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# SOME REGULARITY THEOREMS FOR CARNOT-CARATHÉODORY METRICS

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## 1. Introduction

Let $M$ be a smooth connected $m$-dimensional manifold and $Q$ a smooth $q$-dimensional distribution on $M$ which is *bracket generating*, i.e., for every $p \in M$ the local sections of $Q$ near $p$ span together with all their commutators the tangent space $T_pM$ of $M$ at $p$.

A curve $\varphi$ in $M$ is called *horizontal* if $\varphi$ is tangent almost everywhere to $Q$. It is a classical result of Chow that any two points of $M$ can be joined by a horizontal curve (see e.g. [13, 12]). Thus if $Q$ is equipped with a Riemannian metric $\langle\ ,\ \rangle_Q$, then the function $d_c\colon M \times M \to \mathbb{R}$, $(p, u) \to \inf\{\text{length}(\varphi) | \varphi$ is horizontal and joins $p$ to $u\}$ is a distance on $M$, the *Carnot-Carathéodory metric* induced by $(Q, \langle\ ,\ \rangle_Q)$.

Let $\langle\ ,\ \rangle$ be an extension of $\langle\ ,\ \rangle_Q$ to a Riemannian metric on $M$, and let dist be the induced distance on $M$. Then $d_c \geq \text{dist}$, and any rectifiable curve with respect to $d_c$ is rectifiable with respect to dist, hence differentiable almost everywhere and moreover horizontal [13]. Vice versa every horizontal curve is locally rectifiable with respect to $d_c$; its $d_c$-length coincides with its usual length as a curve in $(M, \langle\ ,\ \rangle)$ (see [17]; this also follows from the general theory of length structures in [6]). Thus $(M, d_c)$ is a locally compact *length space* and complete if this is true for $(M, \text{dist})$.

Let $p \in M$ and $\varepsilon > 0$ be such that the closure of the open $d_c$-ball $B$ of radius $\varepsilon$ around $p$ is compact. Then it follows from the theory of locally compact length spaces [6] that every $u \in M$ with $d_c(p, u) < \varepsilon$ can be joined to $p$ by a minimizing *geodesic* with respect to $d_c$, i.e., a horizontal curve which realizes locally the $d_c$-distance of its curve points (this is also proved in [17]). Strichartz showed that if $Q$ satisfies the *strong bracket generating hypothesis* (see [17]), i.e., if $TM$ is generated by $Q$ and $[X, Q]$ for every nonzero local section $X$ of $Q$, then these geodesics are solutions of a system of Hamilton-Jacobi equations on the cotangent bundle $T'M$ of $M$, in particular they are smooth curves. This

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leads to the definition of an exponential map of an open neighborhood of the zero section of $T'M$ onto $M$; however its restriction to a fiber of $T'M$ is not of maximal rank at 0.

In this paper we give a different approach to the theory of geodesics. We extend $\langle\ ,\ \rangle_Q$ to a Riemannian metric $\langle\ ,\ \rangle$ on $M$ and consider a variational problem in $(M, \langle\ ,\ \rangle)$. We obtain a simple differential equation for the critical points of this variational problem and show that these critical points are geodesics with respect to $d_c$, i.e., they are locally minimizing curves (this answers a question in [17]). On the other hand, Bär [1] showed that every geodesic is a critical point; together this gives a complete description of the geodesics.

This leads to the definition of an exponential map $\exp_p^c$ at a given point $p \in M$ which maps an open neighborhood $\Omega$ of 0 in $T_pM$ onto an open neighborhood of $p$ in $M$. We show that $\exp_p^c$ is of maximal rank on an open and dense subset of $\Omega$. However $\exp_p^c$ depends on the choice of the extension of $\langle\ ,\ \rangle_Q$ to a Riemannian metric on $M$ and moreover on the choice of a local trivialization of $TM$ adapted to our situation. If the distribution $Q$ satisfies the strong bracket generating hypothesis, then $(Q, \langle\ ,\ \rangle_Q)$ determines a unique Riemannian metric $\langle\ ,\ \rangle$ on $M$ extending $\langle\ ,\ \rangle_Q$ and thus $\exp_p^c$ only depends on the local trivilization. Moreover every $d_c$-geodesic emanating from $p$ is uniquely determined by its tangent and the covariant derivative of its tangent at $p$, i.e., $\exp_p^c$ is defined intrinsically.

As an application of the investigation of geodesics we show that any isometry between manifolds with Carnot-Carathéodory (briefly CC-) metrics is necessarily smooth and clearly commutes with the exponential map. We conclude the paper with an example where the geodesics can easily be computed explicitly.

## 2. The space of $H_1$-curves in $M$ through a given point

Let $p \in M$. We consider the Hilbert manifold $H_1^p(I, U)$ of all continuous, absolutely continuous curves $\varphi: I \to U$ through $\varphi(0) = p$ with square integrable derivative, where $U$ is a suitable open neighborhood of $p$.

Fix a Riemannian metric $\langle\ ,\ \rangle$ on $M$ extending $\langle\ ,\ \rangle_Q$. Given $p \in M$ select a local orthonormal basis $\{X^1, \cdots, X^q\}$ of $Q$ and a local orthonormal basis $\{X^{q+1}, \cdots, X^m\}$ of the $\langle\ ,\ \rangle$-orthogonal complement $Q^\perp$ of

$Q$. The local frame $\{X^1, \cdots, X^m\}$, defined on an open $d_c$-ball $U$ of radius $\rho > 0$ around $p$, will be called *admissible*. Let $\theta^1, \cdots, \theta^m$ be the dual coframe and let $\theta = (\theta^1, \cdots, \theta^m)$. $\theta$ is a 1-form on $U$ with values in a Euclidean $m$-space $\mathbb{R}^m$.

The map $\Theta$, defined on $H_1^p(I, U)$ by $(\Theta\varphi)(t) = \theta\varphi'(t)$, has its image in the Hilbert space $H_0(I, \mathbb{R}^m)$ of square integrable curves in $\mathbb{R}^m$.

**Lemma 2.1.** *$\Theta$ is a diffeomorphism of $H_1^p(I, U)$ onto an open neighborhood of 0 in $H_0(I, \mathbb{R}^m)$.*

The proof uses the fact that the Banach-manifold of all continuously differentiable curves in $U$ starting at $p$ is diffeomorphic to an open neighborhood of 0 in the Banach space of continuous curves in $T_pM \sim \mathbb{R}^m$ (see [9]) and a standard completion argument.

There are unique 1-forms $\theta_j^i$ on $U$ such that

(a) $\theta_j^i = -\theta_i^j$,

(b) $d\theta^i = \sum_{k=1}^m \theta^k \wedge \theta_k^i$

(see [16]).

Let $\varphi \in H_1^p(I, U)$ and let $X$ be an element of the tangent space $H_1^p(\varphi)$ of $H_1^p(I, U)$ at $\varphi$, i.e., $X$ is a section of $TM$ over $\varphi$ of class $H_1$ which vanishes at $p = \varphi(0)$. Denote by $\frac{D}{dt}X$ the covariant derivative of $X$ with respect to the Riemannian connection of $\langle\ ,\ \rangle$. Then

**Lemma 2.2.**

$$\theta^i\left(\frac{D}{dt}X\right) = \frac{d}{dt}(\theta^i(X)) + \sum_{j=1}^m \theta_j^i(\varphi')\theta^j(X).$$

Lemma 2.2 is well known and can be found in [16].

Write $d\theta = (d\theta^1, \cdots, d\theta^m)$; $d\theta$ is a 2-form on $U$ with values in $\mathbb{R}^m$. As a corollary of 2.2, the differential $d\Theta_\varphi$ of $\Theta$ at $\varphi$ can be computed as follows:

**Lemma 2.3.** *If $x \in H_1^p(\varphi)$, then $d\Theta_\varphi X = \frac{d}{dt}(\theta X) - 2d\theta(\varphi', X)$.*

*Proof.* Let $\Psi\colon (-\varepsilon, \varepsilon) \times I \to U$ be a variation of $\varphi = \Psi_0$ with variation vector field $X = \frac{\partial}{\partial s}\Psi|_{s=0}$ such that $\Psi(-\varepsilon, \varepsilon) \times \{0\} = p$. Then

$$(d\Theta_\varphi X)(t) = \frac{\partial}{\partial s}\theta\frac{\partial}{\partial t}\Psi(s, t)|_{s=0},$$

and by Lemma 2.2 the $i$th component of $(d\Theta_\varphi X)(t)$ equals

$$\theta^i\left(\frac{D}{\partial s}\frac{\partial}{\partial t}\Psi(s, t)\right)_{s=0} - \sum_{j=1}^m \theta_j^i(X)\theta^j(\varphi').$$

Using $\frac{D}{\partial s}\frac{\partial}{\partial t}\Psi(s, t) = \frac{D}{\partial t}\frac{\partial}{\partial s}\Psi(s, t)$ and again Lemma 2.2 for $\frac{D}{\partial t}\frac{\partial}{\partial s}\Psi(s, t)$ we obtain for the $i$th component of $(d\Theta_\varphi X)(t)$ the value

$$\frac{d}{dt}(\theta^i X)(t) + \sum_{j=1}^{m}(\theta^i_j(\varphi')\theta^j(X) - \theta^i_j(X)\theta^j(\varphi'))$$

which shows the claim. q.e.d.

Now for every $u \in U$ and $X \in T_uM$ the assignment $Y \to 2d\theta(X, Y)$ is a linear mapping of $T_uM$ into $\mathbb{R}^m$. Let $a^*(X)$ be its adjoint with respect to the scalar product $\langle\ ,\ \rangle_u$ on $T_uM$ and the Euclidean scalar product $\langle\ ,\ \rangle$ on $\mathbb{R}^m$. $a^*(X)$ is a linear map of $\mathbb{R}^m$ into $T_uM$ which satisfies $\langle 2d\theta(X, Y), Z\rangle = \langle Y, a^*(X)Z\rangle$ for all $Y \in T_uM$, $Z \in \mathbb{R}^m$. Moreover the assignment $X \to a(X) = \theta a^*(X)$ is a smooth 1-form on $U$ with values in the vector space of linear endomorphisms of $\mathbb{R}^m$. For convenience we will also write $a(X, Z)$ instead of $a(X)Z$.

**Remark 2.4.** The form $a$ can also be computed as follows: Let $b^i_{jk}$ $(i, j, k = 1, \cdots, m)$ be the unique smooth functions on $U$ which satisfy $d\theta^i = \frac{1}{2}\sum_{j,k} b^i_{jk}\theta^j \wedge \theta^k$ and $b^i_{jk} = -b^i_{kj}$. Then an easy computation shows $\theta^i a(X, Z) = \sum_{j=1}^m \sum_{k=1}^m b^j_{ik}\theta^k(X)Z_j$ for all $Z = (Z_1, \cdots, Z_m) \in \mathbb{R}^m$. However we do not need this formula in the sequel (compare [16]).

The pullback via $\Theta$ of the $L^2$-scalar product of $H_0(I, \mathbb{R}^m)$ is a Riemannian structure $g$ on $H_1^p(I, U)$ which induces for every compact neighborhood $A$ of $p$ in $U$ a complete metric on $H_1^p(I, A) \subset H_1^p(I, U)$. If $\varphi \in H_1^p(I, U)$ and $X, Y \in H_1^p(\varphi)$, then $g_\varphi(X, Y) = \int_0^1 \langle d\Theta_\varphi X(t), d\Theta_\varphi Y(t)\rangle\, dt$.

The linear subspace $\{X \in H_1^p(\varphi) | X(1) = 0\} \subset H_1^p(\varphi)$ is closed in $H_1^p(\varphi)$; hence its $g_\varphi$-orthogonal complement $J(\varphi)$ is an $m$-dimensional linear subspace of $H_1^p(\varphi)$. We have

**Lemma 2.5.** $J(\varphi) = \{X \in H_1^p(\varphi) | \frac{d}{dt}(d\Theta_\varphi X)(t) - a(\varphi'(t), (d\Theta_\varphi X)(t)) \equiv 0\}$.

*Proof.* Let $Y \in H_1^p(\varphi)$ be the preimage under $d\Theta_\varphi$ of a curve of class $H_1$ in $\mathbb{R}^m$. By Lemma 2.3 for every $X \in H_1^p(\varphi)$ we have

$$g_\varphi(X, Y) = \langle\theta X(1), (d\Theta_\varphi Y)(1)\rangle - \int_0^1 \left\langle \theta X(t), \frac{d}{dt}(d\Theta_\varphi Y)(t) - a(\varphi'(t), (d\Theta_\varphi Y)(t))\right\rangle dt.$$

Thus any solution $c\colon I \to V$ of the differential equation

$$(1) \qquad c'(t) = a(\varphi'(t))c(t)$$

is the image under $d\Theta_\varphi$ of an element of $J(\varphi)$. Now (1) is a linear differential equation whose coefficients (i.e., the entries of the matrix representing $a(\varphi'(t)))$ are as regular in $t$ as the map $t \to \theta\varphi'(t)$, i.e., they are square integrable. Thus (1) admits precisely $m$ linear independent solutions which shows the lemma. q.e.d.

If $\varphi$ has a continuous derivative, the existence of an $m$-dimensional space of solutions of (1) follows from the standard theory for solutions of ordinary differential equations with continuous coefficients. We include a proof for the general case since it provides us with norm estimates which are needed later.

For a curve $\varphi$ of class $H_1$ in $U$ and an element $c$ of the Banach space $L^\infty(I, \mathbb{R}^m)$ of essentially bounded maps $I \to \mathbb{R}^m$ provided with the norm $|c| = \operatorname{ess\,sup}_{t\in I} \|c(t)\|$, define $T_\varphi c(s) = \int_0^s a(\varphi'(t))c(t)\,dt$. Thus $c$ is a solution of (1) with $c(0) = c_0$ for some $c_0 \in \mathbb{R}^m$ if and only if $c - T_\varphi \equiv c_0$.

Let $\|L\|$ be the operator norm of a linear endomorphism $L$ of the Euclidean space $\mathbb{R}^m$. Then $\varphi \in H_1^p(I, U)$ means $\nu(\varphi) = \int_0^1 \|a(\varphi'(t))\|\,dt < \infty$.

