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DIFFERENTIAL-GEOMETRIC CHARACTERIZATIONS OF COMPLETE INTERSECTIONS

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Abstract

We characterize complete interesections in terms of local differential geometry.

Let $X^n \subset \mathbb{CP}^{n+a}$ be a variety. We first localize the problem; we give a criterion for X to be a complete intersection that is testable at any smooth point of X. We rephrase the criterion in the language of projective differential geometry and derive a sufficient condition for X to be a complete intersection that is computable at a general point $x \in X$. The sufficient condition has a geometric interpretation in terms of restrictions on the spaces of osculating hypersurfaces at x. When this sufficient condition holds, we are able to define systems of partial differential equations that generalize the classical Monge equation that characterizes conic curves in $\mathbb{CP}^2$.

Using our sufficent condition, we show that if the ideal of X is generated by quadrics and $a < \frac{1}{3}[n-(b+1)+3]$, where $b = \dim X_{sing}$, then X is a complete intersection.

0. Introduction

Local and global geometry

Projective differential geometry has been used to study the local geometry of subvarieties of projective space by various authors (e.g. [2], [4], [6], [8], [15]). However, there are few examples where global conclusions are drawn from the local picture.

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One (global) fact about projective varieties that has been encoded into the infinitesimal geometry is the following: if there is a line on a variety $X^n \subset \mathbb{CP}^{n+a}$ along which the embedded tangent space is constant, then X must be singular. Griffiths and Harris realized this fact had implications for the projective second fundamental form (that its singular locus must be empty at general points) which enabled them to reprove Severi's theorem that the only smooth surface in $\mathbb{P}^5$ with degenerate secant variety is the Veronese [6, (6.18)], using local methods.

In [9], we used Zak's theorem on tangencies [16, (1.8)], or equivalently the Fulton-Hansen Connectedness theorem [5, (3.1)] to encode additional global information about projective varieties into the local differential geometry. We proved a *rank restriction theorem* [9, (6.1)] that bounds from below the ranks of quadrics in the projective second fundamental form. A general principle that the rank restriction theorem illustrates is:

In order for a variety of small codimension to be smooth, it must "bend enough".

In the case of the rank restriction theorem, "bending enough" corresponds to genericity of the projective second fundamental form at general points.

The rank restriction theorem enabled us to reprove Zak's theorem on Severi varieties and to prove further results about varieties with degenerate secant varieties using local methods in [10]. One consequence of the rank restriction theorem that is particularly easy to apply is:

Theorem [9, (4.14)]. *Let $X^n \subset \mathbb{CP}^{n+a}$ be a variety with $a < \frac{1}{2}[n-(b+1)]+1$, where $b = \dim X_{sing}$. Then at general points of X, the third fundamental form of X is zero.*

In this paper we apply the results of [9] to obtain information about the quadrics containing a variety. The results on quadrics follow from an in-depth local study that occupies the bulk of this paper.

The local study shows precisely how one can detect the failure of a variety to be a complete intersection from the differential invariants at a general point $x \in X$. Vaguely stated, the motivating principle is:

If X is not a complete intersection, it "bends less" than expected.

Work of others (e.g. [13]) on complete intersections can be understood in these terms when one uses a Kähler metric. In this paper, "bending less" will correspond to certain non-generic behavior of projective differential invariants and the "expectation" from knowledge about the generators of the ideal of X that is also computable from the projective differential invariants.

Osculating hypersurfaces

We use differential invariants to study the hypersurfaces containing X via the *osculating hypersurfaces* at a point $x \in X$. Osculating hypersurfaces are geometric objects of interest in their own right. There are several equivalent definitions of what it means for a hypersurface to osculate to order k at a point $x \in X$, an elementary one is as follows:

Definition 0.1. Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and let $Z \subset \mathbb{P}^{n+a}$ be a hypersurface. Let $x \in X \cap Z$. In an affine open set $\mathbb{A}$ containing x, Z is given locally by a function f which restricts to a function $\bar{f}$ on $X \cap \mathbb{A}$. If we identify the completion of the local ring of $X \cap \mathbb{A}$ at x with $\mathbb{C}[[x^1, ..., x^n]]$, then Z *osculates to order* k *at* x if the power series of $\bar{f}$ at x has no terms of degree less than or equal to k.

Let $V = \mathbb{C}^{n+a+1}$ and let $X^n \subset \mathbb{P}V = \mathbb{CP}^{n+a}$ be a variety of dimension n. Let $I_X \subset S^{\bullet}V^*$ denote the ideal of X and let $I_{X,d} = I_d = S^dV^* \cap I_X$ denote the d-th graded piece of I_X.

I_d, the vector space of all hypersurfaces of degree d containing X, can be characterized geometrically as follows: Let $v_d(X) \subset \mathbb{P}S^dV$ denote the d-th Veronese re-embedding of X, and let $< v_d(X) >$ denote its linear span. $I_d =< v_d(X) >^{\perp} \subset S^dV^*$, the annhilator of the linear span of $v_d(X)$. Similarly, the space of hypersurfaces of degree d osculating to order k at a smooth point $x \in X$ is $\ker \mathbf{FF}^k_{v_d(X)x} \subset S^dV^*$, the kernel of the *k-th fundamental form* of $v_d(X)$ at x. (See §2 for definitions of the fundamental forms.)

Using the language of osculating hypersurfaces, [9, (4.14)], may be restated as:

Restatement of [9, (4.14)]. *Let* $X^n \subset \mathbb{CP}^{n+a}$ *be a variety with* $a < \frac{1}{2}[n-(b+1)]+1$, *where* $b = \dim X_{sing}$. *Let* $x \in X$ *be a general point. If a hyperplane* H *osculates to order two at* x, *then* $X \subset H$.

In contrast, any surface in $\mathbb{P}^6$ has at least one hyperplane osculating to order two at *every* point. There is also a class of smooth surfaces in $\mathbb{P}^5$, the *Legendrian surfaces*, which have the property that at every point there is at least one hyperplane osculating to order two. (This class includes the ruled surfaces.)

Overview

The precise meanings of terms used in this overview are explained in the sections.

In §1 we observe that if X is such that $I_X = (I_d)$ and $I_{d-1} = (0)$, then X is a complete intersection if and only if every hypersurface $Z \in I_d$ is smooth at all the smooth points of X (1.1). We generalize (1.1) to a local characterization of all complete intersections. The characterization is simplified thanks to the following definition due to L'vovsky [12]:

Definition 1.2. Let $X \subset \mathbb{P}V$ be a variety. Let $P \in I_d$ and let $Z = Z_P \subset \mathbb{P}V$ be the corresponding hypersurface. We will say Z *trivially contains* X if $P = l^1 P_1 + ... + l^k P_k$ with $P_1, ..., P_k \in I_{d-1}$ and $l^1, ..., l^k \in V^*$, and otherwise that Z *essentially contains* X.

Proposition 1.6 A local characterization of complete intersections. *Let $X \subset \mathbb{P}V$ be a variety and let X_{sm} denote its smooth points. The following are equivalent:*

1. *X is a complete intersection.*
2. *Every hypersurface essentially containing X is smooth at all $x \in X_{sm}$.*
3. *Let $x \in X_{sm}$. Every hypersurface essentially containing X is smooth at x.*
4. *Let $x \in X_{sm}$. For all k, the map*

$$[d]_k : I_k/(I_{k-1} \circ V^*) \to N_x^* X/d(I_{k-1} \circ V^*)$$
$$[P] \mapsto [dP_x]$$

is injective, which dP denotes the exterior derivative of the polynomial P.

In fact, (1.6) not only localizes the question of whether or not X is a complete intersection, it allows one to study the property one degree at a time.

Definition 1.8. Fix a point $x \in X_{sm}$. We will say X *has no excess equations in degree k at x*, or that *$(CI)_k$ holds at x*, if $[d]_k$ is injective at x. (Note that X is a complete intersection if and only if $(CI)_k$ holds at x for all k.)

Unfortunately, to determine if X satisfies $(CI)_k$ at a point x is not necessarily computable in a predictable number of steps. To avoid this problem, we restrict attention to sufficient conditions for a variety to be a complete intersection in the remainder of the paper. The sufficient conditions are expressed geometrically in terms of the osculating hypersurfaces of X at x, and to deal with osculating hypersurfaces we need to deal with the projective differential invariants of X.

In §2, we review the *projective fundamental forms* of a variety, denoted $\mathbf{FF}^k_X$ in this paper. Of special importance is $\mathbf{FF}^2_X = II_X$, the projective second fundamental form. We also need to work with some subtler differential invariants, which we denote F_k and call the *$(k-2)$-nd variation of II.*

In §3, we study the space of hypersurfaces of degree d osculating to order k at a smooth point $x \in X$, taking advantage of their description as $\ker \mathbf{FF}^k_{v_d(X),x}$ mentioned above. We compute the fundamental forms of the Veronese re-embeddings of X and prove some generalities about the osculating spaces. First we show that the dimension of the space of hypersurfaces of degree d osculating to order k at x is fixed for all $k \leq d$, generalizing the fact that there is an $(a-1)$-dimensional space of hyperplanes osculating to order one at each smooth point:

Proposition 3.16. *Let $X^n \subseteq \mathbb{P}V = \mathbb{CP}^{n+a}$ be a variety and let $x \in X_{sm}$. For all $p \leq d$,*

$$\dim \left\{ \begin{array}{c} \textit{(not necessarily irreducible) hypersurfaces} \\ \textit{of degree } d \textit{ osculating to order } p \textit{ at } x \end{array} \right\}$$
$$= \binom{(n+a+1)+(d-1)}{d} - \left\{ 1+n+\binom{n+1}{2}+\dots+\binom{n+p-1}{p} \right\}.$$

For $k > d$, the dimensions depend on the geometry of X. For $d+1 \leq k \leq 2d-1$ there are *lower* bounds on the dimensions of the space of hypersurfaces of degree d osculating to order k at x. For example:

Proposition 3.17. *Let $X^n \subseteq \mathbb{P}V = \mathbb{CP}^{n+a}$ be any variety, and $x \in X$ any smooth point.*

$$\dim \left\{ \begin{array}{c} \textit{(not necessarily irreducible) hypersurfaces of} \\ \textit{degree } d \textit{ osculating to order } 2d-1 \textit{ at } x \end{array} \right\} \geq \binom{a+d-1}{d} - 1.$$

We rephrase (1.6) in a manner suitable for approximation by osculating hypersurfaces.

We include several examples, including a review of the classical Monge equation; a fifth order differential equation that characterizes conic curves in $\mathbb{P}^2$.

We define a condition denoted by $(CI)_k^{2k}$ under which $(CI)_k$ is satisfied (3.22). If the codimension of X is sufficiently small, $(CI)_k^{2k}$ is a genericity condition. While only being a sufficient condition for $(CI)_k$, $(CI)_k^{2k}$ has the advantage of being computable by taking at most $2k$ derivatives.

Finally we describe generalizations of the classical Monge equation. The classical Monge equation actually only classifies "nondegenerate" conic curves, those that are not pairs of lines. The generalized Monge systems described by (3.23) characterize "nondegenerate" complete intersections whose ideals are generated in degrees $d_1 < \dots < d_r$, by a PDE system of order $2d_r + 1$. In this case "nondegenerate" means that the conditions $(CI)_{d_1}^{2d_1}, \dots, (CI)_{d_r}^{2d_r}$ hold, each of which is a natural genericity condition in small codimension.

In §4 we study the osculating quadrics of X in detail. We utilize a condition slightly stronger than $(CI)_2^4$, which we call *strong genericity in degree two.* Strong genericity in degree two is computable by taking two derivatives at a general point. It is a genericity condition on $II_{X,x}$; namely that the system of quadrics $|II_X|_x$ has no linear syzygies. Our results are as follows:

We define a more precise generalized Monge system for complete intersections of quadrics (4.17), in the sense that (4.17) is expressed in terms of the invariants F_k instead of the fundamental forms of $v_2(X)$.

Theorem (4.18) states that if there are no linear syzygies in II_X at general points, then there are *upper* bounds on the dimensions of the spaces of quadrics osculating to orders three and four, complementing (3.17). Moreover, these bounds are achieved if and only if the (fifth order) generalized Monge system for quadrics holds, and the only way such varieties can be cut out by quadrics is that the generalized Monge system holds.

A consequence of (4.18) is:

Corollary 4.20. *Let $X \subset \mathbb{P}V$ be a variety and $x \in X$ a general point. Assume $III_{X_x} = 0$. If there are no quadric hypersurfaces singular at x that osculate to order four at x, and Q is a quadric hypersurface osculating to order five at x, then $X \subseteq Q$.*

Theorem (4.28) describes a sufficient condition for I_X to be generated by quadrics.

Theorem (4.33) describes the structure of varieties whose projective differential invariants of order greater than two are zero at general points.

We observe that if $III_X = 0$ and the ideal of X is generated by quadrics, it is extremely difficult for X to fail to be a complete intersection without being an intersection of a complete intersection with a rational variety (4.39). We illustrate this principle with some examples.

In §5 we describe the equations of some familiar homogeneous varieties in a manner that illustrates the results in §4. Among the varieties treated are the Severi varieties and the ten-dimensional spinor variety, $\mathbb{S}^{10} \subset \mathbb{P}^{15}$.

In §6 we study systems of quadrics with linear syzygies and systems with nonzero prolongations. In representation-theoretic language the study is as follows: Let $A \subset S^2T^*$ be a system of quadrics. We determine rank restrictions that force the intersection of $A \otimes T^*$ with the two irreducible $Gl(T^*)$ modules in $S^2T^* \otimes T^*$ to be the origin. We combine the results here with the results of [9] and §4 to prove:

Theorem 6.26. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and $x \in X$ a general point. Let $b = \dim X_{sing}$. (Set $b = -1$ if X is smooth.) If $a < \frac{1}{3}[n - (b+1) + 3]$, then*

$$(6.27)\quad \begin{aligned}&\dim\{\textit{quadrics osculating to order three at } x\}\\ &\qquad \leq a + \binom{a+1}{2} - 1,\\ &\dim\{\textit{quadrics osculating to order four at } x\} \leq a-1.\end{aligned}$$

Equality occurs on the first (respectively second) expression of (6.27) *if and only if* (4.9.3)(*resp.*(4.9.4)) *holds at* x. *If the generalized Monge system* (4.17) *holds, then* X *is a complete intersection of the* $(a-1)$-*dimensional family of quadrics osculating to order four.*

Corollary 6.28. *Let* $X^n \subset \mathbb{P}^{n+a}$ *be a variety and* $x \in X$ *a general point. Let* $b = \dim X_{sing}$. *(Set* $b=-1$ *if* X *is smooth.) If* $a < \frac{1}{3}[n-(b+1)+3]$ *then any quadric osculating to order four at* x *is smooth at* x, *and any quadric osculating to order five at* x *contains* X.

