^2(y, u) = 0$.

Write $y' = D_{(z,0)} f_x^1(y,u)$. Then $D_{(z,u)} f_x(y,u) = (y',0) = D_{(z,0)} f_x(y',0)$. But $D_{(z,0)} f_x$ is an isomorphism and therefore $(y,u) = (y',0)$. This proves the claim.

Suppose now that (2) does not happen. Then there is a sequence of points $(x_m, z_m, v_m) \in X \times N \times \mathbb{R}^l \setminus \{0\}$ with

a. $v_m \to 0$,
b. $(0, f^2_{x_m}(z_m, v_m)) \in L_{x_m}(z_m, v_m)$.

Write $w_m = \frac{v_m}{|v_m|} \in \mathbb{S}^{l-1}$, $r_m = |v_m|$, $f_m = f_{x_m}$, and $D^i_m = D_{v_m} f^i_m$, $i = 1, 2$. We can assume that $x_m \to x \in X$, $z_m \to z$ and $w_m \to w \in \mathbb{S}^{l-1}$. It follows that there is a $(s_m, u_m) \in T_{(z_m,v_m)}(N \times \mathbb{S}^{l-1}(r_m))$, $|s_m|^2 + |u_m|^2 = 1$, such that (i) $D^1_m(s_m, u_m) = 0$, and (ii) $D^2_m(s_m, u_m)$ is parallel to $f^2_m(z_m, v_m)$. We have that $\langle u_m, v_m \rangle = 0$. Since $D_m = D_{v_m} f_m$ is an isomorphism, by (i), $D^2_m(s_m, u_m) \neq 0$. By changing the sign of (s_m, u_m) we can assume that $\frac{D^2_m(s_m,u_m)}{|D^2_m(s_m,u_m)|} = \frac{f^2_m(z_m,v_m)}{|f^2_m(z_m,v_m)|}$. Also, we can suppose that $u_m \to u \in \mathbb{R}^l$ and $s_m \to s \in T_z N$.

Claim 2. *We have that* $\frac{f^2_m(z_m,v_m)}{|f^2_m(z_m,v_m)|} \to \frac{D_{(z,0)} f^2_x(0,w)}{|D_{(z,0)} f^2_x(0,w)|}$, *as* $m \to \infty$.

Proof of Claim 2. Since f^2 is continuous, all second-order partial derivatives of the coordinate functions of the f^2_x at v, with, say, $|v| \leq 1$, are bounded by some constant. Hence there is a constant $C > 0$ such that $|f^2_m(z_m, v_m) - D_{(z_m,0)} f^2_m(0, v_m)| = |f^2_m(z_m, v_m) - f^2_m(z_m, 0) - D_{(z_m,0)} f^2_m(0, v_m)| \leq C\,|(0, v_m)|^2 = |v_m|^2$, for sufficiently large m. It follows that $\frac{f^2_m(z_m,v_m)}{|(0,v_m)|} \to lim_{m\to\infty} \frac{D_{(z_m,0)} f^2_m(0,v_m)}{|(0,v_m)|} = D_{(z,0)} f^2_x(0, w)$. Note that, by Claim 1 and $w \neq 0$, $D_{(z,0)} f^2_x(0, w) \neq 0$. This implies that $\frac{|f^2_m(z_m,v_m)|}{|(0,v_m)|} \to |D_{(z,0)} f^2_x(0, w)| \neq 0$, and thus $\frac{|(0,v_m)|}{|f^2_m(z_m,v_m)|} \to \frac{1}{|D_{(z,0)} f^2_x(0,w)|}$. Therefore $lim_{m\to\infty} \frac{f^2_m(z_m,v_m)}{|f^2_m(z_m,v_m)|} = lim_{m\to\infty} \frac{f^2_m(z_m,v_m)}{|(0,v_m)|} \frac{|(0,v_m)|}{|f^2_m(z_m,v_m)|} = D_{(z,0)} f^2_x (0, w)\, \frac{1}{|D_{(z,0)} f^2_x(0,w)|}$. This proves the claim.

But $\frac{D^2_m(s_m,u_m)}{|D^2_m(s_m,u_m)|} \to \frac{D_{(z,0)} f^2_x(s,u)}{|D_{(z,0)} f^2_x(s,u)|}$; therefore $\frac{D_{(z,0)} f^2_x(s,u)}{|D_{(z,0)} f^2_x(s,u)|} = \frac{D_{(z,0)} f^2_x(0,w)}{|D_{(z,0)} f^2_x(0,w)|}$. Consequently, $D_{(z,0)} f^2_x(s, u) = D_{(z,0)} f^2_x(0, w')$, where $w' = \lambda w$, for some $\lambda > 0$. Hence $D_{(z,0)} f^2_x(s, u - w') = 0$, and by Claim 1, $u = w' = \lambda w$ is a contradiction because $|w| = 1$ and $\langle u, w \rangle = 0$. This proves the lemma. q.e.d.

2. Space at infinity of some complete negatively curved manifolds

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A map $f : X_1 \to X_2$ is a quasi-isometric embedding if there are $\epsilon \geq 0$ and $\lambda \geq 1$ such that $\frac{1}{\lambda}\, d_1(x, y) - \epsilon \leq d_2(f(x), f(y)) \leq \lambda\, d_1(x, y) + \epsilon$, for all $x, y \in X_1$. A quasi-isometric embedding f is called a quasi-isometry if there is a constant $K \geq 0$ such that every point in X_2 lies in the K-neighborhood of the image of f. A quasi-geodesic in a metric space (X, d) is a quasi-isometric embedding $\beta : I \to X$, where the interval $I \subset \mathbb{R}$ is considered with the canonical metric $d_{\mathbb{R}}(t, s) = |t - s|$. If $I = [a, \infty)$, β is called a quasi-geodesic ray. If we want to specify the constants λ and ϵ in the definitions above, we will use the prefix (λ, ϵ). It is a simple exercise to prove that the composition of a (λ, ϵ)-quasi-isomeric embedding with a (λ', ϵ')-quasi-isomeric embedding is a $(\lambda\lambda', \lambda'\epsilon + \epsilon')$-quasi-isomeric embedding. Also, if $f : X_1 \to X_2$ is a quasi-isometry and the Hausdorff distance between some subsets $A, B \subset X_1$ is finite, then the Hausdorff distance between $f(A)$ and $f(B)$ is also finite. In this paper a *unit speed geodesic* will always mean an isometric embedding with domain some interval $I \subset \mathbb{R}$. Also, a *geodesic* will mean a function $t \mapsto \alpha(\rho t)$, where α is a unit speed geodesic and $\rho > 0$. Then every geodesic is a quasi-geodesic with $\epsilon = 0$, that is, a $(\lambda, 0)$-quasi-geodesic, for some λ.

Lemma 2.1. *Let g, g' be two complete Riemannian metrics on the manifold Q. Suppose there are constants $a, b > 0$ such that $a^2 \leq g'(v, v) \leq b^2$ for every $v \in TQ$ with $g(v, v) = 1$. Then the identity $(Q, g) \to (Q, g')$ is a $(\lambda, 0)$-quasi-isometry, where $\lambda = max\{\frac{1}{a}, b\}$.*

Proof. The condition above implies $a^2\, g(v, v) \leq g'(v, v) \leq b^2\, g(v, v)$, which in turn implies $\frac{1}{b^2}\, g'(v, v) \leq g(v, v) \leq \frac{1}{a^2}\, g'(v, v)$, for all $v \in Q$. Let d, d' be the intrinsic metrics on Q defined by g, g', respectively. Let $x, y \in Q$ and $\beta : [0, 1] \to Q$ be a path whose endpoints are x, y and such that $d(x, y) = length_g(\beta) = \int_0^1 \sqrt{g(\beta'(t), \beta'(t))}\, dt$. Then $d'(x, y) \leq length_{g'}(\beta) = \int_0^1 \sqrt{g'(\beta'(t), \beta'(t))}\, dt \leq b \int_0^1 \sqrt{g(\beta'(t), \beta'(t))}\, dt = b\, d(x, y)$. In the same way we prove $d \leq \frac{1}{a}\, d'$. Then the identity 1_Q is a quasi-isometry with $\epsilon = 0$ and $\lambda = max\{\frac{1}{a}, b\}$. This proves the lemma. q.e.d.

