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CONSTRUCTION OF CONNECTION INDUCING MAPS BETWEEN PRINCIPAL BUNDLES. PART I

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0. Introduction

Consider two C^∞-smooth principal bundles, say $X \to V$ and $Y \to W$, with the same structure group G and with C^∞ connections Γ on X and Δ on Y, respectively. We look for a C^∞-map $f\colon V \to W$ such that the induced bundle $f^*(Y)$ over V with the induced connection $f^*(\Delta)$ is isomorphic to (X, Γ). This means that f can be covered by (or lifted to) a morphism of bundles, $F\colon X \to Y$, inducing Γ from Δ, which is expressed by $F^*(\Delta) = \Gamma$.

0.1. The problem of inducing connections was first studied by Narasimhan and Ramanan [3] who proved that for a given compact Lie group G and an integer $n = 0, 1, \cdots$, there exists a (universal) bundle (Y, Δ) over some (classifying) compact manifold W, such that every G-bundle X over an n-dimensional manifold V with an arbitrary C^∞-connection Γ can be induced by a C^∞-morphism $F\colon X \to Y$. Furthermore, they give a precise description of the universal connection Δ for the unitary and the orthogonal groups. Namely, if $G = \mathrm{U}(p)$ they take the Grassmann manifold $\mathrm{Gr}_p(\mathbf{C}^q)$ for W and use the standard connection Δ on the canonical bundle $Y \to \mathrm{Gr}_p(\mathbf{C}^q)$ (here Y is the Stiefel manifold of orthogonal p-frames in $\mathbf{C}^q$). The dimension q for which they prove the existence of F is $q = (n+1)(2n+1)p^3$, where $n = \dim V$. Similarly, for $G = \mathrm{O}(p)$, their method provides a connection inducing map into the real Grassmann manifold $\mathrm{Gr}_p(\mathbf{R}^q)$, again for $q = (n+1)(2n+1)p^3$.

0.2. The result by Narasimhan-Ramanan was improved for $G = \mathrm{O}(p)$ by Gromov (see 2.2.6 in [1]) who showed the existence of a connection inducing map $f\colon V \to \mathrm{Gr}_p(\mathbf{R}^q)$ for $q = \max(p(n+1), p(n+2)+n)$. Furthermore, if the manifold V is parallelizable and the bundle $X \to V$ is trivial, then

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$q = p(n+2)$ suffices for the existence of a map $f\colon V \to \mathrm{Gr}_p(\mathbf{R}^q)$ inducing a given connection on X. The improvement was achieved by using the theory of *topological sheaves* (instead of a partition of unity argument used in [3]) to build the (global) map f out of local (connection inducing) maps.

0.3. In this paper, we study connection inducing maps between arbitrary bundles. In particular, in §2 we prove the following:

Theorem. *Let Γ be an arbitrary connection on a trivial $\mathrm{O}(p)$-bundle over a stably parallelizable manifold V. If $q \geqslant p(n+3)/2$, then there exists a connection inducing map $V \to \mathrm{Gr}_p(\mathbf{R}^q)$.*

0.4. Remarks. (A) One may, in principle, apply the Theorem to a nontrivial bundle X over a nonstably parallelizable manifold V. Namely, take the trivial $2p$-dimensional bundle $X' \to V' \supset V$, where V' is a $(2n-1)$-dimensional parallelizable manifold which is the total space of the normal bundle of V and where X' contains X as a subbundle. With this, one easily obtains the existence of the inducing connection map to $\mathrm{Gr}_p(\mathbf{R}^q)$ for $q = 2p(n+1)$. Of course (compare 0.2), this bound on q is too crude and it will be improved in Part II of this paper.

(B) The construction of a connection inducing map F between principal G-bundles X and Y amounts to solving a certain system of $\alpha = \dim V \times \dim G$ partial differential equations imposed on $\beta = \dim W + \dim G$ unknown functions (see 1.1). Therefore (see [2]), for a fixed Δ and for $\alpha > \beta$ a *generic* connection Γ cannot be induced (even locally) from Δ. This means that inducible connections Γ form a meager subset (depending on Δ) in the space of C^∞ connections on V. In particular, if Y is the canonical $\mathrm{O}(p)$-bundle over $\mathrm{Gr}_p(\mathbf{R}^q)$, then

$$\alpha = \tfrac{1}{2}np(p-1), \quad \beta = p(q-p) + \tfrac{1}{2}p(p-1).$$

Hence, a generic connection on V cannot be induced from this Y unless $q \geqslant (p+1)/2 + n \cdot (p-1)/2$. This bound on q asymptotically (for $p, n \to \infty$) agrees with the inequality $q \geqslant p(n+3)/2$ in the above Theorem. In fact, the existence of a (local) connection inducing map for a real analytic connection is established in §2 for $q \geqslant p(n+1)/2$.

(C) If $q \geqslant p(n+1)$, then the P.D.E. system for connection inducing maps $f\colon V \to \mathrm{Gr}_p(\mathbf{R}^q)$ can be reduced to an algebraic system (see [2], [1]). But such a reduction hardly is possible for $q \approx pn/2$. Moreover, an appropriate regularity (see Ω-regularity in §1.2) condition on f makes the linearized P.D.E. equations algebraically solvable. This allows us to apply *Nash's implicit function theorem* for local study of such maps f and to use the theory of topological sheaves for obtaining global results. (Compare 2.2, 2.3 in [1]).