Corollary 6.29. *Let* $X^n \subset \mathbb{P}^{n+a}$ *be a variety with* I_X *generated by quadrics. Let* $b = \dim X_{sing}$. *(Set* $b=-1$ *if* X *is smooth.) If* $a < \frac{1}{3}[n-(b+1)+3]$, *then* X *is a complete intersection.*

Notation. We will use the following conventions for indices

$$\begin{aligned}&0 \leq B, C \leq n+a,\\ &1 \leq \alpha, \beta \leq n,\\ &n+1 \leq \mu, \nu \leq n+a.\end{aligned}$$

Alternating products of vectors will be denoted with a wedge ($\wedge$), and symmetric products will not have any symbol (e.g., $\omega \circ \beta$ will be denoted $\omega\beta$). T_xX denotes the holomorphic tangent space to X at x, and $\tilde{T}_xX$ the embedded tangent space. In general we will supress reference to the base point of our manifold X when we abbreviate the names of bundles, so T should be read as T_pX for a general $p \in X$, N as N_pX etc... . $\{e_i\}$ means the span of the vectors e_i over the index range i. If $Y \subset \mathbb{P}^m$, then $\hat{Y} \subset \mathbb{C}^{m+1}$ will be used to denote the cone over Y (with the exception that the cone over the embedded tangent space $\tilde{T}$ will be denoted $\hat{T}$). We will often ignore twists in bundles, so T will be used to denote both T_xX and $T_xX(1) = \hat{T}/\hat{x}$. If $A \in \mathbb{C}^{n+a+1}$, its projection to $\mathbb{P}^{n+a}$ will be denoted $[A]$. If V is a vector space and W a subspace, and $(e_1 ... e_n)$ a basis of V such that $\{e_1...e_p\} = W$, we write $\{e_{p+1}...e_n\}$ mod$W$ to denote the space V/W. For vector subspaces $W \subset V$, we will use the notation $W^\perp \subset V^*$ for the annihilator of W in V^*. We will use the summation convention throughout (i.e., repeated indices are to be summed over). $\mathfrak{S}_{\alpha\beta\gamma}$ denotes cylic summation over the fixed indices $\alpha\beta\gamma$. In general, X will denote a variety, X_{sm} its smooth points, and X_{sing} its singular points. $\mathbb{CP}^k$ will be denoted $\mathbb{P}^k$. $\mathbf{FF}^k_X$ is the k-th fundamental form of X. We will often denote $\mathbf{FF}^2_X$ by II and $\mathbf{FF}^3_X$ by III. $F_k = F^\mu_k A_\mu \mod \hat{T}$ is the differential invariant which we call the $(k-2)$*-nd variation of* II. By a *general point* $x \in X$ we mean a smooth point of X such that all the discrete information in the differential invariants of X is locally constant. The nongeneral points of X are a codimension one subset of X

1. An elementary characterization of complete intersections

Let $V = \mathbb{C}^{n+a+1}$, and let $X^n \subset \mathbb{P}V = \mathbb{CP}^{n+a}$ be a variety of dimension n. Let X_{sm} denote the smooth points of X. Let $I_X \subset S^\bullet V^*$ denote the ideal of X and let $I_{X,d} = I_d = S^d V^* \cap I_X$ denote the d-th graded piece of I_X. Fixing a smooth point $x \in X$, there is a distinguished subspace of I_d, namely the hypersurfaces of degree d that are singular at x, i.e., $P \in I_d$ such that $(dP)_x = 0$, where dP denotes the exterior derivative of the polynomial P.

Proposition 1.1. *Let $X \subset \mathbb{P}V$ be a variety such that $I_X = (I_d)$ (i.e., I_X is generated by I_d) and $I_{d-1} = (0)$. Then the following are equivalent:*

1. *X is a complete intersection.*
2. *Every hypersurface of degree d containing X is smooth at all $x \in X_{sm}$.*
3. *Let $x \in X_{sm}$. Every hypersurface of degree d containing X is smooth at x.*

Proof. Say $P \in I_d$ is nonzero and such that $dP_x = 0$. Let $x \in X_{sm}$ be a smooth point. Then there exist $P_1, ..., P_a \in I_d$ such that $(dP_1)_x, ..., (dP_a)_x$ span the conormal space $N^*_x X \subset T^*_x \mathbb{P}V$ (actually $N^*_x X(1) \subset T^*_x \mathbb{P}V(1)$). Since $(dP_1)_x, ..., (dP_a)_x$ are linearly independent, P is not in the ideal generated by $P_1, ..., P_a$, and thus X is not a complete intersection. Conversely, if X is not a complete intersection, one can always find such a P. q.e.d.

The following definition is due to L'vovsky [12]:

Definition 1.2. Let $X \subset \mathbb{P}V$ be a variety. Let $P \in I_d$, and let $Z = Z_P \subset \mathbb{P}V$ be the corresponding hypersurface. We will say Z *trivially contains* X if $P = l^1 P_1 + ... + l^m P_m$ with $P_1, ..., P_m \in I_{d-1}$ and $l^1, ..., l^m \in V^*$, and otherwise that Z *essentially contains* X.

Note that the space of hypersurfaces of degree k that trivially contain

X is $I_{k-1} \circ V^*$. Fix $x \in X_{sm}$ and consider the map

$$[d]_k : I_k/(I_{k-1} \circ V^*) \to N_x^*X/d(I_{k-1} \circ V^*) \qquad (1.3.k)$$
$$[P] \mapsto [dP_x],$$

where dP denotes the exterior derivative of the polynomial P. (Again, here and in what follows, we really should be writing $N_x^*X(1)$.)

Definition 1.4. Fix $x \in X_{sm}$. Let $N_k^* := \{dP_x | P \in I_k\} \subseteq N_x^*X$. Let $d_1, ..., d_r$ be the smallest integers such that

$$0 \subsetneq N_{d_1}^* \subsetneq N_{d_2}^* \subsetneq ... \subsetneq N_{d_r}^* = N_x^*X. \qquad (1.5)$$

We will call (1.5) *the natural filtration of* N_x^*X.

(1.1) generalizes to the following statement:

Proposition 1.6 A local characterization of complete intersections. *Let $X \subset \mathbb{P}V$ be a variety. The following are equivalent:*

1. *X is a complete intersection.*
2. *Every hypersurface of degree d containing X is smooth at all $x \in X_{sm}$.*
3. *Let $x \in X_{sm}$. Every hypersurface essentially containing X is smooth at x.*
4. *Let $x \in X_{sm}$ be any smooth point of X. For all k, the map*

$$[d]_k : I_k/(I_{k-1} \circ V^*) \to N_x^*X/d(I_{k-1} \circ V^*)$$
$$[P] \mapsto [dP_x],$$

is injective. (dP denotes the exterior derivative of the polynomial P.)

Proof. Consider the sum of all the maps (1.3.k),

$$[d]^{\oplus} : \oplus_k I_k/(I_{k-1} \circ V^*) \to \oplus_k (N_k^*/N_{k-1}^*), \qquad (1.7)$$

where N_k^* is as in (1.4)

The dimension of the target of $[d]^{\oplus}$ is a, and $[d]^{\oplus}$ is surjective so the dimension of the source is a if and only if $[d]^{\oplus}$ is injective, i.e., all the maps (1.3.k) are injective. On the other hand, the dimension of the source is exactly the number of polynomials needed for a minimal set of generators of I_X, and the kernel of $[d]^{\oplus}$ is exactly the hypersurfaces that essentially contain X and are singular at x. q.e.d.

Definition 1.8. Fix a point $x \in X_{sm}$. We will say X *has no excess equations in degree k at x*, or that $(CI)_k$ *holds at* x, if (1.3.k) is injective. The equivalence 1 $\Leftrightarrow$ 3 of (1.6) may be rephrased as: X is a complete intersection iff $(CI)_k$ holds for all k at some $x \in X_{sm}$.

Although the following is clear from the discussion above, we record it as a proposition.

Proposition 1.9. *Let $X \subset \mathbb{P}V$ be a variety and $x \in X$ a smooth point. Let $N^*_{d_1} \subset N^*_{d_2} \subset \ldots \subset N^*_{d_r} = N^*$ be the natural filtration of $N^*_x X$ described in* (1.4) *and let $a_j = \dim(N^*_{d_j}/N^*_{d_{j-1}})$. Then there is at least an $(a_1 - 1)$-dimensional space of irreducible hypersurfaces of degree d_1 essentially containing X, an $(a_2 - 1)$-dimensional space of irreducible hypersurfaces of degree d_2 essentially containing X, ..., and an $(a_r - 1)$-dimensional space of irreducible hypersurfaces of degree d_r essentially containing X.*

X is a complete intersection if and only if I_X is generated by the equations for these hypersurfaces.

So far we have localized the study of complete intersections to a point, and further, filtered the conormal bundle at that point to enable us to study one degree at a time. Unfortunately, to determine if a hypersurface essentially contains X, one might need to take an arbitrarily high number of derivatives. To have computable conditions, we will work with osculating hypersurfaces rather than the hypersurfaces containing X. The advantage will be that we will only need to study a fixed number of derivatives for each fixed degree of hypersurface; the disadvantage is that we will only obtain sufficient conditions to be a complete intersection.

We first review some notions from projective differential geometry.

2. Frames, fundamental forms

Refer to [6],[9] for more detailed explanations of what follows.

Let $X \subset \mathbb{P}V = \mathbb{CP}^{n+a}$ be a variety and $x \in X$ a general point. We will compute the fundamental forms of X using a method described in [6], which will be useful for the computations of this paper. One may take what follows as the definitions of the fundamental forms, although more geometric definitions are given in [9].

Given $X \subset \mathbb{P}V$ and $x \in X_{sm}$, let $\hat{x} \subset V$ denote the line x determines and let $A_0 \in \hat{x}$. Extend A_0 to a basis $f = (A_0, \ldots, A_{n+a}) = (A_0, A_\alpha, A_\mu)$ of V adapted to the filtration $\hat{x} \subset \hat{T} \subset V$, where $1 \leq \alpha, \beta \leq n$, $n+1 \leq \mu, \nu \leq n+a$, so $\hat{T} = \{A_0, A_\alpha\}$ is the cone over the embedded tangent space to X at x. Let $\mathcal{F}^1 \to X_{sm}$ be the bundle of all such bases.

The *Maurer-Cartan form* of $Gl(V)$, defined by

$$\Omega_f := f^{-1}df \qquad (2.1)$$

restricts to $\mathcal{F}^1 \subset Gl(V)$ and satisfies the *Maurer-Cartan equation*

$$d\Omega = -\Omega \wedge \Omega. \tag{2.2}$$

See ([6],[9]) for more on the Maurer-Cartan form.

$\mathcal{F}^1$ is a principal G_1-bundle over X_{sm} where

$$G_1 := \left\{ g \in Gl(V) \mid g = \begin{pmatrix} g^0_0 & g^0_\beta & g^0_\nu \\ 0 & g^\alpha_\beta & g^\alpha_\nu \\ 0 & 0 & g^\mu_\nu \end{pmatrix} \right\}. \tag{2.3}$$

Define inductively a series of maps:

$$\underline{d}_k A_0 : (T\mathcal{F}^1)^{\otimes k} \to V/\operatorname{Image}(\underline{d}^0, ..., \underline{d}^{k-1}) \tag{2.4}$$

as follows: Let d denote exterior differentiation, let $\underline{d}^0 A_0 = A_0$ and let $\underline{d}^1 A_0 = dA_0 \mod A_0$. If $v_1, ..., v_k \in T_f\mathcal{F}^1$, extend $v_1, ..., v_k$ to holomorphic vector fields in some neighborhood of f which we denote $\tilde{v}_1, ..., \tilde{v}_k$. Let

$$\underline{d}^k A_0(v_1, ..., v_k) := v_1(\rfloor d(\tilde{v}_2 \rfloor ... d(\tilde{v}_k \rfloor dA_0) \mod \pi_k^{-1}(\operatorname{Image} \underline{d}^{k-1}), \tag{2.5}$$

where $\pi_k : V \to V/(\operatorname{Image}\{\underline{d}^0, \underline{d}^1, ..., \underline{d}^{k-1}\})$ is the projection, and $\rfloor$ denotes the contraction $T \times T^{*\otimes l} \to T^{*\otimes l-1}$. (2.5) is independent of the extension of $v_1, ..., v_k$ to holomorphic vector fields. (The proof that (2.5) is independent of the choice of extension to holomorphic vector fields is the same as the standard argument in the real case; see e.g. [14].)

$\underline{d}^k A_0$ descends to be a well defined element of

$$S^k T^*_x X \otimes V/\operatorname{Image}(\underline{d}^0, ..., \underline{d}^{k-1}) \tag{2.6}$$

called the *k-th fundamental form* of X (twisted by $\mathcal{O}(k-1)$) which we will denote $\mathbf{FF}^k_X$, except that we will often denote $\mathbf{FF}^2_X$ by II, and $\mathbf{FF}^3_X$ by III.

To fix notation, we will verify this assertion in the case $k = 2$. Given $f \in \mathcal{F}^1$, let $(\omega^B_C) = \Omega = f^{-1}df$ denote the entries of the Maurer-Cartan form. Write ω^B for ω^B_0. The first two terms of (2.5) expressed in frames are

$$\underline{d}^1 A_0 = \omega^\alpha \otimes A_\alpha \mod \hat{x}, \tag{2.7}$$

$$\underline{d}^2 A_0 = \omega^\alpha \omega^\mu_\alpha \otimes A_\mu \mod \hat{T}. \tag{2.8}$$

To see that $\underline{d}^2 A_0$ descends (modulo twisting) to a section of $S^2T^* \otimes N$, note that for $g \in G_1$, $\Omega_{fg} = g^{-1}\Omega_f g + g^{-1}dg$, where the expression dg is to be understood as the matrix of differentials of the functions (g^A_B), so tensoring (2.8) with A^*_0; the fiber motions of $\omega^\alpha, \omega^\mu_\alpha, A^*_0$ and A_μ cancel.

Note that as a form on $\mathcal{F}^1$, $\omega^\mu = 0$ which implies $d\omega^\mu = 0$. (2.2) implies $d\omega^\mu = -\omega^\mu_\alpha \wedge \omega^\alpha$. Since the forms $\{\omega^\alpha\}$ are all independent, the Cartan Lemma (see e.g. [1]) implies that $\omega^\mu_\alpha = q^\mu_{\alpha\beta}\omega^\beta$ for some functions $q^\mu_{\alpha\beta} = q^\mu_{\beta\alpha}$ defined on $\mathcal{F}^1$. Thus (2.8) may be rewritten:

$$\underline{d}^2 A_0 = q^\mu_{\alpha\beta}\omega^\alpha\omega^\beta \otimes A_\mu \quad \mathrm{mod}\ \hat{T}.$$

For the purposes of this paper it will be useful to consider $\mathbf{FF}^2_X = II \in S^2T^* \otimes N$ as a map

$$II : N^* \to S^2T^*, \tag{2.9}$$

which is dual to the standard way of considering II as a map.

We will consider the k-th fundamental form of X at x as a map

$$\mathbf{FF}^k_{Xx} : \ker \mathbf{FF}^{k-1}_{Xx} \to S^kT^*_x, \tag{2.10}$$

where $\ker \mathbf{FF}^{k-1}_{Xx} \subset V^*$ and $\ker \mathbf{FF}^0_{Xx} = \hat{x}^\perp \subset V^*$. $\mathbb{P}(\ker \mathbf{FF}^k_{Xx}) \subset \mathbb{P}V^*$ is the space of hyperplanes osculating to order k at x.

Remark 2.11. We will often omit reference to the base point x, ignore twists, and use T to denote both $\hat{T}/\hat{x}$ and $T_xX = \hat{x}^* \otimes \hat{T}/\hat{x}$.

Another way of understanding fundamental forms is as follows:

The quotient map

$$V^* \to V^*/\hat{x}^\perp = \mathcal{O}_{\mathbb{P}V}(1)_x \tag{2.12}$$

gives rise to a spectral sequence of a filtered complex by letting

$$\begin{aligned} &F^0K^0 = V^*, \quad F^0K^1 = \mathcal{O}_X(1)_x, \\ &F^1K^0 = 0, \quad F^p = F^pK^1 = \mathfrak{m}^p_x(1). \end{aligned} \tag{2.13}$$

The maps are

$$\begin{aligned} &\underline{d}^0 : V^* \to F^0/F^1 = \mathcal{O}_{X,x}(1)/\mathfrak{m}_x(1) \simeq \mathbb{C}, \\ &\underline{d}^1 : \ker \underline{d}^0 \to F^1/F^2 = \mathfrak{m}_x(1)/\mathfrak{m}^2_x(1) \simeq T^*(1), \\ &\underline{d}^2 : \ker \underline{d}^1 \to F^2/F^3 = \mathfrak{m}^2_x(1)/\mathfrak{m}^3_x(1) \simeq (S^2T^*)(1), \\ &\vdots \end{aligned} \tag{2.14}$$

To study the space of hypersurfaces of degree d osculating to order k at a general point $x \in X$ in terms of local differential invariants, we need to compute the fundamental forms of the d-th Veronese embedding of X, $v_d(X) \subset \mathbb{P}S^dV$. We will do this in §3, and their expression will involve more subtle invariants of $X \subset \mathbb{P}V$ which we now describe.