In what remains of this section (Q, g) will denote a complete Riemannian manifold with sectional curvatures in the interval $[c_1, c_2]$, $c_1 < c_2 < 0$, and $S \subset Q$ a closed totally geodesic submanifold of Q, such that the map $\pi_1(S) \to \pi_1(Q)$ is an isomorphism. Write $\Gamma = \pi_1(S) = \pi_1(Q)$. Also, d will denote the intrinsic metric on Q induced by g. Note that S is convex in Q, and hence $d|_S$ is also the intrinsic metric on S induced by $g|_S$. We can assume that the universal cover $\tilde{S}$ of S is contained in the universal cover $\tilde{Q}$ of Q. We will consider $\tilde{Q}$ with the lifted metric $\tilde{g}$ and the induced distance will be denoted by $\tilde{d}$. The group Γ acts by isometries on $\tilde{Q}$ such that $\Gamma(S) = S$ and $Q = \tilde{Q}/\Gamma$, $S = \tilde{S}/\Gamma$. The covering projection will be denoted by $p : \tilde{Q} \to \tilde{Q}/\Gamma = Q$. Let R be the normal bundle of S, that is, for $z \in S$, $R_z = \{v \in T_zQ \,:\, g(v, u) = 0$, for all $u \in T_zS\} \subset T_zQ$. Write $\pi(v) = z$ if $v \in R_z$, that is, $\pi : R \to S$

is the bundle projection. The unit sphere bundle and unit disc bundle of R will be denoted by N and W, respectively. Note that the normal bundle, normal sphere bundle and the normal disc bundle of $\tilde{S}$ in $\tilde{Q}$ are the liftings $\tilde{R}$, $\tilde{N}$, and $\tilde{W}$ of R, N, and W, respectively. For $v \in T_qQ$ or $v \in T_q\tilde{Q}$, $v \neq 0$, the map $t \mapsto exp_q(tv)$, $t \geq 0$, will be denoted by c_v and its image will be denoted by the same symbol. Since $\tilde{Q}$ is simply connected, c_v is a geodesic ray, for every $v \in \tilde{N}$. We have the following well-known facts.

1. For any closed convex set $C \subset \tilde{Q}$, and a geodesic c, the function $t \mapsto \tilde{d}\,(\,c(t)\,,\,C\,)$ is convex. This implies 2 below.
2. Let c be a geodesic ray beginning at some $z \in \tilde{S}$. Then either $c \subset \tilde{S}$ or $\tilde{d}\,(\,c(t)\,,\,\tilde{S}\,) \to \infty$, as $t \to \infty$.
3. For every $v \in R$, $v \neq 0$, c_v is a geodesic ray. Moreover, for non-zero vectors v_1, $v_2 \in T$, with $\pi(v_1) \neq \pi(v_2)$, we have that the function $t \mapsto d\,(\,c_{v_1}(t)\,,\,c_{v_2}\,)$ tends to ∞ as $t \to \infty$.
4. The exponential map $E : R \to Q$, $E(v) = exp_{\pi(v)}(v)$, is a diffeomorphism. We can define then the submersion $proj : Q \to S$, $proj(q) = z$, if $exp(v) = q$, for some $v \in T_z$. Define $\eta : Q \to [0, \infty)$ by $\eta(q) = |v\,|$. Then we have $\eta(q) = d\,(\,q\,,\,S\,)$. Also, the exponential map $\tilde{E} : \tilde{T} \to \tilde{Q}$, $\tilde{E}(v) = exp_{\pi(v)}(v)$, is a diffeomorphism and $\tilde{E}$ is a lifting of E.
5. Since S is compact, there is a function $\varrho : [0, \infty) \to [0, \infty)$ with the following three properties: (1) for $q_1, q_2 \in Q$ we have
$$\varrho(a)\, d\,(\,proj(q_1)\,,\,proj(q_2)\,) \leq d\,(\,q_1\,,\,q_2,\,)$$
where $a = min\{\eta(q_1), \eta(q_2)\}$; (2) $\varrho(0) = 1$; (3) ϱ is an increasing function which tends to ∞ as $t \to \infty$.
6. Recall that we are assuming that all sectional curvatures of $\tilde{Q}$ are less than $c_2 < 0$. Given $\lambda \geq 1$, $\epsilon \geq 0$, there is a number $K = K(\lambda, \epsilon, c_2)$ such that the following happens. For every (λ, ϵ)-quasi-geodesic c in $\tilde{Q}$ there is a unit speed geodesic β with the same endpoints as c, whose Hausdorff distance from c is less or equal K. Note K depends on λ, ϵ, c_2, but not on the particular manifold $\tilde{Q}$ (see, for instance, [**1**], p. 401; see also Proposition 1.2 on p. 399 of [**1**]).

Recall that the space at infinity $\partial_\infty\tilde{Q}$ of $\tilde{Q}$ can be defined as $\{quasi-geodesic\ rays\ in\ \tilde{Q}\}/\sim$ where the relation $\sim$ is given by $\beta_1 \sim \beta_2$ if their Hausdorff distance is finite. We say that a quasi-geodesic β *converges to* $p \in \partial_\infty\tilde{Q}$ if $\beta \in p$. Fact 6 implies that we can define $\partial_\infty\tilde{Q}$ also by $\{\,geodesics\ rays\ in\ \tilde{Q}\,\}/\sim$. We consider $\partial_\infty\tilde{Q}$ with the usual cone topology (see [**1**], p. 263). Recall that, for any $q \in \tilde{Q}$, the map $\{v \in T_q\tilde{Q} : |v| = 1\} \to \partial_\infty\tilde{Q}$ given by $v \mapsto [c_v]$ is a homeomorphism. Let $\varsigma : [0, 1) \to [0, \infty)$ be a homeomorphism that is the identity near 0. We also have that $\overline{(\tilde{Q})} = \tilde{Q} \cup \partial_\infty \tilde{Q}$ can be given a topology such that the map $\{v \in T_q\tilde{Q} \,:\, |v| \leq 1\} \to \partial_\infty \tilde{Q}$ given by $v \mapsto exp_q(\varsigma(|v|)\frac{v}{|v|})$, for $|v| < 1$ and $v \mapsto [c_v]$ for $v = 1$, is a homeomorphism. We have some more facts or comments.

7. Given $q \in \tilde{Q}$ and $p \in \partial_\infty \tilde{Q}$, there is a unique unit speed geodesic ray β beginning at q and converging to p.
8. Since $\tilde{S}$ is convex in $\tilde{Q}$, every geodesic ray in $\tilde{S}$ is a geodesic ray in $\tilde{Q}$. Therefore $\partial_\infty \tilde{S} \subset \partial_\infty \tilde{Q}$. For a quasi-geodesic ray β we have $[\beta] \in \partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S}$ if and only if β diverges from $\tilde{S}$, that is, $\tilde{d}\,(\,\beta(t)\,,\,\tilde{S}\,) \to \infty$, as $t \to \infty$.
9. For every $p \in \partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S}$ there is a unique $v \in \tilde{N}$ such that c_v converges to p. Moreover, the map $\tilde{A} : \tilde{N} \to \partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S}$, given by $\tilde{A}(v) = [c_v]$ is a homeomorphism. Furthermore, we can extend $\tilde{A}$ to a homeomorphism $\tilde{W} \to \overline{(\tilde{Q})} \setminus \partial_\infty \tilde{S}$ by defining $\tilde{A}(v) = \tilde{E}(\varsigma(|v|)\frac{v}{|v|}) = exp_q(\varsigma(|v|)\frac{v}{|v|})$, for $|v| < 1$, $v \in \tilde{W}_q$ (recall that ς is the identity near zero).

Lemma 2.2. *Let $\beta : [a, \infty) \to \tilde{Q}$. The following are equivalent.*

(i) *β is a quasi-geodesic ray and diverges from $\tilde{S}$.*

(ii) *$p\beta$ is a quasi-geodesic ray, where $p : \tilde{Q} \to Q$ is the covering projection.*

Proof. First note that if a path $\alpha(t)$, $t \geq a$, satisfies the (λ, ϵ)-quasi-geodesic ray condition, for $t \geq a' \geq a$, then $\alpha(t)$ satisfies the (λ, ϵ')-quasi-geodesic ray condition, for all $t \geq a$, where $\epsilon' = \epsilon +$ diameter $(\alpha([a, a']))$.

(i) implies (ii). Let β satisfy (i). Then there are $\lambda \geq 1$, $\epsilon \geq 0$ such that $\frac{1}{\lambda}|t - t'| - \epsilon \leq \tilde{d}(\beta(t), \beta(t')) \leq \lambda|t - t'| + \epsilon$, for every $t, t' \geq a$. Fix $t, t' \geq a$ and let α be the unit speed geodesic segment joining $\beta(t)$ to $\beta(t')$. Then $p\alpha$ joins $p\beta(t)$ to $p\beta(t')$. Therefore $d(p\beta(t), p\beta(t')) \leq length_g(p\alpha) = length_{\tilde{g}}(\alpha) = d(\beta(t), \beta(t')) \leq \lambda|t - t'| + \epsilon$. We proved that $d(p\beta(t), p\beta(t')) \leq \lambda|t - t'| + \epsilon$.