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1. General criteria for the existence of connection inducing maps

1.1. Let us consider the (first order) differential operator $\mathscr{D} = \mathscr{D}_\Delta$ which relates to each morphism (i.e. a bundle homomorphism) F: $X \to Y$ the induced connection $F^*(\Delta)$ on X for a fixed C^∞-connection Δ on Y. We view morphisms $X \to Y$ as sections of the bundle $Z \to V$ associated to the principal bundle $X \to V$ with the fiber $= Y$ for the action of G on Y. This Z naturally fibers over $V \times W$ with the fiber $X_v \times Y_w/G$ canonically isomorphic to the space of G-equivariant maps $X_v \to Y_w$. On the other hand, every C^{r+1}-smooth bundle homomorphism F: $X \to Y$ by definition is given by a C^{r+1}-map f: $V \to W$ and a family of G-equivariant maps F_v: $X_v \to Y_{f(v)}$ which are C^{r+1}-smooth in $v \in V$. Thus F becomes a C^{r+1}-section $V \to Z$ covering the graph $V \to W$ of f. The range of the operator $F \mapsto F^*(\Delta)$ consists of the space of C^r-connections in X. These are C^r-sections of the fibration $H \to V$ whose fiber $H_v \subset H$ for $v \in V$ can be described as follows. Denote by X_v^1 the space of 1-jets (or differentials) of germs of sections $V \to X$ at v. Namely, X_v^1 consists of linear maps $T_v(V) \to T(X)$ which project to the identity Id: $T_v(V) \hookleftarrow$ by the differential (of the projection map) of the fibration $X \to V$. The group G naturally acts on X_v^1 and the fiber H_v equals X_v^1/G.

Observe that $\dim H_v = \alpha = \dim V \times \dim G$ and $\dim Z_v = \beta = \dim W + \dim G$. Therefore, the connection inducing equation $\mathscr{D}_\Delta(F) = \Gamma$ amounts to α equations in β unknown functions.

Our next objective is to describe an open subset A in the space of 1-jets of germs of sections $V \to Z$, such that the connection inducing operator $\mathscr{D}_\Delta$: $F \mapsto F^*(\Delta)$ becomes *infinitesimally invertible* on A. First, recall the pertinent definitions from [1, 2.3.1]. Let $\mathscr{D}$ be a nonlinear first order differential operator between spaces of sections of two arbitrary fibrations Z and H over V. The operator $\mathscr{D}$ (acting from sections of Z to those of H) is called *infinitesimally invertible* on a subset $A \subset Z^1$, where Z^1 stands for the space of 1-jets of germs of sections $V \to Z$ if for every section F: $V \to Z$ whose 1-jet sends V to A the linearization of $\mathscr{D}$ at F, called L_F, admits a right inverse M_F which is a linear differential operator. Here we are interested in the case where M_F is a zero order operator which is (nonlinear) differential of order one in F. The resulting operator $M_F(\cdot)$ (in two variables F and $\cdot$) is called an infinitesimal inversion of order zero and defect (that is the order of M in F) one (see 2.3.1. in [1] for a detailed discussion).

1.2. To describe the pertinent set A for the connection inducing operator $\mathscr{D}(F) = F^*(\Delta)$ we invoke the bundle $\tilde{Y} \to W$ associated to Y whose fiber is the Lie algebra of G with the adjoint action of G. Denote by Ω: $T(W) \otimes T(W) \to \tilde{Y}$ the curvature form of Δ. Call a linear subspace $T' \subset T_w(W)$ for $w \in W$ *Ω-regular* if one of the following three (obviously) equivalent conditions is satisfied:

(i) For some (and hence for every) basis $\tau_1, \cdots, \tau_n$ in T' the linear system

$$\Omega_w(\tau_i, \partial) = l_i, \qquad i = 1, \cdots, n, \tag{1}$$

is solvable in $\partial \in T_w(W)$ for every n-tuple of vectors l_i in the Lie algebra g of G.

(i)′ The homogeneous system

$$\Omega_w(\tau_i, \partial) = 0, \qquad i = 1, \cdots, n, \tag{1$'$}$$

is nonsingular. Namely, the dimension of the space of solutions equals $(\dim W - n \dim g.)$

(ii) The linear map $T_w(W) \to \mathrm{Hom}(T', g)$ given by $\tau \mapsto h_\tau(\tau') = \Omega_w(\tau, \tau')$ is surjective.

1.3. Remark. This definition will be used in §2 for subspaces $T' \subset T$, where T is an arbitrary linear space endowed with some bilinear vector-valued form Ω.

1.4. Example. If Ω is an ordinary (i.e. $\mathbf{R}$-valued) form, then $T' \subset T$ is Ω-regular if and only if $T' \cap \ker \Omega = 0$, where $\ker \Omega = \{t \in T \mid \Omega(t, t') = 0$ for all $t' \in T\}$. In particular, if Ω is nonsingular (symplectic) then every subspace in T is Ω-regular (compare [1, 3.4]). Notice that the curvature Ω of the canonical Ω of the canonical O(2)-bundle over the Grassmannian manifold $\mathrm{Gr}_p(\mathbf{R}^q)$ is symplectic (this Ω can be regarded as an $\mathbf{R}$-form since the Lie algebra of O(2) is $\approx \mathbf{R}$).