Differentiating the equation $\omega^\mu_\alpha - q^\mu_{\alpha\beta}\omega^\beta = 0$ and using the Cartan lemma, one obtains functions $r^\mu_{\alpha\beta\gamma}$ (symmetric in their lower indices) defined on $\mathcal{F}^1$ by the equation:

$$r^\mu_{\alpha\beta\gamma}\omega^\gamma = -dq^\mu_{\alpha\beta} - q^\mu_{\alpha\beta}\omega^0_0 - q^\nu_{\alpha\beta}\omega^\mu_\nu + q^\mu_{\alpha\delta}\omega^\delta_\beta + q^\mu_{\beta\delta}\omega^\delta_\alpha, \tag{2.15}$$

(see [9]). The form

$$F_3 := r^\mu_{\alpha\beta\gamma}\omega^\alpha\omega^\beta\omega^\gamma \otimes A_\mu \mod \hat{T} \tag{2.16}$$

is a section of $S^3(T^*\mathcal{F}^1) \otimes \pi^*(NX)$ (again, ignoring twists). F_3 was defined for hypersurfaces in [6] and called the *cubic form* there. In [9], F_3 was denoted ∂II. We will also call F_3 *the first variation of II.* F_3 is actually a section of $S^3(SB) \otimes \pi^*NX$ where $SB \subset T^*\mathcal{F}^1$ is the subbundle of semi-basic forms, that is those that annhilate vertical tangent vectors. As defined, F_3 is a *relative invariant* in the sense that it is defined as a section of a bundle over $\mathcal{F}^1$ instead of a bundle over X. One can define F_3 as a section of a bundle over X (see [9]). However it is not advantageous to do so because there is a whole series of relative invariants, of which F_3 is the first, and none of the others can be defined as a section of a bundle over X. (Given a particular variety, one can make normalizations that enable one to define the relative invariants as a section of a bundle over X, but the normalizations will not be canonical.) On the other hand, certain combinations of relative invariants can be defined as sections of natural bundles over X. In this paper we will deal with combinations that are the fundamental forms of the Veronese re-embeddings of X.

F_3 is the projective analogue of the covariant derivative of the second fundamental form of a metric connection, which we will denote ∇II. ∇II is defined as a section of a bundle over the original variety. One may think of the projective structure as specifying an equivalence class of connections, and the necessity of defining F_3 over a principal bundle as corresponding to the ambiguity in the choice of compatible connection.

Differentiating (2.15), one obtains functions $r^\mu_{\alpha\beta\gamma\delta}$ defined on $\mathcal{F}^1$ by

$$\begin{aligned} r^\mu_{\alpha\beta\gamma\delta}\omega^\delta =& - dr^\mu_{\alpha\beta\gamma} - 2r^\mu_{\alpha\beta\gamma}\omega^0_0 - r^\nu_{\alpha\beta\gamma}\omega^\mu_\nu + \mathfrak{S}_{\alpha\beta\gamma} r^\mu_{\alpha\beta\delta}\omega^\delta_\gamma \\ & - \mathfrak{S}_{\alpha\beta\gamma} q^\mu_{\alpha\delta} q^\nu_{\beta\gamma}\omega^\delta_\nu + \mathfrak{S}_{\alpha\beta\gamma} q^\mu_{\alpha\beta}\omega^0_\gamma, \end{aligned} \tag{2.17}$$

which leads to a form

$$\begin{aligned} F_4 =& r^\mu_{\alpha\beta\gamma\delta}\omega^\alpha\omega^\beta\omega^\gamma\omega^\delta \otimes A_\mu \mod \hat{T} \\ & \in \Gamma(S^4(T^*\mathcal{F}^1) \otimes \pi^*NX). \end{aligned} \tag{2.18}$$

Say we have defined F_{k-1} by differentiating the equations defining the coefficients of F_{k-2}. Then we can define F_k by differentiating the defining equations of the coefficients of F_{k-1}.

We will call F_k the $(k-2)$*-nd variation of* II. F_k measures how X is infinitesimally leaving its embedded tangent space to order $(k-1)$ at x. We will use the notation F_0 for the quotient map $V^* \to V^*/\hat{x}^\perp$, F_1 for the quotient map $\ker F_0 \to T^*$, and $F_2 = \mathbf{FF}^2_X$. On $\mathcal{F}^1$, F_k is a section of $S^kT^*\mathcal{F}^1 \otimes \pi^*NX$, in fact a section of $S^k(SB) \otimes \pi^*NX$.

Proposition 2.19. *The coefficients* $r^\mu_{\alpha_1\ldots\alpha_l}$ *of* F_l *are defined by the formula*

$$
\begin{aligned}
r^\mu_{\alpha_1\ldots\alpha_l}\omega^{\alpha_l} = &- dr^\mu_{\alpha_1\ldots\alpha_{l-1}} - lr^\mu_{\alpha_1\ldots\alpha_{l-1}}\omega^0_0 - r^\nu_{\alpha_1\ldots\alpha_{l-1}}\omega^\mu_\nu \\
&+ \mathfrak{S}_{\alpha_1,\ldots,\alpha_{l-1}} r^\mu_{\alpha_1\ldots\alpha_{l-2}\beta}\omega^\beta_{\alpha_{l-1}} \\
&+ l\mathfrak{S}_{\alpha_1,\ldots,\alpha_{l-1}} r^\mu_{\alpha_1\ldots\alpha_{l-2}}\omega^0_{\alpha_{l-1}} \\
&- \Sigma_{p=1}^{l-3}\mathfrak{S}_{\alpha_1,\ldots,\alpha_{l-2}} r^\mu_{\alpha_1\ldots\alpha_p\beta} r^\nu_{\alpha_{p+1}\ldots\alpha_{l-2}}\omega^\beta_\nu \\
&- \Sigma_{p=2}^{l-3}\mathfrak{S}_{\alpha_1,\ldots,\alpha_{l-1}}(p-2+l) r^\mu_{\alpha_1\ldots\alpha_p} r^\nu_{\alpha_{p+1}\ldots\alpha_{l-1}}\omega^0_\nu .
\end{aligned} \tag{2.20}
$$

Proof. Say we have computed the coefficients for F_{k+1}, i.e., that

$$
\begin{aligned}
0 = &dr^\mu_{\alpha_1\ldots\alpha_k} - a_k r^\mu_{\alpha_1\ldots\alpha_k}\omega^0_0 - b_k r^\nu_{\alpha_1\ldots\alpha_k}\omega^\mu_\nu \\
&+ c_k\mathfrak{S}_{\alpha_1\ldots\alpha_k} r^\mu_{\alpha_1\ldots\alpha_{k-1}\beta}\omega^\beta_{\alpha_k} + e_k\mathfrak{S}_{\alpha_1,\ldots,\alpha_k} r^\mu_{\alpha_1\ldots\alpha_{k-1}}\omega^0_{\alpha_k} \\
&+ \Sigma_{p=1}^{k-2} f_{p,k}\mathfrak{S}_{\alpha_1,\ldots,\alpha_k} r^\mu_{\alpha_1\ldots\alpha_p\beta} r^\nu_{\alpha_{p+1}\ldots\alpha_k}\omega^\beta_\nu \\
&+ \Sigma_{p=2}^{k-2} g_{p,k}\mathfrak{S}_{\alpha_1,\ldots,\alpha_k} r^\mu_{\alpha_1\ldots\alpha_p} r^\nu_{\alpha_{p+1}\ldots\alpha_k}\omega^0_\nu - r^\mu_{\alpha_1\ldots\alpha_{k+1}}\omega^{\alpha_{k+1}},
\end{aligned} \tag{2.21}
$$

where $a_k, \ldots, g_{p,k}$ are some constants independent of X. Taking the exterior derivative of (2.21) the terms with semi-basic coefficients (we can ignore the others since they cancel out) are

$$a_k r^\mu_{\alpha_1\dots\alpha_{k+1}}\omega^{\alpha_{k+1}}\omega^0_0 + a_k r^\mu_{\alpha_1\dots\alpha_k}\omega^0_\beta\omega^\beta$$

$$+b_k r^\nu_{\alpha_1\dots\alpha_{k+1}}\omega^{\alpha_{k+1}}\omega^\mu_\nu + b_k r^\nu_{\alpha_1\dots\alpha_k}\omega^\mu_\beta\omega^\beta_\nu$$

$$+c_k\mathfrak{S}_{\alpha_1,\dots,\alpha_k}\{r^\mu_{\alpha_1\dots\alpha_{k-1}\beta\alpha_{k+1}}\omega^{\alpha_{k+1}}\omega^\beta_{\alpha_k}$$

$$-r^\mu_{\alpha_1\dots\alpha_{k-1}\beta}(\omega^\beta_0\omega^0_{\alpha_k} + \omega^\beta_\nu\omega^\nu_{\alpha_k})\}$$

$$-e_k\mathfrak{S}_{\alpha_1,\dots,\alpha_k}\{r^\mu_{\alpha_1\dots\alpha_{k-1}\beta}\omega^\beta\omega^0_{\alpha_k} - r^\mu_{\alpha_1\dots\alpha_{k-1}}\omega^0_\nu\omega^\nu_{\alpha_k}\}$$

$$-\Sigma_{p=1}^{k-2} f_{p,k}\mathfrak{S}_{\alpha_1,\dots,\alpha_k}\{r^\mu_{\alpha_1\dots\alpha_p\beta\delta}r^\nu_{\alpha_{p+1}\dots\alpha_k}\omega^\delta\omega^\beta_\nu$$

$$+r^\mu_{\alpha_1\dots\alpha_p\beta}r^\nu_{\alpha_{p+1}\dots\alpha_k\delta}\omega^\delta\omega^\beta_\nu - r^\mu_{\alpha_1\dots\alpha_p\beta}r^\nu_{\alpha_{p+1}\dots\alpha_k}\omega^\beta\omega^0_\nu\}$$

$$-\Sigma_{p=2}^{k-1} g_{p,k}\mathfrak{S}_{\alpha_1,\dots,\alpha_k}\{r^\mu_{\alpha_1\dots\alpha_p\beta}r^\nu_{\alpha_{p+1}\dots\alpha_k}\omega^\beta\omega^0_\nu$$

$$+r^\mu_{\alpha_1,\dots,\alpha_p}r^\nu_{\alpha_{p+1}\dots\alpha_k\beta}\omega^\beta\omega^0_\nu\}$$

$$-dr^\mu_{\alpha_1\dots\alpha_{k+1}}\omega^{\alpha_{k+1}} + r^\mu_{\alpha_1\dots\alpha_{k+1}}(\omega^{\alpha_{k+1}}\omega^0_0 + \omega^{\alpha_{k+1}}_\beta\omega^\beta).$$

Collecting terms, the coeffeicient of $\omega^{\alpha_{k+1}}$ is

$$\begin{aligned}
&- (a_k+1)r^\mu_{\alpha_1\dots\alpha_{k+1}}\omega^0_0 - b_k r^\nu_{\alpha_1\dots\alpha_{k+1}}\omega^\mu_\nu\\
&+ c_k\mathfrak{S}_{\alpha_1,\dots,\alpha_k} r^\mu_{\alpha_1\dots\alpha_{k+1}\beta}\omega^\beta_{\alpha_k} + r^\mu_{\alpha_1\dots\alpha_k\beta}\omega^\beta_{\alpha_{k+1}}\\
&+ a_k r^\mu_{\alpha_1\dots\alpha_k}\omega^0_{\alpha_{k+1}} + c_k\mathfrak{S}_{\alpha_1,\dots,\alpha_k} r^\mu_{\alpha_1\dots\alpha_{k-1},\alpha_{k+1}}\omega^0_{\alpha_k}\\
&+ e_k\mathfrak{S}_{\alpha_1,\dots,\alpha_k} r^\mu_{\alpha_1\dots\alpha_{k-1},\alpha_{k+1}}\omega^0_{\alpha_k}\\
&+ b_k r^\nu_{\alpha_1\dots\alpha_k} q^\mu_{\beta\alpha_{k+1}}\omega^\beta_\nu + c_k\mathfrak{S}_{\alpha_1,\dots,\alpha_k} r^\mu_{\alpha_1\dots\alpha_{k-1},\beta} q^\nu_{\alpha_k\alpha_{k+1}}\omega^\beta_\nu\\
&+ \Sigma f_{p,k}\mathfrak{S}_{\alpha_1,\dots,\alpha_k}\{r^\mu_{\alpha_1\dots\alpha_p\beta\alpha_{k+1}} r^\nu_{\alpha_{p+1}\dots\alpha_k}\omega^\beta_\nu\\
&\qquad\qquad + r^\mu_{\alpha_1\dots\alpha_p\beta} r^\nu_{\alpha_{p+1}\dots\alpha_k\alpha_{k+1}}\omega^\beta_\nu\}\\
&+ \mathfrak{S}_{\alpha_1,\dots,\alpha_k}\{e_k r^\mu_{\alpha_1\dots\alpha_{k-1}} q^\nu_{\alpha_k\alpha_{k+1}}\omega^0_\nu\\
&\qquad\qquad + \Sigma f_{p,k} r^\mu_{\alpha_1\dots\alpha_p\alpha_{k+1}} r^\nu_{\alpha_{p+1}\dots\alpha_k}\omega^0_\nu\\
&\qquad\qquad + \Sigma g_{p,k} r^\mu_{\alpha_1\dots\alpha_p\alpha_{k+1}} r^\nu_{\alpha_{p+1}\dots\alpha_k}\omega^0_\nu\}.
\end{aligned} \tag{2.22}$$

Thus, comparing (2.22) with (2.15),(2.17) and (2.20), we have $a_{k+1} = a_k + 1 = k+1$, $b_{k+1} = b_k = 1$, $c_{k+1} = c_k = 1$, $e_{k+1} = a_k = k$, $f_{p,k} = 1$, $g_{p,k} = p - 2 + k$. q.e.d.

To determine the actual geometric information in the invariants F_k we must know how they vary in the fiber. F_3, F_4 vary in the fiber as follows: If $(\tilde{A}_0, \tilde{A}_\alpha, \tilde{A}_\mu)$ is a new frame with

$$\begin{aligned} \tilde{A}_\mu &= A_\mu + g^0_\mu A_0 + g^\alpha_\mu A_\alpha, \\ \tilde{A}_\alpha &= A_\alpha + g^0_\alpha A_0, \end{aligned} \tag{2.23}$$

then

$$\begin{aligned} \tilde{r}^\mu_{\alpha\beta\gamma} =& r^\mu_{\alpha\beta\gamma} + \mathfrak{S}_{\alpha\beta\gamma} g^0_\alpha q^\mu_{\beta\gamma} + \mathfrak{S}_{\alpha\beta\gamma} g^\delta_\nu q^\nu_{\alpha\beta} q^\mu_{\gamma\delta}, \\ \tilde{r}^\mu_{\alpha\beta\gamma\delta} =& r^\mu_{\alpha\beta\gamma\delta} + \mathfrak{S}_{\alpha\beta\gamma\delta} g^0_\alpha r^\mu_{\beta\gamma\delta} \\ &+ \mathfrak{S}_{\alpha\beta\gamma\delta} g^\epsilon_\nu (r^\nu_{\alpha\beta\gamma} q^\mu_{\delta\epsilon} + q^\nu_{\alpha\beta} r^\mu_{\gamma\delta\epsilon}) + g^0_\nu q^\mu_{\alpha\beta} q^\nu_{\gamma\delta}. \end{aligned} \tag{2.24}$$

We do not consider motions by $g^\mu_\nu, g^\alpha_\beta, g^0_0$ because they just conjugate the coefficients by invertible matrices. We will use the notation $\Delta r^\mu_{\alpha\beta\gamma}$ to denote the change in $r^\mu_{\alpha\beta\gamma}$ by a fiber motion of the type in (2.23). By (2.16), we have

$$\Delta r^\mu_{\alpha\beta\gamma} = \mathfrak{S}_{\alpha\beta\gamma}(g^0_\alpha q^\mu_{\beta\gamma} + g^\delta_\nu q^\nu_{\alpha\beta} q^\mu_{\gamma\delta}). \tag{2.25}$$

We will occasionally write $F_k = F^k_\mu A_\mu = r^\mu_{\alpha_1\ldots\alpha_k}\omega^{\alpha_1}\ldots\omega^{\alpha_k}A_\mu$ with the ambiguity understood. Because they are only relative invariants, the invariants F_k are more difficult to deal with than the fundamental forms.