**Lemma 2.6.** *For every $\varphi \in H_1^p(I, U)$, $\mathrm{Id} - T_\varphi$ is a continuous invertible linear automorphism of $L^\infty(I, \mathbb{R}^m)$. The operator norm of $(\mathrm{Id} - T_\varphi c)^{-1}$ does not exceed $(2\nu(\varphi) + 2)^{2\nu(\varphi)+1}$.*

*Proof.* Let $c \in L^\infty(I, V)$; then $\|T_\varphi c(s)\| = \|\int_0^s a(\varphi'(t))c(t)\,dt\| \le \nu(\varphi)|c|$, i.e., $T_\varphi$ is a continuous linear endomorphism of $L^\infty(I, \mathbb{R}^m)$ whose operator norm does not exceed $\nu(\varphi)$.

Let $k > 0$ be the smallest integer which is not smaller than $2\nu(\varphi)$ and choose a partition $0 = s(0) < s(1) < \cdots < s(k) = 1$ of $I$ such that $\int_{s(j)}^{s(j+1)} \|a(\varphi'(t))\|\,dt \le \frac{1}{2}$ for all $j < k$. Define $\psi_j(t) = s(j) + t(s(j+1) - s(j))$ and $\varphi_j(t) = \varphi(\psi_j(t))$, for $t \in I$, and let $c \in L^\infty(I, \mathbb{R}^m)$, $c_j(t) = c(\psi_j(t))$. Since the operator norm of $T_{\varphi_j}$ is not larger than $\frac{1}{2}$, $\mathrm{Id} - T_{\varphi_j}$ is invertible (see [15, p. 231]) and $(\mathrm{Id} - T_{\varphi_j})^{-1} = \sum_{i=0}^\infty T_{\varphi_j}^i$, in particular the operator norm of $(\mathrm{Id} - T_{\varphi_j})^{-1}$ does not exceed $\sum_{i=0}^\infty 2^{-i} = 2$. Hence there is a unique $\alpha \in L^\infty(I, \mathbb{R}^m)$ such that we have $(\mathrm{Id} - T_{\varphi_j})\alpha_j = c_j + \int_0^{s(j)} a(\varphi'(t))\alpha(t)\,dt$ with $\alpha_j(t) = \alpha(\psi_j(t))$ $(j < k)$. Then

$$
\begin{aligned}
c(\psi_j(t)) = c_j(t) &= \alpha_j(t) - \int_0^t a(\varphi_j'(s))\alpha_j(s)\,ds - \int_0^{s(j)} a(\varphi'(s))\alpha(s)\,ds \\
&= \alpha(\psi_j(t)) - \int_0^{\psi_j(t)} a(\varphi'(s))\alpha(s)\,ds,
\end{aligned}
$$

which means $(\mathrm{Id}-T_\varphi)\alpha = c$. This shows that $\mathrm{Id}-T_\varphi$ is invertible. Moreover we have

$$|\alpha_j| \le 2\left(|c_j| + \left\|\int_0^{s(j)} a(\varphi'(t))\alpha(t)\,dt\right\|\right) \le 2\left(|c| + \nu(\varphi)\sup_{i<j}|\alpha_i|\right),$$

and inductively $|\alpha| = \sup_{j<k}|\alpha_j| \le 2^k(1+\nu(\varphi))^k|c|$. This means that the operator norm of $(\mathrm{Id}-T_\varphi)^{-1}$ does not exceed $2^k(1+\nu(\varphi))^k$ which is the claim.

**Remark 2.7.** Let $S \subset U$ be a smooth $k$-dimensional submanifold with tangent bundle $TS$. Then $\{\varphi \in H_1^p(I, U) | \varphi(1) \in S\}$ is a smooth submanifold of $H_1^p(I, U)$ of codimension $m-k$. Its tangent space at $\varphi$ consists of all $X \in H_1^p(\varphi)$ with $X(1) \in TS$. Lemma 2.5 thus shows that the $g_\varphi$-orthogonal complement of this tangent space is just the $(m-k)$-dimensional vector space $\{X \in J(\varphi) | (d\Theta_\varphi X)(1) \in (TS)^\perp\}$.

**Remark 2.8.** If $M$ is a Lie group with identity $e = p$, and the vector fields $X^1, \cdots, X^m$ are left-invariant, then the Lie algebra $\mathfrak{M}$ of $M$ can naturally be identified with $\mathbb{R}^m$. With this identification, $\theta$ is the canonical left-invariant 1-form on $M$ with values in $\mathfrak{M}$ (see [9]). Thus $2d\theta(X, Y) = (\mathrm{ad}\,X)(Y)$, where as usual ad denotes the adjoint representation of $\mathfrak{M}$. Let Ad be the adjoint representation of $M$ in $\mathfrak{M}$, and denote by $\mathrm{Ad}_u^*$ the adjoint of $\mathrm{Ad}_u$ for $u \in M$. If $\varphi \in H_1^e(I, M)$, then for every $c_0 \in \mathfrak{M}$ the curve $t \to \mathrm{Ad}_{\varphi(t)}^* c_0$ satisfies the differential equation (1) of Lemma 2.5. Thus in this case $J(\varphi) = \{X \in H_1^e(\varphi) | (d\Theta_\varphi X)(t) = \mathrm{Ad}_{\varphi(t)}^* c_0$ for some $c_0 \in \mathfrak{M}\}$.

## 3. The manifold of curves tangent to $Q$

In this section we begin to investigate the submanifold $HQ$ of $H_1^p(I, U)$ of curves which are tangent almost everywhere to $Q$.

Identify $Q$ with the subspace $\theta(Q) \cong \mathbb{R}^q$ of $\mathbb{R}^m$. The set $HQ = \Theta^{-1}H_0(I, Q)$ of curves which are tangent almost everywhere to $Q$ is a closed submanifold of $H_1^p(I, U)$. If $\widetilde{P}: V \to Q$ denotes the $\langle\ ,\ \rangle$-orthogonal projection, then for every $\varphi \in HQ$ the $g_\varphi$-orthogonal projection $P$ of $H_1^p(\varphi)$ onto the tangent space $HQ_\varphi$ of $HQ$ at $\varphi$ is defined by $PX = (d\Theta_\varphi)^{-1}\widetilde{P}d\Theta_\varphi X$.

Let $K(\varphi) \sim H_0(I, Q^\perp)$ be the kernel of the projection $P$, and define $\Omega(\varphi) = \{X \in HQ_\varphi | X(1) = 0\}$. Then $H_1^p(\varphi) = \Omega(\varphi) \oplus (J(\varphi) + K(\varphi))$, and the $g_\varphi$-orthogonal complement $\Omega(\varphi)^\perp$ of $\Omega(\varphi)$ in $HQ_\varphi$ is contained in $P(J(\varphi) + K(\varphi)) = PJ(\varphi)$. Thus $\Omega(\varphi)^\perp = \{X \in HQ_\varphi | (d\Theta_\varphi X)(t) = \widetilde{P}c(t)$ for $c \in H_1(I, V)$ with $c'(t) = a(\varphi'(t))c(t)\}$.

Let $R\colon HQ \to U$, $\varphi \to \varphi(1)$ be the endpoint map. Then the rank of $R$ at $\varphi$ equals the dimension of $\Omega(\varphi)^\perp$, and this dimension varies between $q = \dim Q$ at the constant curve $\varphi(I) = p$ and $m = \dim M$. In particular for $u \in U$ the closed subset $R^{-1}(u)$ of $HQ$ may not be a submanifold.

However the set $\{\varphi \in HQ | \operatorname{rank} R_\varphi = m\}$ is clearly open in $HQ$. If $M$ is a Lie group, then Remark 2.8 shows that it is even open as a subset of $HQ$ with the $C^0$-topology. A similar property holds in general. For its formulation let dist again be the distance on $M$ induced by the Riemannian metric, and recall that the space $C^0(I, M)$ of continuous curves in $M$ with the distance $d_\infty(\varphi, \psi) = \sup\{\operatorname{dist}(\varphi(t), \psi(t)) | t \in I\}$ is a Banach manifold, in particular a locally complete metric space. Let $E\colon HQ \to \mathbb{R}$ be the restriction to $HQ$ of the energy function $\varphi \to \frac{1}{2}\int_0^1 \|\varphi'(t)\|^2\,dt$. First we have

**Lemma 3.1.** *Let* $\mu > 0$ *and* $\varphi \in HQ \cap E^{-1}[0, \mu)$. *Then for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $\sup_{t\in I} \|\int_0^t (\theta\gamma'(s) - \theta\varphi'(s))\,ds\| < \varepsilon$ *for all* $\gamma \in HQ \cap E^{-1}[0, \mu)$ *with* $d_\infty(\gamma, \varphi) < \delta$.

*Proof.* Let $U, X^1, \cdots, X^m, \theta$ be as before and assume without loss of generality that there is a diffeomorphism $\Psi$ of $\mathbb{R}^m$ onto $U$ with $\Psi(0) = p$. Define $c(t) = \Psi^{-1}\varphi(t)$ and denote by $\|L\|$ its operator norm for a linear map $L$ between Euclidean vector spaces.

Let $A$ be a compact neighborhood of $c(I)$ in $\mathbb{R}^m$ and let

$$\rho = \sup\{\|d\Psi_u\|, \|d\Psi^{-1}_{\Psi(u)}\| | u \in A\} < \infty.$$

By the smoothness of $\Psi$ there is then $\sigma > 0$ such that for every $t \in I$ and every $u \in U$ with $\operatorname{dist}(\varphi(t), u) < \sigma$

$$\|\theta(d\Psi)_{c(t)} - \theta(d\Psi)_{\Psi^{-1}u}\| < \varepsilon/8\rho\sqrt{\mu}.$$

Choose $n \geq 2$ such that $\operatorname{dist}(\varphi(s), \varphi(t)) < \sigma$ for $|s - t| < 1/n$, and let $\delta < \sigma$ be sufficiently small that $d_\infty(\varphi, \gamma) < \delta$ implies $\|c(t) - \Psi^{-1}\gamma(t)\| < \varepsilon/16n\rho$ for all $t \in I$. Let $\gamma \in HQ \cap E^{-1}[0, \mu)$ with $d_\infty(\gamma, \varphi) < \delta$ and define $\overline{c}(t) = \Psi^{-1}\gamma(t)$. Then

$$\begin{aligned}
\sup_{t\in I}&\left\|\int_0^t (\theta\gamma'(s) - \theta\varphi'(s))\,ds\right\| \\
&\le \sup_{t\in I}\left\|\int_0^t (\theta(d\Psi)_{\overline{c}(s)} - \theta(d\Psi)_{c(s)})\overline{c}'(s)\,ds\right\| \\
&\quad + \sup_{t\in I}\left\|\int_0^t \theta(d\Psi)_{c(s)}(\overline{c}'(s) - c'(s))\,ds\right\| \\
&\le \int_0^1 \|(\theta(d\Psi)_{\overline{c}(s)} - \theta(d\Psi)_{c(s)})\overline{c}'(s)\|\,ds \\
&\quad + \sup_{k\le n-1}\left\|\sum_{j=0}^k \theta(d\Psi)_{c(j/n)}\int_{j/n}^{(j+1)/n} (\overline{c}'(s) - c'(s))\,ds\right\| \\
&\quad + \sup_{k<n}\sup_{r<1/n}\left\|\theta(d\Psi)_{c(k/n)}\int_{k/n}^{k/n+r} (\overline{c}'(s) - c'(s))\,ds\right\| \\
&\quad + \sum_{j\ge 0}\int_{j/n}^{(j+1)/n} \|(\theta(d\Psi)_{c(s)} - \theta(d\Psi)_{c(j/n)})(\overline{c}'(s) - c'(s))\|\,ds \\
&\le \frac{\varepsilon\rho\sqrt{\mu}}{8}\int_0^1 \|\overline{c}'(s)\|\,ds \\
&\quad + 2\sum_{j=0}^n \rho\|\overline{c}(j/n) - c(j/n)\| \\
&\quad + \rho\sup_{j\le n}\sup_{r<1/n}(\|\overline{c}(\tau + j/n) - c(\tau + j/n)\| + \|\overline{c}(j/n) - c(j/n)\|) \\
&\quad + \frac{\varepsilon\rho\sqrt{\mu}}{8}\int_0^1 \|\overline{c}'(s) - c'(s)\|\,ds.
\end{aligned}$$

Since $\int_0^1 \|c'(s)\|\,ds \le \rho\sqrt{\mu}$ and $\int_0^1 \|\overline{c}'(s)\|\,dx \le \rho\sqrt{\mu}$, the latter sum does not exceed $\varepsilon$ which yields the claim. q.e.d.

Recall the definition of the automorphisms $T_\gamma$ of $L^\infty(I, V)$ $(\gamma \in H_1^p(I, U))$ preceding Lemma 2.6. From Lemma 3.1 we obtain

**Corollary 3.2.** *Let $\mu > 0$ and $\varphi \in HQ \cap E^{-1}[0, \mu)$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that $|((\mathrm{Id} - T_\varphi)^{-1} - (\mathrm{Id} - T_\gamma)^{-1}c| < \varepsilon$ for all $c \in \mathbb{R}^m$ with $\|c\| = 1$ and all $\gamma \in HQ \cap E^{-1}[0, \mu)$ with $d_\infty(\gamma, \varphi) < \delta$.*

*Proof.* Choose a compact neighborhood $B$ of $\varphi(I)$ in $U$. Since $a$ is a smooth 1-form on $U$ with values in the linear space of endomorphisms of $\mathbb{R}^m$, there is $\alpha > 0$ such that $\|a(X)\| \le \alpha\|X\|$ for all $u \in B$ and

$X \in T_uM$. This means $\nu(\gamma) \le \alpha\sqrt{\mu}$ for all $\gamma \in HQ \cap E^{-1}[0, \mu) \cap H_1^p(I, B) = \widetilde{H}$.

