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NORMAL FORMS FOR GENERIC MANIFOLDS AND HOLOMORPHIC EXTENSION OF CR FUNCTIONS

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1. Introduction

The main results of this paper give normal forms for coordinates on generic CR manifolds of arbitrary codimension, conditions for local holomorphic extendability, and decomposition of CR functions defined on these manifolds. More precisely, we consider a manifold M defined near the origin of $\mathbb{C}^{n+l}$ by

$$\Im m\, w = \phi(z, \bar{z}, \Re e\, w), \tag{1.1}$$

where $z \in \mathbb{C}^n$, $w \in \mathbb{C}^l$, and ϕ is a smooth function defined in a neighborhood of the origin in $\mathbb{R}^{2n+l}$ and valued in $\mathbb{R}^l$. We shall always assume

$$\phi(0) = 0, \qquad d\phi(0) = 0. \tag{1.2}$$

A *wedge* of *edge* M is an open set of $\mathbb{C}^{n+l}$ of the form

$$\mathscr{W}(\mathscr{O}, \mathscr{C}) = \{(z,w) \in \mathscr{O};\ \Im m\, w - \phi(z, \bar{z}, \Re e\, w) \in \mathscr{C}\}, \tag{1.3}$$

where $\mathscr{O}$ is a neighborhood of 0 in $\mathbb{C}^{n+l}$, and $\mathscr{C}$ is a convex open cone in $\mathbb{R}^l$.

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If $l = 1$, i.e. M is a hypersurface in $\mathbb{C}^{n+1}$, and if M is of finite type (see §2 below) it is known that any CR function extends to be holomorphic on one side of M (see Baouendi-Treves [6] which generalizes the classical result of Lewy [12]). In Baouendi-Rothschild-Treves [4] M is called *rigid* if ϕ in (1.1) can be chosen to be independent of $\Re e\, w$. One of the results of that paper is that if M is a rigid generic CR manifold of finite type, then any CR function on M extends to be holomorphic in a wedge of the form (1.3).

Here we generalize these results to generic CR manifolds which can be viewed as perturbations of rigid ones. The new class, called *semi-rigid*, contains in particular all hypersurfaces. In appropriate local coordinates, these perturbations are terms of higher homogeneity in the Taylor expansions of the defining functions; however semi-rigidity will be defined by an invariant condition. Theorem 8 states that if M is semi-rigid and of finite type, then every CR function on M extends to be holomorphic in a wedge with edge M. We also give new results in this paper on the question of decomposing a CR function into a finite sum of boundary values of holomorphic functions in wedges of the form (1.3).

The paper is organized as follows. In §2 we define semi-rigidity (Definition (2.5)) in an invariant fashion using commutators of the holomorphic and antiholomorphic vector fields. Next, we give normal forms for coordinates for general generic CR manifolds and for semi-rigid ones. We reprove and generalize a result of Bloom-Graham [7], but our methods are completely different. As a corollary we give a geometric condition which is sufficient for the existence of a CR function which does not extend to be holomorphic in any wedge. In §§3 and 4 we prove these results using group-theoretic methods and a theorem of Helffer-Nourrigat [8]. In §5 we state and prove uniqueness of the normal forms up to certain transformations.

We use the notion of microlocal hypoanalyticity introduced in Baouendi-Chang-Treves [2]. In §6 we define the hypoanalytic wave front set for CR distributions by means of the *mini*-FBI (Fourier-Bros-Iagolnitzer) transform, which is a slight variation of the FBI transform used in [2]. It is more closely related to the one used in [4] in the rigid case. The material covered in §6 evolved from several discussions with F. Treves during the Fall of 1984; more details will appear in his forthcoming book [17]. In §7 we prove the result, Theorem 8, mentioned above, on extendability of CR distributions on semi-rigid CR manifolds of finite type. Some results on microlocal hypoanalyticity and extendability of CR distributions on general generic manifolds (not necessarily semi-rigid) are given in §8. These results are new even in the hypersurface case.

In §9 we study the question of holomorphic decomposition of CR functions. In [4] it was shown that for a rigid generic CR manifold, not necessarily of finite type, any CR distribution is a finite sum of boundary values of holomorphic functions in some wedges. Trépreau [15] has recently given an example which shows that this is impossible in general. Here we prove that such a decomposition exists whenever the CR distribution has hypoanalytic wave front contained in a disjoint union of strictly convex closed cones in $\mathbb{R}^l \setminus \{0\}$. This, in particular, gives a different proof of the result of Andreotti-Hill [1] for hypersurfaces.

We wish to thank François Treves for many useful conversations on the subject of this paper. Also we are indebted to Bernard Helffer for his help in formulating the condition of semi-rigidity. Some of the main results of this work were announced in [3].