Example 2.26. Invariants of curves in $\mathbb{P}^2$.

(This computation is originally due Monge, and in this language to Cartan [2].) Take an adapted frame $f = (A_0, A_1, A_2)$ defined up to the action of G_1. As long as our curve is not a line, at a general point we can rescale the second fundamental form so that $q^2_{11} \equiv 1$. (e.g. by scaling A_0). We restrict to frames on which $q^2_{11} \equiv 1$. The defining equation for F_3 becomes

$$r^2_{111}\omega^1 = (-\omega^0_0 - \omega^2_2 + 2\omega^1_1) \tag{2.27}$$

(ommiting the $\otimes$ symbols here and in what follows) which we normalize to zero by sending

$$A_1 \mapsto A_1 - \frac{1}{2} r^2_{111} A_0 \tag{2.28}$$

(or by moving A_2 by A_1) and restricting to frames where $r^2_{111} = 0$. Thus, for a curve in $\mathbb{P}^2$, F_3 contains no geometric information. On our restricted frame bundle,

$$r^2_{1111}\omega^1 = (\omega^0_1 + \omega^1_2), \tag{2.29}$$

which we normalize to zero by sending

$$A_2 \mapsto A_2 - r^2_{1111}A_0, \tag{2.30}$$

so F_4 does not contain any geometric information either. Restricting to frames where $r^2_{1111} = 0$,

$$r^2_{11111}\omega^1 = \omega^0_2. \tag{2.31}$$

r^2_{11111} cannot be normalized to zero, but if it is not zero, it can be made constant by scaling A_2. Making $r^2_{11111} = 1$ sets

$$r^2_{111111}\omega^1 = \omega^0_1, \tag{2.32}$$

which cannot be normalized, so F_6 represents the first non-discrete differential invariant of a nondegenerate curve in $\mathbb{P}^2$. Note that $F_5 = 0$ implies $F_6 = 0$, and in fact that all higher $F_k = 0$. In §4 we will continue this example.

Example 2.33. Surfaces in $\mathbb{P}^5$ with zero third fundamental form.

We may adapt frames such that $II_X = \omega^1\omega^1 \otimes A_3 + \omega^1\omega^2 \otimes A_4 + \omega^2\omega^2 \otimes A_5 \operatorname{mod} \hat{T}$. The variability in F_3 is

(2.34)

$$\begin{array}{ccc}
\Delta r^3_{111} = g^0_1 + g^1_3, & \Delta r^4_{111} = g^2_3, & \Delta r^5_{111} = 0, \\
\Delta r^3_{112} = g^0_2 + g^1_4, & \Delta r^4_{112} = g^0_1 + g^2_4 + g^1_3, & \Delta r^5_{112} = g^2_1, \\
\Delta r^3_{122} = g^1_5, & \Delta r^4_{122} = g^0_2 + g^1_4 + g^2_3, & \Delta r^5_{122} = g^0_1 + g^2_4, \\
\Delta r^3_{222} = 0, & \Delta r^4_{222} = g^1_5, & \Delta r^5_{222} = g^0_2 + g^2_5.
\end{array}$$

Using the motions g^α_μ in (2.23), we can normalize six of the twelve terms $r^\mu_{\alpha\beta\gamma}$ to zero. This uses up all our normalizations (the g^0_α cannot contribute any additional normalizations in this case). Informally, we will say F_3 contains 6 functions worth of geometric information.

More generally, for $X^n \subset \mathbb{P}^{n+\binom{n+1}{2}}$ with $III_X = 0$, i.e., $|II_X| = \mathbb{P}S^2T^*$, there are $\binom{n+1}{2}n$ effective normalizations g^α_ν so there are $\binom{n+1}{2}\binom{n+2}{3} - \binom{n+1}{2}n$ functions worth of geometric information in F_3.

For an n-dimensional variety of codimension a, if $F_3 = 0$, the equations for the coefficients of F_4 are

$$r^\mu_{\alpha\beta\gamma\delta}\omega^\delta = \mathfrak{S}_{\alpha\beta\gamma}(q^\mu_{\alpha\beta}\omega^0_\gamma + q^\mu_{\alpha\epsilon}q^\nu_{\beta\gamma}\omega^\epsilon_\nu) \quad \forall\mu, \alpha \le \beta \le \gamma, \tag{2.35}$$

which is $a\binom{n+2}{3}$ equations relating the n independent forms $\{\omega^\delta\}$ to the $n + an$ forms $\{\omega^0_\gamma, \omega^\epsilon_\nu\}$. (If one does not use the normalizations g^0_α, the forms $\{\omega^\alpha, \omega^0_\beta\}$ are all independent, and the $a\binom{n+2}{3}$ equations express the an forms ω^ϵ_ν in terms of the $2n$ forms $\{\omega^\alpha, \omega^0_\gamma\}$.) Thinking of II as given, the equations for the coefficients of F_3 are extremely overdetermined, but sometimes they have solutions.

Example 2.36. Griffiths and Harris showed that for $n \geq 2$ and $a = 1$, if one can restrict to a sub-bundle $\mathcal{F}^3 \subset \mathcal{F}^2$ on which the functions $r^\mu_{\alpha\beta\gamma}$ are all identically zero, then one can restrict to a further subbundle $\mathcal{F}^4 \subseteq \mathcal{F}^3$, on which the coefficients of all the higher forms are also identically zero ([6, B16]). To prove the $n = 2$ case one must compute the equations of F_5 to determine that the coefficients of F_4 are zero.

Example 2.37. Let $X^n \subset \mathbb{P}^{n+2}$ and let $x \in X$ be a general point. Assume there is a nonsingular quadric in $|II_X|$. (This will be the case iff X has a nondegenerate dual variety. X^* is nondegenerate if X is smooth and $n \geq 4$ by [3].) We may normalize II_X such that

$$(2.38) \qquad \begin{aligned} II_X = \;& ((\omega^1)^2 + \ldots(\omega^n)^2) \otimes A_{n+1} \\ & +(\lambda_1(\omega^1)^2 + \ldots\lambda_n(\omega^n)^2) \otimes A_{n+2}, \end{aligned}$$

where $\lambda_i \in \mathbb{C}$. This case is the closest to the hypersurface case and is still extremely overdetermined. If $F_3 = 0$ we may let

$$(2.39) \qquad r^\mu_{\alpha\beta\gamma\delta} = \mathfrak{S}_{\alpha\beta\gamma\delta} b^\mu_{\nu\tau} q^\nu_{\alpha\beta} q^\tau_{\gamma\delta},$$

where $b^\mu_{\nu\tau} = b^\mu_{\tau\nu} \in \mathbb{C}$ are constants. Some, but not all of these terms can be normalized to zero. (E.g. one can set $b^\mu_{\mu\mu} = 0$ via g^0_μ.) One can check that (2.39) is a consistent solution to the equations. i.e.,

There exist varieties of arbitrary dimension and codimension two such that it is possible to reduce to frames where the coefficients of F_3 are all identically zero, but that there is no frame such that the coefficients of F_4 are also identically zero.

We will show that the equations above are consistent in (4.34) with an easier, indirect proof. The reader may wish to contrast this with the results of [8].

Remark 2.40. In the metric situation, if $\nabla II = 0$ at a general point, then it is always the case that $\nabla^k II = 0$, $\forall k$, and the variety must be a locally symmetric space. The reward of dealing with our subtler invariants is that their vanishing will detect more general properties than being a locally symmetric space.

Although example (2.37) shows that being able to reduce to frames where the coefficients of F_k are all zero does not imply that there is a further reduction of frame such that the coefficients of F_{k+1} are also zero, one does have the following result:

Proposition 2.41. *Let $X \subset \mathbb{P}V$ be a variety, and $x \in X$ any smooth point. If there exists a framing where the coefficients of $F_k, ..., F_{2k}$ are all zero at x, then the coefficients of F_l are also zero at x $\forall l > 2k$.*

Proof. Examining (2.20), we see that if the coefficients $r^{\mu}_{\alpha_1 \ldots \alpha_k}, \ldots,$ $r^{\mu}_{\alpha_1 \ldots \alpha_{2k}}$ are zero, then the $r^{\mu}_{\alpha_1 \ldots \alpha_{2k+1}}$ are as well.

3. Observations about the osculating hypersurfaces of X

As mentioned in the introduction,

$$\ker \mathbf{FF}^k_{v_d(X)x} = \{\text{hypersurfaces of degree } d \text{ osculating to order } k \text{ at } x\}.$$

We have $\ker \mathbf{FF}^k_{v_d(X)x} \subseteq \ker \mathbf{FF}^{k+1}_{v_d(X)x}$ and for sufficiently large k, the maps $\mathbf{FF}^k_{v_d(X)}$ are zero and the kernels stabilize to $I_d \subset S^d V^*$.

In this section, we compute $\ker \mathbf{FF}^k_{v_d(X)x}$ in terms of the invariants F_l of X, make some observations about osculating hypersurfaces, and describe a generalization of the classical Monge equation that characterizes conic curves in $\mathbb{P}^2$.

The natural representation $\rho_d : Gl(V) \to Gl(S^d V)$ allows us to compute frames for $v_d(X)$ using $\mathcal{F}^1_{\mathbb{P}V}$ instead of $\mathcal{F}^1_{\mathbb{P}(S^d V)}$. More precisely, one may compute the fundamental forms of $v_d(X)$ using any adapted frame bundle over $v_d(X)$ that respects the filtration $\hat{x} \subset \hat{T}_x v_d(X) \subset S^d V^*$. In particular, we may restrict to the frame bundle $\rho_d(\mathcal{F}^1_X) \subset \mathcal{F}^1_{v_d(X)}$. We compute the k-th fundamental form of $v_d(X)$ by computing $\underline{d}^k(A_0^d)$.

In other words, we compute the fundamental forms of $v_d(X)$ by applying the operator $\underline{d}$ to sections of $\mathcal{O}(d) \to X$. In the spectral sequence perspective, one uses the quotient maps

$$S^d V^* \to S^d V^* / (\hat{x}^{\perp})^d = \mathcal{O}_X(d)_x \tag{3.1}$$

and the analogous filtrations. The equality $\mathcal{O}_X(d) = v_d^*(\mathcal{O}_{v_d(X)}(1))$ connects the perspectives.

Example 3.2. The fundamental forms of $v_2(X)$.

Assume II_X is injective, and write A_0^2 for $A_0 A_0$. The fundamental forms of $v_2(X)$ expressed in frames are:

$$
\begin{aligned}
&\underline{d}^1 A_0^2 = 2\omega^\alpha A_\alpha A_0 \mod \{A_0^2\},\\
&\underline{d}^2 A_0^2 = 2\omega^\alpha\omega^\beta(q^\mu_{\alpha\beta}A_\mu A_0 + A_\alpha A_\beta) \mod \{A_\alpha A_0, A_0^2\},\\
&\underline{d}^3 A_0^2 = 2\omega^\alpha\omega^\beta\omega^\gamma(r^\mu_{\alpha\beta\gamma}A_\mu A_0 + 3\mathfrak{S}_{\alpha\beta\gamma}q^\mu_{\alpha\beta}A_\mu A_\gamma),\\
&\qquad \mod \{q^\mu_{\alpha\beta}A_\mu A_0 + A_\alpha A_\beta, A_\alpha A_0, A_0^2\},\\
&\underline{d}^4 A_0^2 = 2\omega^\alpha\omega^\beta\omega^\gamma\omega^\delta(r^\mu_{\alpha\beta\gamma\delta}A_\mu A_0 + 4\mathfrak{S}_{\alpha\beta\gamma\delta}r^\mu_{\alpha\beta\gamma}A_\mu A_\delta\\
&\qquad\qquad + 3\mathfrak{S}_{\alpha\beta\gamma\delta}q^\mu_{\alpha\beta}q^\nu_{\gamma\delta}A_\mu A_\nu),\\
&\qquad \mod \{\mathrm{Image}(\underline{d}^3, \underline{d}^2, \underline{d}^1, \underline{d}^0)\},\\
&\qquad \vdots
\end{aligned}
\tag{3.3}
$$

or in invariant notation,

$$
\begin{aligned}
\mathbf{FF}^1_{v_2(X)} &= 2F_1F_0|_{\hat{x}^{2\perp}},\\
\mathbf{FF}^2_{v_2(X)} &= 2(F_2F_0 + F_1F_1)|_{(\hat{x}\hat{T})^\perp},\\
\mathbf{FF}^3_{v_2(X)} &= 2(F_3F_0 + 3F_2F_1)|_{\ker \mathbf{FF}^2_{v_2(X)}},\\
\mathbf{FF}^4_{v_2(X)} &= 2(F_4F_0 + 4F_3F_1 + 3F_2F_2)|_{\ker \mathbf{FF}^3_{v_2(X)}},\\
\mathbf{FF}^5_{v_2(X)} &= 2(F_5F_0 + 5F_4F_1 + 10F_3F_2)|_{\ker \mathbf{FF}^4_{v_2(X)}},\\
&\ \ \vdots
\end{aligned}
\tag{3.4}
$$

Remark 3.5. If we had not assumed that II_{X_x} was injective, then the coefficients of III_X would have appeared beginning in $\underline{d}^3$, and its infinitesimal variations would have appeared in higher order terms.

3.6. The classical Monge equation; Example 2.26 continued. Examining (3.4), we see that $F_5 = 0$ implies $\mathbf{FF}^5_{v_2(C)} = 0$. If x is a general point of C, this implies $I_2 = \ker \mathbf{FF}^4_{v_2(X)}$. On the other hand, $\dim(\ker \mathbf{FF}^j_{v_2(C)}) = 5 - j$ for $j \le 4$, and therefore there is a $Q \in S^2V^*$ osculating to all orders, i.e., C is a conic curve. $F_5 = 0$ is exactly the classical Monge equation for a curve in a plane to be a conic.

In fact, if we work in local affine coordinates $[1, x, y]$ where the curve is $y = y(x)$ we can recover the original Monge equation. First take a lift

$$
f = (A_0, A_1, A_2) = \begin{pmatrix} 1 & 0 & 0\\ x & 1 & 0\\ y & 0 & 1 \end{pmatrix},
$$

and then solve for $g \in G_1$ such that

$$(3.7) \qquad (fg)^{-1}d(fg) = \begin{pmatrix} \omega_0^0 & \omega_1^0 & \omega_2^0 \\ \omega_0^1 & \omega_1^1 & \omega_2^1 \\ \omega_0^2 & \omega_1^2 & \omega_2^2 \end{pmatrix}$$

satisfies $\omega_0^2 = \omega_1^2 - \omega_0^1 = \omega_2^1 + \omega_1^0 = \omega_0^0 + \omega_2^2 - 2\omega_1^1 = 0$. Let $\tilde{f} = fg$. In our lift, $\omega_0^1 = dx$ and $\omega_2^0 = (*)((y^{''})^{-\frac{2}{3}})^{'''}dx$, where $(*)$ is a nonzero term of lower order. Setting $\omega_2^0 = 0$ yields the Monge equation

$$(3.8) \qquad ((y^{''})^{-\frac{2}{3}})^{'''} = 0.$$

The computation above is a little involved, and is substantially simplified if one instead lifts such that $\omega^1 = (y'')^{-\frac{3}{2}}dx$ (this makes ω^1 have unit "affine arc-length"). The computation in affine space is carried out in detail in [2] and [14, Vol. II].

3.9 Example 2.36 continued. The actual Griffiths-Harris result mentioned in (2.36) is that for $n > 1$, hypersurfaces with $F_3 = 0$ are quadrics.

Proposition 3.10. *The fundamental forms of $v_d(X)$ are*

$$(3.11) \qquad \begin{aligned} \mathbf{FF}^k_{v_d(X)} = \ & \Sigma_{l_1+\ldots+l_d=k} c_{l_1\ldots l_d} F_{l_1} \ldots F_{l_d}, \\ & \operatorname{mod}(\Sigma_{l<k} \mathbf{FF}^l_{v_d(X)}(S^l T))|_{\ker \mathbf{FF}^{k-1}_{v_d(X)}}, \end{aligned}$$

where the $c_{l_1\ldots l_d}$ are nonzero constants.