Let $\varepsilon > 0$ and choose $\bar{\varepsilon} < \varepsilon/2(2\sqrt{\mu}+2)^{2\sqrt{\mu}+1}$. Let $\Sigma$ be the unit sphere in $\mathbb{R}^m$. Then the map $\psi : \Sigma \times I \to V$, $(c, t) \to ((\mathrm{Id} - T_\varphi)^{-1}c)(t)$ is continuous. Hence there is $\rho > 0$ such that $\|\psi_c(t) - \psi_c(s)\| < \bar{\varepsilon}/(\alpha\sqrt{\mu}+1)$ for all $c \in \Sigma$ and $s, t \in I$ with $|s-t| < \rho$. Moreover

$$\sigma = \sup\{\|\psi_c(t)\| \,|\, c \in \Sigma,\, t \in I\} < \infty.$$

Let $k > 1/\rho$ and define $\tilde{\psi}_c(t) = \psi_c([kt]/k)$ for $c \in \Sigma$. Then $\tilde{\psi}_c \in L^\infty(I, \mathbb{R}^m)$ and $|\tilde{\psi}_c - \psi_c| < \bar{\varepsilon}/(\alpha\sqrt{\mu}+1)$, where $|\ |$ is the norm in $L^\infty(I, \mathbb{R}^m)$ as before. Let $\gamma \in \widetilde{H}$; since the operator norm of $\mathrm{Id} - T_\gamma$ does not exceed $\alpha\sqrt{\mu}+1$, we have $|(\mathrm{Id} - T_\gamma)(\psi_c - \tilde{\psi}_c)| < \bar{\varepsilon}$.

For $\gamma \in \widetilde{H}$ and $t \in I$, define a linear endomorphism $A_\gamma(t)$ of $\mathbb{R}^m$ by $A_\gamma(t) = \int_0^t a(\gamma'(s))\,ds$ which means $\lambda A_\gamma(t) = \int_0^t \lambda(a(\gamma'(s)))\,ds$ for every linear functional $\lambda$ on the vector space of linear endomorphisms of $\mathbb{R}^m$. Then

$$\begin{aligned} T_\gamma \tilde{\psi}_c(t) &= \int_{[tk]/k}^t a(\gamma'(t))\psi_c([tk]/k)\,dt \\ &\quad + \sum_{j=0}^{[tk]-1} \int_{j/k}^{(j+1)/k} a(\gamma'(t))\psi_c(j/k)\,dt \\ &= (A_\gamma(t) - A_\gamma([tk]/k))\psi_c([tk]/k) \\ &\quad + \sum_{j=0}^{[tk]-1} (A_\gamma(j+1)/k - A_\gamma(j/k))\psi_c(j/k). \end{aligned}$$

Since $a(X_u, \theta Y_u) \in \mathbb{R}^m$ depends smoothly on $u \in U$ for smooth vectors fields $X$, $Y$ on $U$, Lemma 3.1 shows that there is $\delta > 0$ such that for all $\gamma \in HQ \cap E^{-1}[0, \mu)$ with $d_\infty(\varphi, \gamma) < \delta$ we have $\gamma \in \widetilde{H}$ and

$$\sup_{t \in I} \|A_\gamma(t) - A_\varphi(t)\| < \bar{\varepsilon}/2(k+1)\sigma.$$

By the definition of $\sigma$ this means

$$|T_\varphi \tilde{\psi}_c - T_\gamma \tilde{\psi}_c| \le 2\sum_{j=0}^k \|A_\varphi(j/k) - A_\gamma(j/k)\|\sigma < \bar{\varepsilon},$$

hence

$$\begin{aligned} |(\mathrm{Id} - T_\gamma)\tilde{\psi}_c - c| &\le |(\mathrm{Id} - T_\gamma)\tilde{\psi}_c - (\mathrm{Id} - T_\varphi)\tilde{\psi}_c| \\ &\quad + |(\mathrm{Id} - T_\varphi)\tilde{\psi}_c - (\mathrm{Id} - T_\varphi)\psi_c| < 2\bar{\varepsilon}. \end{aligned}$$

Now by Lemma 2.6 the operator norm of $(\mathrm{Id}-T_\gamma)^{-1}$ does not exceed $(2\alpha\sqrt{\mu}+2)2^{2\sqrt{\mu}+1}$; from this we obtain

$$|(\mathrm{Id}-T_\gamma)^{-1}c-\psi_c|\leq|(\mathrm{Id}-T_\gamma)^{-1}c-\tilde{\psi}_c|+|\tilde{\psi}_c-\psi_c|<\varepsilon$$

which is the claim.

**Corollary 3.3.** *For every* $\mu>0$ *and every* $k\leq m$ *the set* $\{\varphi\in HQ|\operatorname{rank}R_\varphi\geq k\}\cap E^{-1}[0,\mu)$ *is open in* $HQ\cap E^{-1}[0,\mu)\subset(C^0(I,U),d_\infty)$.

*Proof.* Let $\varphi\in HQ$ with $E(\varphi)<\mu$ and $\operatorname{rank}R_\varphi=k$. Let $\Sigma\subset\mathbb{R}^m$ be the unit sphere in the orthogonal complement of the intersection of $V$ with $(\mathrm{Id}-T_\varphi)L^\infty(I,Q^\perp)$. Then there is by Lemma 2.5 a number $\varepsilon>0$ such that $\sup_{t\in I}\|\widetilde{P}(\mathrm{Id}-T_\varphi)^{-1}c_0\|>2\varepsilon$ for all $c_0\in\Sigma$. By Lemma 3.2 we can find $\delta>0$ such that $|(\mathrm{Id}-T_\gamma)^{-1}c_0-(\mathrm{Id}-T_\varphi)^{-1}c_0|<\varepsilon$ for all $c_0\in\Sigma$ and all $\gamma\in HQ\cap E^{-1}[0,\mu)$ with $d_\infty(\varphi,\gamma)<\delta$. For such a $\gamma$ we have $\sup_{t\in I}\|\widetilde{P}(\mathrm{Id}-T_\gamma)^{-1}c_0\|>\varepsilon$ for all $c_0\in\Sigma$, which means by Lemma 2.5 that $\operatorname{rank}R_\gamma=m$.

## 4. Critical points of the energy function

Let $\psi\colon(-\varepsilon,\varepsilon)\to HQ$ be a variation of $\varphi=\psi_0$ with variation vector field $X=\frac{\partial}{\partial s}\psi|_{s=0}$. For the derivative at $s=0$ of the energy function $E$ on $HQ$ we obtain

$$\frac{\partial}{\partial s}E(\psi_s)|_{s=0}=\int_0^1\langle(d\Theta_\varphi X)(t),\theta\varphi'(t)\rangle\,dt,$$

i.e., $(d\Theta_\varphi)^{-1}\theta\varphi'$ is the gradient of $E$ at $\varphi$.

Call $\varphi\in HQ$ a *critical point* of $E$ if $(d\Theta_\varphi)^{-1}\theta\varphi'\in\Omega(\varphi)^\perp$. If the rank of $R$ at $\varphi$ is maximal, then there is a neighborhood $A$ of $\varphi$ in $HQ$ such that $R^{-1}(\varphi(1))\cap A$ is a smooth submanifold of $A$, and $\varphi$ is thus critical for the restriction of $E$ to this submanifold in the usual sense.

This immediately shows that every minimizing $d_c$-geodesic $\varphi$ with rank $R_\varphi=m$ is necessarily a critical point for $E$.

Define a smooth $(2,1)$-tensor field $\bar{a}$ on $U$ by $\theta\bar{a}_u(X,Y)=a(X)\theta Y$ $(u\in U,X,Y\in T_uM)$.

**Lemma 4.1.** *The critical points of* $E$ *are smooth curves parametrized proportional to arc length.*

*Proof.* If $\varphi\in HQ$ is a critical point of $E$, then $(d\Theta_\varphi)^{-1}\theta\varphi'$ is the projection in $HQ_\varphi$ of an element of $J(\varphi)$. Thus by Lemma 2.5 there is a function $\alpha\colon I \to Q^{\perp} \subset \mathbb{R}^m$ such that

$$\frac{d}{dt}\theta\varphi'(t) + \frac{d}{dt}\alpha(t) - a(\varphi'(t), \theta\varphi'(t) + \alpha(t)) \equiv 0. \tag{2}$$

If, by abuse of notation, we denote by $\widetilde{P}$ the $\langle\ ,\ \rangle$-orthogonal projection of $TM$ onto $Q$, then (2) transforms to

$$\frac{d}{dt}\theta c(t) = \theta\overline{a}(\widetilde{P}c(t), c(t)) \tag{2'}$$

which is a system of first order differential equations on $TU$ with $C^{\infty}$-coefficients. Thus every solution of (2′) is smooth, and moreover (2) shows $\langle \frac{d}{dt}\theta\varphi'(t), \theta\varphi'(t)\rangle \equiv 0$, i.e., critical points of $E$ are parametrized proportional to arc length. q.e.d.

Now for every initial condition $X \in T_pM$ there is a unique maximal solution $\tilde{\lambda}(X)$ of (2′) which depends smoothly on $X$. $\tilde{\lambda}(X)$ projects onto a smooth curve $\lambda(X)$ in $U$ which is tangent to $Q$ and parametrized proportional to arc length. Since, by definition, $U$ is just the open $d_c$-ball of radius $\rho$ around $p = \lambda(X)(0)$, for every $X \in T_pM$ with $\|\widetilde{P}X\| < \rho$ the curve $\lambda(X)$ is defined on $I$ and $\lambda(X)|_I \in HQ$ is a critical point of $E$. Hence $X \to \lambda(X)(1)$ defines a smooth map $\exp_p^c$ of $\widetilde{U} = \{X \in T_pM | \|\widetilde{P}X\| < \rho\}$ into $U$. Now $\theta\frac{d}{dt}\lambda(X)(bt) = b\theta\lambda(X)'(bt)$ shows $\theta\tilde{\lambda}(bX)(t) = b\tilde{\lambda}(X)(bt)$ or $\lambda(bX)(t) = \lambda(X)(bt)$ for all $b \in \mathbb{R}$. Thus for every $X \in \widetilde{U}$ we have $\exp_p^c(tX) = \lambda(X)(t)$ $(t \in I)$, i.e., $t \to \exp_p^c(tX)$ $(t \in I)$ is a critical point of $E$.

**Lemma 4.2.** *Let $X \in T_pM$ be such that $\lambda(X)$ is defined on $I$ and* $\operatorname{rank} R_\lambda(X) = m$. *Then for every $\mu > \|\widetilde{P}X\|$ there is $\delta > 0$ and $\beta < \infty$ such that $d_\infty(\lambda(X), \lambda(Y)) < \delta$ and $\|\widetilde{P}Y\| < \mu$ implies $\|Y\| < \beta$.*

*Proof.* Let $\Sigma$ be the unit sphere in $\mathbb{R}^m$. Since rank $R_{\lambda(X)} = m$, there is then $\alpha > 0$ such that $|\widetilde{P}(\mathrm{Id} - T_{\lambda(X)})^{-1}c| > \alpha|(\mathrm{Id} - T_{\lambda(X)})^{-1}c|$ for all $c \in \Sigma$.

Let $\sigma_1 = \inf\{|(\mathrm{Id} - T_{\lambda(X)})^{-1}c| | c \in \Sigma\}$ and $\sigma_2 = \sup\{|(\mathrm{Id} - T_{\lambda(X)})^{-1}c| | c \in \Sigma\}$. By Corollary 3.2 we can choose $\delta > 0$ such that

$$\sup_{c\in\Sigma} |((\mathrm{Id} - T_{\lambda(X)})^{-1} - (\mathrm{Id} - T_\gamma)^{-1})c| < \alpha\sigma_1/2$$

for all $\gamma \in HQ \cap E^{-1}[0, \mu^2)$ with $d_\infty(\lambda(X), \gamma) < \delta$. For such a $\gamma$ we then have

$$|\widetilde{P}(\mathrm{Id} - T_\gamma)^{-1}c| > \alpha\sigma_1/2 > \alpha\sigma_1\sigma_2|(\mathrm{Id} - T_\gamma)^{-1}c|/4$$

for all $c \in \Sigma$. Hence if $Y \in T_pM$ with $\|\widetilde{P}Y\| < \mu$ and $d_\infty(\lambda(X), \lambda(Y)) < \delta$, then $E(\lambda(Y)) < \mu^2$ and

$$|\widetilde{P}(\mathrm{Id} - T_{\lambda(Y)})^{-1}\theta Y| = \|\widetilde{P}Y\| > \alpha\sigma_1\sigma_2|(\mathrm{Id} - T_{\lambda(Y)})^{-1}\theta Y|/4$$
$$\geq \alpha\sigma_1\sigma_2\|Y\|/4.$$

This yields the claim. q.e.d.

Recall that a *$d_c$-geodesic* is a curve $\varphi$ in $M$, which is parametrized proportional to arc length and realizes locally the $d_c$-distance of its curve points. If the closure of $U$ in $M$ is compact (which is always true if we choose $U$ small enough), then every $u \in U$ can be joined to $p$ by a minimizing $d_c$-geodesic (see [6, 17]).

Any such geodesic which is parametrized on $I$ is necessarily a critical point of $E$. This was stated in [17], however the proof provided there is only valid in the strong bracket generating case (where it also follows from the fact that the map $R$ is of maximal rank on each nontrivial curve in $H_1^p(I, U)$). The general case was established by Bär [1]. In particular the map $\exp_p^c\colon \widetilde{U} \to U$ is surjective.

**Corollary 4.3.** *Let $u \in U$ be a regular value of $\exp_p^c$. Then the set $(\exp_p^c)^{-1}(u) \cap \{Y \mid \|\widetilde{P}Y\| = d_c(p, u)\}$ is finite.*

*Proof.* The set $A = (\exp_p^c)^{-1}(u) \cap \{Y \mid \|\widetilde{P}Y\| = d_c(p, u)\}$ is nonempty and closed in $T_pM$. Since $u$ is a regular value for $\exp_p^c$, $A$ is moreover discrete. Assume that there is a sequence $\{X_k\} \subset A$ such that $\|X_k\| \to \infty$ $(k \to \infty)$. Then $\lambda(X_k)$ is a minimizing geodesic joining $p$ to $u$; its energy equals $d_c(p, u)^2$. Thus by passing to a subsequence we may assume that the curves $\lambda(X_k)$ converges in $(C^0(I, U), d_\infty)$ to a curve $\varphi$ which is necessarily a minimizing $d_c$-geodesic joining $p$ to $u$. Then $\varphi = \lambda(X)$ for some $X \in A$ and Lemma 4.2 shows that there is $k_0 > 0$ and $\beta < \infty$ such that $\|X_k\| < \beta$ for all $k > k_0$. This contradicts the assumption that $\|X_k\| \to \infty$ and shows that $A$ is bounded, hence finite.