1.5. Take a linear map φ: $T_v(V) \to T_w(W)$ and let a 1-jet $\Phi \in Z^1$ lie over φ. (If Φ is the 1-jet $J^1_F(v)$ for a morphism F: $X \to Y$, then φ is the differential D_f of the underlying map f: $V \to W$ at $v \in V$.) Call φ *Ω-regular* if it is injective and if the image $\varphi(T_v(V)) \subset T_w(W)$ is Ω-regular for all $v \in V$. Call Φ *Ω-regular* if the underlying map φ is Ω-regular. Then define the subset $A \subset Z^1$ as the set of the *Ω-regular* 1-jets Φ. According to this terminology, we say that F: $X \to Y$ (as well as the underlying map f: $V \to W$) is Ω-regular if the 1-jet J^1_F: $V \to Z^1$ sends V into A. This is equivalent to the *Ω-regularity of the differential D_f*: $T(V) \to T(W)$ at every point $v \in V$.

1.6. Proposition. *The connection inducing operator $\mathscr{D}$: $F \mapsto F^*(\Delta)$ on Ω-regular morphisms F admits an infinitesimal inversion M of order zero and defect one.*

Proof. First, we must linearize the operator $\mathscr{D}$ at some morphism F. To do this, we take a smooth 1-parametric family of morphisms F_t: $X \to Y$ for $t \in [0,1]$, such that $F_0 = F$ and study the family of the induced connection $\Gamma_t = \mathscr{D}(F_t)$. The derivative $\frac{d}{dt}\Gamma_t$ is a 1-form on V with the values in the vector bundle $\tilde{X}$ induced from $\tilde{Y}$ by the map $f_0 = f$: $V \to W$ corresponding to $F_0 = F$. Let us express this form in terms of Ω. Let $V' = V \times [0,1]$ and $X' = X \times [0,1] \to V'$. Consider the connection Δ' on X' induced by the morphism $X' \to Y$ defined by $(x,t) \mapsto F_t(x)$. Denote by ∂ the field $\partial/\partial t$ on $V' = V \times [0,1]$ for $t \in [0,1]$, let $\tilde{\partial}$ be the corresponding field on X', and let $\tilde{\partial}^{\vee}$ be the Δ'-vertical component of $\tilde{\partial}$. Then the value of the 1-form $\Gamma_t' \overset{\text{def}}{=} \frac{d}{dt}(\Gamma_t)$ on every $\tau \in T(V = V \times t)$ is given by

$$\Gamma_t'(\tau) = \Omega'(\tau, \partial) + d\tilde{\partial}^{\vee}(\tau), \tag{2}$$

where d stands for the Δ'-horizontal differential, and Ω' is the curvature of Δ'. Now let us denote by L_F the linearization of $\mathscr{D}$ at F and, assuming the map f is Ω-regular, let us resolve the linearized equation

$$L_F(\tilde{\partial}) = l, \tag{3}$$

where l is a given section $V \to \tilde{X}$ and $\tilde{\partial}$ is the unknown infinitesimal deformation (vector field) of F. We shall seek a solution of (3) among Δ'-horizontal fields $\tilde{\partial}$. In terms of the connection Δ', this *horizontality* is expressed (with a slight abuse of notations) by

$$\tilde{\partial}^{\vee} = 0. \tag{4}$$

Next we introduce another linear algebraic equation (or rather a system of equations) for the projection ∂ of $\tilde{\partial}$ to $T(V \times [0,1])$,

$$\Omega_0'(\tau, \partial) = l \tag{5}$$

for all $\tau \in T(V_0 = V \times 0)$, where $\Omega_0' = \Omega' | V \times 0$. According to (2) every $\tilde{\partial}$ satisfying (4) and (5) is a solution of (3). Since f is Ω-regular, the relation (5) can be expressed, at every point $v \in V_0$, by the following *nonsingular* system of linear *algebraic* equations:

$$\Omega_0'(\tau_i, \partial) = l(\tau_i), \qquad i = 1, \cdots, n, \tag{6}$$

for a fixed basis $\tau_1, \cdots, \tau_n$ in $T_v(V_0)$. Hence, solutions of (4) and (5) form an affine (sub)-space of dimension $d = \dim W - n \dim g$ and the solutions of (3) are sections of a d-dimensional affine (sub)-bundle over V_0. Such a bundle always admits a section. Moreover, one can easily choose a specific section, say ∂_0, with an appropriate partition of unity or with an auxiliary Riemannian metric in the ambient vector bundle (see [1, 2.31]). Finally, we define the infinitesimal inversion $M = M_F$ of $\mathscr{D}$ by $M_F(l) = \partial_0$ which, according to our construction, satisfies the desired (infinitesimal invertibility) relation $L_F(M_F(l)) = l$.

1.7. Now, by specializing analytic results in [1, 2.3.2] to our $\mathscr{D}$, we obtain the following

Corollaries. Denote by $\{\Gamma\}^r$ the space of connections Γ on X with the *fine* C^r-topology (if V is compact, this is the ordinary C^r-topology).