Proof. The two main facts are

$$(3.12) \qquad \underline{d}^r(A \circ B) = \Sigma_{a+b=r}(\underline{d}^a A) \circ (\underline{d}^b B) \qquad \text{(Leibinitz rule)},$$

and

$$(3.13) \qquad \underline{d}^r A_0 = F_r.$$

Thus computing $\underline{d}^k(A_0)^d$ is just like computing the k-th derivative of a function of one variable raised to the d-th power. I.e., let $f(x)$ be a function of one variable. Then

$$(3.14) \qquad (\frac{d}{dx})^k f^d(x) = (f^d)^{(k)} = \Sigma_{l_1+,\ldots,l_d=k} c_{l_1\ldots l_d} f^{(l_1)} \ldots f^{(l_d)}.$$

In particular, all the $c_{l_1\ldots l_d}$ with $l_1 + \ldots + l_d = k$ are nonzero. One way of computing the coefficients is to use induction via the formula

$$(3.15) \qquad (f^d)^{(r)} = \Sigma_{k=0}^r \binom{r}{k} f^{(r-k)}(f^{(d-1)})^{(k)}.$$

q.e.d.

Note that as long as $p \leq d$, $|\mathbf{FF}^p_{v_d(X)}|_x = \mathbb{P}(S^pT^*)$ for all $x \in X$ because of the $F_1^pF_0^{d-p}$ term in (3.11). This property is unchanged if the higher fundamental forms of $X \subset \mathbb{P}V$ are nonzero. This observation is a generalization of the statement that at smooth points $x \in X$ there is always exactly an $(a-1)$-dimensional family of hyperplanes osculating to first order. It implies

Proposition 3.16. *Let $X^n \subseteq \mathbb{P}V = \mathbb{CP}^{n+a}$ be a variety and let $x \in X_{sm}$. For all $p \leq d$,*

$$\dim\left\{\begin{array}{c}\text{(not necessarily irreducible) hypersurfaces}\\ \text{of degree } d \text{ osculating to order } p \text{ at } x\end{array}\right\}$$
$$= \binom{(n+a+1)+(d-1)}{d} - \left\{1 + n + \binom{n+1}{2} + \ldots + \binom{n+p-1}{p}\right\}.$$

Proof.

$$\begin{aligned}\dim(\ker \mathbf{FF}^p_{v_d(X)_x})&\\ &= (\dim(S^dV^*) - \Sigma_{k=0}^{p-1}\dim(\operatorname{Image}\mathbf{FF}^k_{v_d(X)_x})\\ &\quad - \dim(\operatorname{Image}\mathbf{FF}^p_{v_d(X)_x}).\end{aligned}$$

q.e.d.

The pattern in (3.16) does not continue. For example, it is not true that the space of hypersurfaces of degree d osculating to order $d+1$ is of dimension

$$\max\left\{0, \binom{(n+a+1)+(d-1)}{d} - \left\{n + \binom{n+1}{2} + \ldots + \binom{n+(d+1)-1}{d+1}\right\}\right\}.$$

The actual dimensions of the spaces of osculating hypersurfaces depend on properties of the differential invariants of X that we will discuss shortly. However, independent of the structure of the invariants, there are *lower* bounds on the dimensions for $d+1 \leq k \leq 2d-1$.

To see this, first note that the filtration $\hat{x} \subset \hat{T} \subset V$ induces a $(2d+1)$-step filtration of S^dV. For example, the seven-step filtration of S^3V induced is

$$\begin{aligned}&\{\hat{x}^3\} \subset \{\hat{x}^2\hat{T}\} \subset \{\hat{x}^2V, \hat{x}\hat{T}^2\} \subset \{\hat{x}\hat{T}V, \hat{T}^3\}\\ &\subset \{\hat{x}V^2, \hat{T}^2V\} \subset \{\hat{T}V^2\} \subset \{V^3\},\end{aligned}$$

where $\hat{x}^3 = \hat{x} \circ \hat{x} \circ \hat{x}, \hat{x}\hat{T} = \hat{x} \circ \hat{T}$ etc...

Considering the fundamental forms dually as maps

$$\mathbf{FF}^{*k}_{v_d(X)} : S^kT^* \to N_{v_d(X)} / \operatorname{Image} \mathbf{FF}^{k-1}_{v_d(X)}$$

the image of $\mathbf{FF}^{*k}_{v_d(X)}$ lies in a quotient of the $(k+1)$-st term of the filtration of S^dV. Since the filtration has $(2d+1)$ steps, it is not possible for $\ker \mathbf{FF}^k_{v_d(X)}$ to be zero for $k \leq 2d$. This leads to *lower* bounds on $\dim(\ker \mathbf{FF}^k_{v_d(X)})$ for $d < k \leq 2d-1$. For example we have

Proposition 3.17. *Let $X^n \subseteq \mathbb{P}V = \mathbb{CP}^{n+a}$ be any variety, and $x \in X$ be any smooth point. Then*

$$\dim \left\{ \begin{array}{c} \textit{(not necessarily irreducible) hypersurfaces of} \\ \textit{degree } d \textit{ osculating to order } 2d-1 \textit{ at } x \end{array} \right\} \geq \binom{a+d-1}{d} - 1.$$

Proof. The first term in any $\mathbf{FF}^k_{v_d(X)}$ for which S^dN^* is not in the kernel is $(F_2)^d$, which appears in $\mathbf{FF}^{2d}_{v_d(X)}$. Thus $S^dN^* \subsetneq \ker \mathbf{FF}^{2d-1}_{v_d(X)}$. Finally note that $\dim S^dN^* = \binom{a+d-1}{d}$. q.e.d.

Another way to phrase the upper bounds on the dimensions of the space of osculating hypersurfaces is in terms of lower bounds on the dimensions of the osculating spaces. For example,

$$\dim \tilde{T}^2_x v_2(X) = n + \binom{n+1}{2},$$

$$\dim \tilde{T}^3_x v_2(X) \leq n + \binom{n+1}{2} + a(1+n),$$

$$\dim \tilde{T}^4_x v_2(X) \leq n + \binom{n+1}{2} + a(1+n) + \binom{a+1}{2} \tag{3.18}$$

$$= \dim \mathbb{P}S^2V,$$

and

$$\dim \tilde{T}^{(3)}_x v_3(X) = n + \binom{n+1}{2} + \binom{n+2}{3},$$

$$\dim \tilde{T}^{(4)}_x v_3(X) \leq n + \binom{n+1}{2} + \binom{n+2}{3} + a(1+n),$$

$$\dim \tilde{T}^{(5)}_x v_3(X) \leq n + \binom{n+1}{2} + \binom{n+2}{3} + a(1+n) + \binom{a+1}{3},$$
$$\begin{aligned}\dim \tilde{T}^{(6)}_x v_3(X) &\leq n + \binom{n+1}{2} + \binom{n+2}{3} + a(1+n)\\ &\quad + \binom{a+1}{3} + \binom{a+1}{3}\\ &= \dim \mathbb{P}S^3V.\end{aligned}$$

We now rephrase (1.6) in a manner that will allow us to use projective differential invariants to approximate it and interpret it geometrically.

There is a natural codimension a subspace of $N^*_{v_d(X)}$ namely $(N\hat{x}^{d-1})^{\perp}$, which in particular contains all the hypersurfaces in I_d that are singular at x.

Proposition 3.19. *Let $x \in X \subset \mathbb{P}V$ be a smooth point. X is a complete intersection if and only if for all k,*

$$I_k \cap (N\hat{x}^{k-1})^{\perp} \equiv 0 \mod (I_{k-1} \circ V^*) \cap (N\hat{x}^{k-1})^{\perp}.$$

Proof. By (1.6), it is sufficient to show that

$$\ker[d]_k = I_k \cap (N\hat{x}^{k-1})^{\perp} \operatorname{mod}(I_{k-1} \circ V^*) \cap (N\hat{x}^{k-1})^{\perp}. \tag{3.20}$$

Let (x^0, x^α, x^μ) be a basis of V^* (which we also think of as linear coordinates on V) such that $[1, 0, ..., 0] = x$, and $\hat{T} = \{x^\mu = 0\}$. Then N^*_x is spanned by $\{dx^\mu\}$. Expressed in terms of our basis, elements of $(N\hat{x}^{k-1})^{\perp}$ have no $x^\mu (x^0)^{k-1}$ terms. If $P \in I_k$ is such that $dP_x \neq 0$, then $P = y(x^0)^{k-1} + ...$ where $y = c_\mu x^\mu$ for some constants c_μ, and therefore $P \notin (N\hat{x}^{k-1})^{\perp}$. If $[dP_x] = 0$, but $dP_x \neq 0$, then $P = x^0 P' + P''$ where $P' \in I_{k-1}$ and $dP''_x = 0$. q.e.d.

One can define conditions that imply the condition $(CI)_d$ by requiring that for some fixed k, the analog of (3.19) holds for the hypersurfaces of degree d osculating to order k. More precisely, that

$$\ker \mathbf{FF}^k_{v_d(X)} \cap (N\hat{x}^{d-1})^{\perp} \equiv 0 \operatorname{mod}(I_{d-1} \circ V^*) \cap (N\hat{x}^{d-1})^{\perp}. \tag{3.21.k}$$

Definition 3.22. Let $X \subseteq \mathbb{P}V$ be a variety and $x \in X_{sm}$. We will say *X is known to have no excess equations of degree d after taking k derivatives at x*, or that *X satisfies $(CI)^k_d$ at x*, if (3.21.k) holds at x.

We will work with the condition $(CI)^{2d}_d$. By (3.10), $k = 2d$ is the smallest possible k for which (3.21.k) could hold. We have no reason to believe this is the "correct" condition but it provides a useful starting point. (By "correct" we mean that it will be possible to prove some global theorem that forces the condition to hold under suitable codimension and smoothness hypotheses about X.) We will require these restrictions hold at *general points* instead of all smooth points. This will enable us to compute using only a fixed, finite number of derivatives.

Proposition 3.23 (generalized Monge systems). *Fix positive integers $d_1 < d_2 < ... < d_r$ and $a_1, a_2, ..., a_r$, where $a_1 + ... + a_r = a$.*

If $X^n \subset \mathbb{P}V = \mathbb{CP}^{n+a}$ is a complete intersection and $x \in X$ is a general point such that for all d_j and for all k such that $d_{j-1} < k \leq d_j$ (set $d_0 = 0$),

$$\dim\{\ker \mathbf{FF}^k_{v_k(X)}\} \operatorname{mod}(N\hat{x}^{d_j-1})^{\perp} = a_1 + ... + a_j \tag{3.24}$$

and

$$(CI)^{2k}_k \text{ holds at } x, \tag{3.25}$$

then X is a complete intersection of a_1 hypersurfaces of degree d_1, a_2 hypersurfaces of degree d_2, ... , a_r hypersurfaces of degree d_r.

Moreover, $\{dP_x | P \in \ker \mathbf{FF}^{2d_j}_{v_{d_j}(X)}\} = N^*_{d_j}$ where $N^*_{d_j}$ is the j-th term in the natural filtration of $N^*_x X$ (1.4).

Proof. The assumption of $(CI)^{2k}_k$ at x implies that the only possible elements of $\ker \mathbf{FF}^{2k-1}_{v_k(X),x}$ correspond to smooth hypersurfaces. (3.24) implies $\mathbf{FF}^{2d_j+1}_{v_d(X),x} = 0$ so $I_d = \ker \mathbf{FF}^{2d_j}_{v_d(X),x}$. Furthermore, (3.24) implies $\dim(I_{d_j}/(I_{d_j-1} \circ V^*)) = a_j$. q.e.d.

Proposition (3.23) needs some explanation. (3.24) may be understood as systems of partial differential equations. In fact, if one writes out the differential invariants in terms of local coordinates, then (3.24) is a system of PDE of order $2d_r + 1$ which specializes to the classical Monge equation (3.6) when $n = a = 1$ and $d = 2$.

The conditions $(CI)^{2d_j}_{d_j}$ expressed in terms of local coordinates are also conditions on the derivatives, but at least in small codimension, they are open conditions and one might hope to prove that the open conditions must be satisfied in certain geometric situations. The proposition implies:

Among complete intersections that satisfy suitable genericity conditions, if d is the largest degree of a hypersurface essentially containing X, then the ideal of X can be recovered by taking $2d + 1$ derivatives at a general point.

In practice, it is actually useful to work with a stronger condition than $(CI)^{2d}_d$. To understand the stronger condition, interpret $(CI)^{2d}_d$ as saying that $\ker \mathbf{FF}^{2d}_{v_d(X)}$ is as small as possible modulo the differentials of the polynomials that are smooth at x. The stronger condition will be for $d < k \leq 2d$, that $\ker \mathbf{FF}^k_{v_d(X)}$ be as small as possible modulo the differentials of the polynomials that are smooth at x. In this paper we will only discuss this stronger condition in the case $d = 2$ and will explain it in the next section.

4. Osculating quadrics

We now discuss the quadrics containing $X \subset \mathbb{P}V$ in detail. We assume throughout this section that $III_X = 0$ at general points.

Consider the fundamental forms of $v_2(X)$ at a point x restricted to the conormal directions corresponding to the osculating quadrics singular at x. Using the notation $T = \hat{T}/\hat{x}, N = V/\hat{T}$, we have

$$
\begin{aligned}
&\mathbf{FF}^1_{v_2(X)} : T^*\hat{x}^* \to T^* \qquad \text{(Identity on the first factor)},\\
&\mathbf{FF}^2_{v_2(X)}|_{(N\hat{x})^\perp} : S^2T^* \to S^2T^* \quad \text{(Identity map)},\\
&\mathbf{FF}^3_{v_2(X)}|_{(N\hat{x})^\perp} : N^*T^* \to S^3T^* \quad (II \circ \text{Identity map}),\\
&\mathbf{FF}^4_{v_2(X)}|_{(N\hat{x})^\perp} : (\ker \mathbf{FF}^3_{v_2(X)}|_{(N\hat{x})^\perp} \cap N^*T^*) + N^*N^* \to S^4T^*\\
&\qquad\qquad (F_3 \circ \text{ Identity map } + II \circ II).
\end{aligned}
\tag{4.1}
$$

$\ker \mathbf{FF}^3_{v_2(X)}|_{(N\hat{x})^\perp}$ will be as small as possible if

$$F_2F_1 : (NT)^* \to S^3T^* \tag{4.2}$$

is injective, and assuming (4.2) is injective, $\ker \mathbf{FF}^4_{v_2(X)}|_{(N\hat{x})^\perp}$ will be empty if

$$F_2F_2 : N^{2*} \to S^4T^* \tag{4.3}$$

is also injective.

We will say that X satisfies *strong genericity in degree two* at x if (4.2) and (4.3) are injective. One can similarly define strong genericity in degree d at x, but it is more complicated because one needs to take I_{d-1} into account.

It will be useful to study (4.2), (4.3) in bases. Let (x^0, x^α, x^μ) be an adapted basis of V^*, dual to (A_0, A_α, A_μ). We also think of (x^0, x^α, x^μ) as linear coordinates on V.

The induced basis of S^2V^* is $(x^0x^0, x^\alpha x^0, x^\mu x^0, x^\alpha x^\beta, x^\mu x^\beta, x^\mu x^\nu)$. (3.4) implies

$$
\begin{aligned}
\ker \mathbf{FF}^0_{v_2(X)} &= \{x^\alpha x^0, x^\mu x^0, x^\alpha x^\beta, x^\mu x^\beta, x^\mu x^\nu\},\\
\ker \mathbf{FF}^1_{v_2(X)} &= \{x^\mu x^0, x^\alpha x^\beta, x^\mu x^\beta, x^\mu x^\nu\},\\
\ker \mathbf{FF}^2_{v_2(X)} &= \{x^\mu x^0 - q^\mu_{\alpha\beta} x^\alpha x^\beta, x^\mu x^\beta, x^\mu x^\nu\}.
\end{aligned}
\tag{4.4}
$$

Beyond this, the spaces of osculating quadrics depend on the structure of the invariants of X.

