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A NOTE ON A COUNTEREXAMPLE OF DELGADO

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In this note we correct some incorrect analysis appearing in the paper of J. A. Delgado [1].

The example concerns two plane curves γ_1, γ_2, which both are regular and complete, and have nonnegative curvature κ, i.e., $\kappa(\gamma_1) \geqslant 0$, $\kappa(\gamma_2) \geqslant 0$.

In this example Delgado intended to show that γ_1 and γ_2 are internally tangent at 0 and that $\kappa(\gamma_1(t)) \geqslant \kappa(\gamma_2(s))$ whenever $N_1(t) = N_2(s)$ where N_1 (resp. N_2) is the unit outward normal of γ_1 (resp. γ_2). He also showed that γ_1 is not contained in the convex region formed by γ_2, thus showing that Blaschke's theorem does not apply to curves with nonnegative rather than positive curvature. However his analysis is incorrect. The example should go as follows:

$$\gamma_1(t) = (pt, t^4), \quad t \in \mathbf{R}, \quad p > 1,$$

$$\gamma_2(s) = \begin{cases} \left(s, (s-1)^4\right), & s \in \mathbf{R}, \quad s \geqslant 1, \\ (s, 0), & s \in \mathbf{R}, \quad |s| \leqslant 1, \\ \left(s, (s+1)^4\right), & s \in \mathbf{R}, \quad s \leqslant -1, \end{cases}$$

$$N_1(t) = \frac{1}{\left(p^2 + 16t^2\right)^{1/2}}(4t^3, -p),$$

$$N_2(s) = \begin{cases} \dfrac{1}{\left(1 + 16(s-1)^6\right)^{1/2}}\left(4(s-1)^3, -1\right), & \text{if } s \geqslant 1, \\ (0, -1), & \text{if } |s| \leqslant 1, \\ \dfrac{1}{\left(1 + 16(s+1)^6\right)^{1/2}}\left(4(s+1)^3, -1\right), & \text{if } s \leqslant -1. \end{cases}$$

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Hence $N_1(t) = N_2(s)$ iff $s > 1$, and $t = \sqrt[3]{p}(s-1)$, $-1 < s < 1$ and $t = 0$ or $s \leqslant -1$ and $t = \sqrt[3]{p}(s+1)$. We have

$$\kappa(\gamma_1(t)) = \frac{12pt^2}{(p^2 + 16t^6)^{3/2}},$$

$$\kappa(\gamma_2(s)) = \begin{cases} \dfrac{12(s-1)^2}{\left(1 + 16(s-1)^6\right)^{3/2}}, & \text{if } s \geqslant 1, \\ 0, & \text{if } |s| \leqslant 1, \\ \dfrac{12(s+1)^2}{\left(1 + 16(s+1)^6\right)^{3/2}}, & \text{if } s \leqslant -1, \end{cases}$$

(and not as appeared in [1]). So in fact we have

$$\kappa(\gamma_1(0)) = \kappa(\gamma_2(s)) = 0, \quad |s| \leqslant 1,$$
$$\kappa(\gamma_1(t)) < \kappa(\gamma_2(s)) \quad \text{for } N_1(t) = N_2(s), t \neq 0$$

(and not $\kappa(\gamma_1(t)) \geqslant \kappa(\gamma_2(s))$ as appeared in [1]).

Hence it is no surprise that γ_1 eventually leaves the convex region formed by γ_2. However, looking at the conjecture the other way around we should have that γ_2 lies in the convex region formed by γ_1. In fact what we find is that in no neighborhood of the origin does it do so. Thus the conjecture fails rather strongly. The fact that γ_1 "cuts" γ_2 for points $t \neq 0$ is now irrelevant. This point is made even more clear by the fact that if $p = 1$ then $\gamma_1 \cap \gamma_2 = \{0\}$ and the example still works.

References

[1] J. A. Delgado, *Blaschke's theorem for convex hypersurfaces*, J. Differential Geometry **14** (1979) 489–496.

[2] M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, NJ, 1976.

[3] J. A. Thorpe, *Elementary topics in differential geometry*, Undergraduate Texts in Math., Springer, New York, 1979.

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