1.8. Corollary. *For every Ω-regular C^∞-morphism F: $X \to Y$ there exists a neighborhood $\mathscr{U} \subset \{\Gamma\}^2$ of the induced connections $\mathscr{D}(F) \in \{\Gamma\}^\infty \subset \{\Gamma\}^2$, such that every C^r-connection $\Gamma' \in \mathscr{U}$ for $r \geqslant 2$ can be induced by a C^r-morphism F': $X \to Y$. Moreover, if Δ and Γ' are real analytic, then F' also can be chosen real analytic.*

1.9. Corollary. *Suppose, for some $v \in V$, there exists an Ω-regular homomorphism φ: $T_v(V) \to T_w(W)$ for some $w \in W$, and a Lie algebra isomorphism $\tilde{\Phi}$: $\tilde{X}_v \to \tilde{Y}_w$ such that the curvature Ω of Δ at w induces the curvature form Ω'_v of a given C^r-connection Γ on X. That is,*

$$\tilde{\Phi}(\Omega'_v(\tau_1, \tau_2)) = \Omega_w(\varphi(\tau_1), \varphi(\tau_2)) \tag{7}$$

for all τ_1 and τ_2 in $T_v(V)$. Then for $r \geqslant 2$ the connection Γ near $v \in V$ can be induced from Δ. Namely, there exists a neighborhood $\mathscr{U} \subset V$ of v such that the connection Γ over $\mathscr{U}$ can be induced by a C^r-morphism F of the bundle X (now restricted to $\mathscr{U}$) to Y. Moreover, one may choose F such that the differential of the underlying map f satisfies $Df \mid T_v(V) = \varphi$.

1.10. The following global version of 1.9 follows from the theory of flexible sheaves (see 2.2 in [1]). Consider a continuous morphism Φ: $X \to Y$ and let φ: $T(V) \to T(W)$ be a fiberwise injective homomorphism whose underlying map $V \to W$ equals that of Φ. Denote by $\tilde{\Phi}$: $\tilde{X} \to \tilde{Y}$ the fiberwise Lie algebra isomorphism associated to Φ and let

$$(\varphi, \tilde{\Phi})^*(\Omega) = \Omega', \tag{8}$$

where Ω is the curvature of Δ and Ω' is the curvature of a given C^r-connection Γ in X. (The relation (8) means that

$$\tilde{\Phi}\Omega'(\tau_1, \tau_2) = \Omega(\varphi(\tau_1), \varphi(\tau_2))$$

for all tangent fields τ_1 and τ_2 on V.)

If the homomorphism φ is Ω-regular at all $v \in V$ and if $r \geqslant 2$ (recall that the connection Δ in Y is C^∞) then, under the following condition $(*)$ there exists a C^r-morphism F: $X \to Y$ such that $F^*(\Delta) = \Gamma$.

$(*)$ There exists a G-bundle $\overline{X}$ over some manifold $\overline{V}$ with a C^r-connection $\overline{\Gamma}$ and a morphism P: $X \to \overline{X}$, such that:

(a) $P^*(\overline{\Gamma}) = \Gamma$,

(b) the underlying map p: $V \to \overline{V}$ is a submersion such that the pull-back $p^{-1}(\bar{v})$ is an *open* manifold of positive dimension for all $\bar{v} \in \overline{V}$. (Recall a manifold is open if it contains no component which is a compact manifold without boundary.)

1.11. Remark. Condition $(*)$ may seem rather restrictive. However, it can be applied to any V and Γ as follows. Take $V' = V \times \mathbf{R}$ with the obvious projection $p\colon V' \to V$ and with the induced connection Γ' on $X' = X \times \mathbf{R} \to V'$. Then Γ' does satisfy $(*)$. On the other hand, $\Gamma' | V \times 0 = \Gamma$. Thus, by inducing Γ' from Δ we also induce Γ from Δ.

1.12. In order to apply (1.10) and (1.11) let us state the following algebraic Lemma which follows from 2.3.

Lemma. *Let $\varphi^0\colon T(V) \to T(W)$ be a continuous Ω-isotropic homomorphism (which means $\Omega | \varphi^0(T_v(V)) = 0$ for all $v \in V$), such that the bundle induced from Y by the continuous map $\psi\colon V \to V \times W$ underlying φ^0 is isomorphic to X. If φ^0 is Ω-regular then, for any arbitrary 2-form $\Omega'\colon T(V) \otimes T(V) \to \tilde{X}$, there exists an Ω-regular homomorphism $\varphi\colon T(V) \to T(W)$ and a morphism $\Phi\colon X \to Y$, both lying over ψ, such that $(\varphi, \tilde{\Phi})^*(\Omega) = \Omega'$.*

1.13. Corollary. *Denote by T' the Whitney sum of $T(V)$ with the trivial line bundle and let $\varphi^0\colon T' \to T(W)$ be an Ω-regular and Ω-isotropic homomorphism, such that the bundle induced from Y by the underlying continuous map $V \to W$ is isomorphic to X. Then an arbitrary C^r-connection Γ on X for $r \geqslant 2$ can be induced by a C^r-morphism $F\colon X \to Y$.*