## 5. Calculus of variation

Since the rank of $\exp_p^c$ at $0 \in T_pM$ is not maximal the above considerations do not necessarily imply that $\lambda(X)$ is a $d_c$-geodesic for every $X \in \widetilde{U}$, i.e., is locally minimizing with respect to $d_c$. To show that this is nevertheless true we compute the variation of the energy at the critical point $\lambda(X)$ $(X \in \widetilde{U})$.

**Lemma 5.1** *(First variational formula). Let $X \in T_pM$ be such that $\lambda(X)$ is defined on $I$ and let $\psi\colon (-\varepsilon, \varepsilon) \to HQ$ be a variation of $\lambda(X) = \psi_0$ with variation vector field $Y = \frac{\partial}{\partial s}\psi|_{s=0}$. Then $\frac{d}{ds}E(\psi_s)|_{s=0} = \langle Y(1), \tilde{\lambda}(X)(1)\rangle$.*

*Proof.* With $\varphi = \lambda(X)$ we have

$$\begin{aligned}\frac{d}{ds}E(\psi_s)|_{s=0} &= \int_0^1 \langle (d\Theta_\varphi Y)(t), \theta\varphi'(t)\rangle\, dt \\ &= \int_0^1 \langle (d\Theta_\varphi Y)(t), \theta\tilde{\lambda}(X)(t)\rangle\, dt \\ &= \int_0^1 \left\langle \frac{d}{dt}\theta Y(t), \theta\tilde{\lambda}(X)(t)\right\rangle dt \\ &\quad + \int_0^1 \langle \theta Y(t), a(\varphi'(t), \theta\tilde{\lambda}(X)(t))\rangle\, dt \\ &= \langle Y(1), \tilde{\lambda}(X)(1)\rangle. \quad \text{q.e.d.}\end{aligned}$$

Recall that $d\theta$ is a smooth 2-form on $U$ with values in $\mathbb{R}^m$. Hence for $u \in U$ and every tangent vector $Y \in T_uM$ the derivative $Y(d\theta)$ of $d\theta$ in the direction of $Y$ is a bilinear mapping of $T_uM$ into $\mathbb{R}^m$ depending smoothly on $Y$.

It will be convenient, furthermore, to use the following notational convention: Recall that for every $u \in U$ the restriction of $\theta$ to $T_uM$ is a linear isomorphism of $T_uM$ onto $\mathbb{R}^m$, i.e., for every $W \in \mathbb{R}^m$ there is a unique $W(u) \in T_uM$ such that $\theta W(u) = W$. Thus whenever no confusion about the base point $u$ is possible we can write $d\theta(W, Z)$ or $d\theta(W, Z(u))$ or $d\theta(W(u), Z)$ to denote the vector $d\theta(W(u), Z(u)) \in \mathbb{R}^m$. Similarly we denote by $a(W)$ the linear map $a(W(u))$ ($u \in U$, $W$, $Z \in \mathbb{R}^m$). With this convention the second variational formula for $E$ can be expressed as follows:

**Lemma 5.2** (*Second variational formula*). *Let $X \in T_pM$ be such that $\lambda(X)$ is defined on $I$, and let $\psi\colon (-\varepsilon, \varepsilon)^2 \to HQ$ be a 2-parameter variation of $\varphi = \psi(0, 0)$ with fixed endpoints $\psi(-\varepsilon, \varepsilon)^2(1) = \varphi(1)$ and variation vector fields $Y = \frac{\partial}{\partial s}\psi|_{u=s=0}$, $Z = \frac{\partial}{\partial u}\psi|_{u=s=0}$. Then*

$$\begin{aligned}&\frac{\partial}{\partial u}\frac{\partial}{\partial s}E(\psi(u, s))|_{u=s=0} \\ &\quad = \int_0^1 \langle (d\Theta_\varphi Z)(t), (d\Theta_\varphi Y)(t) - a(Y(t), \theta\tilde{\lambda}(X)(t))\rangle\, dt \\ &\quad\quad + \int_0^1 2\langle (Z(t)\, d\theta)(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t)\rangle\, dt.\end{aligned}$$

*Proof.* Since

$$\frac{\partial}{\partial u}\frac{\partial}{\partial s}E(\psi(u,s))|_{u=s=0} = \int_0^1 \left\langle \frac{\partial}{\partial u}\frac{\partial}{\partial s}\theta\frac{\partial}{\partial t}\psi, \theta\frac{\partial}{\partial t}\psi\right\rangle_{u=s=0} dt + \int_0^1 \langle (d\Theta_\varphi Y)(t), (d\Theta_\varphi Z(t))\rangle\, dt,$$

we have to transform the first integral. Define $W(t) = \frac{\partial}{\partial u}\theta\frac{\partial}{\partial s}\Psi|_{u=s=0}$; then

$$\frac{\partial}{\partial s}\theta\frac{\partial}{\partial t}\Psi = \frac{\partial}{\partial t}\theta\frac{\partial}{\partial s}\Psi + 2d\theta\left(\frac{\partial}{\partial t}\Psi, \frac{\partial}{\partial s}\Psi\right)$$

yields

$$\frac{\partial}{\partial u}\frac{\partial}{\partial s}\theta\frac{\partial}{\partial t}\Psi|_{u=s=0} = \frac{d}{dt}W(t) + 2d\theta(\varphi'(t), W(t)) + 2d\theta(d\Theta_\varphi Z(t), Y(t)) + 2(Z(t)d\theta)(\varphi'(t), Y(t)).$$

Since $W(1) = 0$ we have $\int_0^1 \langle (d\Theta_\varphi W)(t), \theta\tilde{\lambda}(X)(t)\rangle\, dt = 0$, i.e.,

$$\int_0^1 \langle (d\Theta_\varphi W)(t), \theta\varphi'(t)\rangle\, dt = -\int_0^1 \langle (d\Theta_\varphi W)(t), (\mathrm{Id} - \widetilde{P})\theta\tilde{\lambda}(X)(t)\rangle\, dt.$$

But $\frac{\partial}{\partial u}\frac{\partial}{\partial s}\theta\frac{\partial}{\partial t}\psi \subset Q = \widetilde{P}\mathbb{R}^m$ shows

$$(\widetilde{P} - \mathrm{Id}(d\Theta_\varphi W)(t) = (\mathrm{Id} - \widetilde{P})(2d\theta(d\Theta_\varphi Z(t), Y(t)) + 2(Z(t)d\theta)(\varphi'(t), Y(t)));$$

hence

$$\int_0^1 \left\langle \frac{\partial}{\partial u}\frac{\partial}{\partial s}\theta\frac{\partial}{\partial t}\Psi|_{u=s=0}, \theta\varphi'(t)\right\rangle dt = \int_0^1 \langle 2d\theta(d\Theta_\varphi Z(t), Y(t)) + 2(Z(t)\,d\theta)(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t)\rangle\, dt,$$

and from this the lemma follows.

**Remark 5.3.** Assume again that $M$ is a Lie group and that the vector fields $X^1, \cdots, X^m$ are left-invariant. By Remark 2.8 we have $J(\varphi) = \{X \in H_1^e(\varphi) | (d\Theta_\varphi X)(t) = Ad^*_{\varphi(t)}c_0$ for some $c_0 \in \mathfrak{M}\}$. For $Y \in \mathfrak{M}$ let $(\mathrm{ad}(Y))^*$ be the adjoint of the linear endomorphism $\mathrm{ad}(Y)$ of $\mathfrak{M}$. Then the formula of Lemma 5.2 reduces to

$$\frac{\partial}{\partial u}\frac{\partial}{\partial s}E(\Psi(u,s))|_{u=s=0} = \int_0^1 \langle (d\Theta_\varphi Z)(t), (d\Theta_\varphi Y)(t) - (\mathrm{ad}(\theta Y(t)))^*(\theta\tilde{\lambda}(X)(t))\rangle\, dt.$$

Thus if $Y$ is contained in the zero space of the Hessian of $E$ at $\varphi$, then $Y$ is a solution of the differential equation

$$(3) \qquad (d\Theta_\varphi Y)(t) = \widetilde{P}(\mathrm{Ad}^*_{\varphi(t)} c_0 + (\mathrm{ad}(\theta Y(t)))^*(\theta\tilde{\lambda}(X)(t)))$$

for some $c_0 \in \mathfrak{M}$. Every solution of (3) is uniquely determined by the choice of $c_0$ and the initial condition $Y(0)$; in particular the dimension of the vector space of solutions of (3) vanishing at $t = 0$ equals rank $R_\varphi = \dim \Omega(\varphi)^\perp$ which in contrast to the fact that the Riemannian situation may be strictly smaller than $\dim M$.

Next we want to compute the zero space of the Hessian of $E$. For this the following notation will be useful: Given $\varphi \in HQ$ and $Z \in H_0(I, \mathbb{R}^m)$ there is a unique vector field $\fint Z \in H_1^p(\varphi)$ such that $Z(t) = \frac{d}{dt}\theta(\fint Z)(t) - a(\varphi'(t), \theta(\fint Z)(t))$. Write also $(\fint Z)(t) = \fint_0^t Z(\tau)\,d\tau$. For every $W \in H_1^p(\varphi)$ we then have

$$\frac{d}{dt}\langle W(t), (\fint Z)(t)\rangle = \langle d\Theta_\varphi W(t), \theta(\fint Z)(t)\rangle + \langle \theta W(t), Z(t)\rangle.$$

Now if $\omega$ is a $(2, 0)$-tensor on $U$ with values in $\mathbb{R}^m$, then for each $u \in U$ and $X \in T_uM$ the assignment $Y \to \omega(X, Y)$ is a linear map of $T_uM$ into $\mathbb{R}^m$. We denote by $(\omega(X))^*$ its adjoint. With these notation we obtain

**Corollary 5.4.** *Under the assumptions of Lemma* 5.2 *we have*

$$\begin{aligned}\frac{\partial}{\partial u}\frac{\partial}{\partial s}E(\psi(u, s))|_{u=s=0} \\ = \int_0^1 \Big\langle d\Theta_\varphi Z(t), d\Theta_\varphi Y(t) - \theta \fint_0^t a(d\Theta_\varphi Y(\tau), \theta\tilde{\lambda}(X)(\tau))\,d\tau \\ - \theta \fint_0^t 2((Y(\tau)\,d\theta)(\varphi'(\tau)))^*\theta\tilde{\lambda}(X)(\tau)\,d\tau\Big\rangle\,dt.\end{aligned}$$

*Proof.* The claim follows from Lemma 5.2 and the following computation:

$$\begin{aligned}-\int_0^1 \langle d\Theta_\varphi Y(t), a(Z(t), \theta\tilde{\lambda}(X)(t))\rangle\,dt \\ = \int_0^1 \langle \theta Z(t), a(d\Theta_\varphi Y(t), \theta\tilde{\lambda}(X)(t))\rangle\,dt,\end{aligned}$$

and since $Z(1) = 0$, integration by parts shows that the latter integral equals

$$-\int_0^1 \Big\langle d\Theta_\varphi Z(t), \theta \fint_0^t a(d\Theta_\varphi Y(\tau), \theta\tilde{\lambda}(X)(\tau))\,d\tau\Big\rangle\,dt.$$

Analogously

$$\int_0^1 \langle ((Y(t)\,d\theta)(\varphi'(t), Z(t)), \theta\tilde{\lambda}(X)(t)\rangle\,dt$$
$$= \int_0^1 \langle \theta Z(t), ((Y(t)\,d\theta)(\varphi'(t)))^* \theta\tilde{\lambda}(X)(t)\rangle\,dt$$
$$= -\int_0^1 \left\langle d\Theta_\varphi Z(t), \theta \int_0^t ((Y(\tau)\,d\theta)(\varphi'(\tau)))^* \theta\tilde{\lambda}(X)(\tau)\,d\tau \right\rangle dt. \quad \text{q.e.d.}$$

Corollary 5.4 shows that if a field $Y \in HQ_\varphi$ is contained in the zero space of the Hessian of $E$ at $\varphi$, then there is $\tilde{Y} \in J(\varphi)$ such that

$$\begin{aligned} d\Theta_\varphi Y(t) = \tilde{P}\Big(d\Theta_\varphi \tilde{Y}(t) + \theta \int_0^t a(d\Theta_\varphi Y(\tau), \theta\tilde{\lambda}(X)(\tau))\,d\tau \\ + \theta \int_0^t 2((Y(\tau)\,d\theta)(\varphi'(\tau)))^* \theta\tilde{\lambda}(X)(\tau)\,d\tau\Big). \end{aligned} \tag{4}$$

This differential equation can be transformed to a differential equation of the form $c'(t) = f(t, c(t))$ for some smooth function $f: I \times V \to V$ which is linear in the second variable as follows (then $c(t)$ is interpreted as $\theta Y(t)$): We have

$$\begin{aligned} &a(d\Theta_\varphi Y(t), \theta\tilde{\lambda}(X)(t)) \\ &\quad = \frac{d}{dt} a(Y(t), \theta\tilde{\lambda}(X)(t)) - (\varphi'(t)a)(Y(t), \theta\tilde{\lambda}(X)(t)) \\ &\quad - a(Y(t), a(\varphi'(t), \theta\tilde{\lambda}(X)(t))) \\ &\quad + 2a(d\theta(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t)), \end{aligned}$$

and hence

$$\begin{aligned} &\theta \int_0^t a(d\Theta_\varphi Y(t), \theta\tilde{\lambda}(X)(\tau))\,d\tau \\ &\quad = a(Y(t), \theta\tilde{\lambda}(X)(t)) + \theta \int_0^t \Big(a(\varphi'(\tau), a(Y(\tau), \theta\tilde{\lambda}(X)(\tau))) \\ &\qquad - (\varphi'(\tau)a)(Y(\tau), \theta\tilde{\lambda}(X)(\tau)) \\ &\qquad - a(Y(\tau), a(\varphi'(\tau), \theta\tilde{\lambda}(X)(\tau))) \\ &\qquad + 2a(d\theta(\varphi'(\tau), Y(\tau)), \theta\tilde{\lambda}(X)(\tau))\Big)\,d\tau \\ &\quad = \tilde{f}(t, \theta Y(t)), \end{aligned}$$

where $\tilde{f}\colon I \times V \to V$ is clearly linear in the second variable. Thus

$$\begin{aligned}\frac{d}{dt}\theta Y(t) = \tilde{P}\Big(d\Theta_\varphi \tilde{Y}(t) + \tilde{f}(t, \theta Y(t))\\ + \theta \int_0^t ((Y(\tau)\, d\theta)(\varphi'(\tau)))^* \theta\tilde{\lambda}(X)(\tau)\, d\tau\Big)\\ - 2d\theta(\varphi'(t), Y(t))\end{aligned} \tag{4'}$$

is clearly an equation of the required form.