We show the other inequality. By item 6, β is at finite Hausdorff distance (say, $K \geq 0$) from a geodesic ray α. Since β (hence α) gets far away from $\tilde{S}$, it converges to a point at infinity in $\partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S}$. Therefore we can assume that $\alpha(t) = c_{\tilde{v}}(t) = exp_{\tilde{z}}(t\tilde{v})$ for some $\tilde{v} \in \tilde{R}_{\tilde{z}}$, with $|\tilde{v}| = 1$. It follows that $p\beta$ is at Hausdorff distance $K' = K + d(\beta(a), \tilde{S})$ from c_v, where $v \in R_z$ is the image of $\tilde{v}$ by the derivative $Dp(\tilde{z})$, and $z = p(\tilde{z})$. Note that c_v is a geodesic ray in Q (see item 3). Let U denote the K neighborhood of c_v in Q and $\tilde{U}$ the K neighborhood of $c_{\tilde{v}}$ in $\tilde{Q}$. We claim that $p : \tilde{U} \to U$ satisfies: $d(p(x), p(y)) \geq \tilde{d}(x, y) - 4K$, for $x, y \in \tilde{U}$. To prove this let $t, t' \geq 0$ such that $d(x, c(t)) = d(x, c_v) \leq K$ and $d(y, c_{\tilde{v}}(t')) = d(y, c_{\tilde{v}}) \leq K$. We have $\tilde{d}(x, y) \leq \tilde{d}(x, c_{\tilde{v}}(t)) +$

$\tilde{d}(c_{\tilde{v}}(t), c_{\tilde{v}}(t')) + \tilde{d}(c_{\tilde{v}}(t'), y) \leq 2K + |t - t'| = 2K + d(c_v(t), c_v(t')) \leq 2K + d(c_v(t), p(x)) + d(p(x), p(y)) + d(p(y), c_v(t')) \leq 4K + d(p(x), p(y))$. This proves our claim. Consequently, $d(p\beta(t), p\beta(t')) \geq \tilde{d}(\beta(t), \beta(t')) - 4K \geq \frac{1}{\lambda}|t - t'| - (\epsilon + 4K)$.

(ii) implies (i). Let β satisfy (ii). Since $p\beta$ is a proper map, its distance to S must tend to infinity. Hence the distance of β to $\tilde{S}$ also tends to infinity.

Let $p\beta$ satisfy $\frac{1}{\lambda}|t - t'| - \epsilon \leq d(p\beta(t), p\beta(t')) \leq \lambda|t - t'| + \epsilon$, for some $\lambda \geq 1$, $\epsilon \geq 0$. Fix $t, t' \geq a$ and let α be the unit speed geodesic segment joining $\beta(t)$ to $\beta(t')$. Then $p\alpha$ joins $p\beta(t)$ to $p\beta(t')$. Therefore $\tilde{d}(\beta(t), \beta(t')) = length_{\tilde{g}}(\alpha) = length_g(p\alpha) \geq d(p\beta(t), p\beta(t')) \geq \frac{1}{\lambda}|t - t'| - \epsilon$. It follows that $\frac{1}{\lambda}|t - t'| - \epsilon \leq \tilde{d}(\beta(t), \beta(t'))$.

We prove the other inequality. Since S is compact and by item 5, the radius of injectivity of a point in Q tends to infinity as the points get far from S. Hence there is $a' \geq a$ such that for every $t \geq a'$, the ball of radius $e = \lambda + \epsilon$ centered at $\beta(t)$ is convex. Let $t' > t > a'$ and n be an integer such that $n < t' - t \leq n + 1$. Let α_k, $k = 1, \ldots, n$, be the unit speed geodesic segment from $p\beta(t+k-1)$ to $p\beta(t+k)$, and α_{n+1} the unit speed geodesic segment from $p\beta(t+n)$ to $p\beta(t')$. Note that $length_g(\alpha_k) = d(p\beta(t+k-1), p\beta(t+k)) \leq \lambda + \epsilon = e$. Therefore $p\beta|_{[t+k-1,t+k]}$ is homotopic, rel endpoints, to α_k (analogously for α_{n+1}). Let α be the concatenation $\alpha_1 * \ldots * \alpha_{n+1}$. Then α is homotopic, rel endpoints, to $p\beta|_{[t,t']}$. Note that the length of α is $\leq (n+1)e$. Let $\tilde{\alpha}$ be the lifting of α beginning at $\beta(a')$. Then $\tilde{\alpha}$ is homotopic, rel endpoints, to $\beta|_{[t,t']}$. Hence $\tilde{d}(\beta(t), \beta(t')) \leq length(\tilde{\alpha}) \leq (n+1)e = ne + e < e(t' - t) + e$. We showed that $\frac{1}{\lambda}|t - t'| - \epsilon \leq \tilde{d}(\beta(t), \beta(t')) < (\lambda + \epsilon)|t' - t| + (\lambda + \epsilon)$. This proves the lemma. q.e.d.

Let Q_1, Q_2 be two complete simply connected negatively curved manifolds. If β is a quasi-geodesic in Q_1 and $f : Q_1 \to Q_2$ is a quasi-isometry, then $f(\beta)$ is also a quasi-geodesic. Also, if two subsets of Q_1 have finite Hausdorff distance, their images under f will have finite Hausdorff distance as well. Therefore f induces a map $f_\infty : \partial_\infty Q_1 \to \partial_\infty Q_2$. Hence f extends to $\bar{f} : \overline{Q}_1 \to \overline{Q}_2$ by $\bar{f}|_{\partial_\infty Q_1} = f_\infty$ and $\bar{f}|_{Q_1} = f$. We have:

10. For every quasi-isometry $f : Q_1 \to Q_2$, $f_\infty : \partial_\infty Q_1 \to \partial_\infty Q_2$ is a homeomorphism. In addition, if f is a homeomorphism, then $\bar{f}$ is a homeomorphism.
11. Let g' be another complete Riemannian metric on $\tilde{Q}$ whose sectional curvatures are also $\leq c_2 < 0$, and is such that there are constants $a, b > 0$ with $a^2 \leq g'(v, v) \leq b^2$ for every $v \in T\tilde{Q}$ with $\tilde{g}(v, v) = 1$, and such that $\tilde{S}$ is also a convex subset of $(\tilde{Q}, g')$. Then $\partial_\infty \tilde{Q}$ is the same if defined using $\tilde{g}$ or g'. Moreover item 9 above also holds for $(\tilde{Q}, g')$ (with respect to all proper concepts defined using g' instead of $\tilde{g}$). This is because the identity $(\tilde{Q}, \tilde{g}) \to (\tilde{Q}, g')$

induces the homeomorphism $\partial_\infty \tilde{Q} \to \partial_\infty \tilde{Q}$ that preserves $\partial_\infty \tilde{S}$ (see Lemma 2.1 and item 10).

Since Γ acts by isometries on $\tilde{Q}$, we have that Γ acts on $\partial_\infty \tilde{Q}$ (see item 10). Also, since Γ preserves $\tilde{S}$, Γ also preserves $\partial_\infty \tilde{S}$. Hence Γ acts on $\partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S}$. Since S is closed, we have:

12. For every $\gamma \in \Gamma$, $\gamma : \partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S} \to \partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S}$ has no fixed points. Therefore the action of Γ on $\overline{(\tilde{Q})} \setminus \partial_\infty \tilde{S}$ is free. Moreover, the action of Γ on $\overline{(\tilde{Q})} \setminus \partial_\infty \tilde{S}$ is properly discontinuous.

We now define the space at infinity $\partial_\infty Q$ of Q as $\{$*quasi-geodesic rays in* $Q\}/\sim$. As before, the relation $\sim$ is given by $\beta_1 \sim \beta_2$ if their Hausdorff distance is finite. We can define a topology on $\partial_\infty Q$ in the same way as for $\partial_\infty \tilde{Q}$, but we can take advantage of the already-defined topology of $\partial_\infty \tilde{Q}$.

Lemma 2.3. *There is a one-to-one correspondence between* $\partial_\infty Q$ *and* $\left(\partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S}\right) / \Gamma$.

Proof. By path lifting and Lemma 2.2 there is a one-to-one correspondence between the sets $\{$ *quasi-geodesic rays in* $Q\}$ and $\{$ *quasi-geodesic rays in* $\tilde{Q}$ *that diverge from* $\tilde{S}\} / \Gamma$. Then the correspondence $[\beta] \mapsto p(\beta)$, for quasi-geodesic rays in $\tilde{Q}$ that diverge from $\tilde{S}$, is one-to-one (see item 8). This proves the lemma. q.e.d.

We define then the topology of $\partial_\infty Q$ such that the one-to-one correspondence mentioned in the proof of the lemma is a homeomorphism. Also, we define the topology on $\overline{Q} = Q \cup \partial_\infty Q$ such that $\left(\overline{(\tilde{Q})} \setminus \partial_\infty \tilde{S}\right) / \Gamma \to \overline{Q}$ is a homeomorphism. It is straightforward to verify that Q and $\partial_\infty Q$ are subspaces of $\overline{Q}$ (see also item 12). The next lemma is a version of item 9 for Q.

Lemma 2.4. *For every* $p \in \partial_\infty Q$ *there is a unique* $v \in N$ *such that* c_v *converges to* p. *(Recall that* N *is the unit sphere bundle of the normal bundle* S.*) Moreover, the map* $A : N \to \partial_\infty Q$, *given by* $A(v) = [c_v]$, *is a homeomorphism. Furthermore, we can extend* A *to a homeomorphism* $W \to \partial_\infty Q$ *by defining* $A(v) = E((\varsigma(|v|)\frac{v}{|v|}))$, *for* $|v| < 1$. *(Recall* ς *is the identity near 0.) Also,* $\tilde{A}$ *is a lifting of* A.

Proof. The first statement follows from items 4 and 5. Define $A(v) = p\tilde{A}(\tilde{v})$, where $Dp(\tilde{v}) = v$. Items 9 and 12 imply the lemma. See also item 4. q.e.d.