Proof. Compose φ^0 with the obvious (fiberwise injective) projection $T(V \times \mathbf{R}) \to T'$ and apply the Lemma to the resulting Ω-regular and Ω-isotropic homomorphism $T(V \times \mathbf{R}) \to T(W)$. Thus we get an Ω-regular homomorphism $\varphi'\colon T(V \times \mathbf{R}) \to T(W)$ and a morphism $\Phi'\colon X' \to Y$ for $X' = X \times \mathbf{R} \to V \times \mathbf{R}$ such that $(\varphi', \tilde{\phi}')^*(\Omega) = \Omega'$ for the curvature form Ω' of the connection Γ' induced from Γ by the projection $V \times \mathbf{R} \to V$. Now, by applying 1.10, we induce Γ' from Δ and, by restricting Γ' to $V = V \times 0$ (compare 1.11) we induce Γ as well.

1.14. Let us specialize 1.13 to the case of the *trivial* vector bundle $X = G \times V \to V$ over a stably *parallelizable* manifold V. Assume, there is an $(n + 1)$-dimensional Ω-regular and Ω-*isotropic* subspace $T'_w \subset T_w(W)$ for some $w \in W$, where the Ω-isotropy means $\Omega(\tau'_1, \tau'_2) = 0$ for all τ'_1, τ'_2 in T'. Then the required φ^0 does exist. Indeed, since the bundle T' is trivial it can be induced by a (constant) map $\psi_w\colon V \to w \in W$ from the subpace T'_w viewed as a bundle over $\{w\}$ and since X is trivial it is isomorphic to $\psi^*_w(Y)$. Thus we conclude:

1.15. Proposition. *Let Γ be a C^r-connection in a trivial bundle X over a stably parallelizable n-dimensional manifold V. If the connection Δ in Y admits an $(n + 1)$-dimensional Ω-regular and Ω-isotropic subspace in some tangent space $T_w(W)$ then, for $r \geqslant 2$, there exists a C^r-morphism $X \to Y$ which induces Γ from Δ.*

1.16. Remark (compare 1.9). Extend a connection Γ on X to a connection Γ' on $X' = X \times \mathbf{R} \to V' = V \times \mathbf{R}$ and take a Γ-inducing Ω-regular morphism F of $X = X \times 0$ to Y. Let us try to extend F to a Γ'-inducing morphism $X' \to Y$. Denote by ∂^h the Γ'-horizontal lift to X' of the field $\partial = \partial/\partial t$ on V' and observe that the equation $(F')^*(\Delta) = \Gamma'$ implies (compare (2))

$$\Omega\left(\frac{\partial f'}{\partial t}, \partial_\tau f'\right) = \tilde{F}'\Omega'(\partial, \tau) \tag{9}$$

for all tangent fields τ on V', where the following notations are used:

Ω' is the curvature of Γ',

$\tilde{F}'$ is the associated homomorphism of the Lie algebra bundle, $\tilde{F}'$: $\tilde{X}' \to \tilde{Y}$,

$\partial f'/\partial t$ and $\partial_\tau f'$ are the images of the fields $\partial/\partial t$ and τ correspondingly under the differential of the map f': $V \to W$ underlying F'.

If $F' \mid X = F$, then the Ω-regularity of F allows one to resolve equation (9) in $\partial f'/\partial t$ and to bring it to the evolution (or Cauchy-Kowalewsky) form. We write it as

$$\frac{\partial f'}{\partial t} = \Omega^{-1}\left(\partial_\tau f', \tilde{F}'\Omega'(\partial, \tau)\right), \tag{10}$$

where the "inverse" Ω^{-1} is defined at (v, t) in so far as $f \mid V \times t$ is Ω-regular at (v, t).

Now, one can easily see that

1.17. Lemma. *If a map F' satisfies equation* (10) (*which, whenever defined, is equivalent to* (9)) *and the conditions* (a) $F' \mid X = F$ *and* (b) *the differential of F' sends ∂^h to a Δ-horizontal field, then $F'^*(\Delta) = \Gamma'$.*

The proof follows by reversing the computation which brought up equation (9). The same conclusion equally applies to small neighborhoods $\mathcal{U}' \subset V \times \mathbf{R}$ of $V \times 0$. Therefore, the extension of F to $\mathcal{U}'$ reduces to solving the evolution system expressed by (10) and the above condition (b) with the initial data (a).

1.18. Corollary. *If the connections Γ' and Δ and the morphism F are real analytic, then for some neighborhood $\mathcal{U}'$ of $V \times 0$ in $V \times \mathbf{R}$, there exists a real analytic morphism of $X' \mid \mathcal{U}$ to Y which induces Γ' on $X' \mid \mathcal{U}'$, where $X' \mid \mathcal{U}'$ denotes the restriction of X' to $\mathcal{U}'$.*

Proof. Apply the Cauchy-Kowalewsky theorem (compare [2]).

Now let us return to the assumptions of Corollary 1.9 where we had an Ω-regular linear map φ: $T_v(V) \to T_w(W)$ inducing the curvature of Γ via some $\tilde{\Phi}$. In the real analytic case, the above Corollary 1.18 insures the existence of some neighborhood $\mathcal{U}'$ of $(v, 0) \in V \times \mathbf{R}$ and of a Γ'-inducing morphism of $X' \mid \mathcal{U}'$ to Y for every (real analytic!) connection Γ' on $V \times \mathbf{R}$. (This result, as was told to me by Professor M. Gromov, is due to E. Cartan.)