$\mathbf{FF}^3_{v_2(X)}$ maps its domain as follows:

$$\begin{aligned} x^\mu x^0 - q^\mu_{\alpha\beta} x^\alpha x^\beta &\mapsto 2F^\mu_3, \\ x^\mu x^\gamma &\mapsto 6F^\mu_2 \omega^\gamma, \\ x^\mu x^\nu &\mapsto 0, \end{aligned} \tag{4.5}$$

where we have written $F^\mu_3 = r^\mu_{\alpha\beta\gamma}\omega^\alpha\omega^\beta\omega^\gamma$ etc ... (Here, since we have chosen a particular frame, the coefficients $r^\mu_{\alpha\beta\gamma}$ etc... are well defined).

The failure of (4.2) to be injective means that there exists a nontrivial equation $l_\mu q^\mu = 0$ with $l_\mu \in T^*$, $q^\mu \in |\widehat{II_x}|$. We will call such an equation a *linear syzygy* in II_x.

Similarly, (4.3) will fail to be injective if there exists a nontrivial equation $k_{\mu\nu} q^\mu q^\nu = 0$ with $k_{\mu\nu} = k_{\nu\mu} \in \mathbb{C}$. We will refer to such an equation as a *quadratic relation among the quadrics in* $|II_X|$.

In fact, requiring (4.3) to be injective is redundant:

Lemma 4.6. *Let T be a vector space. If a system of quadrics $A \subseteq S^2T^*$ satisfies a quadratic relation among the quadrics in A, i.e., an equation of the form $c_{\mu\nu}q^\mu q^\nu = 0$ with q^μ a collection of independent elements of A and $c_{\mu\nu} = c_{\nu\mu}$ constants (not all zero), then A has linear syzygies.*

Proof. Let $v \in T$. Then

$$v \rfloor (c_{\mu\nu} q^\mu q^\nu) = 0,$$

i.e.,

$$c_{\mu\nu}(v \rfloor q^\mu) q^\nu = 0,$$

which is a nontrivial linear syzygy with $l_\nu = c_{\mu\nu}(v \rfloor q^\mu)$ as long as $v \notin \operatorname{Singloc}(c_{\mu\nu}q^\mu)$ for some ν. Since not all the quadrics $c_{\mu\nu}q^\mu$ are zero, we can always find such a v.

In summary, we have

Proposition 4.7. *Let $X \subset \mathbb{P}V$ be a variety. Let $x \in X$ be a general point and assume $III_{X\,x} = 0$ and $II_{X\,x}$ has no linear syzygies. Then X satisfies $(CI)_2$ at x, i.e., X has no excess equations in degree 2 at x.*

Remark 4.8. One can compare (4.7) with the following two varieties: The twisted cubic curve $C \subset \mathbb{P}^3$ is cut out by quadrics and is not a complete intersection, but $III_C \neq 0$. The Segre $X = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, or any generic hyperplane section of it, is also cut out by quadrics and not a complete intersection, but there is a linear syzygy among the quadrics in II_X at any point due to the point in the base locus (see 5.4 for more details).

We now derive a more refined version of (3.23) for intersections of quadrics.

In order that N_x^* be spanned by differentials of quadratic polynomials, it is necessary that

$$\{dP_x | P \in \ker \mathbf{FF}^k_{v_2(X)}\} = N_x^* \tag{4.9.k}$$

for all k. (We supress reference to the base point x in what follows.) For $k \leq 2$, (4.9.k) automatically holds; for $k = 3$, (4.9.3) will hold if and only if

$$F_3^\mu = 3a^\mu_{\nu\gamma}\omega^\gamma F_2^\nu \tag{4.10}$$

for some constants $\alpha^\mu_{\nu\gamma} \in \mathbb{C}$. Notice that if $r^\mu_{\alpha\beta\gamma} = \mathfrak{S}_{\alpha\beta\gamma} a^\mu_{\nu\gamma} q^\nu_{\alpha\beta}$ in some frame, it holds in any choice of frame (with different constants $a^\mu_{\nu\gamma}$), so the expression (4.10) has intrinsic meaning. If (4.8) holds, then

$$\ker \mathbf{FF}^3_{v_2(X)} = \{x^\mu x^0 - q^\mu_{\alpha\beta}x^\alpha x^\beta - \alpha^\mu_{\nu\beta}x^\nu x^\beta, x^\mu x^\nu\}. \tag{4.11}$$

Assuming (4.10), and that there are no linear syzygies in II_X, $\mathbf{FF}^4_{v_2(X)}$ maps its domain as follows:

$$\begin{aligned} x^\mu x^0 - q^\mu_{\alpha\beta}x^\alpha x^\beta - a^\mu_{\nu\alpha}x^\nu x^\alpha & \\ \mapsto 2F_4^\mu - 8a^\mu_{\nu\alpha}\omega^\alpha F_3^\nu, & \\ x^\mu x^\nu \mapsto 6F_2^\mu F_\nu^2. & \end{aligned} \tag{4.12}$$

(4.9.4) is the condition

$$F_4^\mu = 4a^\mu_{\nu\alpha}\omega^\alpha F_3^\nu + 3b^\mu_{\nu\tau}F_2^\nu F_2^\tau \tag{4.13}$$

for some constants $b^\mu_{\nu\tau} = b^\mu_{\tau\nu} \in \mathbb{C}$ (this expression also has intrinsic meaning, in that if it holds in one choice of frame, it will hold in all choices). If (4.10), (4.13) hold and there are no linear syzygies in II_X, then

$$\ker \mathbf{FF}^4_{v_2(X)} = \{x^\mu x^0 - q^\mu_{\alpha\beta}x^\alpha x^\beta - a^\mu_{\nu\gamma}x^\nu x^\gamma - b^\mu_{\nu\tau}x^\nu x^\tau\}. \tag{4.14}$$

$\mathbf{FF}^5_{v_2(X)}$ maps its domain as follows:

$$\begin{aligned} x^\mu x^0 - q^\mu_{\alpha\beta}x^\alpha x^\beta - a^\mu_{\nu\gamma}x^\nu x^\gamma - b^\mu_{\nu\tau}x^\nu x^\tau & \\ \mapsto 2F_5^\mu - 10a^\mu_{\nu\gamma}\omega^\gamma F_4^\nu - 20b^\mu_{\nu\tau}F_\nu^3 F_2^\tau, & \end{aligned} \tag{4.15}$$

(4.9.5) is the condition

$$F_5^\mu = 5a^\mu_{\nu\gamma}\omega^\gamma F_4^\nu + 10b^\mu_{\nu\tau}F_3^\nu F_2^\tau \tag{4.16}$$

(which also has intrinsic meaning).

If (4.10),(4.13),(4.16) and strong genericity all hold, then $\mathbf{FF}^5_{v_2(X)} = 0$. Since we are at a general point, this implies all higher fundamental forms are zero and $I_2 = \ker \mathbf{FF}^4_{v_2(X)}$.

In summary: if

$$\begin{aligned} F_3^\mu &= 3a^\mu_{\nu\gamma}\omega^\gamma F_2^\nu, \\ F_4^\mu &= 4a^\mu_{\nu\alpha}\omega^\alpha F_3^\nu + 3b^\mu_{\nu\tau}F_2^\nu F_2^\tau, \\ F_5^\mu &= 5a^\mu_{\nu\gamma}\omega^\gamma F_4^\nu + 10b^\mu_{\nu\tau}F_3^\nu F_2^\tau, \end{aligned} \tag{4.17}$$

where $a^\mu_{\nu\alpha}, b^\mu_{\nu\tau} = b^\mu_{\tau\nu} \in \mathbb{C}$, and strong genericity in degree 2 holds at x, then N^*_x is spanned by the differentials of a set of generators of I_2, i.e., the only hypersurfaces that essentially contain X are of degree two. In this case, we will call (4.17) the *generalized Monge system for quadrics.*

In summary, we have

Theorem 4.18. *Let $X \subset \mathbb{P}V$ be a variety and $x \in X$ a general point. Assume $III_{X_x} = 0$ and that there are no linear syzygies in $|II_X|_x$. Then*

(4.19)

$$\begin{aligned} &\dim\{\textit{quadrics osculating to order three at } x\} \\ &\qquad \le a + \binom{a+1}{2} - 1, \\ &\dim\{\textit{quadrics osculating to order four at } x\} \le a - 1. \end{aligned}$$

If the generalized Monge system (4.17) *holds, then*

$$\begin{aligned} I_2 = \ker \mathbf{FF}^4_{v_2(X)x} = \{&x^\mu x^0 - q^\mu_{\alpha\beta}x^\alpha x^\beta - a^\mu_{\nu\gamma}x^\nu x^\gamma \\ &- b^\mu_{\nu\tau}x^\nu x^\tau,\ n+1 \le \mu \le n+a\}, \end{aligned}$$

where $q^\mu_{\alpha\beta}$ are the coefficients of the second fundamental form at x, and $a^\mu_{\nu\alpha}, b^\mu_{\nu\tau}$ are coefficients expressing F_3, F_4 in terms of F_2.

Equality occurs on the first (respectively second) expression of (4.19) *if and only if the first (resp. second) line of* (4.17) *holds at x. If the generalized Monge system does not hold, then I_X is not generated by quadrics.*

Theorem (4.18) implies:

Corollary 4.20. *Let $X \subset \mathbb{P}V$ be a variety and $x \in X$ a general point. Assume $III_{X_x} = 0$. If there are no quadric hypersurfaces singular at x that osculate to order four at x, and Q is a quadric hypersurface osculating to order five at x, then $X \subseteq Q$.*

Remark 4.21. The assumption of small codimension (which is implicit in the hypotheses that $\mathbf{FF}^3_{X_x} = 0$ and that there are no linear syzygies in $|II_X|_x$) is essential to being able to determine the quadratic equations of X by taking only five derivatives. One already needs more derivatives for a curve in $\mathbb{P}^3$.

Corollary 4.22. *X as in (4.18) is determined by II_X, F_3, F_4 at one general point. In fact, the higher variations of II are given by the formula:*

$$\begin{aligned}(4.23)\qquad c_{k0}F^\mu_k =&\frac{1}{2}\left(c_{(k-1)1}a^\mu_{\nu\alpha}\omega^\alpha F^\nu_{k-1}\right.\\&\left.+b^\mu_{\nu\tau}(\Sigma_{l+m=k}c_{lm}F^\nu_l F^\tau_m)\right)\ \forall k>2.\end{aligned}$$

Example 4.24. Let $X^6 \subset \mathbb{P}^{12}$, let $x \in X$ be a smooth point and let

$$\begin{aligned}(4.25)\qquad II_{Xx} =&\omega^1\omega^4 A_7 + \omega^2\omega^5 A_8 + \omega^3\omega^6 A_9 + \omega^1\omega^5 A_{10}\\&+\omega^2\omega^6 A_{11} + \omega^3\omega^4 A_{12}.\end{aligned}$$

Say in addition that $F_3 = F_4 = 0$. Then I_X is generated by

(4.26)

$$\begin{aligned}&\{x^0x^7 - x^1x^4, x^0x^8 - x^2x^5, x^0x^9 - x^3x^6, x^0x^{10}\\&- x^1x^5, x^0x^{11} - x^2x^6, x^0x^{12} - x^3x^4, x^7x^8x^9 - x^{10}x^{11}x^{12}\}.\end{aligned}$$

The presence of relations or syzygies among the quadrics in $|II_X|$ can produce equations of higher degrees, that are not elements of $I_2 \circ S^kV^*$, as in (4.26). Not all syzygies and relations actually produce such equations, there are tautological ones, which we will call the *Koszul syzygies.* For example, let $|II_X| = \{Q^\mu\}$, where $Q^\mu = q^\mu_{\alpha\beta}\omega^\alpha\omega^\beta$. For each $\mu < \nu$ there are Koszul syzygies

$$(4.27)\qquad q^\nu_{\alpha\beta}\omega^\alpha\omega^\beta Q^\mu - q^\mu_{\alpha\beta}\omega^\alpha\omega^\beta Q^\nu = 0.$$

Theorem 4.28. *Let $X \subset \mathbb{P}V$ be a variety. A sufficient condition for I_X to be generated by quadrics is that at a smooth point $x \in X$ the following hold:*

1. $\{dP_x | P \in I_2\} = N^*_x$.

2. Any sygyzies or polynimials satisfied by $|II_X|_x$ of the form

$$l_{\mu_1,\dots,\mu_k,\alpha_1,\dots,\alpha_p}q^{\mu_1}\dots q^{\mu_k}\omega^{\alpha_1,\dots,\alpha_p} = 0,$$

where $l_{\mu_1,\dots,\mu_k,\alpha_1,\dots,\alpha_p} \in \mathbb{C}$, other than the Koszul syzygies, are generated by the linear syzygies and the quadratic relations among the quadrics in $|II_X|_x$.

In particular, if $X \subset \mathbb{P}V$ is a variety, $x \in X$ is a general point, there are no linear syzygies in $|II_X|_x$, no polynomials satisfied by the quadrics in $|II_X|_x$, and the generalized Monge system (4.17) holds, then I_X is generated by quadrics; in fact X is a complete intersection of quadrics.

Proof. Let (x^0, x^α, x^μ) be an adapted basis of V^* dual to $(A_0, A\alpha, A\mu)$.

Consider the case of cubics containing X. Condition 1 implies that I_2 contains quadrics of the form

$$x^\mu x^0 + ... \tag{4.29}$$

for each $n+1 \le \mu \le n+a$, so $I_2 \circ V^*$ has cubics of the form

$$x^B(x^\mu x^0 + ...) \tag{4.30}$$

$\forall 0 \le B \le n+a$. Given $P \in I_3$, we may modify P by elements of $I_2 \circ V^*$ so that we may assume

$$P(A_{B\mu 0}) = 0 \, \forall 0 \le B \le n+a. \tag{4.31}$$

That P vanishes to all orders on X implies that $P(\mathbf{FF}^k_{v_3(X)}) = 0 \, \forall k$, where we have considered the pairing $S^3V^* \times \mathbf{FF}^k_{v_3(X)} \to S^kT^*$. In particular,

$$\begin{aligned}
P(\mathbf{FF}^0_{v_3(X)}) &= 0 \Rightarrow P(A_{000}) = 0,\\
P(\mathbf{FF}^1_{v_3(X)}) &= 0 \Rightarrow P(A_{\alpha 00}) = 0 \quad \forall \alpha,\\
P(\mathbf{FF}^2_{v_3(X)}) &= 0 \Rightarrow P(A_{\alpha\beta 0}) = 0 \quad \forall \alpha, \beta,\\
P(\mathbf{FF}^3_{v_3(X)}) &= 0 \Rightarrow P(A_{\alpha\beta\gamma}) = 0 \quad \forall \alpha, \beta, \gamma,\\
P(\mathbf{FF}^4_{v_3(X)}) &= 0 \Rightarrow P(\mathfrak{S}_{\alpha\beta\gamma\delta} q^\mu_{\alpha\beta} A_{\mu\gamma\delta})\\
&= \mathfrak{S}_{\alpha\beta\gamma\delta} q^\mu_{\alpha\beta} P(A_{\mu\gamma\delta}) = 0 \quad \forall \alpha, \beta, \gamma, \delta,\\
P(\mathbf{FF}^5_{v_3(X)}) &= 0 \Rightarrow P(\mathfrak{S}_{\alpha\beta\gamma\delta\epsilon} q^\mu_{\alpha\beta} q^\nu_{\gamma\delta} A_{\mu\nu\epsilon})\\
&= 0 \quad \forall \alpha, \beta, \gamma, \delta, \epsilon,\\
P(\mathbf{FF}^6_{v_3(X)}) &= 0 \Rightarrow P(\mathfrak{S}_{\alpha_1 ... \alpha_6} q^\mu_{\alpha_1\alpha_2} q^\nu_{\alpha_3\alpha_4} q^\tau_{\alpha_5\alpha_6} A_{\mu\nu\tau})\\
&= 0 \quad \forall \alpha_1 ... \alpha_6,
\end{aligned} \tag{4.32}$$

where on each line we have used the line above and (4.31) to reduce to only having one nonzero term to worry about. Using (4.32) and the absense of polynomials and linear syzygies, we see that the modified P is zero, i.e., that $I_3 \equiv 0 \bmod I_2 \circ V^*$.