We will write $\eta([c_v]) = \infty$ and $E(\infty v) = [c_v]$, for $v \in N$ (see item 4).

Lemma 2.5. *Let* $v \in N$ *and* $q_n = E(t_n v_n)$, $t_n \in [0, \infty]$, $v_n \in R$ *and* $|v_n|$ *bounded away from both* 0 *and* $+\infty$. *Then* $q_n \to [c_v]$ *(in* $\partial_\infty Q$*) if and only if* $t_n \to \infty$ *and* $v_n \to v$.

Proof. It follows from Lemma 2.4. q.e.d.

We also have a version of item 11 for Q.

Lemma 2.6. *Let g' be another complete Riemannian metric on Q whose sectional curvatures are also $\leq c_2 < 0$, and such that there are constants $a, b > 0$ with $a^2 \leq g'(v,v) \leq b^2$ for every $v \in TQ$ with $g(v,v) = 1$, and such that S is also a convex subset of (Q, g'). Then $\partial_\infty Q$ is the same if defined using g or g'. Moreover Lemmas 2.4 and 2.5 above also hold for (Q, g') (with respect to all proper concepts defined using g' instead of g).*

Proof. It follows from item 11 and Lemma 2.5. Note that the liftings $\tilde{g}$, $\tilde{g}'$ of g and g' satisfy $a^2 \leq \tilde{g}'(v,v) \leq b^2$ for every $v \in T\tilde{Q}$ with $\tilde{g}(v,v) = 1$. This proves the lemma. q.e.d.

3. Proof of Theorem 1

Let the metric g and the closed simple curve α be as in the statement of Theorem 1. Write $N = \mathbb{S}^1 \times \mathbb{S}^{n-2}$ and $\Sigma^M = \Lambda_g \Phi^M$, where $\Lambda_g : DIFF(M) \to \mathcal{MET}^{sec<0}(M)$ and $\Phi^M = \Phi^M(\alpha, V, r) : DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial) \to DIFF(M)$ are the maps defined in the Introduction. The base point of the k-sphere $\mathbb{S}^k$ will always be the point $u_0 = (1, 0, \ldots, 0)$. Let $\theta : \mathbb{S}^k \to DIFF(N \times I, \partial)$, $\theta(u_0) = 1_{N \times I}$, represent an element in $\pi_k(\, DIFF(N \times I, \partial)\,)$.

We will prove that if $\pi_k(\Sigma^M)([\theta])$ is zero, then $\pi_k(\iota_N)([\theta])$ is also zero. Equivalently, if $\Sigma^M \theta$ extends to the $(k+1)$-disc $\mathbb{D}^{k+1}$, then $\iota_N \theta$ also extends to $\mathbb{D}^{k+1}$. So, suppose that $\Sigma^M \theta : \mathbb{S}^k \to \mathcal{MET}^{sec<0}(M)$ extends to a map $\sigma' : \mathbb{D}^{k+1} \to \mathcal{MET}^{sec<0}(M)$. We can assume that this map is smooth.

Remark. Originally σ' may not be smooth, but it is homotopic to a smooth map. By "σ' is smooth" we mean that the map $\mathbb{D}^{k+1} \times (TM \oplus TM) \to \mathbb{R}$, given by $(u, v_1, v_2) \mapsto \sigma'(u)_x(v_1, v_2)$, $v_1, v_2 \in T_xM$, is smooth. To homotope a given σ' to a smooth one σ'', we can use classical averaging techniques: just define $\sigma_x(u)''(v_1, v_2) = \int_{\mathbb{R}^{k+1}} \eta(u - w)\, \sigma'(w)_x(v_1, v_2)\, dw$, which is smooth. Here, (1) η is a smooth ϵ-bump function, i.e., $\int_{\mathbb{R}^{k+1}} \eta = 1$ and $\eta(w) = 0$, for $|w| \geq \epsilon$ and, (2) we are extending σ' (originally defined on $\mathbb{D}^{k+1}$) to all $\mathbb{R}^n$, radially. Since σ' is continuous, the second-order derivatives of $\sigma'_x(u)$ and $\sigma'_x(u')$ are close for u close to u'. Therefore the second-order derivatives of $\sigma'_x(u)$ are close to the second-order derivatives of $\sigma''_x(u)$. Hence, if ϵ is sufficiently small, we will also have $\sigma''(u) \in \mathcal{MET}^{sec<0}(M)$.

Also, by deforming σ', we can assume that it is radial near $\partial\, \mathbb{D}^{k+1}$. Thus $\sigma'(u)$, $u \in \mathbb{D}^{k+1}$, is a negatively curved metric on M. Also, $\sigma'(u) = \Sigma^M \theta(u)$, for $u \in \mathbb{S}^k$, and $\sigma'(u_0) = g$. Since σ' is continuous, there is a constant $c_2 < 0$ such that all sectional curvatures of the

Riemannian manifolds $(M, \sigma'(u))$, $u \in \mathbb{D}^{k+1}$, are less or equal c_2. Write $\varphi_u = \Phi^M(\theta(u))$, $u \in \mathbb{S}^k$. Hence we have that $\sigma'(u) = (\varphi_u)_*\sigma'(u_0) = (\varphi_u)_*g$, for $u \in \mathbb{S}^k$. Note that φ_u is, by definition, the identity outside the closed normal geodesic tubular neighborhood U of width $2r$ of α. Also, φ_u is the identity on the closed normal geodesic tubular neighborhood of width r of α. Note that $\varphi_u : M \to M$ induces the identity at the π_1-level and hence φ_u is freely homotopic to 1_M.

Since σ' is continuous and $\mathbb{D}^{k+1}$ is compact, we can find constants $a, b > 0$ such that $a^2 \leq \sigma'(u)(v,v) \leq b^2$ for every $v \in TM$ with $g(v,v) = 1$, $u \in \mathbb{D}^{k+1}$.

Let Q be the covering space of M with respect to the infinite cyclic subgroup of $\pi_1(M, \alpha(1))$ generated by α. Denote by $\sigma(u)$ the pull-back on Q of the metric $\sigma'(u)$ on M. For the lifting of g on Q we use the same letter g. Note that α lifts to Q and we denote this lifting also by α. Let $\phi_u : Q \to Q$ be diffeomorphism which is the unique lifting of φ_u to Q with the property that $\phi_u|_\alpha$ is the identity. We have some comments.

(i) $\sigma(u) = (\phi_u)_*\sigma(u_0) = (\phi_u)_*g$, for $u \in \mathbb{S}^k$.

(ii) The tubular neighborhood U lifts to a countable number of components, with exactly one being diffeomorphic to U. We call this lifting also by U. All other components U_1, $U_2, \ldots$ are diffeomorphic to $\mathbb{D}^{n-1} \times \mathbb{R}$. Note that ϕ_u is the identity outside the union of $\bigcup U_i$ and U and inside the closed normal geodesic tubular neighborhood of width r of α.

(iii) Since $\varphi_u : M \to M$ induces the identity at the π_1-level, and $\mathbb{S}^k$ is compact, there is a constant C such that $d_{\sigma(u')}(\,p\,,\,\phi_u(p)\,) < C$, for any $u, u' \in \mathbb{S}^k$, where $d_{\sigma(u')}$ denotes the distance in the Riemannian manifold $(Q, \sigma(u'))$.

(iv) $(\phi_u)|_U = \left[\Phi^Q(\alpha, V', r)\theta(u)\right]|_U$, for $u \in \mathbb{S}^k$. Here V' is the lifting of V.

(v) We have that $a^2 \leq \sigma(u)(v,v) \leq b^2$ for every $v \in TQ$ with $g(v,v) = 1$, $u \in \mathbb{D}^{k+1}$. It follows that $\frac{a^2}{b^2} \leq \sigma(u)(v,v) \leq \frac{b^2}{a^2}$ for every $v \in TQ$ with $\sigma(u')(v,v) = 1$, $u, u' \in \mathbb{D}^{k+1}$.

(vi) All sectional curvatures of the Riemannian manifolds $(Q, \sigma(u))$, $u \in \mathbb{D}^{k+1}$, are less or equal c_2.

Since $(M, \sigma'(u))$ is a closed negatively curved manifold, it contains exactly one immersed closed geodesic which is freely homotopic to $\alpha \subset M$. Therefore $(Q, \sigma(u))$ contains exactly one embedded closed geodesic α_u which is freely homotopic to $\alpha \subset Q$. Note that α_u is unique up to affine reparametrizations. Also, α_u depends continuously on u (see [**2**] and [**17**]). Write $\alpha_0 = \alpha_{u_0}$ and note that $\alpha_u = \phi_u(\alpha_0)$, for all $u \in \mathbb{S}^k$.