2. Ω-regular subspaces

2.1. Consider linear spaces T, T', and g and let Ω be an antisymmetric bilinear form $T \otimes T \to G$. For every homomorphism φ: $T' \to T$ let $\delta(\varphi)$ denote the induced form $\varphi^*(\Omega)$ on T'. Recall that a homomorphism φ is called Ω-regular if the linear map $T \to \operatorname{Hom}(T', g)$ given by $(\tau \mapsto h_\tau(\tau') = \Omega(\tau, \varphi(\tau'))$ is surjective. Restrict the above map $\varphi \mapsto \delta(\varphi)$ to the space of Ω-regular homomorphisms $T' \to T$.

Lemma. *The map*

$$\delta\colon \operatorname{Reg}\operatorname{Hom}(T', T) \to \big(\textit{the space of antisymmetric 2-forms} T' \otimes T' \to g\big)$$

is a submersion.

Proof. Fix a basis $\tau_i' \in T'$, $i = 1, \cdots, n$, and let $\tau_i = \varphi(\tau_i')$. The surjectivity of the differential of δ at φ is equivalent to solvability in $x_i \in T$, $i = 1, \cdots$, of the linear system

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\big[\Omega(\tau_i + \varepsilon x_i, \tau_j + \varepsilon x_j) - \Omega(\tau_i, \tau_j)\big] = \omega_{ij} \tag{11}$$

for any given antisymmetric matrix of vectors $\omega_{ij} \in g$, $1 \leqslant i, j \leqslant n$. If φ is Ω-regular, one can solve for every fixed j the system

$$\Omega(\tau_i, x_j) = \tfrac{1}{2}\omega_{ij}, \qquad i = 1, \cdots, n.$$

Since Ω and ω are antisymmetric, these solution x_j satisfy the system of equations

$$\Omega(\tau_i, x_j) + \Omega(x_i, \tau_j) = \omega_{ij}, \qquad 1 \leqslant i, j \leqslant n,$$

which is equivalent to (11).

2.2. Corollary. *Consider vector bundles T, T', and g over V and a bilinear antisymmetric form Ω: $T \otimes T \to g$. If φ: $T' \to T$ is an Ω-regular (continuous) homomorphism, then for every sufficiently small antisymmetric form ω on T' there exists an Ω-regular homomorphism φ': $T' \to T$ such that $(\varphi')^*(\Omega) = \varphi^*(\Omega) + \omega$.*

(If V is noncompact, "small" refers to the fine C^0-topology in the space of sections $\operatorname{Hom}(T' \otimes T' \to g)$.)

Proof. The map δ which now sends (the total space of) the bundle $\operatorname{Hom}(T' \to T)$ to the bundle of g-valued forms on T', is a topological submersion on the subset $\operatorname{Reg}\operatorname{Hom}(T', T) \subset \operatorname{Hom}(T', T)$ because it is a submersion over every point $v \in V$ by the Lemma in 2.1.

2.3. Corollary. *Let T, T', g and Ω be as in 2.2 and let φ^0: $T' \to T$ be an Ω-regular and Ω-isotropic homomorphism. That is, $(\varphi^0)^*(\Omega) \equiv 0$. Then, every g-valued 2-form ω on T' can be induced from Ω by an Ω-regular homomorphism φ: $T' \to T$.*

Proof. By 2.2 there exists a small positive function ε on V and an Ω-regular homomorphism φ', such that $(\varphi')^*(\Omega) = \varepsilon\omega$. Then the homomorphism $\varepsilon^{-2}\varphi'$ induces ω.

2.4. In order to make sure that the general results in §1 are nonvacuous, we need examples of connections Δ on W such that the tangent bundle $T(W)$ contains sufficiently many Ω-regular subspaces.

2.5. Lemma. *Let T and g be linear spaces and let T' be a subspace in T. Set $m = \dim T$, $n = \dim T'$, and $k = \dim g$. Then, in the following three cases (and, as one can easily see, only in these cases) there exists a g-valued 2-form Ω on T (i.e. an antisymmetric bilinear map $T \otimes T \to g$) for which T' is Ω-regular.*

(i) $n = 1$, $m > k$.

(ii) $k = 1$ *and m is even.*

(iii) $m \geqslant n \cdot k$ *and* $m > n$.

Proof. Take a subspace $S \subset T$ complementary to T'. Then, in case (i) take any surjective linear map Ω_0: $S \otimes T' \to g$ and define Ω by

$$\Omega(s_1 + t_1', s_2 + t_2') = \Omega_0(s_1 \otimes t_2') - \Omega_0(s_2 \otimes t_1')$$

for all $s_1, s_2 \in S$ and $t_1', t_2' \in T'$.