Thus for every $\tilde{Y} \in J(\varphi)$ and every $Y_0 \in T_pM$ there is a unique solution $Y$ of (4) with initial condition $Y(0) = Y_0$. Such a field is called a *Jacobi field* along $\varphi$.

By the linearity of $(4')$ the Jacobi fields along $\varphi$ form a vector space of dimension $m + \operatorname{rank} R_\varphi$, and the zero space of the Hessian of $E$ at $\varphi$ consists exactly of the space of Jacobi fields along $\varphi$ vanishing at $t = 0$ and $t = 1$.

As in the Riemannian situation the space of Jacobi fields vanishing at $t = 0$ equals the space of variational vector fields along $\varphi$ of variations by geodesics.

**Lemma 5.5.** *Let $\psi(s, t) = \lambda(X_1 + sX_2)(t)$ for some $X_1, X_2 \in V$. Then $\frac{\partial}{\partial s}\psi|_{s=0}$ is the Jacobi field $Y$ along $\varphi = \psi_0$ with initial condition $Y(0) = 0$ which is determined by the field $\tilde{Y} \in J(\varphi)$ with $d\Theta_\varphi \tilde{Y}(0) = X_2$.*

*Proof.* Let $\alpha(s, t) = (1 - \tilde{P})\theta\tilde{\lambda}(X_1 + sX_2)(t)$ and define $Y(t) = \frac{\partial}{\partial s}\psi(s, t)|_{s=0}$. Since

$$\frac{\partial}{\partial t}\theta\frac{\partial}{\partial t}\psi + \frac{\partial}{\partial t}\alpha - a\left(\frac{\partial}{\partial t}\psi, \theta\frac{\partial}{\partial t}\psi + \alpha\right) \equiv 0$$

and

$$\frac{\partial}{\partial s}\theta\frac{\partial}{\partial t}\psi|_{s=0} = d\Theta_\varphi Y,$$

we have

$$\begin{aligned}\left(\frac{\partial}{\partial s}\frac{\partial}{\partial t}\theta\frac{\partial}{\partial t}\psi + \frac{\partial}{\partial s}\frac{\partial}{\partial t}\alpha\right)_{s=0} &= \frac{d}{dt}d\Theta_\varphi Y + \frac{\partial}{\partial t}\frac{\partial}{\partial s}\alpha|_{s=0}\\ &= (Y(t)a)(\varphi'(t), \theta\varphi'(t) + \alpha(0, t))\\ &\quad + a(d\Theta_\varphi Y(t), \theta\varphi'(t) + \alpha(0, t))\\ &\quad + a\left(\varphi'(t), d\Theta_\varphi Y(t) + \frac{\partial}{\partial s}\alpha|_{s=0}\right),\end{aligned}$$

which means

$$\frac{d}{dt}\left(d\Theta_\varphi Y(t)+\frac{\partial}{\partial s}\alpha|_{s=0}\right)-a\left(\varphi'(t),\,d\Theta_\varphi Y(t)+\frac{\partial}{\partial s}\alpha|_{s=0}\right)$$
$$=(Y(t)a)(\varphi'(t),\,\theta\tilde{\lambda}(X_1)(t))+a(d\Theta_\varphi Y(t),\,\theta\tilde{\lambda}(X_1)(t)).$$

Let $\widetilde{Y}\in J(\varphi)$ be such that $d\Theta_\varphi\widetilde{Y}(0)=X_2=d\Theta_\varphi Y(0)+\frac{\partial}{\partial s}\alpha(0,0)$. Since by Lemma 2.5 $\frac{d}{dt}d\Theta_\varphi\widetilde{Y}(t)-a(\varphi'(t),\,d\Theta_\varphi\widetilde{Y}(t))=0$, it follows from the above equation that

$$d\Theta_\varphi Y(t)+\frac{\partial}{\partial s}\alpha=\theta\int_0^t((Y(\tau)a)(\varphi'(\tau),\,\theta\tilde{\lambda}(X_1)(\tau)))\,d\tau$$
$$+\theta\int_0^t a(d\Theta_\varphi Y(\tau),\,\theta\tilde{\lambda}(X_1)(\tau))\,d\tau+d\Theta_\varphi\widetilde{Y}(t),$$

hence we only have to show that $(Y(t)a)(Z,W)=2((Y(t)\,d\theta)Z)^*W$ for all $Z$, $W\in\mathbb{R}^m$. Let $X\in\mathbb{R}^m$. Then

$$\begin{aligned}\frac{\partial}{\partial s}\langle a_{\psi(s,t)}(Z,W),X\rangle_{s=0}&=\langle(Y(t)a)(Z,W),X\rangle_{s=0}\\&=2\frac{\partial}{\partial s}\langle W,d\theta_{\psi(s,t)}(Z,X)\rangle_{s=0}\\&=2\langle W,(Y(t)d\theta)(Z,X)\rangle\\&=2\langle((Y(t)\,d\theta)Z)^*W,X\rangle,\end{aligned}$$

which implies $(Y(t)a)(Z,W)=2((Y(t)\,d\theta)Z)^*W$ as required.

**Remark 5.6.** (a) For $X\in T_pM$ let null$(X)$ be the dimension of the vector space of Jacobi fields along $\lambda(X)$ vanishing at $\lambda(X)(0)$ and $\lambda(X)(1)$. It then follows from Corollary 5.4 and Lemma 5.5 that the rank of $\exp_p^c$ at $X$ equals rank $R_{\lambda(X)}-\text{null}(X)$. In particular if $\widetilde{X}\in T_{\lambda(X)(1)}M$ is such that $\lambda(\widetilde{X})(t)=\lambda(X)(1-t)$, then the rank of $\exp^c_{\lambda(X)(1)}$ at $\widetilde{X}$ equals the rank of $\exp_p^c$ at $X$.

(b) By Sard's theorem almost every $u\in U$ is a regular value for $\exp_p^c$. If $u\in U$ is such a point, then (a) shows that $p$ is a regular value for $\exp_u^c$. Let $X\in T_pM$ be such that $t\to\exp_p^c tX$ is a minimizing geodesic joining $p$ to $u$. Then $\exp_p^c$ has maximal rank at $X$ and hence by (a) and 5.4 and 5.5 the zero space of the Hessian of $E$ at $\lambda(X)$ vanishes. Since $\lambda(X)$ is minimizing, this means that the Hessian of $E$ at $\lambda(X)$ is positive definite.

We now have a closer look at the Hessian of $E$ at the critical point $\varphi$. Assume that $\rho\le 2$ as in the beginning of this section is sufficiently small such that the closed $d_c$-ball of radius $\rho$ around $p$ is compact. For

$X \in T_pM$ with $\|\widetilde{P}X\| < \rho$, $\varphi = \lambda(X)$ is defined on $I$ and thus can be viewed as an element of $HQ$. For $Y, Z \in HQ_\varphi$ define

$$I_X(Y, Z) = \int_0^1 \langle (d\Theta_\varphi Z)(t), (d\Theta_\varphi Y)(t) - a(Y(t), \theta\tilde{\lambda}(X)(t))\rangle\, dt$$
$$+ \int_0^1 2\langle (Z(t)\, d\theta)(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t)\rangle\, dt.$$

Then we have

**Lemma 5.7.** *There is $\kappa \in (0, \rho/2]$ such that $I_X(Y, Y) > 0$ for all $X \in T_pM$ with $\|X\| < \kappa$ and all $0 \neq Y \in HQ_{\lambda(X)}$.*

*Proof.* Since $\lambda(sX)(t) = \lambda(X)(st)$ for all $X \in T_pM$ and $s, t \in I$, it suffices to show that there is $\kappa \in (0, \rho/2]$ such that for all $X \in T_pM$ with $\|\widetilde{P}X\| = \rho/2$, all $\delta < \kappa/\|X\|$, and all $Y \in HQ_{\lambda(X)}$ which do not vanish identically on $[0, \delta]$ we have

$$I_\delta(Y, Y) = \int_0^\delta \langle (d\Theta_\varphi Y)(t), (d\Theta_\varphi Y)(t) - a(Y(t), \theta\tilde{\lambda}(X)(t))\rangle\, dt$$
$$+ \int_0^\delta 2\langle (Y(t)\, d\theta)(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t)\rangle\, dt > 0,$$

where as before $\varphi = \lambda(X)$. To show this let $B$ be the compact $d_c$-ball of radius $\rho/2$ around $p$. Then there is $c \geq 1$ such that for all $u \in B$, all $W, \widetilde{W} \in \mathbb{R}^m$, and $Z \in T_uM$

(i) $\|d\theta_u(W, \widetilde{W})\| \leq c\|W\|\,\|\widetilde{W}\|$,
(ii) $\|a_u(W, \widetilde{W})\| \leq c\|W\|\,\|\widetilde{W}\|$,
(iii) $\|(Z\, d\theta)(W, \widetilde{W})\| \leq c\|Z\|\,\|W\|\,\|\widetilde{W}\|/2$.

Now if $Y \in HQ_\varphi$, then $Y(0) = 0$ and consequently $\theta Y(t) = \int_0^t \frac{d}{ds}\theta Y(s)\, ds$ for all $t \in I$. Since $\varphi(I) \subset B$ and $\frac{d}{ds}\theta Y(s) = (d\Theta_\varphi Y)(s) - 2\, d\theta(\varphi'(s), Y(s))$ it follows from (i) and $\|\theta\varphi'(s)\| = \rho/2 \leq 1$ for all $s \in I$ that

$$\int_0^t \left\| \frac{d}{ds}\theta Y(s) \right\| ds \leq \int_0^t \|(d\Theta_\varphi Y)(s)\|\, ds + c\int_0^t \|Y(s)\|\, ds.$$

For $s \leq t$ we have $\|Y(s)\| \leq \int_0^t \|\frac{d}{du}\theta Y(u)\|\, du$, hence

$$\int_0^t \left\| \frac{d}{ds}\theta Y(s) \right\| ds \leq \int_0^t \|(d\Theta_\varphi Y)(s)\|\, ds + ct\int_0^t \left\| \frac{d}{ds}\theta Y(s) \right\| ds.$$

Thus if $t \leq 1/c$, then

$$\begin{aligned}\|Y(t)\|^2 &\leq \left(\int_0^t \left\|\frac{d}{ds}\theta Y(s)\right\| ds\right)^2 \\ &\leq (1-ct)^{-2}\left(\int_0^t \|(d\Theta_\varphi Y)(s)\|\,ds\right)^2 \\ &\leq \frac{t}{(1-ct)^2}\int_0^t \|(d\Theta_\varphi Y)(s)\|^2\,ds.\end{aligned}$$

Now (ii) and (2) for $\theta\tilde{\lambda}(X)$ show $\|\frac{d}{dt}\tilde{\lambda}(X)(t)\| \leq c\|\tilde{\lambda}(X)(t)\|$ and consequently $\|\tilde{\lambda}(X)(t)\| \leq e^{ct}\|X\|$. Thus for $\delta < 1/c$ we obtain

$$\begin{aligned}&\left|\int_0^\delta \langle (d\Theta_\varphi Y)(t), ad^*(\theta Y(t), \theta\tilde{\lambda}(X)(t))\rangle\,dt\right| \\ &\quad\leq \left(\int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2\,dt\right)^{1/2}\left(\int_0^\delta \|ad^*(\theta Y(t), \theta\tilde{\lambda}(X)(t))\|^2\,dt\right)^{1/2} \\ &\quad\leq ce^{c\delta}\|X\|\left(\int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2\,dt\right)^{1/2}\left(\int_0^\delta \|Y(t)\|^2\,dt\right)^{1/2}.\end{aligned}$$

Inserting $\|Y(t)\|^2 \leq (\delta/(1-c\delta)^2)\int_0^\delta \|(d\Theta_\varphi Y)(s)\|\,ds$ $(t \leq \delta)$, this yields

$$\begin{aligned}&\left|\int_0^\delta \langle (d\Theta_\varphi Y)(t), ad^*(\theta Y(t), \theta\tilde{\lambda}(X)(t))\rangle\,dt\right| \\ &\qquad\leq (\delta/(1-c\delta))ce^{c\delta}\|X\|\int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2\,dt.\end{aligned}$$

On the other hand it follows from (iii) that

$$\begin{aligned}&\left|\int_0^\delta 2\langle (Y(t)\,d\theta)(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t)\rangle\,dt\right| \\ &\qquad\leq c\left(\int_0^\delta \|Y(t)\|^2\,dt\right)\left(\int_0^\delta \|\tilde{\lambda}(X)(t)\|\,dt\right) \\ &\qquad\leq (\delta^3/(1-c\delta)^2)ce^{c\delta}\|X\|\int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2\,dt.\end{aligned}$$

Thus if we choose $\delta > 0$ sufficiently small that

$$\delta/(1-c\delta) < \min\{1, (2c\|X\|)^{-1}\},$$

then $I_\delta(Y, Y)$ is not smaller than a positive multiple of $\int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2\,dt$ which is positive for all $Y \in HQ_\varphi$ not vanishing identically on $[0, \delta]$. This is the claim.