Since $n \geq 5$, we can find a compactly supported smooth isotopy $s : Q \times I \to Q$ with $s_0 = 1_Q$ and $s_1(\alpha_0) = \alpha$. Using s, we get a homotopy $(s_t)^{-1}\phi_u s_t$ between ϕ_u and $\psi_u = (s_1)^{-1}\phi_u s_1$. Therefore we can assume that for $u \in \mathbb{S}^k$ we have $\sigma(u) = (\psi_u)_* g$. Note that (ii) above still holds with $U' = (s_1)^{-1}U$, $U_i' = (s_1)^{-1}U_i$ instead of U, U_i, respectively. Note that U_i' coincides with U_i outside a compact set. Also, since s is compactly supported, (iii) holds too. For (iv), we assume that U' is the closed normal geodesic tubular neighborhood of width $2r$ of α_0 and s_1 sends any geodesic of length $2r$ beginning orthogonally at α_0 isometrically to geodesic of length $2r$ beginning orthogonally at α (we may have to consider a much smaller $r > 0$ here). Note that (v) and (vi) still hold. The following version of (iv) is true:

(iv') $(\psi_u)|_{U'} = \left[\Phi^Q(\alpha_0, V'', r)\theta(u)\right]|_{U'}$, for $u \in \mathbb{S}^k$. Here $V'' = (s_1^{-1})_* V'$.

Now, by [6, Prop. 5.5] α_u depends smoothly on $u \in \mathbb{D}^{k+1}$. Hence we have a smooth map $h : \mathbb{D}^{k+1} \times \mathbb{S}^1 \to Q$, given by $h_u = \alpha_u$. Note that h is radial near ∂. We have the following facts:

1. We can identify $\mathbb{S}^1$ with its image α_0 and, using the exponential map orthogonal to $\mathbb{S}^1$, with respect to $g = \sigma(u_0)$ and the trivialization V'', we can identify Q to $\mathbb{S}^1 \times \mathbb{R}^{n-1}$. With this identification V'' becomes just the canonical base $E = \{e_1, \ldots, e_{n-1}\}$ and (iv') above has now the following form: $(\psi_u)|_{U'} = \left[\Phi^Q(\alpha_0, E, r)\theta(u)\right]|_{U'}$, for $u \in \mathbb{S}^k$.
2. Because of the argument above (using the homotopy s), we cannot guarantee that all metrics $\sigma(u)$ are lifted metrics from M, but we do have that all liftings of the $\sigma(u)$ to the universal cover $\tilde{Q} = \tilde{M}$ are all quasi-isometric.

The next claim says that we can assume all $h_u = \alpha_u : \mathbb{S}^1 \to Q$ to be equal to α_0.

Claim 1. We can modify σ (hence also α_u and h) on $int\,(\mathbb{D}^{k+1})$ such that:

a. The liftings of the metrics $\sigma(u)$ to the universal cover $\tilde{Q} = \tilde{M}$ are all quasi-isometric.
b. $\alpha_u = \alpha_0$, for all $u \in \mathbb{D}^{k+1}$.

Proof of Claim 1. Let H be as in Lemma 1.4. Then the required new metrics are just $[(H_u)_1]^*\sigma(u)$, that is, the pull-backs of $\sigma(u)$ by the inverse of the diffeomorphism given by the isotopy H_u at time $t = 0$. Note that the metrics do not change outside a compact set of Q. Just one more detail. In order to be able to apply Lemma 1.4 for $k = 0$, we have to know that the loop $\beta : \mathbb{D}^1 \to Q$ given by $\beta(u) = h(u, 1)$ is homotopy trivial. But if this is not the case, let l be such that β is homotopic (rel base point) to α_0^{-l}. Then just replace h by $h\,\vartheta$, where $\vartheta : \mathbb{D}^1 \times \mathbb{S}^1 \to \mathbb{D}^1 \times \mathbb{S}^1$, $\vartheta(u, z) = (u, e^{\pi l(u+1)i}\, z)$. Note that h_u and $(h\,\vartheta)_u$

represent the same geodesic, but with different basepoint. This proves Claim 1.

Hence, from now on, we assume that all α_u are equal to $\alpha_0 : \mathbb{S}^1 \to Q$. Note that the new metrics $\sigma(u)$, $u \in int\,(\mathbb{D}^{k+1})$, are not necessarily pull-back from metrics in M. Recall that we are identifying Q with $\mathbb{S}^1 \times \mathbb{R}^{n-1}$, and the rays $\{z\} \times \mathbb{R}^+ v$, $v \in \mathbb{S}^{n-2}$, are geodesics (with respect to $g = \sigma(u_0)$) emanating from $z \in \mathbb{S}^1 \subset Q$ and normal to $\mathbb{S}^1$. Denote by $W_\delta = \mathbb{S}^1 \times \mathbb{D}^{n-1}(\delta)$ the closed normal tubular neighborhood of $\mathbb{S}^1$ in Q of width $\delta > 0$, with respect to the metric $\sigma(u_0)$. Note that $\partial W_\delta = \mathbb{S}^1 \times \mathbb{S}^{n-2}(\delta)$.

For each $u \in \mathbb{D}^{k+1}$ and $z \in \mathbb{S}^1$, let $T^u(z)$ be the orthogonal complement of the tangent space $T_z\mathbb{S}^1 \subset T_zQ$ with respect to the $\sigma(u)$ metric and denote by $exp^u_z : T^u(z) \to Q$ the normal exponential map, also with respect to the $\sigma(u)$ metric. Note that the map $exp^u : T^u \to Q$ is a diffeomorphism, where T^u is the bundle over $\mathbb{S}^1$ whose fibers are $T^u(z)$, $z \in \mathbb{S}^1$. We will denote by N^u the sphere bundle of T^u. The orthogonal projection (with respect to the $\sigma(u_0)$ metric) of the tangent vectors $(z, e_1), \dots, (z, e_{n-1}) \in T_zQ = \{z\} \times \mathbb{R}^{n-1}$ (here $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots$) into $T^u(z)$ gives a base of $T^u(z)$. Applying the Gram-Schimidt orthogonalization process, we obtain an orthonormal base $v^1_u(z), \dots, v^{n-1}_u(z)$ of $T^u(z)$. Clearly, these bases are continuous in z, and hence they provide a trivialization of the normal bundle T^u. We denote by $\chi_u : T^u \to \mathbb{S}^1 \times \mathbb{R}^{n-1}$ the bundle trivializations given by $\chi_u(v^i_u(z)) = (z, e_i)$. Note that these trivializations are continuous in $u \in \mathbb{D}^{k+1}$.

For every $(u, z, v) \in \mathbb{D}^{k+1} \times \mathbb{S}^1 \times (\mathbb{R}^{n-1} \setminus \{0\})$, define $\tau_u(z, v) = (z', v')$, where $\chi_u \circ (exp^u)^{-1}(z, v) = (z', w)$ and $v' = \frac{w}{|w|}$. Then $\tau_u : \mathbb{S}^1 \times (\mathbb{R}^{n-1} \setminus \{0\}) \to \mathbb{S}^1 \times \mathbb{S}^{n-2}$ is a smooth map. The restriction of τ_u to any $\partial W_\delta \subset \mathbb{S}^1 \times \mathbb{R}^{n-1}$ will be denoted also by τ_u. From now on we assume $\delta < r$.

Claim 2 There is $\delta > 0$ such that the map $\tau_u : \partial W_\delta \to \mathbb{S}^1 \times \mathbb{S}^{n-2}$ is a diffeomorphism.

Proof of Claim 2. Just apply Lemma 1.6 to the map $\chi_u \circ (exp^u)^{-1}$. This proves Claim 2.

Note that τ_u depends continuously on u. Note also that Claim 2 implies that every normal geodesic (with respect to any metric $\sigma(u)$) emanating from α_0 intersects ∂W_δ transversally in a unique point. Denote by $\rho_u : \partial W_\delta \to (0, \infty)$ the smooth map given by $\tau_u(z, v) = |w|$, where we are using the notation before the statement of Claim 2.

To simplify our notation we take $\delta = 1$ and write $W = W_1$. Thus $\partial W = N = \mathbb{S}^1 \times \mathbb{S}^{n-2}$ and we write $N \times [1, \infty) = Q \setminus int\,W$. Now, for each $u \in \mathbb{D}^{k+1}$ we define a self-diffeomorphism $f_u \in DIFF(N \times [1,\infty), N \times \{1\})$ by

$$f_u((z,v),t) = exp^u_{z'}(\ [\chi_u]^{-1}(\ z',\ \rho_u(z,v)\, tv'\ )\ )$$

where $\tau_u(z,v) = (z', v')$. It is not difficult to show that $f_u((z,v),1) = ((z,v),1)$ and that f_u is continuous in $u \in \mathbb{D}^{K+1}$.

Here is an alternative interpretation of f_u. For $(u,z,v) \in \mathbb{D}^{k+1} \times \mathbb{S}^1 \times T^u(z)$, denote by $c^u_{(z,v)} : [0,\infty) \to Q$ the $\sigma(u)$ geodesic ray given by $c^u_{(z,v)}(t) = exp^u_z(tv)$. Then f_u sends $c^{u_0}_{(z,v)}$ to $c^u_{(z',s)}$, where $exp^u_{z'}(s) = (z,v) \in Q$. Explicitly, we have $f_u(\, c^{u_0}_{(z,v)}(t)\,) = c^u_{(z',s)}(|s|t)$, for $t \geq 1$. Using Claim 2, it is not difficult to prove that $f_u(\, N \times [1,\infty)\,) = N \times [1,\infty)$ and that f_u is a diffeomorphism.