Next, let $\dim g = 1$ and $\dim T$ be even. Then take any nonsingular $\mathbf{R}$-valued form on T for Ω. This concludes case (ii). Furthermore, if $\dim T' < \dim T$ this applies to an even dimensional subspace $T_0 \supset T'$ in T and thus yields case (iii) for $\dim g = 1$. Now, let $\dim g \geqslant 2$ and $\dim T' \geqslant 2$. Then there exists a g-valued 2-form Ω' on T' for which the homomorphism h': $T \to \mathrm{Hom}(T', g)$ given by $t' \to h'_{t'}(t'') = \Omega(t', t'')$ is injective. Indeed, take $\Omega' = (\omega_1, \omega_2, \cdots, \omega_k)$ for $k = \dim g$ where the $\mathbf{R}$-valued forms ω_1 and ω_2 have ranks $\geqslant \dim T' - 1$ and $\ker \omega_1 \cap \ker \omega_2 = 0$, where, by definition,

$$\ker \omega = \{t' \in T' \mid \omega(t', t'') = 0 \text{ for all } t'' \in T'\}.$$

If $\dim T \geqslant \dim T' \cdot \dim g$, then there exists a linear map h: $S \to \mathrm{Hom}(T', g)$ such that the images $h(S)$ and $h'(T)$ span $\mathrm{Hom}(T', g)$. Finally, we define

$$\Omega(s_1 + t_1', s_2 + t_2') = \Omega'(t_1', t_2') + h_{s_1}(t_2') - h_{s_2}(t_1').$$

2.6. Remark. If g is the Lie algebra of G and T is the tangent space of a manifold W at a point $w \in W$, then for any g-valued 2-form Ω_0 on T there obviously exists a C^∞ (and even real analytic) connection in any given G-bundle $Y = G \times W \to W$ whose curvature at w equals Ω_0.

2.7. Let us return to the G-bundles $X \to V$ and $Y \to W$ where $\dim G = k$, $\dim V = n$, and $\dim W = m$, and assume the bundle X to be trivial. We combine the results in §1 with 2.5 and 2.6 and obtain the following

Corollary. *If the dimensions k, m, and n satisfy one of the above conditions* (i), (ii), *or* (iii), *then there exist subsets* $\{\Gamma\}_0$ *and* $\{\Delta\}_0$ *in the space of connections on X and on Y corresponding which satisfy the following conditions*:

(1) *The subsets* $\{\Gamma\}_0$ *and* $\{\Delta\}_0$ *are nonempty.*

(2) *The subset* $\{\Gamma\}_0$ *is open in the fine C^2-topology in the space of connections in X.*

(3) *The subset* $\{\Delta\}_0$ *is open in the fine C^1-topology in the space of connections in Y.*

(4) *For every C^∞-connection* $\Delta \in \{\Delta\}_0$ *and every C^r-connection* $\Gamma \in \{\Gamma\}_0$ *for* $r \geqslant 2$ *there exists a C^r-morphism* $F\colon X \to Y$ *such that* $F^*(\Delta) = \Gamma$.

(5) *Let* $\Delta \in \{\Delta\}_0$ *and* $\Gamma \in \{\Gamma\}_0$ *be real analytic and let* Γ' *be a real analytic connection in the bundle* $X' = X \times \mathbf{R} \to V \times \mathbf{R}$ *such that* $\Gamma' | V = V \times 0$ *equals* Γ. *Then, there exists a neighborhood* $\mathscr{U} \subset X \times \mathbf{R}$ *of* $X \times 0$ *such that the connection* Γ' *over* $\mathscr{U}$ *can be induced from* Δ *by a* C^{an}*-morphism of* $X' | \mathscr{U}$ *to* Y.

2.8. Example. Let $k = \dim g \geqslant 2$ and $n = \dim V \geqslant 0$. Then the above applies for $m \geqslant kn$, for $m = \dim W$. In particular, we obtain a nonempty open set of real analytic connections Γ' on $V \times \mathbf{R}$ which can be locally induced from some fixed Δ. By Remark (B) in 0.4, such a Δ does not exist for $m < kn$.

2.9. Lemma. *Let* T, g, *and* $T' \subset T$ *be linear spaces of dimension* m, k, *and* n (*compare* 2.5). *If*

$$m \geqslant n(k+1), \tag{12}$$

then there exists a g*-valued* 2*-form* Ω *on* T *for which* $T' \subset T$ *is* Ω*-regular and* Ω*-isotropic.*

Proof. Take $S \subset T$ as in the proof of 2.5 and a surjective linear map h: $S \to \mathrm{Hom}(T', g)$. Then the form $\Omega(s_1 + t_1', s_2 + t_2') = h_{s_1}(t_2') - h_{s_2}(t_1')$ is the required one.

2.10. Remarks. (a) Inequality (12) obviously is the best possible.

(b) Every sufficiently small perturbation Ω' on Ω admits an n-dimensional Ω-regular and Ω-isotropic subspace $T'' = T''(\Omega') \subset T$.