**Corollary 5.8.** $\exp_p^c$ *is of maximal rank on an open and dense subset of* $\{X \in T_pM \| \|X\| < \kappa\}$.

*Proof.* We argue by contradiction and assume that there is an open subset $U$ of $\{X \in T_pM \| \|X\| < \kappa\}$ such that $\exp_p^c$ is singular at every $X \in U$. By ev. diminishing the size of $U$ we may assume that the rank of $\exp_p^c$ is constant on $U$ and that $\exp_p^c U$ is a smooth embedded submanifold $N$ of $M$ of dimension $n = \operatorname{rank} \exp_p^c|_U < m$. For $X \in U$ the tangent space of $N$ at $u = \exp_p^c X$ equals the vector space of all endpoints of Jacobi fields along $\lambda(X)$, which vanish at $p = \lambda(X)(0)$, and by 5.6, 5.7, and the choice of $U$ this space is just $\{Y(1) | Y \in HQ_{\lambda(X)}\}$. This means in particular that $Q_u$ is contained in $T_uN$ for all $u \in N$. Hence for all $u \in N$, $T_uN$ contains the span at $u$ of the Lie algebra generated by $Q_u$. This span is $T_uM$ since $Q$ is bracket generating which implies the contradiction $m > \dim N = \dim T_uN \geq \dim T_uM = m$.

**Corollary 5.9.** *Every critical point of $E$ is a geodesic with respect to $d_c$.*

*Proof.* Let $X \in T_pM$ with $\|X\| = \kappa$, let $\varphi = \lambda(X)$, and assume $\operatorname{rank} R_\varphi = n \leq m$. Then $W = \{d\Theta_\varphi Y(1)_{\varphi(1)} | Y \in J(\varphi), PY = 0\}$ is an $(m-n)$-dimensional subspace of $T_{\varphi(1)}M$.

Choose a smooth $(m-n)$-dimensional submanifold $S$ of $U$ containing $\varphi(1)$ with the property that $W$ is the tangent space of $S$ at $\varphi(1)$. Then $\Lambda = \{\psi \in H_1^p(I, U) | \psi(1) \in S\}$ is a smooth submanifold of $H_1^p(I, U)$. By Remark 2.7 for every $\psi \in \Lambda$ the $g_\psi$-orthogonal complement of the tangent space $\Lambda_\psi$ of $\Lambda$ at $\psi$ consists of all $Y \in J(\psi)$ with $d\Theta_\psi Y(1)_{\psi(1)} \in (TS)^\perp$. By the definition of $W$ this means that $HQ$ meets $\Lambda$ transversally at $\varphi$. Moreover the $g_\varphi$-orthogonal complement of the intersection $\Lambda_\varphi \cap HQ_\varphi$ in $HQ_\varphi$ is just $\Omega(\varphi)^\perp$. Thus there is an open neighborhood $A$ of $\varphi$ in $HQ$ such that $A \cap \Lambda$ is a smooth submanifold of $HQ$ and $\varphi$ is a critical point of the restriction of $E$ to $A \cap \Lambda$.

By Lemma 5.7 the Hessian of $E|_{A\cap\Lambda}$ at $\varphi$ is positive definite. Hence there is an open neighborhood $B$ of $\varphi$ in $A \cap \Lambda$ such that $E(\psi) > E(\varphi)$ for all $\psi \in B$, say $B$ is the intersection of $A \cap \Lambda$ with the preimage under $\Theta$ of a $2\varepsilon$-neighborhood of $\theta\varphi'$ in the Hilbert space $H_0(I, Q)$ for some $\varepsilon > 0$.

Assume that there is $\tau \in [0, 1-\varepsilon]$ such that $d_c(\varphi(\tau), \varphi(\tau+\varepsilon)) < \|\widetilde{P}X\|\varepsilon$. Let $\psi\colon [\tau, \tau+\varepsilon] \to U$ be a minimizing geodesic joining $\psi(\tau) = \varphi(\tau)$ to $\psi(\tau+\varepsilon) = \varphi(\tau+\varepsilon)$. Then the curve

$$t \to \begin{cases} \varphi(t) & \text{if } t \in [0, \tau] \cup [\tau+\varepsilon, 1], \\ \psi(t) & \text{if } t \in [\tau, \tau+\varepsilon] \end{cases}$$

is contained in $B$ (recall $\|X\| = \kappa \leq 1$) and its energy is strictly smaller than $E(\varphi)$. This is a contradiction.

## 6. Isometries

In this section we investigate isometries of CC-metrics and show that they are necessarily smooth maps. Let $f\colon (M, d_c) \to (\widetilde{M}, \tilde{d}_c)$ be an isometry. Then $f$ is a homeomorphism which maps the space of $H_1$-curves in $M$ which are tangent almost everywhere to $Q$ onto the space of $H_1$-curves in $\widetilde{M}$ which are tangent almost everywhere to the distribution $\widetilde{Q}$ inducing $\tilde{d}_c$.

Let $p \in M$ and let $U$ be an open neighborhood of $p$ in $M$ such that $TM|_U$ and $T\widetilde{M}|_{f(U)}$ admit admissible trivializations $X^1, \cdots, X^m$ and $\widetilde{X}^1, \cdots, \widetilde{X}^m$ as before. Then the assignment $\varphi \to f \circ \varphi$ is a bijection of $HQ = \{\varphi \in H_1^p(I, U) | \varphi'(t) \in Q$ for almost all $t \in I\}$ onto

$$H\widetilde{Q} = \{\varphi \in H_1^{f(p)}(I, f(U)) | \varphi'(t) \in \widetilde{Q} \quad \text{for almost all } t \in I\}.$$

Now if $\varphi\colon I \to U$ is an element of $HQ$, then $\varphi$ is rectifiable with respect to $d_c$, and moreover $\|\varphi'(t)\|$ equals the dilation of $\varphi$ at $t$ for almost every $t \in I$, i.e.,

$$\|\varphi'(t)\| = \limsup_{\varepsilon \to 0} \frac{d_c(\varphi(t), \varphi(t+\varepsilon))}{\varepsilon}$$

(see [14, 17]). Since $f$ is an isometry, this means $\|(f\varphi)'(t)\|_{\widetilde{Q}} = \|\varphi'(t)\|$ for almost every $t \in I$, i.e., the map $\varphi \to f \circ \varphi$ commutes with the energy function.

The above trialization of $TM$ on $U$ gives rise to an exponential map $\exp_p^c$ at $p$ as before, which is defined on an open star-shaped neighborhood $W$ of 0 in $T_pM$. In the same way an exponential map $\exp_{f(p)}^c$ at $f(p)$ is defined on an open neighborhood $\widetilde{W}$ of 0 in $T_{f(p)}\widetilde{M}$. For $X \in W$, define $\lambda(X) \in HQ$ by $\lambda(X)(t) = \exp_p^c tX$ $(t \in I)$, i.e., we assume in the sequel always that $\lambda(X)$ is parametrized on $I$. It follows from Corollary 5.9 that for every $X \in W$ there is $\widetilde{X} \in \widetilde{W}$ such that $F\lambda(X) = f \circ \lambda(X) = \lambda(\widetilde{X})$, i.e., the map $F\colon \lambda(X) \to f \circ \lambda(X)$ is a bijection of the space of geodesics in $U$, which emanate from $p$ and are parametrized on $I$ onto the space of geodesics in $f(U)$ which emanate from $f(p)$ and are parametrized on $I$. Notice that $\widetilde{X} \in T_{f(p)}\widetilde{M}$ with $\lambda(\widetilde{X}) = F\lambda(X)$ is not necessarily unique.

**Lemma 6.1.** *Let $X \in T_pM$, $\widetilde{X} \in T_{f(p)}\widetilde{M}$, and $F\lambda(X) = \lambda(\widetilde{X})$. If $\exp_p^c$ is of maximal rank at $X$ and $\exp_{f(p)}^c$ is of maximal rank at $\widetilde{X}$, then:*

(i) *$f \exp_p^c tX = \exp_{f(p)}^c t\widetilde{X}$ for all $t$ for which $\exp_p^c tX$ is defined,*

(ii) *there is an open neighborhood $\Omega$ of $X$ in $T_pM$, which is mapped by $\exp_p^c$ diffeomorphically into $U$, and a diffeomorphism $\Psi$ of $f(\exp_p^c \Omega)$ into $T_{f(p)}\widetilde{M}$ such that $F\lambda(Y) = \lambda(\Psi \circ f \circ \exp_p^c Y)$ for all $Y \in \Omega$.*

*Proof.* (i) Assume that there is $\nu > 1$ such that $f \exp_p^c \nu X \neq \exp_{f(p)}^c \nu \widetilde{X}$. Since $t \to f \exp_p^c tX$ $(t \in [0, \nu])$ is a geodesic in $\widetilde{M}$ there is $Y \in T_{f(p)}\widetilde{M}$ such that $f \exp_p^c tX = \exp_{f(p)}^c tY$ for all $t \in [0, \nu]$. Now $\exp_{f(p)}^c \nu Y \neq \exp_{f(p)}^c \nu \widetilde{X}$ shows $\widetilde{X} \neq Y$. On the other hand $\lambda(Y) = F\lambda(X) = \lambda(\widetilde{X})$ which means $\widetilde{P}(\mathrm{Id} - T_{\lambda(\widetilde{X})})^{-1} Y = \widetilde{P}(\mathrm{Id} - T_{\lambda(\widetilde{X})})^{-1} \widetilde{X}$ (compare Lemma 2.6) or $\widetilde{P}(\mathrm{Id} - T_{\lambda(\widetilde{X})})^{-1}(Y - \widetilde{X}) = 0$ in contradiction to rank $R_{\lambda(\widetilde{X})} = m$.

(ii) By the assumptions on $X$ and $\widetilde{X}$ there are open neighborhoods $B$ of $X$ in $T_pM$ and $\widetilde{B}$ of $\widetilde{X}$ in $T_{f(p)}\widetilde{M}$ with the following properties:

(i) $\exp_p^c$ maps $B$ diffeomorphically onto an open neighborhood $A$ of $u$ in $U$.

(ii) There is a diffeomorphism $\overline{\Psi}$ of $fA$ onto $\widetilde{B}$ such that $\exp_{f(p)}^c \circ \overline{\Psi} = \mathrm{Id}_{fA}$.

Assume that $\widetilde{B}$ contains the $2\delta$-neighborhood of $\widetilde{X}$ in $T_p\widetilde{M}$. If the lemma does not hold, then there is a sequence $\{X_k\} \subset B$ with $X_k \to X$ $(k \to \infty)$ such that $F\lambda(X_k) = \lambda(Y_k)$ for some $Y_k \in T_{f(p)}\widetilde{M}$ with $\|Y_k - \widetilde{X}\| > \delta$ for all $k > 0$. Since $d_\infty(\lambda(\widetilde{X}), \lambda(Y_k)) \to 0$ $(k \to \infty)$, Lemma 4.2 shows that the sequence $\{Y_k\} \subset T_{f(p)}\widetilde{M}$ is bounded; hence passing to a subsequence we may assume that $\{Y_k\}_k$ converges to some $Y \in T_{f(p)}\widetilde{M}$. Clearly $\|Y - \widetilde{X}\| \geq \delta$. Since $d_\infty(\lambda(Y_k), \lambda(Y)) \to 0$ $(k \to \infty)$, it follows that $\lambda(Y) = \lambda(\widetilde{X})$. But this means $\widetilde{P}(\mathrm{Id} - T_{\lambda(\widetilde{X})})^{-1} Y = \widetilde{P}(\mathrm{Id} - T_{\lambda(\widetilde{X})})^{-1} \widetilde{X}$ or $\widetilde{P}(\mathrm{Id} - T_{\lambda(\widetilde{X})})^{-1}(Y - \widetilde{X}) = 0$ in contradiction to rank $R_{\lambda(\widetilde{X})} = m$. This shows the claim. q.e.d.

Now we are ready to show

**Theorem 6.2.** *An isometry $(M, d_c) \to (\widetilde{M}, \tilde{d}_c)$ is smooth.*

*Proof.* We show first that an isometry $f\colon (M, d_c) \to (\widetilde{M}, \tilde{d}_c)$ is smooth on an open dense subset of $M$. This follows from a successive application of the following.

**Sublemma.** *Let $N \subset M$ be a smooth embedded submanifold of dimension $n < m$. Assume that the restriction of $f$ to $N$ is smooth. Then for* *every point $p$ of an open dense subset of $N$ there is an open neighborhood $U$ of $p$ in $N$ and a smooth $(n+1)$-dimensional embedded submanifold $\overline{N}$ of $M$ containing $U$ such that the restriction of $f$ to $\overline{N}$ is smooth.*

To show the sublemma observe that since $Q$ is bracket generating, $N$ contains an open dense subset with the property that for every $p$ of this set the tangent space $T_pN$ of $N$ at $p$ does not contain $Q_p$.

Since, as an isometry of $(M, d_c)$ onto $(\widetilde{M}, \tilde{d}_c)$, $f$ is absolutely continuous with respect to Lebesgue measure (compare [12]), there is $Y \in T_pM$ such that $\widetilde{P}Y$ is transversal to $N$ at $p$, $u = \lambda(Y)(1)$ is a regular value for $\exp_p^c$, and $f(u)$ is a regular value for $\exp_{f(p)}^c$. Let $X \in T_uM$ be such that $\lambda(X)(t) = \lambda(Y)(1-t)$. By Remark 5.6 $p$ is a regular value for $\exp_u^c$, and $f(p)$ is a regular value for $\exp_{f(u)}^c$. Choose $\Omega \subset T_uM$ and $\Psi\colon f(\exp_u^c \Omega) \to T_{f(u)}\widetilde{M}$ as in Lemma 6.1. Then $\Omega \cap (\exp_u^c)^{-1}(N) = W$ is a smooth submanifold of $\Omega$. Since the restriction of $f$ to $N$ is smooth, the same is true for the restriction of $Df = \Psi \circ f \circ \exp_u^c$ to $W$. Since $\lambda(X)$ meets $N$ transversally at $\lambda(X)(1)$, there is an open neighborhood $B$ of $X$ in $W$ and a number $\varepsilon > 0$ such that $\overline{N} = \{\exp_u^c tY | Y \in B, t \in (1-\varepsilon, 1+\varepsilon)\}$ is a smooth embedded submanifold of $M$. But $f\exp_u^c tY = \exp_{f(u)}^c t(DfY)$ for all $Y \in B$ and $t \in (1-\varepsilon, 1+\varepsilon)$ shows that the restriction of $f$ to $\overline{N}$ is smooth. This finishes the proof of the sublemma.