We denote by $\partial_\infty Q$ the space at infinity of Q with respect to the $\sigma(u_0)$ metric. Recall that the elements of $\partial_\infty Q$ are equivalence classes $[\beta]$ of $\sigma(u_0)$ quasi-geodesic rays $\beta : [a,\infty) \to Q = \mathbb{S}^1 \times \mathbb{R}^{n-1}$ (see Section 2). Note that, since all metrics $\sigma(u)$ are quasi-isometric, a $\sigma(u)$ quasi-geodesic ray is a $\sigma(u')$ quasi-geodesic ray, for any $u, u' \in \mathbb{D}^{k+1}$. Hence $\partial_\infty Q$ is independent of the metric $\sigma(u)$ used (see (v) and Lemma 2.6). Still, the choice of a $u \in \mathbb{D}^{k+1}$ gives canonical elements in each equivalence class in $\partial_\infty Q$: just choose the unique unit speed $\sigma(u)$ geodesic ray that "converges" (that is,"belongs") to the class, and that emanates $\sigma(u)$-orthogonally from $\mathbb{S}^1 \subset Q$. If we choose the $\sigma(u_0)$ metric, this set of geodesic rays is in one-to-one correspondence with $N = \mathbb{S}^1 \times \mathbb{S}^{n-2} \subset Q$. We identify $N \times \{\infty\}$ with $\partial_\infty Q$ by $((z,v),\infty) \mapsto [c^{u_0}_{(z,v)}]$. Hence we can write now $(Q \setminus int\, W\,) \cup \partial_\infty Q = (\, N \times [1,\infty)\,) \cup \partial_\infty Q = N \times [1,\infty]$ (see Lemma 2.5).

We now extend each f_u to a map $f_u : N \times [1,\infty] \to N \times [1,\infty]$ in the following way. For $((z,v),\infty) = [c^{u_0}_{(z,v)}]$, define $f_u(\,[c^{u_0}_{(z,v)}\,]\,) = [\, f_u(\, c^{u_0}_{(z,v)}\,)\,]$. Recall that, as we mentioned before, we have $f_u(\, c^{u_0}_{(z,v)}(t)\,) = c^u_{(z',s)}(|s|t)$, for $exp^u_{z'}(s) = (z,v) \in Q$, $t \geq 1$. That is, $f_u(\, c^{u_0}_{(z,v)}\,)$ is a $\sigma(u)$ geodesic ray, and hence it is a $\sigma(u_0)$ quasi-geodesic ray. Therefore $[\, f_u(\, c^{u_0}_{(z,v)}\,)\,]$ is a well-defined element in ∂_∞.

We will write $exp = exp^{u_0}$. Also, as in Section 2, we will write $exp\,(\infty v) = [c_v]$, for $v \in N$.

Claim 3. $f_u : N \times [1,\infty] \to N \times [1,\infty]$ is a homeomorphism.

Proof of Claim 3. Note that f_u is already continuous (even differentiable) on Q. We have to prove that f_u is continuous on points in $\partial_\infty Q$. Let $q_n = exp(t_n v_n) \to [c_v]$, $v, v_n \in N$, $t_n \in [0,\infty]$. Then, by Lemma 2.5, $v_n \to v$ and $t_n \to \infty$. Let $u \in \mathbb{D}^{k+1}$ and write $f = f_u$. We have to prove that $q'_n = f(q_n)$ converges to $f([c_v]) = [f(c_v)]$. Write $w_n = (exp^u)^{-1}(v_n)$. Then $w_n \to w = (exp^u)^{-1}(v) \neq 0$. Note that $f([c_v]) = [f(c_v)] = [c^u_w]$, where c^u_w is the $\sigma(u)$ geodesic ray $t \mapsto exp^u(tw)$.

Note also that, by definition, $f(q_n) = exp^u(t_n w_n)$. The claim follows now from Lemmas 2.5 and 2.6.

Claim 4. f_u is continuous in $u \in \mathbb{D}^{k+1}$.

Proof of Claim 4. Note that we know that $u \mapsto f_u|_Q$ is continuous. Let $q_n = exp(t_n v_n) \to [c_v]$, $v, v_n \in N$, $t_n \in [0, \infty]$. Then, by Lemma 2.5, $v_n \to v$ and $t_n \to \infty$. Let also $u, u_n \in \mathbb{D}^{k+1}$ with $u_n \to u$. To simplify our notation we assume that $u = u_0$ (the proof for a general u is obtained by properly writing the superscript u on some symbols; see also Lemma 2.6). Hence, by the previous identifications, $exp^{u_0} = exp : T = Q \to Q$ is just the identity and f_{u_0} is also the identity . Write $f_n = f_{u_n}$ and $w_n = (exp^{u_n})^{-1}(v_n)$. Then $w_n \to (exp^{u_0})^{-1}(v) = v$. We have to prove that $q_n' = f_n(q_n) = exp^{u_n}(t_n w_n) = c^{u_n}_{w_n}(t_n)$ converges to $f([c_v]) = [c_v]$. Note that $c^{u_n}_{w_n}(1) = exp^{u_n}(w_n) = v_n \to v$. To prove that $q_n' \to [c_v]$ we will work in $\tilde{Q}$ instead of Q. Therefore we "lift" everything to $\tilde{Q}$ and we express this by writing the superscript *tilde* over each symbol. Hence we have $\tilde{v}, \tilde{w}_n \in \tilde{N}$, $u, u_n \in \mathbb{D}^{k+1}$, $t_n > 0$ satisfying

1. $\tilde{w}_n \to \tilde{v}$ and $c^{u_n}_{\tilde{w}_n}(1) = exp^{u_n}(\tilde{w}_n) \to \tilde{v}$,
2. $u_n \to u_0$, hence $\tilde{\sigma}(u_n) \to \tilde{\sigma}(u_0) = \tilde{g}$.

We have then that $c_{\tilde{v}}$ is a $\tilde{g}$ geodesic ray and the $c^{u_n}_{\tilde{w}_n}$ are $\tilde{\sigma}(u)$ geodesic rays. Write $c^n = c^{u_n}_{\tilde{w}_n}$ and $\tilde{q}_n' = c^n(t_n)$. We have to prove that $\tilde{q}_n' \to [c_{\tilde{v}}]$. Since $u_n \to u_0$, the maps $exp^{u_n} \to exp = 1_{\tilde{Q}}$ (in the compact-open topology). Therefore

(*) for any r, $\delta > 0$ there is n_0 such that $\tilde{d}(c^n(t), c_{\tilde{v}}(t)) < \delta$, for $t \leq r$, and $n \geq n_0$.

Since $c_{\tilde{v}}$ is a unit speed geodesic (i.e., a (1,0)-quasi-geodesic ray), by (1) and (2), for large n we have that $c^n = c^{u_n}_{\tilde{w}_n}$ is a $\tilde{\sigma}(u)$ $(2,0)$-quasi-geodesic ray. By (v) above and Lemma 2.1 the identity $(\tilde{Q}, \tilde{\sigma}(u)) \to (\tilde{Q}, \tilde{g})$ is a $(\lambda, 0)$-quasi isometry, where $\lambda = max\{\frac{a^2}{b^2}, \frac{b^2}{a^2}\}$. Therefore, we have that c^n is a $\tilde{g}$ $(2\lambda, 0)$-quasi-geodesic ray. Let $K = K(2\lambda, 0, c_2)$ be as in item 6 of Section 2, and c_2 is as in (vi) above. Then there is a unit speed $\tilde{g}$ geodesic ray $\beta_n(t)$, $t \in [1, a_n]$, that is at K Hausdorff distance from c^n, $t \in [1, t_n]$, and has the same endpoints: $\beta_n(1) = c^n(1) \to \tilde{v}$ and $\beta_n(a_n) = c^n(t_n) = \tilde{q}_n'$. Note that $a_n \to \infty$ because $t_n \to \infty$. We have that (*) above (take $\delta = 1$ in (*)) implies that

(**) given an $r > 0$ there is a n_0 such that $\tilde{d}(c_{\tilde{v}}(t), \beta_n) \leq C = K + 1$, for $t \leq r$ and $n \geq n_0$.

Since $\tilde{Q}$ is complete and simply connected, we can extend each β_n to a geodesic ray $\beta_n : [1, \infty] \to \tilde{Q}$. Then $[\beta_n] \in \partial_\infty \tilde{Q}$. Let $\beta_n'(t)$, $t \in [1, \infty]$ be the unit speed $\tilde{g}$ geodesic ray with $\beta_n'(1) = \tilde{v}$, $\beta_n'(\infty) = \beta_n(\infty)$.