The proof for equations of degree d is the same. One may assume that P contains no terms of the form $x^{B_1} ... x^{B_{d-2}} x^\mu x^0$, and use the first $2d+1$ fundamental forms of $v_d(X)$. q.e.d.

We now explain what happens in the special case of (4.17) when $F_3 = F_4 = 0$:

Theorem 4.33. *Let $X \subset \mathbb{P}V$ be a variety and $x \in X$ a general point. Assume $III_{X_x} = 0$. If $F_3 = F_4 = 0$ at x, then the minimal number of generators of I_X is a + the minimal number of generators of* *syzygies and polynomials in* $|II_X|_x$ *modulo the Koszul syzygies. Furthermore, either* X *is a smooth homogeneous space or* $X_{sing} \subset H$*, where* H *is a hyperplane, and* $X \backslash (X \cap H)$ *is a homogeneous variety in the affine space* $\mathbb{P}V \backslash H$*. In either case* X *is birationally equivalent to* $\mathbb{P}^n$*, and* I_X *is generated by quadrics if and only if all the syzygies and polynomials of* $|II_X|$ *are generated by the linear syzygies and quadratic relations among the quadrics in* $|II_X|$ *and the Koszul syzygies and relations.*

Proof. X is given by the following construction (see [11], [9]):

Fix any smooth point $x \in X$. Let $Y = \text{Baseloc}|II_X|_x \subset \mathbb{P}T_x$. Linearly embed $T \subset \mathbb{C}^{n+1}$. Let $y^0 ... y^n$ be linear coordinates on $\mathbb{C}^{n+1}$ such that $T = \{y^0 = 0\}$. Consider the rational map:

$$(4.34) \quad Bl_Y\mathbb{P}^n - - - \rightarrow \mathbb{P}\{\widehat{|II_X|}, y^0 \circ T^*\}^* \subseteq \mathbb{P}\{\mathcal{O}_{\mathbb{P}^n} \otimes \mathcal{I}_Y\} \subset \mathbb{P}(S^2\mathbb{C}^{n+1}).$$

To see that the image of this map is isomorphic to X, note that it has the correct codimension and that all differential invariants at x are the same (only II_X is nonzero for both X and the image). Thus the varieties are isomorphic.

Remark 4.35. If X is as in (4.33), let $\mathbb{P}^{a-1} = \{x^0 = 0, x^\alpha = 0\}$. X will be singular in the space $X \cap \mathbb{P}^{a-1}$ unless the number of generators of syzygies and polynomials modulo Koszul is at least $a - 1$. In fact, if there are no polynomials among the quadrics, then $\mathbb{P}^{a-1} \subset X$ (see (4.39)).

4.36. Example 2.37 continued. (2.39) may be rephrased as $F_4 = b^\mu_{\nu\tau} F^\nu_2 F^\tau_2$. After normalizing $b^{n+1}_{(n+1)(n+1)} = b^{n+2}_{(n+2)(n+2)} = 0$, all our effective normalizations that keep $F_3 = 0$ are used up and the equations of X are

$$\begin{aligned}\{&x^{n+1}x^0 - \Sigma_\alpha x^\alpha x^\alpha - b^{n+1}_{(n+1)(n+2)} x^{n+1} x^{n+2} - b^{n+1}_{(n+2)(n+2)} x^{n+2} x^{n+2}, \\ &x^{n+2}x^0 - \Sigma_\alpha \lambda_\alpha x^\alpha x^\alpha - b^{n+2}_{(n+1)(n+1)} x^{n+1} x^{n+1} - b^{n+2}_{(n+1)(n+2)} x^{n+1} x^{n+2}\}.\end{aligned}$$

Note that since it is impossible to normalize both the $a^\mu_{\nu\alpha}$'s and the $b^\mu_{\nu\tau}$'s to zero simultaneously, X cannot be rational.

Example 4.37. We describe the equations of varieties as in (4.33) whose ideals are generated by quadrics. Write the map (4.34) as

$$x \mapsto [y^0y^0, y^0y^1, ..., y^0y^n, Q^1(x), ..., Q^a(x)] = [x^0, ..., x^n, ..., x^{n+a}].$$

Say there are r linear syzygies $k^l_{\mu\alpha} x^\alpha Q^\mu$, $1 \leq l \leq r$, and s quadratic relations among the quadrics in $|II_X|$, $k^m_{\mu\nu} Q^\mu Q^\nu$, $1 \leq m \leq s$ and assume they and the Koszul syzygies generate the space of syzygies and relations. The $a+r+s$ equations of X are

$$\{x^0x^\mu - q^\mu_{\alpha\beta}x^\alpha x^\beta,\ k^l_{\mu\alpha}x^\mu x^\alpha,\ k^m_{\mu\nu}x^\mu x^\nu\}. \tag{4.38}$$

4.39. An apparent dichotomy for varieties cut out by quadrics.

It is extremely difficult for a variety $X \subset \mathbb{P}V$ with $III_{X x} = 0$ at general points $x \in X$, and satisfying (4.17) to have extra equations without X being rational. For example, say there is a linear syzygy of the form

$$k_{\mu\alpha}\omega^\alpha F_2^\mu = 0, \tag{4.40}$$

where $k_{\mu\alpha} \in \mathbb{C}$. In order for this linear syzygy to produce an extra equation, the constants $a^\mu_{\nu\alpha}, b^\mu_{\nu\tau}$ must satisfy the equations

$$\begin{aligned} &k_{\mu\alpha}\omega^\alpha a^\mu_{\nu\beta}\omega^\beta F_2^\nu = 0, \\ &k_{\mu\alpha}\omega^\alpha(a^\mu_{\nu\beta}\omega^\beta a^\nu_{\tau\gamma}\omega^\gamma F_2^\tau + b^\mu_{\nu\tau}F_2^\nu F_2^\tau) = 0, \\ &k_{\mu\alpha}\omega^\alpha(a^\mu_{\nu\beta}\omega^\beta(a^\nu_{\tau\gamma}\omega^\gamma a^\tau_{\sigma\delta}\omega^\delta F_2^\sigma + b^\nu_{\tau\sigma}F_2^\tau F_2^\sigma) \\ &\qquad\qquad + b^\mu_{\nu\tau}a^\nu_{\sigma\delta}\omega^\delta F_2^\sigma F_2^\tau) = 0, \end{aligned} \tag{4.41}$$

which are severely overdetermined, on the order of n^5 equations for the $a^2n + \binom{a+1}{2}a$ coefficients $a^\mu_{\nu\alpha}, b^\mu_{\nu\tau}$.

Question 4.42. *Let X be a variety and let $x \in X$ be a general point. If I_X is generated by quadrics and $\mathbf{FF}^3_{X x} = 0$, is it necessarily the case that $X = Z_1 \cap Z_2$ where Z_1 is a complete intersection and Z_2 is rational, and both I_{Z_1}, I_{Z_2} are generated by quadrics?*

Remark 4.43. Note that if $III_X \neq 0$ then equations (4.41) are replaced with a much less overdetermined system, so the validity of the question is heavily dependent on $III_X = 0$. If $a \leq \frac{1}{2}[n-(b+1)]$, then we are guaranteed $III_X = 0$ by [9, (4.15)].

We will show the answer to question (4.42) is *yes* in the codimension range $a < \frac{1}{3}[n-(b+1)+2]$ in §6. If the answer to (4.42) is no, then (4.41) gives a guide to all potential counter-examples to Hartshorne's conjecture on complete intersections whose ideals are generated by quadrics. The first intersesting case is for 11-folds in $\mathbb{P}^{15}$.

The system one must solve to get a non-complete intersection is as follows:

Let $1 \leq i,j,k \leq r$, $r+1 \leq s,t \leq n$. One must find constants

$$q^{n+i}_{jk},\ q^{n+i}_{js},\ a^{n+i}_{n+j,k},$$

$a^{n+i}_{n+j,s}, b^{n+i}_{n+j,n+k}$ as follows: Let $r^{n+i}_{\alpha\beta\gamma} = \mathfrak{S}_{\alpha\beta\gamma} a^{n+i}_{\mu\alpha} q^{\mu}_{\beta\gamma}$ etc... be given as in (4.17). One needs

$$\begin{aligned} &\mathfrak{S}_{ijk} q^{n+i}_{jk} = 0,\ q^{n+i}_{js} + q^{n+j}_{is} = 0, \\ &\mathfrak{S}_{ijkl} r^{n+i}_{jkl} = 0,\ \mathfrak{S}_{ijk} r^{n+i}_{jks} = 0,\ r^{n+i}_{jst} + r^{n+j}_{ist} = 0, \end{aligned} \tag{4.44}$$

and the analogous equations for F_4 and F_5 to hold. If one is looking for smooth non-complete intersections, one also needs to check smoothness, which amounts to genericity conditions on $a^{\mu}_{\nu\alpha}, b^{\mu}_{\nu\tau}$.

Proposition 4.45. *The answer to* (4.42) *is yes for surfaces in* $\mathbb{P}^4$ *and* $\mathbb{P}^5$ *and for* 3*-folds in* $\mathbb{P}^5$.

Proof. Case of surfaces in $\mathbb{P}^4$. $|II_X|$ must consist of two quadrics if $III_X = 0$. In order for X to fail to be a complete intersection, there must be a linear syzygy in $|II_X|$. This implies that we can choose frames such that $II_X = \omega^1\omega^1 \otimes A_3 + \omega^1\omega^2 \otimes A_4$ in a neighborhood of our general point x. The variablility in the coefficients of F_3 is as follows:

$$\begin{array}{ll} \Delta r^3_{111} = 3g^1_3 + g^0_1, & \Delta r^4_{111} = 3g^2_3, \\ \Delta r^3_{112} = 2g^1_4 + g^0_2, & \Delta r^4_{112} = g^1_3 + g^2_4 + g^0_1, \\ \Delta r^3_{122} = 0, & \Delta r^4_{122} = g^1_4 + g^0_2, \\ \Delta r^3_{222} = 0, & \Delta r^4_{222} = 0. \end{array} \tag{4.46}$$

Using g^{α}_{μ}, set $r^3_{111}, r^3_{112}, r^4_{111}, r^4_{112} = 0$. If I_X is generated by quadrics, (4.17) implies

$$\begin{aligned} &r^3_{111} = 3a^3_{31},\ r^3_{112} = a^3_{32} + 2a^3_{41},\ r^3_{122} = 2a^3_{42}, r^3_{222} = 0, \\ &r^4_{111} = 3a^4_{31},\ r^3_{112} = a^4_{32} + 2a^3_{41},\ r^4_{122} = 2a^4_{42}, r^3_{222} = 0, \end{aligned} \tag{4.47}$$

which implies $a^3_{31} = a^3_{32} + 2a^3_{41} = a^3_{42} = a^4_{31} = a^4_{32} + 2a^4_{41} = 0$ in these frames.

To have the syzygy persist, the relation $\omega^1 F^4_3 - \omega^2 F^3_3 = 0$ must also hold, which forces all the $a^{\mu}_{\nu\alpha} = 0$ and thus $F_3 = 0$.

Similarly, if I_X is generated by quadrics, then $F^{\mu}_4 = b^{\mu}_{\nu\tau} F^{\nu}_2 F^{\tau}_2$ which, after normalizing by g^0_3, g^0_4 and requiring $\omega^1 F^4_4 - \omega^2 F^3_4 = 0$, implies $F_4 = 0$.

The case of a 3-fold in $\mathbb{P}^5$ is similar; only there are two possibilities for $|II_X|$, namely $\{\omega^1\omega^1, \omega^1\omega^3\}$ and $\{\omega^1\omega^2, \omega^1\omega^3\}$. The case of a surface in $\mathbb{P}^5$ was proved in [10] (assuming even less than we assumed here).

5. Some homogeneous examples

In this section we write out the equations of some homogeneous varieties in a manner that illuminates the computations of §4.

Example 5.1, The Veronese. $v_2(\mathbb{P}W) \to \mathbb{P}S^2W$.

Let W have basis (B_0, B_α), $1 \leq \alpha, \beta \leq n$. Let $A_0 = B_0 \circ B_0, A_\alpha = B_0 \circ B_\alpha$. Let $\mu = (\alpha, \beta)$, $\alpha \leq \beta$ index the normal directions, i.e., $A_\mu = B_\alpha \circ B_\beta$. If $\gamma \leq \delta$ we may adapt frames such that

$$II_X = \omega^\alpha \omega^\beta A_{(\alpha\beta)} \operatorname{mod}\{A_0, A_\alpha\}, \tag{5.2}$$

i.e., $q^{(\alpha,\beta)}_{\gamma\delta} = \delta^\alpha_\gamma \delta^\beta_\delta$. The equations are

$$\begin{gathered} \{x^0 x^{(\alpha,\beta)} - x^\alpha x^\beta, x^{(\alpha,\beta)} x^\gamma - x^{(\alpha,\gamma)} x^\beta - x^{(\beta,\gamma)} x^\alpha, \\ x^{(\alpha,\beta)} x^{(\gamma,\delta)} - x^{(\alpha,\gamma)} x^{(\beta,\delta)}\} \subset S^2 V^*, \end{gathered} \tag{5.3}$$

(where in the second term we require $\alpha < \beta < \gamma$) of which there are $a + \binom{n}{3} + [\binom{n}{4} + 3\binom{n}{3}] = \dim \Lambda^2(S^2W)$. The first set of equations comes from $|II_X|$, the second linear syzygies in $|II_X|$, and the third quadratic relations among the quadrics in $|II_X|$.

Example 5.4, The Segre. $\mathbb{P}W_1 \times \mathbb{P}W_2 \to \mathbb{P}(W_1 \otimes W_2)$.

Let (B_0, B_α), (C_0, C_j) be respective adapted framings of W_1, W_2, $1 \leq i, j \leq m$, $1 \leq \alpha, \beta \leq n$. Let $A_0 = B_0 \otimes C_0$, $A\alpha = B_\alpha \otimes C_0$, $A_j = C_j \otimes B_0$ and $A_{\alpha j} = B_\alpha \otimes C_j$. So $\{A_0\} \subset \{A_0, A_\alpha, A_j\} \subset \{A_0, A_\alpha, A_j, A_{\alpha j}\}$ is a first order adapted framing. Let $\{\omega^\alpha\}$, $\{\phi^i\}$ be the pullbacks of the semi basic forms on $\mathbb{P}W_1$ and $\mathbb{P}W_2$ respectively. Then

$$II_X = \omega^\alpha \phi^j \otimes A_{\alpha j} \quad \operatorname{mod} \{A_0, A_\alpha, A_j\}. \tag{5.5}$$

The equations of the Segre are

$$\begin{aligned} \{x^0 x^{\alpha j} - x^\alpha x^j,\ x^j x^{\alpha k} - x^k x^{\alpha j},\quad & x^\beta x^{\alpha k} - x^\alpha x^{\beta k}, \\ & x^{\alpha j} x^{\beta k} - x^{\beta j} x^{\alpha k}\}, \end{aligned} \tag{5.6}$$

where the first set of equations comes from II_X, the second and third from linear syzygies in $|II_X|$, and the last set from quadratic relations among the quadrics in $|II_X|$, a total of $mn + [\binom{n}{2}m + \binom{m}{2}n] + \binom{n}{2}\binom{m}{2}$ equations.

Example 5.7, The Grassmannian $G(2, W) \subset \mathbb{P}(\Lambda^2 W)$.

Identify the tangent space to $G(2, W)$ with the $2 \times (m-2)$ matrices $(\dim W = m)$ and index everything accordingly. Write the normal indices as $\mu = (ij), i < j, 3 \leq i, j \leq m$ and the tangent indicies as $\alpha = (1_j)$ or (2_j). Then

$$II_X = (\omega^{1j}\omega^{2k} - \omega^{2j}\omega^{1k}) \otimes A_{jk} \quad \operatorname{mod} \{A_0, A_\alpha\}. \tag{5.8}$$

The equations are

$$\begin{aligned}\{\ & x^0x^{ij} - (x^{1i}x^{2j} - x^{1j}x^{2i}),\ x^{1k}x^{ij} + x^{1i}x^{jk} - x^{1j}x^{ik}, \\ & x^{2k}x^{ij} + x^{2i}x^{jk} - x^{2j}x^{ik},\ x^{ij}x^{kl} + x^{il}x^{jk} - x^{ik}x^{jl}\},\end{aligned} \tag{5.9}$$

where the first set of equations comes from II_X, the second and third from linear syzygies in $|II_X|$, (which comes from picking two columns and a tangent vector) and the last set from quadratic relations among the quadrics in $|II_X|$ (which comes from picking pairs out of four columns), a total of $a + 2\binom{m}{3} + \binom{m}{4}$ equations.