To finish the proof of the theorem let $p \in M$ be arbitrary. Then there is a regular value $w \in M$ of $\exp_p^c$ such that $f(w)$ is a regular value of $\exp_{f(p)}^c$ and $f$ is smooth near $w$. Let $\lambda(Y)$ $(Y \in T_pM)$ be a minimizing geodesic joining $p$ to $w$, and let $\Omega$ and $\Psi$ be as in Lemma 6.1. If we choose $\Omega$ sufficiently small, then $Df = \Psi \circ f \circ \exp_p^c$ is a diffeomorphism of $\Omega$ into $T_{f(p)}\widetilde{M}$ such that $F\lambda(Z) = \lambda(DfZ)$ for all $Z \in \Omega$. Lemma 6.1 shows $f\exp_p^c tZ = \exp_{f(p)}^c t(DfZ)$ for all $Z \in \Omega$ and all $t$ for which both sides are defined.

For $Z \in \Omega$, define $\alpha(Z) = -\tilde{\lambda}(Z)(1)$ and $\beta(Z) = -\tilde{\lambda}(DfZ)(1)$, where $\tilde{\lambda}(Z)$ is as before. Then $\alpha$ and $\beta$ are smooth maps of $\Omega$ into $TM$ and $T\widetilde{M}$, resp., and $\exp_{\lambda(Z)(1)}^c$ is of maximal rank at $\alpha(Z)$ and $\exp_{f\lambda(Z)(1)}^c$ is of maximal rank at $\beta(Z)$. Hence there is a compact neighborhood $K$ of $Y$ in $\Omega$ and $\varepsilon > 0$ such that, for every $Z \in K$, $\exp_{\lambda(Z)(1)}^c$ and $\exp_{f\lambda(Z)(1)}^c$ are of maximal rank at $(1+\varepsilon)\alpha(Z)$ and $(1+\varepsilon)\beta(Z)$ respectively.

Since $\{\varepsilon Z | -Z \in K\}$ has nonempty interior, Corollary 5.8 shows that there is $W \in K$ such that $\lambda(-\varepsilon W)(1) = u$ is a regular value for $\exp_p^c$, and $f(u)$ is a regular value for $\exp_{f(p)}^c$.

Let $X = -(1+\varepsilon)\tilde{\lambda}(W)(-\varepsilon)$ and $\widetilde{X} = -(1+\varepsilon)\tilde{\lambda}(DfW)(-\varepsilon)$. We have $\lambda(X)(t) = \exp_p^c((1+\varepsilon)t-\varepsilon)W$ and $\lambda(\widetilde{X})(t) = \exp_{f(p)}^c((1+\varepsilon)t-\varepsilon)(DfW)$, and consequently $F\lambda(X) = \lambda(\widetilde{X})$.

By Lemma 6.1 there is an open neighborhood $B$ of $X$ in $T_uM$ and a diffeomorphism $\Phi$ of $B$ into $T_{f(u)}\widetilde{M}$ such that $F\lambda(Z) = \lambda(\Phi Z)$ for all $Z \in B$. Since $\exp_u^c$ is of maximal rank at $\varepsilon X/(1+\varepsilon)$, for sufficiently small $B$ the map $Z \to \lambda(Z)(\varepsilon/(1+\varepsilon))$ is a diffeomorphism of $B$ onto an open neighborhood of $p$. But $f\lambda(Z)(t) = \lambda(\Phi Z)(t)$ for all $t \in I$ and $Z \in B$ then implies that $f$ is smooth near $p$. Hence the proof of the theorem is finished.

## 7. The strong bracket generating case

In this section we investigate the group of isometries of a CC-metric $d_c$ which is induced by a distribution $Q$ satisfying the strong bracket generating hypothesis (see [17]), i.e., for every nonzero section $X$ of $Q$, $TM$ is generated by $Q$ and $[X, Q]$.

Let $N$ be the annihilator of $Q$ in the cotangent bundle $T^*M$ of $M$. $N$ is a smooth $k = (m-q)$-dimensional subbundle of $T^*M$.

**Lemma 7.1.** *Every Riemannian metric on $Q$ gives rise to a unique Riemannian metric on $N$.*

*Proof.* Let $p \in M$, $0 \neq \omega_p \in N_p$, and let $\omega$ be a local section of $N$ through $\omega_p$. If $X$, $Y$ are local sections of $Q$ near $p$, then

$$d\omega(X, Y) = \tfrac{1}{2}\{X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])\} = -\tfrac{1}{2}\omega([X, Y]),$$

in particular $d\omega(X_p, Y_p)$ only depends on $\omega_p$, not on the choice of the extension $\omega$. Since the commutators of sections of $Q$ span $TM$, the restriction of $d\omega$ to $Q_p$ does not vanish. This means that there is a natural injective bundle map $J$ of $N$ into the exterior product $Q^* \wedge Q^*$. Since a Riemannian metric on $Q$ induces a Riemannian metric on $Q^* \wedge Q^*$, this metric can be pulled back via $J$ to a metric on $N$. q.e.d.

For a section $\omega$ of $N$, define a map $J\omega\colon Q \to T^*M$ by $(J\omega(X))(Y) = d\omega(X, Y)$. Since $Q$ satisfies the strong bracket generating hypothesis, $J\omega$ is an injective bundle map and $J\omega(Q)$ is complementary to $N$.

**Lemma 7.2.** *Let $\omega^1, \cdots, \omega^k$ be a local orthonormal basis of $N$ with respect to the metric of 7.1. Then for every $i \in \{1, \cdots, k\}$ the $(m-1)$-dimensional subspace of $T_p^*M$, which is spanned by $J\omega^i(Q_p) \cup \{\omega_p^j | i \neq j\}$, only depends on $\omega_p^1, \cdots, \omega_p^k$ and is transversal to $\{\lambda\omega_p^i | \lambda \in \mathbb{R}\}$.*

*Proof.* Clearly $A_p^i = \operatorname{span}(J\omega^i(Q_p) \cup \{\omega_p^j | i \neq j\})$ is transversal to $\omega_p^i$. To show that $A_p^i$ only depends on $\omega_p^1, \cdots, \omega_p^k$ let $\overline{\omega}^1, \cdots, \overline{\omega}^k$ be another local orthonormal basis of $N$ near $p$ with $\overline{\omega}_p^i = \omega_p^i$. Then there is a smooth function $(g_{ij})$ of a neighborhood of $p$ in $M$ into the special orthonormal group $\mathrm{SO}(k)$ such that $(g_{ij})(p) = \mathrm{Id}$ and $\overline{\omega}^i = \sum_j g_{ij}\omega^j$. We have $(dg_{jj})_p = 0$ for $j = 1, \cdots, n$. Let $X^1, \cdots, X^q$ be a local orthonormal basis of $Q$ near $p$ and let $\sigma^j = J\omega^i(X^j)$. Then $dg_{ij} = \sum_\alpha a_\alpha^{ij}\sigma^\alpha + \sum_\beta b_\beta^{ij}\omega^\beta$ with smooth functions $a_\alpha^{ij}$ and $b_\beta^{ij}$, and

$$\begin{aligned} d\overline{\omega}^i &= \sum_j dg_{ij} \wedge \omega^j + \sum_j g_{ij} d\omega^j \\ &= \sum_{\alpha, j} a_\alpha^{ij}\sigma^\alpha \wedge \omega^j + \sum_{\beta, j} b_\beta^{ij}\omega^\beta \wedge \omega^j + \sum g_{ij} d\omega^j. \end{aligned}$$

Since $a_\alpha^{jj} = 0$ for $j = 1, \cdots, n$, this implies $(J\overline{\omega}^i)(X) = (J\omega^i)(X) + \sum_{i \neq j}\sum_\alpha a_\alpha^{ij}(X)\omega^j$ for every $X \in Q_p$, i.e., $(J\overline{\omega}^i)(X) - (J\omega^i)(X) \in \operatorname{span}\{\omega^j | i \neq j\}$ as claimed.

**Corollary 7.3.** *If $Q$ satisfies the strong bracket generating hypothesis, then every Riemannian metric $\langle\ ,\ \rangle_Q$ on $Q$ can intrinsically be extended to a Riemannian metric on $M$.*

*Proof.* Let $p \in M$. By 7.2 the choice of an orthonormal basis $\omega_p^1, \ldots \omega_p^k$ of $N_p$ determines for every $i \in \{1, \cdots, k\}$ an $(m-1)$-dimensional subspace of $T_p^*M$, which annihilates a 1-dimensional subspace $A^i$ of $T_pM$ transversal to the kernel of $\omega_p^i$. Let $Z^i \in A^i$ be such that $\omega_p^i(Z_i) = 1$.

The vectors $Z^1, \cdots, Z^k$ span a $k$-dimensional subspace of $T_pM$ which is complementary to $Q_p$; we thus can define an extension $g(\omega_p^1, \cdots, \omega_p^k)$ of $(\langle\ ,\ \rangle_Q)_p$ by choosing the vectors $Z^1, \cdots, Z^k$ orthonormal and perpendicular to $Q_p$.

Now the space of orthonormal bases of $N_p$ can be identified with the orthogonal group $\mathrm{O}(k)$. Let $\mu$ be the normalized Haar measure on $\mathrm{O}(k)$ (which satisfies $\mu(\mathrm{O}(k)) = 1$), and define $\langle X, Y\rangle_p = \int_{\mathrm{O}(k)} g(\xi)(X, Y)d\mu(\xi)$ for $X, Y \in T_pM$. Then $\langle\ ,\ \rangle_p$ is a scalar product on $T_pM$ extending the product on $Q_p$ and moreover is defined intrinsically by $(Q, \langle\ ,\ \rangle_Q)$. q.e.d.

The Riemannian metric $\langle\ ,\ \rangle$ on $M$ defined in 7.3 will be called the *canonical extension* of $\langle\ ,\ \rangle_Q$.

Since every isometry of $d_c$ is smooth, 7.3 yields

**Corollary 7.4.** *The group of isometries of $d_c$ is a closed subgroup of the Lie group of isometries of the canonical extension of $\langle\ ,\ \rangle_Q$.*

**Lemma 7.5.**

$$\theta \frac{D}{dt}\lambda(X)'(t)|_{t=0} = \tilde{P}ad^*(\tilde{P}\theta X, (1-\tilde{P})\theta X) - (1-\tilde{P})ad^*(\tilde{P}\theta X, \tilde{P}\theta X).$$

*Proof.* Let $\varphi = \lambda(X)$. Then $\theta\varphi'(0) = \tilde{P}\theta X$, $\frac{d}{dt}\theta\varphi'(t)|_{t=0} = \tilde{P}ad^*(\tilde{P}\theta X, \theta X)$, and, moreover by Lemma 2.3, $\theta\frac{D}{dt}\varphi'(t) = \frac{d}{dt}\theta\varphi'(t) - ad^*(\theta\varphi'(t), \theta\varphi'(t))$. Together this yields the claim.

**Corollary 7.6.** *If $Q$ satisfies the strong bracket generating hypothesis, then every $d_c$-geodesic through $p$ is uniquely determined by its tangent and its covariant derivative at $p$. In particular $\exp_p^c$ is intrinsically defined.*

*Proof.* For $Y \in Q_p$ the map $\alpha_Y : Q_p^\perp \to Q_p$, $Z \to \tilde{P}a_p(\theta Y, \theta Z)$ does not depend on the choice of the local trivialization of $TM$ near $p$, and is injective. The corollary thus follows from 7.5 and the fact that every Riemannian metric on a strong bracket generating distribution can intrinsically be extended to a Riemannian metric on $M$.

## 8. Nilpotent homogeneous Lie groups

In this section we investigate the group of isometries of a left-invariant CC-metric $d_c$ on a nilpotent homogeneous Lie group. Thus let $N$ be a nilpotent homogeneous simply connected Lie group whose Lie algebra $\mathfrak{M}$ is generated by a complement $Q$ of its derived algebra $[\mathfrak{M}, \mathfrak{M}]$, i.e., if $Q^1 = Q$ and $Q^{i+1} = [Q, Q^i]$, then there is $k \geq 1$ such that $\mathfrak{M} = \bigoplus_{i=1}^k Q^i$ (direct sum). For every $r > 0$ the assignment $\delta_r : \sum_{i=1}^k X^i \to \sum_{i=1}^k r^i X^i$ ($X^i \in Q^i$) is a Lie algebra automorphism of $\mathfrak{M}$ which integrates to an automorphism $\Delta_r$ of $N$. Let $d_c$ be a left-invariant CC-metric on $N$ induced by a scalar product $\langle\ ,\ \rangle_Q$ on $Q$. Then $\{\Delta_t | t > 0\}$ is a 1-parameter group of homotheties with respect to $d_c$, i.e., $d_c(\Delta_r p, \Delta_r u) = r d_c(p, u)$ for all $p$, $u \in N$, $r > 0$ (compare [13]).

Choose an extension of $\langle\ ,\ \rangle_Q$ to a scalar product $\langle\ ,\ \rangle$ on $\mathfrak{M}$ such that the decomposition $\mathfrak{M} = \bigoplus_{i=1}^k Q^i$ is $\langle\ ,\ \rangle$-orthogonal and a left-invariant $\langle\ ,\ \rangle$ -orthonormal trivialization of $TN$. Denote as before by $\lambda(X)$ ($X \in \mathfrak{M}$) the geodesic through the identity $\lambda(X)(0) = e$ with respect to these data. Since $\theta\frac{d}{ds}(\Delta_t\lambda(X)(s)) = t\theta(\frac{d}{ds}\lambda(X)(s))$ for all $t > 0$ and $s \in \mathbb{R}$, we have $\Delta_t\lambda(X) = \lambda(t^2\delta_{1/t}X)$.

**Lemma 8.1.** *Every isometry of $(N, d_c)$ fixing the identity $e$ permutes the 1-parameter subgroups of $N$ which are tangent to $Q$.*

*Proof.* For every $X \in Q$ the geodesic $\lambda(X)$ is the 1-parameter subgroup in $N$ defined by $X$ and is globally minimizing since the 1-parameter subgroups tangent to $Q$ are globally minimizing geodesics with respect to the Riemannian metric $\langle\ ,\ \rangle$ on $N$. Since $f$ is smooth, $f\lambda(X)$ is a globally minimizing geodesic in $(N, d_c)$ with $\operatorname{rank} R_{f\lambda(Y)} = \operatorname{rank} R_{\lambda(X)} = k$ for some $k \le m$.