Therefore $\tilde{d}(\beta_n(t), \beta'_n(t)) \leq \tilde{d}(\beta_n(1), \beta'_n(1)) = \tilde{d}(c^n(1), \tilde{v}) \to 0$. We can assume then that $\tilde{d}(\beta_n(t), \beta'_n(t)) \leq 1$, for all n and $t \geq 1$. Hence, a version of (**) holds with β'_n instead of β_n and $C+1$ instead of C. This new version of (**) implies that $[\beta'_n] \to [c_{\tilde{v}}]$, and this together with condition (1) implies $\beta'_n(t) \to c_{\tilde{v}}(t)$, for every $t \in [1, \infty]$. Since $[\beta'_n] \to [c_{\tilde{v}}]$ and $a_n \to \infty$, we have that $\beta'_n(a_n) \to [c_{\tilde{v}}]$. But $\tilde{d}(\tilde{q}'_n, \beta'_n(a_n)) = \tilde{d}(\beta_n(a_n), \beta'_n(a_n)) \leq 1$; therefore $\tilde{q}'_n \to [c_{\tilde{v}}]$. This proves the claim.

Claim 5. For all $u \in \mathbb{S}^k$ we have $f_u|_{Q \setminus W} = (\psi_u)|_{Q \setminus W}$ and $(f_u)|_{\partial_\infty} = 1_{\partial_\infty}$.

Proof of Claim 5. Let $u \in \mathbb{S}^k$. Since $\sigma(u) = g$ on W, then $T^u = T^{u_0} = \mathbb{S}^1 \times \mathbb{R}^{n-1}$ and $exp^u_z(v) = (z, v)$ for all $z \in \mathbb{S}^1$ and $|v| \leq 1$. It follows that $f_u(\, c_{u_0}(z,v)(t)\,) = c_u(z,v)(t)$, for $t \geq 1$. On the other hand, since $\sigma(u) = (\phi_u)_*\sigma(u_0)$ we have that $\psi : (Q, \sigma(u_0)) \to (Q, \sigma(u))$ is an isometry. Hence $\psi_u(c_{u_0}(z,v)(t))$, $t \geq 0$, is a $\sigma(u)$ geodesic. Since ψ_u is the identity in $W \subset U'$, we have $\psi_u(z) = z$ and $(\psi_u)_* v = v$. Therefore $\psi_u(\, c_{u_0}(z,v)(t)\,)$, $t \geq 0$ is the $\sigma(u)$ geodesic that begins at z with direction v. Thus $\psi_u(c_{u_0}(z,v)(t)) = c_u(z,v)(t)$, for $t \geq 0$. Consequently, $f_u(\, c_{u_0}(z,v)(t)\,) = \psi_u(\, c_{u_0}(z,v)(t)\,)$, $t \geq 1$. This proves $f_u|_{Q \setminus W} = (\psi_u)|_{Q \setminus W}$ because every point in $Q \setminus W$ belongs to some $\sigma(u_0)$ geodesic $c_{u_0}(z,v)(t)$. Now, since ψ_u is at bounded distance from the identity (recall that (iii) above holds for ψ), then $f_u(\, c_{u_0}(z,v)\,)$ is at bounded distance from $c_{u_0}(z,v)$, and thus they define the same point in ∂_∞. Therefore $f_u(\, [c_{u_0}(z,v)]\,) = [\, c_{u_0}(z,v)\,)\,]$. Hence $(f_u)|_{\partial_\infty} = 1_{\partial_\infty}$. This proves the claim.

By means of an orientation-preserving homeomorphism $[1, \infty] \to [0,1]$, we can identify $[1, \infty]$ with $[0,1]$. It follows from Claim 3 that we can consider $f_u \in P(N)$. And we obtain, by Claim 4, a continuous map $f : \mathbb{D}^{k+1} \to P(N)$. We choose this identification map to be linear when restricted to the interval $[r, 2r]$ with image the interval $[\frac{1}{3}, \frac{2}{3}]$. The next claim proves Theorem 1.

Claim 6. $f|_{\mathbb{S}^k}$ is homotopic to $\iota_N \theta$.

Proof of Claim 6. Let $u \in \mathbb{S}^k$. Recall that ψ_u is the identity outside the union of $\bigcup U'_i$ and U' and inside the closed normal geodesic tubular neighborhood of width r of $\alpha_0 = \mathbb{S}^1$ (see (iii) above). In particular, ψ_u is the identity on W. From (iv') (and (1)) above we have

$$(\psi_u)|_{U'} = \left[\Phi^Q(\alpha_0, E, r)\theta(u)\right]|_{U'}, \text{ for } u \in \mathbb{S}^k.$$

Recall also that each U'_i is diffeomorphic to $\mathbb{D}^{n-1} \times \mathbb{R}$. Let $\bar{\alpha}_0$ be the (not necessarily embedded) closed g geodesic which is the image of $\alpha_0 \subset Q$ by the covering map $Q \to M$. Note that U_i is the $2r$ normal geodesic tubular neighborhood of a lifting β_i of $\alpha \subset M$ which is diffeomorphic to $\mathbb{R}$. Since $\alpha \subset M$ is freely homotopic to the closed geodesic $\bar{\alpha}_0 \subset M$, we have that β_i is at finite distance from some embedded geodesic line which is a lifting of $\bar{\alpha}_0$. Therefore the closure of U_i in $Q \cup \partial_\infty$ is formed exactly by the two points at infinity determined by this geodesic line. Consequently, the closure $\bar{U}_i$ of each U_i is homeomorphic to $\mathbb{D}^n$ and intersects ∂_∞ in exactly two different points. Now, applying Alexander's trick to each $\psi|_{\bar{U}_i}$, we obtain an isotopy (rel U') that isotopes ϕ_u to a map that is the identity outside $U' \setminus int\,(W)$, and coincides with ψ_u on U', that is, coincides with $\Phi^Q(\alpha_0, E[\frac{1}{3}, \frac{2}{3}], r)\theta(u)$ on U'. (Note that this isotopy can be defined because the diameters of the closed sets $\bar{U}_i$ in $(Q \setminus int\,W) \cup \partial_\infty = N \times [1, \infty]$ converge to zero as $i \to \infty$.) Here we refer to any metric compatible with the topology of $N \times [1, \infty]$.) Therefore ψ_u is canonically isotopic to a map ϑ_u that is the identity outside U' and on U' coincides with $\Phi^Q(\alpha_0, E, r)\theta(u)$. In fact, ϑ_u is the identity outside $N \times [r, 2r] \subset U \setminus W \subset N \times [1, \infty]$. That is, for $t \in [1, r] \cup [2r, \infty]$, $\vartheta_u((z, v), t) = ((z, v), t)$, $(z, v) \in N$.

On the other hand, we can deform θ_u to θ'_u, where θ'_u is the identity on $N \times (\,[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]\,)$ and $\theta'_u((z, v), t) = \theta'_u((z, v), 3t - 1)$, for $t \in [\frac{1}{3}, \frac{2}{3}]$. Finally, using the identification mentioned before this claim, we obtain that $\theta' = \vartheta$. This proves Claim 6 and Theorem 1.

4. Proof of Theorem 2

First, we recall some definitions and introduce some notation. For a compact manifold M, the spaces of smooth and topological pseudo-isotopies of M are denoted by $P^{diff}(M)$ and $P(M)$, respectively. Both $P^{diff}(M)$ and $P(M)$ are groups with composition as the group operation. We have stabilization maps $\Sigma : P(M) \to P(M \times I)$. The direct limit of the sequence $P(M) \to P(M \times I) \to P(M \times I^2) \to \dots$ is called the space of stable topological pseudo-isotopies of M, and it is denoted by $\mathcal{P}(M)$. We define $\mathcal{P}^{diff}(M)$ in a similar way. The inclusion $P^{diff}(M) \to P(M)$ induces an inclusion $\mathcal{P}^{diff}(M) \to \mathcal{P}(M)$. We mention two important facts:

1. $\mathcal{P}^{diff}(-)$, $\mathcal{P}(-)$ are homotopy functors.
2. The maps $\pi_k(P^{diff}(M)) \to \pi_k(\mathcal{P}^{diff}(M))$, $\pi_k(P(M)) \to \pi_k(\mathcal{P}(M))$ are isomorphisms for $max\{2k + 9, 3k + 7\} \leq dim\, M$ (see [**16**]).

Lemma 4.1. *For every k and every compact smooth manifold M, the kernel and the cokernel of $\pi_k(\mathcal{P}^{diff}(M)) \to \pi_k(\mathcal{P}(M)))$ are finitely generated.*

Proof. We have a long exact sequence (see [**13**], p.12): $\ldots \to \pi_{k+1}(\mathcal{P}_S(M))) \to \pi_k(\mathcal{P}^{diff}(M))) \to \pi_k(\mathcal{P}(M))) \to \pi_k(\mathcal{P}_{\mathcal{S}}(M)) \to \ldots$, where $\mathcal{P}_S(M) = lim_n \Omega^n \mathcal{P}(S^n M)$. An important fact here is that $\pi_*(\mathcal{P}_S(M))$ is a homology theory with coefficients in $\pi_{*-1}(\mathcal{P}^{diff}(*))$. Since these groups are finitely generated (see [**4**]), the lemma follows. q.e.d.