Example 5.10, The Severi Varieties. (In particular, $E_6/P \subseteq \mathbb{P}^{26}$).

Refer to [10] for notation. Let $\mathbb{A}$ be a division algebra over $\mathbb{C}$. Let $\mathbb{A}^2$ have division algebra valued coordinates (u, v). For X Severi, $T_xX \simeq \mathbb{A}^2$ and

$$|II_X| = \mathbb{P}\{u\overline{v}, u\overline{u}, v\overline{v}\}, \tag{5.11}$$

where the first expression gives $\dim_{\mathbb{C}}\mathbb{A}$ quadrics, and the other two expressions give one each. The linear sygyzies are

$$\overline{u}(u\overline{v}) - \overline{v}(u\overline{u}), \quad v(u\overline{v}) - u(v\overline{v}), \tag{5.12}$$

where each expression gives $\dim_{\mathbb{C}}\mathbb{A}$ linear relations. There is a unique quadratic relation among the quadrics in $|II_X|$;

$$(u\overline{v})(\overline{u}v) - (u\overline{u})(v\overline{v}), \tag{5.13}$$

where $\overline{u}v$ is the same set of quadrics as $u\overline{v}$. In fact we can see how these fit into the set of all equations by writing an element of $y \in \mathcal{H} = V$ as

$$y = \begin{pmatrix} r_1 & \bar{u}_1 & \bar{u}_2 \\ u_1 & r_2 & \bar{u}_3 \\ u_2 & u_3 & r_3 \end{pmatrix}, \qquad r_i \in \mathbb{C},\ u_i \in \mathbb{A}, \tag{5.14}$$

and taking

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The equations for the Severi variety are just the 2×2 minors. Here r_1 plays the role of x^0 in the previous examples. The equations are:

$$\begin{aligned}\{& r_1r_2 - u_1\overline{u_1}, r_1r_3 - u_2\overline{u_2}, r_1u_3 - \overline{u_1}u_2, r_2u_2 - u_1u_3, r_3u_1 \\ & - u_2\overline{u_3}, r_2r_3 - u_3\overline{u_3}\},\end{aligned} \tag{5.15}$$

where the first three terms ($2+\dim_{\mathbb{C}}\mathbb{A}$ equations) come from $|II_X|$, and using them to rewrite the rest without r_1, r_2, u_3, we see the fourth and fifth terms ($2\dim_{\mathbb{C}}\mathbb{A}$ equations) are the linear syzygies in $|II_X|$, and the last term (1 equation) is the quadratic relation among the quadrics in $|II_X|$. The reader may find it amusing to explicitly correlate this description of $v_2(\mathbb{P}^2)$, Segre$(\mathbb{P}^2\times\mathbb{P}^2)$, and $G(2,6)$ with the ones given above.

Example 5.16, The Spinor variety $\mathbb{S}^{10}\subset\mathbb{P}^{15}$.

Let $V=\mathbb{C}^5$. Write $\mathbb{C}^{16}=\Lambda^{even}V=\Lambda^0V\oplus\Lambda^2V\oplus\Lambda^4V$ with basis (A_0, A_{ij}, A_i), $1\le i<j\le 5$. Fixing $x\in\mathbb{S}$, we may identify $x\simeq[A_0]\simeq\mathbb{P}(\Lambda^0V)$, $\hat{T}_x\mathbb{S}\simeq\{A_0, A_{ij}\}\simeq(\Lambda^0V\oplus\Lambda^2V)$, and the system of quadrics obtained from the second fundamental form is the complete system of quadrics with base locus the Grassmanian $G(2,5)\subset\mathbb{P}(\Lambda^2V)$. We may take

$$\begin{aligned} II_X \equiv&(\omega^{12}\omega^{34}-\omega^{13}\omega^{24}-\omega^{14}\omega^{23})\otimes A_5\\ &+(\omega^{12}\omega^{35}-\omega^{13}\omega^{25}-\omega^{15}\omega^{23})\otimes A_4\\ &+(\omega^{12}\omega^{45}-\omega^{14}\omega^{25}-\omega^{15}\omega^{24})\otimes A_3\\ &+(\omega^{13}\omega^{45}-\omega^{14}\omega^{35}-\omega^{15}\omega^{34})\otimes A_2\\ &+(\omega^{23}\omega^{45}-\omega^{24}\omega^{35}-\omega^{25}\omega^{34})\otimes A_1 \operatorname{mod}\{A_0, A_{ij}\}. \end{aligned}\tag{5.17}$$

Here there are only linear syzygies. The equations are

$$\begin{aligned} \{&x^5x^0-(x^{12}x^{34}-x^{13}x^{24}-x^{14}x^{23}),\\ &x^4x^0-(x^{12}x^{35}-x^{13}x^{25}-x^{15}x^{23})\\ &x^3x^0-(x^{12}x^{45}-x^{14}x^{25}-x^{15}x^{24}),\\ &x^2x^0-(x^{13}x^{45}-x^{14}x^{35}-x^{15}x^{34}),\\ &x^1x^0-(x^{23}x^{45}-x^{24}x^{35}-x^{25}x^{34}),\\ &x^{15}x^5+x^{14}x^4+x^{13}x^3+x^{12}x^2,\\ &x^{25}x^5+x^{24}x^4+x^{23}x^3-x^{12}x^1,\\ &x^{35}x^5+x^{34}x^4-x^{23}x^3-x^{13}x^1,\\ &x^{45}x^5-x^{34}x^3-x^{24}x^2-x^{14}x^1,\\ &x^{45}x^4+x^{35}x^3+x^{25}x^2+x^{15}x^1\}. \end{aligned}\tag{5.18}$$

The reader may find it amusing to explicitly correlate these equations with the system of quadrics (5.11) with $\mathbb{A}=\mathbb{O}$.

6. Properties of systems of quadrics

In this section we study what the existence of a linear syzygy implies about a system of quadrics and the implications of a companion condition. We then combine these observations with the results of [9] and §3 to draw some global conclusions.

Given $\widehat{|II_X|} \subset S^2T^*$, a system of quadrics arising from the second fundamental form at a general point of a variety, any cubic in the third fundamental form must be in its prolongation, $\widehat{|II_X|}^{(1)} := (\widehat{|II_X|} \otimes T^*) \cap S^3T^*$ (see [6, (1.47)], [9, (3.12)]). We will now define a complement to $\widehat{|II_X|}^{(1)}$ in $\widehat{|II_X|} \otimes T^*$ consisting of the space of linear syzygies. Recall that as a $Gl(T^*)$ module, $T^{*\otimes 3}$ naturally splits into three factors; $S^3T^* \oplus \Lambda^3T^* \oplus (S^{(21)}T^*)^{\oplus 2}$ where the last factor is two copies of the irreducible $Gl(T^*)$ module obtained from the Young diagram with two boxes in the first row and one in the second (hence the notation (21)). We can choose two such copies as follows: Let

$$S^{(21)}_{\circ}T^* := \ker(S^2T^* \otimes T^* \to S^3T^*), \tag{6.1}$$

$$S^{(21)}_{\Lambda}T^* := \ker(\Lambda^2T^* \otimes T^* \to \Lambda^3T^*). \tag{6.2}$$

Then

$$S^2T^* \otimes T^* = S^3T^* \oplus S^{(21)}_{\circ}T^*. \tag{6.3}$$

Given $A \subset S^2T^*$, define

$$A^{[1]} := (A \otimes T^*) \cap S^{(21)}_{\circ}T^*, \tag{6.4}$$

so

$$A \otimes T^* = A^{(1)} \oplus A^{[1]}. \tag{6.5}$$

If $\widehat{|II_X|}^{(1)} = 0$, then $III_X = 0$, and if $\widehat{|II_X|}^{[1]} = 0$ and $\widehat{|II_X|}^{(1)} = 0$, then by (4.18) the space of quadrics containing X is at most a-dimensional.

We now show that if the codimension of X is sufficiently small, then $\widehat{|II_X|}^{[1]} = 0$ and $\widehat{|II_X|}^{(1)} = 0$; i.e., that $\widehat{|II_X|} \otimes T^*$ is disjoint from the two $Gl(T)$ invariant linear spaces in $S^2T^* \otimes T^*$.

Theorem 6.6. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and let $x \in X$ be any smooth point. Let $b = \dim X_{sing}$. (Set $b = -1$ if X is smooth.) If $a < \frac{1}{2}[n+1-(b+1)]$, then $\widehat{|II_X|}_x^{(1)} = 0$, where $\widehat{|II_X|}_x^{(1)} = S^3T^* \cap (\widehat{|II_X|}_x \otimes T^*)$ is the prolongation of $\widehat{|II_X|}_x$.*

Proof. Say there were a nonzero $P \in \widehat{|II_X|}^{(1)}$. Consider

$$\rfloor P : T \to \widehat{|II_X|}. \tag{6.7}$$

Applying[9, (6.1)] with $A_x = \text{image}(\rfloor P)$ gives

$$\begin{aligned} &\dim(\text{Singloc}(\text{image}(\rfloor P)) \\ &\quad \leq 2(a-1) - (\dim \text{image}(\rfloor P) - 1) + (b+1). \end{aligned} \tag{6.8}$$

Observe that

$$n = \dim(\ker(\rfloor P)) + \dim(\text{image}(\rfloor P)) \tag{6.9}$$

and

$$\ker(\rfloor P) \subseteq \text{Singloc}(\text{image}(\rfloor P)), \tag{6.10}$$

so

$$n - \dim(\text{image}(\rfloor P)) \leq 2a - 1 - \dim(\text{image}(\rfloor P)) + b + 1, \tag{6.11}$$

i.e.,

$$a \geq \frac{1}{2}[n + 1 - (b+1)]. \tag{6.12}$$

Remark 6.13. (6.6) gives a new proof of [9, (4.15)] stated in the introduction with refined information about what the structure of the second and third fundamental forms of a variety must be in small codimension.

To study $\widehat{|II_X|}^{[1]}$ is a bit more difficult. An $R \in \widehat{|II_X|}^{[1]}$ is a map $R : T \to \widehat{|II_X|}$ and $\ker R \subseteq \text{Baseloc}\widehat{|II_X|}$, so if we had a theorem directly restricting the size of base loci of subsystems instead of singular loci, we would be in better shape.

In general, we can think of II_X as

$$\begin{aligned} ii : T &\to T^* \otimes N \\ w &\mapsto II_X(w, \cdot). \end{aligned} \tag{6.14}$$

If we restrict ii to $\ker R$, we get a map

$$ii' : \ker R \to (\ker R)^{\perp} \otimes N_R, \tag{6.15}$$

where $N_R := ii(\ker R)(T^*)$. Now $R \in \ker(\widehat{|II_X|} \otimes T^* \to S^3T^*)$ implies $(\ker R)^{\perp} \simeq N_R$ and moreover that ii' descends to a map

$$ii' : \ker R \to \Lambda^2(\ker R)^{\perp} \simeq \Lambda^2 N_R. \tag{6.16}$$

In bases, the situation is as follows: Let $L := \ker R^{\perp} \subset T^*$ and let $M \subset T^*$ be any complement to L. We may write $S^2T^* = S^2L \oplus (L \otimes M) \oplus S^2M$.

Let $A = R(T) \subset \widehat{|II_X|}$. Let $\{q^j\}$ be a basis of A and $\{l^j\}$ a basis of L. Write $q^j = b^j + c^j + d^j$ reflecting the decomposition of S^2T^*. Then

$$b^j = b^i_{jk} l^j l^k, c^j = m^j_k l^k,\ d^j = 0, \tag{6.17}$$
$$\text{with } b^i_{jk} \in \mathbb{C},\ \mathfrak{S}_{ijk} b^i_{jk} = 0,\ m^i_j = -m^j_i \in M.$$

By rechoosing M if necessary, we may also assume

$$b^i_{jk} = b^j_{ik}. \tag{6.18}$$

Lemma 6.19. *Let $A^p \subset S^2T^*$ be an p-dimensional system of quadrics on an n-dimensional vector space. Say there is a linear syzygy*

$$l^1Q_1 + \dots + l^pQ_p = 0,$$

where both $l^i \in T^*$ and $Q_i \in A$ are independent sets of vectors. Then $\forall Q \in A$,

$$\text{rank } Q \leq 2(p-1).$$

Proof. Let

$$Q = \lambda^1 Q_1 + \dots + \lambda^p Q_p, \quad \lambda^i \in \mathbb{C}. \tag{6.20}$$

Let $l_p, \dots, l_p$ be a dual basis to $l^1, \dots, l^p$. Observe that

$$(\lambda^1 l_1 + \dots + \lambda^p l_p) \rfloor Q = 0. \tag{6.21}$$

On the other hand, we may write

$$Q = l^1\alpha^1 + \dots l^p\alpha^p \tag{6.22}$$

for some $\alpha^1 \dots \alpha^p \in T^*$. Now change bases in L such that $l^p = (\lambda^1 l_1 + \dots \lambda^p l_p)^*$. Then $\alpha^p = 0$ and we see rank $Q \leq 2(p-1)$.

Taking the m_{ij} all independent (for $i < j$), produces quadrics of rank $2(p-1)$, so the result is sharp. q.e.d.

Applying [9, (6.1)] to (6.19),(4.18) yields:

Lemma 6.23. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and let $x \in X$ be a general point. Let $b = \dim X_{sing}$. (Set $b = -1$ if X is smooth.) If $a < \frac{1}{3}[n - (b+1) + 3]$, then there are no linear syzygies in $|II_X|_x$.*

Proof. Lemma 6.20 implies that if there is such a syzygy, then there is a p-dimensional subsystem $A_x \subseteq |II_X|_x$ with the property that no quadric in A_x is of rank greater than $2(p-1)$. Now [9, (6.1)] applied to A_x gives

$$2(a-1)+(b+1) \geq n+p-1-2(p-1), \tag{6.24}$$

i.e.,

$$2(a-1)+p \geq n-(b+1)+1. \tag{6.25}$$

Finally just notice that $p \leq a$. q.e.d.

By Lemma 6.23 and Theorem 4.18 we thus obtain

Theorem 6.26. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and $x \in X$ a general point. Let $b = \dim X_{sing}$. (Set $b=-1$ if X is smooth.) If $a < \frac{1}{3}[n-(b+1)+3]$, then*

$$\begin{aligned} &\dim\{\textit{quadrics osculating to order three at } x\} \\ &\qquad \leq a + \binom{a+1}{2} - 1, \end{aligned} \tag{6.27}$$

$$\dim\{\textit{quadrics osculating to order four at } x\} \leq a-1.$$

Equality occurs in the first (respectively second) expression of (6.27) *if and only if* (4.9.3)*(resp.*(4.9.4)*) hold at x. If the generalized Monge system* (4.17) *holds, then X is a complete intersection of the $(a-1)$-dimensional family of quadrics osculating to order four.*

Proof. It only remains to show there cannot be any hypersurfaces of higher degree generating new elements of the ideal of X. The only way to have an equation of higher degree is if there exists a nontrivial polynomial $P(Q^1, ..., Q^a)=0$, which is not possible in this codimension range as the existence of such a polynomial would imply the secant variety of X is degenerate. q.e.d.

Corollary 6.28. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and $x \in X$ a general point. Let $b = \dim X_{sing}$. (Set $b=-1$ if X is smooth.) If $a < \frac{1}{3}[n-(b+1)+3]$ then any quadric osculating to order four at x is smooth at x, and any quadric osculating to order five at x contains X.*

Corollary 6.29. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety with I_X generated by quadrics. Let $b = \dim X_{sing}$. (Set $b=-1$ if X is smooth.) If $a < \frac{1}{3}[n-(b+1)+3]$, then X is a complete intersection.*

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