For $Y \in Q$, define $\operatorname{rank}(Y)$ to be the dimension of the smallest $\operatorname{ad} Y$-invariant subspace of $\mathfrak{M}$ containing $Q$. Remark 2.8 shows $\operatorname{rank} R_\varphi \ge \operatorname{rank}(\varphi'(0))$ for every smooth curve $\varphi: I \to N$ through $\varphi(0) = e$ which is tangent to $Q$, and moreover the rank is preserved by every diffeomorphism of $N$ which leaves $Q$ invariant and fixes $e$. If $\varphi$ is a 1-parameter subgroup of $N$ tangent to $Q$ then $\operatorname{rank} R_\varphi = \operatorname{rank}(\varphi'(0))$.

For every $r > 0$, $f(r) = \Delta_r f \Delta_{1/r}$ is an isometry of $(N, d_c)$ fixing $e$. Let $B$ be the compact $d_c$-ball of radius $2\|X\|$ around $e$. By Ascoli's theorem there is a sequence $r_i$ $(i > 0)$ such that $r_i \to 0$ $(i \to \infty)$ and that the sequence of maps $f(r_i)$ converges uniformly on $B$ to a map $\overline{f}$. Since $\overline{f}$ is an isometry of $(B, d_c)$ fixing $e$, by 7.2 a diffeomorphism of $B$, and $\overline{f}\lambda(X)$ is a minimizing geodesic in $(N, d_c)$ with $\operatorname{rank} R_{\overline{f}\lambda(X)} = \operatorname{rank}(\overline{f}\lambda(X)'(0)) = k$.

Let $A \subset \mathfrak{M}$ be the $\langle\ ,\ \rangle$-orthogonal complement in $\mathfrak{M}$ of $\mathfrak{M} \cap (\mathrm{Id} - T_{\overline{f}\lambda(X)})L^\infty(I, Q^\perp)$, and let $\overline{P}: \mathfrak{M} \to A$ be the $\langle\ ,\ \rangle$-orthogonal projection. Since $\operatorname{rank} R_{\overline{f}\lambda(X)} = \operatorname{rank}(\overline{f}\lambda(X)'(0))$, Remark 2.8 shows that $A$ equals the smallest $\operatorname{ad}(\overline{f}\lambda(X)'(0))$-invariant subspace of $\mathfrak{M}$ containing $Q$ and hence is invariant under the automorphisms $\delta_r$ $(r > 0)$. By 3.2, $A$ is transversal to $\mathfrak{M} \cap (\mathrm{Id} - T_{f(r_i)\lambda(X)})L^\infty(I, Q^\perp)$ for all sufficiently large $i > 0$; we may assume that this is true for all $i$. This means that there is a unique $Y \in A$ such that $f\lambda(X) = \lambda(Y)$. Now $f(r)\lambda(X) = \lambda(r\delta_{1/r}Y)$ for all $r > 0$, and since $A$ is invariant under the automorphisms $\delta_r$ $(r > 0)$, we have $\|\overline{P}(r\delta_{1/r}Y)\| = \|r\delta_{1/r}Y\|$, and hence by Remark 4.3 the sequence $\|r_i\delta_{1/r_i}Y\|$ is uniformly bounded. But if $Y = Y_1 + Y_2$ with $Y_1 \in Q$ and $Y_2 \in Q^\perp$, we have $\|r\delta_{1/r}Y\| \ge \|Y_2\|/r$ for all $r > 0$ and consequently since $r_i \to 0$ necessarily $Y_2 = 0$, i.e., $\lambda(Y)$ is a 1-parameter subgroup in $N$ as claimed. q.e.d.

For $i \ge 1$, denote by $H_i$ the Lie subgroup of $N$ whose Lie algebra is the ideal $\mathfrak{h}_i = \bigoplus_{j=i}^k Q^j$. $\langle\ ,\ \rangle_Q$ induces a left-invariant CC-metric $d_i$

on the factor group $N/H_i$ in such a way that the canonical projection $\pi_i\colon (N, d_c) \to (N/H_i, d_i)$ is distance decreasing.

Let exp be the exponential map of $N$ and call two pairs $(u, X)$, $(v, Y) \in N \times Q$ *parallel* if the function $t \to d_c(u \exp tX, v \exp tY)$ is bounded on $\mathbb{R}$.

**Lemma 8.2.** *If $(u, X)$ and $(v, Y)$ are parallel, then $X = Y$.*

*Proof.* Let $\Psi$ be the restriction to $Q$ of the map $\pi_2 \circ \exp$. $\Psi$ is an isometry of $(Q, \langle\ ,\ \rangle_Q)$ onto $(N/H_2, d_2)$. The Campbell-Hausdorff formula [5] shows $\pi_2(u \exp tX) = \Psi((\Psi^{-1}\pi_2 u) + tX)$ for all $u \in N$ and $X \in Q$, and hence

$$\begin{aligned} &d_2(\pi_2(u \exp tX), \pi_2(v \exp tY)) \\ &\quad = \|(\Psi^{-1}\pi_2 u - \Psi^{-1}\pi_2 v) + t(X - Y)\| \geq |t|\,\|X - Y\| - \text{const}. \end{aligned}$$

Since the latter expression is uniformly bounded for all $t \in \mathbb{R}$ whenever $(u, X)$ and $(v, Y)$ are parallel, the lemma follows. q.e.d.

Let $Z$ be the center of $N$. Then we have

**Lemma 8.3.** *$p \in Z$ if and only if $(p, X)$ is parallel to $(e, X)$ for all $X \in Q$.*

*Proof.* If $p \in Z$ then $d_c(\exp tX, p \exp tX) = d_c(e, p)$ for all $t \in \mathbb{R}$, i.e., $(p, X)$ is parallel to $(e, X)$ for all $X \in Q$.

On the other hand, if $u = \exp(\sum_{i=1}^k X^i) \notin Z (X^i \in Q^i)$, then there is $X \in Q$ such that $[X, \sum_{i=1}^k X^i] \neq 0$. Let $j - 2 = \min\{i \geq 1 | [X, X^i] \neq 0\}$ and let $\Psi$ be the restriction to $S = \bigoplus_{i=1}^{j-1} Q^i$ of the map $\pi_j \circ \exp$. Under the identification of $S$ with $\mathfrak{M}/\mathfrak{h}_j$, $\Psi$ can be viewed as the exponential map of $N/H_j$, i.e., group multiplication in $N/H_j$ can be computed via the Campbell-Hausdorff formula in $S$. This means

$$d_j(\pi_j \exp tX, \pi_j(u \exp tX)) = d_j(\pi_j e, \Psi((\Psi^{-1}\pi_j u) + t[X^j, X])),$$

which is unbounded in $t \in \mathbb{R}$. Since $\pi_j$ is distance-decreasing, $(u, X)$ is not parallel to $(e, X)$. q.e.d.

A special case of the following corollary is due to Pansu [13, Proposition 18.5]:

**Corollary 8.4.** *Every isometry of $(N, d_c)$ fixing the identity $e$ is a Lie group automorphism of $N$.*

*Proof.* By a theorem of Pansu [13] there is an automorphism $\Psi$ of $N$ such that $d_e(\Psi \circ f)|_Q = \mathrm{Id}|_Q$. $\Psi$ is necessarily an isometry with respect to $d_c$; hence we only have to show that there is no nontrivial isometry $f$ of $(N, d_c)$ with $f(e) = e$ and $d_e f|_Q = \mathrm{Id}|_Q$.

We proceed by induction on the degree of nilpotency of $N$. If this degree equals 1, then $(N, d_c)$ is Euclidian and hence there is nothing to show. Thus let $k \geq 2$ and assume the claim is known for all groups of degree $\leq k-1$. Let $N$ be a group of degree $k$ and let $f$ be as above. By 8.1, $f$ permutes the integral curves of left-invariant vector fields and hence maps parallel elements $(u, X)$ and $(v, X)$ of $N \times Q$ onto parallel elements. Since $f(e) = e$ and $f \exp tX = \exp tX$ for all $X \in Q$, $t \in \mathbb{R}$, this implies by 8.2 and 8.3 the following:

(i) $f$ preserves the center $Z$ of $N$,

(ii) $f(p \exp tX) = f(p) \exp tX$ for all $p \in Z$, $X \in Q$, and $t \in \mathbb{R}$.

For every $p \in Z$ the map $f_p \colon u \to f(p)^{-1} f(up)$ is an isometry of $(N, d_c)$ fixing $e$ which by (ii) satisfies $d_e f_p|_Q = \mathrm{Id}|_Q$. By (i) $f_p$ preserves the center $Z$ of $N$ and hence induces a transformation $\overline{f}_p$ of the factor group $N/Z$ which is an isometry with respect to the induced CC-metric. Since the degree of nilpotency of $N/Z$ equals $k-1$, by the induction hypothesis $\overline{f}_p$ equals the identity of $N/Z$. This is true for every $p \in Z$. Thus the differential of $f$ preserves the left-invariant vector fields tangent to $Q$. Since $Q$ generates $\mathfrak{M}$, $f$ is the identity. Hence the proof is finished. q.e.d.

We conclude this work with the following example.

**Example 8.5.** (a) The Lie algebra $\mathfrak{H}$ of the 3-dimensional Heisenberg group $H^3$ is spanned by vectors $X$, $Y$, and $Z$ which satisfy the relations $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$. Let $\langle\ ,\ \rangle$ be the scalar product on $\mathfrak{H}$ for which this basis is orthonormal, and let $d_c$ be the left-invariant Carnot-Carathéodory metric on $H^3$ induced by $Q = \mathrm{span}\{X, Y\}$ and $\langle\ ,\ \rangle_Q = \langle\ ,\ \rangle|_Q$. We want to compute the geodesics $\lambda(W) \colon t \to \exp_e^c tW$ of $d_c$ through the identity $e$ (compare [10]). For $x_0, y_0, z_0 \in \mathbb{R}$ write $\theta\tilde{\lambda}(x_0X + y_0Y + z_0Z)(t) = x(t)X + y(t)Y + z(t)Z$. By using the relations $(\mathrm{ad}\,X)^*Z = Y$ and $(\mathrm{ad}\,Y)^*Z = -X$ equation (2) of Lemma 4.1 transforms into the following system of differential equations for the coordinate functions $x, y, z$:

$$\begin{aligned} x'(t) &= -y(t)z(t), & x(0) &= x_0, \\ y'(t) &= x(t)z(t), & y(0) &= y_0, \\ z'(t) &= 0, & z(0) &= z_0. \end{aligned}$$

Hence

$$\begin{aligned} &\theta\lambda(x_0X + y_0Y + z_0Z)'(t) \\ &\qquad = (x_0 \cos z_0t - y_0 \sin z_0t)X + (x_0 \sin z_0t + y_0 \cos z_0t)Y. \end{aligned}$$

The Lie group exponential map exp of $H^3$ induces global coordinates on $H^3$ via the identification of $\exp(x_1X+x_2Y+x_3Z)$ with $(x_1, x_2, x_3) \in \mathbb{R}^3$. In these coordinates the vector fields $X$, $Y$, and $Z$ are given by [14]

$$X = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2\frac{\partial}{\partial x_3}, \qquad Y = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1\frac{\partial}{\partial x_3}, \qquad Z = \frac{\partial}{\partial x_3}.$$

Thus for $z_0 \neq 0$ the geodesic $\lambda(X + z_0Z)$ has the coordinate representation

$$\lambda(X + z_0Z)(t) = z_0^{-1}(\sin z_0t, 1 - \cos z_0t, t/2 - \sin z_0t/2z_0),$$

in particular $\lambda(X + z_0Z)(2\pi/z_0) = (0, 0, \pi/z_0^2)$. For every $\alpha \in S^1 \sim [0, 2\pi]$ the isometry of $(\mathfrak{H}, \langle\ ,\ \rangle)$ which fixes the center of $\mathfrak{H}$ and acts as a rotation of angle $\alpha$ in the plane $Q$ is an automorphism of $\mathfrak{H}$ which integrates to an automorphism $\Psi_\alpha$ of $H^3$. $\Psi_\alpha$ is an isometry with respect to $d_c$ which maps for every $W \in \mathfrak{H}$ the geodesic $\lambda(W)$ onto $\lambda(d\Psi_\alpha W)$. Thus for every $s > 0$ there is a $S^1$-family $\{\lambda(d\Psi_\alpha(X + \sqrt{\pi/s}Z))|\alpha \in S^1\}$ of minimizing $d_c$-geodesics joining $e$ to $\lambda(d\Psi_\alpha(X + \sqrt{\pi/s}Z))(2\sqrt{\pi s}) = (0, 0, s)$. In particular each of the $d_c$-geodesics $\{\lambda(W)|\|\tilde{P}W\| = 1\}$ minimizes exactly on the interval $[0, 2\pi(\|W\|^2 - 1)^{-1/2}]$.

(b) Let $\overline{H} = H^3 \times H^3$ be the direct product of two copies of $H^3$ with Lie algebra $\overline{\mathfrak{H}} = \mathfrak{H} \times \mathfrak{H}$, equipped with the left-invariant Riemannian metric $\langle\ ,\ \rangle^0$ which is the product of the metrics $\langle\ ,\ \rangle$ on $\mathfrak{H}$ above. Let $\overline{d}_c$ be induced by $(Q \times Q, \langle\ ,\ \rangle^-|_{Q\times Q})$. Then the map $(H^3, d_c) \to (\overline{H}, \overline{d}_c)$, $u \to (u, e)$ is an isometric embedding. In particular for every $X \in Q$ the 1-parameter subgroup in $\overline{H}$, which is tangent to $(X, 0)$ at $e$, is a minimizing $\overline{d}_c$-geodesic in $\overline{H}$. However the rank of $R$ along this geodesic equals 5, i.e., the exponential map $\exp_e^c$ of $(\overline{H}, \overline{d}_c)$ at $e$ is singular along $\{(tX, 0) \in \overline{\mathfrak{H}}|X \in Q, t \in \mathbb{R}\}$.

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