Lemma 4.1 together with (2) imply:

Corollary 4.2. *For every k and smooth manifold M^n the kernel and the cokernel of $\pi_k(P^{diff}(M)) \to \pi_k(P(M)))$ are finitely generated for $max\{2k+9, 3k+7\} \leq dim\, M$.*

Write $\iota' : DIFF((\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times I, \partial) \to P^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2})$. Since $\iota_{\mathbb{S}^1 \times \mathbb{S}^{n-2}} : DIFF((\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times I, \partial) \to P(\mathbb{S}^1 \times \mathbb{S}^{n-2})$ factors through ι', Corollary 4.2 implies that to prove Theorem 2 it is enough to prove:

Theorem 4.3 *Let p be a prime integer ($p \neq 2$) such that $6p-5 < n$. Then for $k = 2p-4$ we have that $\pi_k(DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial))$ contains $(\mathbb{Z}_p)^\infty$ and $\pi_k(\iota')$ restricted to $(\mathbb{Z}_p)^\infty$ is one-to-one. When $p = 2$, n needs to be ≥ 10. Also, if $n \geq 14$, then $\pi_1(DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial))$ contains $(\mathbb{Z}_2)^\infty$ and $\pi_1(\iota')$ restricted to $(\mathbb{Z}_2)^\infty$ is one-to-one.*

We will need a little more structure. There is an involution " $-$ " defined on $P^{diff}(M)$ by turning a pseudo-isotopy upside down. For M closed we can define this involution easily in the following way. Let $f \in P^{diff}(M)$. Define $\bar{f} = (\,(f_1)^{-1} \times 1_I\,) \circ \hat{f}$, where $\hat{f} = r \circ f \circ r$, $r(x,t) = (x, 1-t)$, and $(f_1(x), 1) = f(x,1)$. This involution homotopy anti-commutes with the stabilization map Σ; hence the involution can be extended to $\mathcal{P}(M)$. This involution induces an involution $- : \pi_k(\mathcal{P}(M)) \to \pi_k(\mathcal{P}(M))$ at the k-homotopy level. We define now a map $\Xi : P^{diff}(M) \to P^{diff}(M)$ by $\Xi(f) = f \circ \bar{f}$, and extend this map to $\mathcal{P}^{diff}(M)$. We have four comments:

i. For $f \in P^{diff}(M)$, $\Xi(f)|_{M \times \{1\}} = 1_{M \times \{1\}}$. Therefore $\Xi(f) \in DIFF(M \times I, \partial)$. Hence the map $\Xi : P^{diff}(M) \to P^{diff}(M)$ factors through $DIFF(M \times I, \partial)$.
ii. Since $P^{diff}(M)$ is a topological group, for $x \in \pi_k(P^{diff}(M))$ we have that $\pi_k(\Xi)(x) = x + \bar{x}$.
iii. The following diagram commutes

$$\begin{array}{ccc} P^{diff}(M) & \to & P^{diff}(M) \\ \downarrow & & \downarrow \\ \mathcal{P}^{diff}(M) & \to & \mathcal{P}^{diff}(M) \end{array}$$

where the horizontal lines are both either " $-$ " or Ξ. Hence we have an analogous diagram at the homotopy group level.
iv. We mentioned in (1) that $\mathcal{P}^{diff}(-)$ is a homotopy functor. But the conjugation "$-$" defined on $\mathcal{P}^{diff}(M)$ depends on M. In any

event, we have that $\mathcal{P}^{diff}(-)$ preserves the conjugation "$-$" up to multiplication by ± 1.

Note that (i) above implies that $\pi_k(\Xi) : \pi_k(P^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2})) \to \pi_k(P^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2}))$ factors through $\pi_k(\, DIFF((\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times I, \partial)\,)$. Therefore, to prove Theorem 4.3 it is enough to prove:

Proposition 4.4. *For every $k = 2p - 4$, p prime integer ($p \neq 2$), $6p - 5 < n$, we have that $\pi_k(P^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2}))$ contains $(\mathbb{Z}_p)^\infty$. Also $\pi_1(P^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2}))$ contains $(\mathbb{Z}_2)^\infty$, provided $n \geq 14$, and $\pi_0(P^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2}))$ contains $(\mathbb{Z}_2)^\infty$, provided ≥ 10. Moreover, in all cases above, $\pi_k(\Xi)$ restricted these subgroups is one-to-one.*

By (2) and (iii), to prove Proposition 4.4 it is enough to prove the following stabilized version:

Proposition 4.5. *For every $k = 2p - 4$, p prime integer ($p \neq 2$), $6p - 5 < n$, we have that $\pi_k(\mathcal{P}^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2}))$ contains $(\mathbb{Z}_p)^\infty$. Also $\pi_1(\mathcal{P}^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2}))$ contains $(\mathbb{Z}_2)^\infty$, provided $n \geq 14$, and $\pi_0(\mathcal{P}^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2}))$ contains $(\mathbb{Z}_2)^\infty$, provided ≥ 10. Moreover, in all cases above, $\pi_k(\Xi)$ restricted these subgroups is one-to-one.*

Since $\mathbb{S}^1$ is a retract of $\mathbb{S}^1 \times \mathbb{S}^{n-2}$, (1) implies that $\pi_k(\mathcal{P}^{diff}(\mathbb{S}^1))$ is a direct summand of $\pi_k(\mathcal{P}^{diff}(\mathbb{S}^1 \times \mathbb{S}^{n-2}))$. Therefore, by (ii) and (iv), to prove Proposition 4.5 it is enough to prove the following version for $\mathbb{S}^1$:

Proposition 4.6. *For every $k = 2p - 4$, p prime integer, we have that $\pi_k(\mathcal{P}^{diff}(\mathbb{S}^1))$ contains $(\mathbb{Z}_p)^\infty$. Also $\pi_1(\mathcal{P}^{diff}(\mathbb{S}^1))$ contains $(\mathbb{Z}_2)^\infty$. Moreover, in these cases, the two group endomorphisms $x \mapsto x + \bar{x}$ and $x \mapsto x - \bar{x}$ are both one-to-one when restricted to these subgroups.*

Proof. For a finite complex X, Waldhausen [**19**] proved that the kernel of the split epimorphism

$$\zeta_k : \pi_k(A(X)) \to \pi_{k-2}(\mathcal{P}^{diff}(X))$$

is finitely generated. Recall that the conjugation in $\mathcal{P}^{diff}(X)$ is defined by turning a pseudo-isotopy upside down. It is also possible to define a conjugation "$-$" on $A(X)$ such that ζ_k preserves conjugation up to multiplication by ± 1 (see [**18**]). The induced map at the k-homotopy level will also be denoted by "$-$".

We recall a result proved in [**14**]. For a space X, we have that $\pi_k(A(X \times \mathbb{S}^1))$ naturally decomposes as a sum of four terms,

$$\pi_k(A(X \times \mathbb{S}^1)) = \pi_k(A(X)) \oplus \pi_{k-1}(A(X)) \oplus \pi_k(N_- A(X)) \oplus \pi_k(N_+ A(X)),$$

and the conjugation leaves invariant the first two terms and interchanges the last two.

The following result is crucial to our argument:

Theorem (p-torsion of $\pi_{2p-2}A(\mathbb{S}^1)$). For every prime p the subgroup of $\pi_{2p-2}(A(\mathbb{S}^1))$ consisting of all elements of order p is isomorphic to $(\mathbb{Z}_p)^\infty$.

A proof of this result was given by J. Grunewald, J. R. Klein, and T. Macko in [**11**]. (It should be noted that in a personal communication Tom Goodwillie had previously given us a sketch of a proof of this theorem. We are grateful to him for this.)

Also, Igusa ([**15**], Part D, Theorem 2.1), building on work of Waldhausen [**19**], proved the following:

Addendum. $\pi_3A(\mathbb{S}^1)$ *contains* $(\mathbb{Z}_2)^\infty$.

Remark. The special case of the p-torsion Theorem above, when $p = 2$, is also due to Igusa (see [**15**], Theorem 8.a.2).

Now, take $X = *$ in the decomposition formula above. Recall that Dwyer showed that $\pi_k(A(*))$ is finitely generated for all k. Therefore the theorem above implies that at least one of the summands $\pi_k(N_-A(*))$, $\pi_k(N_+A(*))$ in the above formula contains $(Z_p)^\infty$, for $k = 2p - 2$, and contains $(\mathbb{Z}_2)^\infty$ when $k = 3$ by the addendum. Hence $y \mapsto y + \bar{y}$ and $y \mapsto y - \bar{y}$, $y \in (\mathbb{Z}_p)^\infty$, are both one-to-one. Since $\zeta_k : \pi_k(A(X)) \to \pi_{k-2}(\mathcal{P}^{diff}(X))$ has finitely generated kernel, we can assume (by passing to a subgroup of finite index) that $y \mapsto \zeta_k(y + \bar{y})$ and $y \mapsto \zeta_k(y - \bar{y})$, $y \in (\mathbb{Z}_p)^\infty$, are also one-to-one. It follows that $x \mapsto x+\bar{x}$ and $x \mapsto x-\bar{x}$, $x \in \zeta_k((\mathbb{Z}_p)^\infty)$, are one-to-one. Finally, the same argument shows that $x \mapsto x + \bar{x}$ and $x \mapsto x - \bar{x}$, $x \in \zeta_3((\mathbb{Z}_2)^\infty)$, are one-to-one. q.e.d.

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