2.11. Now, as in 2.7, we obtain, with 1.15, the following:

Corollary. *Let the manifold V be stably parallelizable, let* $X = V \times G \to V$ *be the trivial bundle and let* $Y \to W$ *be an arbitrary G-bundle. If* $\dim W \geqslant (\dim V + 1)(\dim g + 1)$, *then there exists a C^∞-connection* Δ *on Y such that every C^r-connection* Γ *on V for* $r \geqslant 2$ *can be induced from* Δ *by a C^r-morphism* $F\colon X \to Y$. *Furthermore, every small C^1-perturbation of* Δ *has the same property.*

2.12. Let us turn to the canonical $\mathrm{O}(p)$-bundle Y over the Grassmann manifold $\mathrm{Gr}_p(\mathbf{R}^q)$ with the standard ($\mathrm{O}(q)$-invariant) connection Δ. At every point $w \in \mathrm{Gr}_p(\mathbf{R}^q)$, the tangent space $T_w(\mathrm{Gr}_p(\mathbf{R}^q))$ is identified with the space

$T = \mathrm{Hom}(\mathbf{R}^p, \mathbf{R}^{q-p})$, the Lie algebra g of $\mathrm{O}(p)$ is represented by antisymmetric $(p \times p)$-matrices, and the curvature form Ω of the connection Δ is given by the formula (see [1, 3.2.1])

$$\Omega(A_1, A_2) = A_1'A_2 - A_2'A_1 \tag{13}$$

for $A_1, A_2 \in T$ where the prime denotes transposition of matrices. Observe that $\dim T = p(q-p)$ and $\dim g = p(p-1)/2$.

2.13. Lemma. *Let Ω: $T \otimes T \to g$ be the g-valued 2-form defined by* (13) *where T is the space of $p \times q'$ matrices for $q' = q - p$ and g is the space of antisymmetric matrices of order p. Then, there exists an Ω-regular and Ω-isotropic subspace $T' \subset T$ of dimension $n = 2 \operatorname{ent} q'/p$.*

Proof. Start with the case $p = q'$ and consider the 2-dimensional space generated by the matrices

$$I = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}, \quad I_\lambda = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ 0 & & & & \lambda_p \end{pmatrix}.$$

Since $\Omega(I, I_\lambda) = II_\lambda - I_\lambda I = 0$, this space is Ω-isotropic. It is also Ω-regular, if

$$\lambda_i \neq \lambda_j, \quad \text{for } i \neq j. \tag{14}$$

To see this, look at the linear homogeneous system

$$\Omega(X, I) = 0, \qquad \Omega(X, I_\lambda) = 0 \tag{15}$$

in the unknown $X \in T$, $T = \mathrm{Hom}(\mathbf{R}^p, \mathbf{R}^p)$, and show the space of solutions to have dimension $\leqslant p = p^2 - p(p-1)$, where p^2 is the number of unknowns in (15) and $p(p-1)$ is the number of equations. The equation $\Omega(X, I) = 0$ amounts to $X = X'$ (i.e. the matrix X is symmetric) and the equation $\Omega(X, I_\lambda) = X'I_\lambda - I_\lambda' X = I_\lambda X - XI_\lambda = 0$ implies that every element of X, say x_{ij}, satisfies $(\lambda_i - \lambda_j)x_{ij} = 0$. Hence, every solution of (15) is a diagonal matrix and the proof is concluded for $p = q'$. Now, for $q' = sp + p'$ for $0 \leqslant p' < p$, we take:

$$I^1 = (I, 0, \cdots, 0, 0'),\ I^2 = (0, I, 0, \cdots, 0, 0'), \cdots, I^s = (0, \cdots, 0, I, 0'),$$

$$I_\lambda^1 = (I_\lambda, 0, \cdots, 0, 0'),\ I_\lambda^2 = (0, I_\lambda, 0, \cdots, 0, 0'), \cdots, I_\lambda^s = (0, \cdots, 0, I_\lambda, 0'),$$

where the zeros stand for the zero $(p \times p)$-matrices and $0'$ is the zero $(p \times p')$-matrix. The $\mathrm{Span}(I^i, I_\lambda^i)$ (of dimension $2s = 2 \operatorname{ent} q'/p$) is obviously Ω-isotropic and, if condition (14) holds, it is also Ω-regular. Indeed, the system

$$\begin{aligned} \Omega(X, I^i) &= 0, & i &= 1, \cdots, s, \\ \Omega(X, I_\lambda^i) &= 0, & i &= 1, \cdots, s \end{aligned} \tag{16}$$

(which has $sp(p-1)$ equations in pq' unknowns $X \in T$, $T = \mathrm{Hom}(\mathbf{R}^p, \mathbf{R}^{q'})$), obviously divides into s independent subsystems of the type (15). Hence, by the above, the space is solutions of (16) is $(sp + pp')$-dimensional.

2.14. Now we can prove the results stated in 0.3. and in the last remark in (B) of 0.4. Let X be a trivial $O(p)$-bundle over a stably parallelizable manifold V. Then the existence of the Ω-isotropic and Ω-regular subspace in $T(\mathrm{Gr}_p(\mathbf{R}^q))$ established above and Proposition 1.15 insure the existence of the connection inducing map $f\colon V \to \mathrm{Gr}_p(\mathbf{R}^q)$ for all $q \geqslant p(n+3)/2$. Furthermore, in the real analytic case, we apply 1.18 to a small tubular neighborhood $\mathcal{U} \subset V$ of a hypersurface $V_0 \subset V$ and obtain a connection inducing map $f\colon \mathcal{U} \to \mathrm{Gr}_p(\mathbf{R}^q)$ for all $q \geqslant p(n+1)/2$.

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