2. Homogeneity, normal forms, and semi-rigidity

Let Ω be a smooth manifold of real dimension $2n + l$, and $\mathscr{V}$ a vector subbundle of $\mathbb{C}T\Omega$, the complexified tangent bundle to Ω. Denote by $\mathscr{V}_\omega$ the fiber of $\mathscr{V}$ at $\omega \in \Omega$. We shall assume that for all $\omega \in \Omega$:

$$\mathscr{V}_\omega \cap \overline{\mathscr{V}}_\omega = (0), \qquad \dim_{\mathbb{C}} \mathscr{V}_\omega = n, \tag{2.1}$$

and that $\mathscr{V}$ satisfies the Frobenius condition

$$[\mathscr{V}, \mathscr{V}] \subset \mathscr{V}. \tag{2.2}$$

Under these conditions we shall call $\mathscr{V}$ a *generic CR bundle*. Let $\mathbf{L} = C^\infty(\Omega, \mathscr{V})$ be the C^∞ sections of $\mathscr{V}$ over Ω. The *characteristic set* of $\mathbf{L}$ at ω is defined as the set Σ_ω of all $\xi \in T^*_\omega(\Omega) \setminus \{0\}$ for which the symbols $\sigma(L)(\omega, \xi)$ vanish for all $L \in \mathbf{L}$. We say that $\mathscr{V}$ (or Ω) is of *finite type* at ω (see Kohn [11] and Bloom-Graham [7]) if for any characteristic vector $\xi \in \Sigma_\omega$ there exists a commutator

$$L^{(k)} = [M_1, [M_2, \cdots, [M_{k-1}, M_k] \cdots]], \tag{2.3}$$

with each $M_j \in \mathbf{L} \oplus \overline{\mathbf{L}}$, such that the symbol $\sigma(L^{(k)})$ satisfies

$$\sigma(L^{(k)})(\omega, \xi) \neq 0. \tag{2.4}$$

More generally we say that $\mathscr{V}$ (or Ω) is of *finite type* at (ω, ξ) if (2.4) holds for some $L^{(k)}$.

We let $m(\omega, \xi)$ be the smallest integer k for which there is a commutator of length k satisfying (2.4). If there is no $L^{(k)}$ satisfying (2.4) we take $m(\omega, \xi) = \infty$. The *Hörmander numbers* at $\omega_0 \in \Omega$ are the r distinct integers

$$2 \leqslant m_1 < m_2 < \cdots < m_r \leqslant \infty$$

obtained as $m(\omega_0, \xi)$ for some $\xi \in \Sigma_{\omega_0}$. It is clear that we have $1 \leqslant r \leqslant l$, and that $\mathscr{V}$ is of finite type at ω_0 if and only if $m_r < \infty$.

For $1 \leqslant j \leqslant r$, let E_j be the subspace of $\mathbf{C}T_{\omega_0}(\Omega)$ spanned by all $L^{(k)}_{\omega_0}$, where $L^{(k)}$ is any commutator of the form (2.3) and $1 \leqslant k \leqslant m_j$. The multiplicity l_j of m_j is defined by

$$l_1 = \dim E_1 - 2n, \qquad l_j = \dim E_j - \dim E_{j-1}, \quad 1 < j \leqslant r.$$

Equivalently, l_j is the real dimension of any maximal subspace Σ_j of $\Sigma_{\omega_0} \cup \{0\}$ with the property that

$$\Sigma_j \subset E_{j-1}^{\perp} \quad \text{if } j > 1, \quad \text{and} \quad \Sigma_j \cap E_j^{\perp} = \{0\}.$$

It is clear that we have $l_j \geqslant 1$ for $1 \leqslant j < r$, and $l_r \geqslant 1$ if and only if $m_r < \infty$. Also we have $\Sigma_{j=1}^{r} l_j \leqslant l$, and equality holds if and only if Ω is of finite type at ω_0.

We need to introduce the following definition.

(2.5) Definition. *The manifold Ω equipped with the CR structure $\mathscr{V}$ is semi-rigid at $\omega_0 \in \Omega$ if for all $\xi \in \Sigma_{\omega_0}$*

$$\sigma([L^{(\alpha)}, L^{(\beta)}])(\omega_0, \xi) = 0 \tag{2.6}$$

for all commutators $L^{(\alpha)}$, $L^{(\beta)}$ of the form (2.3) *of lengths α and β respectively, whenever $\alpha \geqslant 2$, $\beta \geqslant 2$, and $\alpha + \beta \leqslant m(\omega_0, \xi)$.*

More generally, we shall say that Ω is semi-rigid at ω_0 up to the jth Hörmander number m_j if (2.6) *holds for all $\xi \in \Sigma_{\omega_0}$ such that $m(\omega_0, \xi) \leqslant m_j$.*

Note that by the definition of the number $m(\omega_0, \xi)$, the left-hand side of (2.6) is always zero if $\alpha + \beta < m(\omega_0, \xi)$.

We have the following examples of semi-rigid structures.

(2.7) Proposition. *The CR structure $(\Omega, \mathscr{V})$ is semi-rigid at ω_0 in any of the following cases hold*:

(i) $l = 1$.

(ii) *The largest finite Hörmander number at ω_0 is at most three.*

(iii) $n = 1$, *and the largest finite Hörmander number at ω_0 is at most four.*

(iv) *All finite Hörmander numbers at ω_0 are equal or, more generally, the difference between any two finite m_i is at most one.*

Proof. First let us show that (2.5) holds when $l = 1$. Assume the unique Hörmander number m_1 at ω_0 is finite. Suppose that $L^{(\alpha)}$ and $L^{(\beta)}$ are commutators of the form (2.3) with $\alpha, \beta \geqslant 2$ and $\alpha + \beta = m_1$. We write

$$L^{(\alpha)} = M + - aT, \qquad L^{(\beta)} = M' + bT$$

with $M, M' \in \mathbf{L} \oplus \overline{\mathbf{L}}$, T a vector field (missing direction, here $\dim \Sigma_{\omega_0} \cup \{0\} = 1$) with $\sigma(T)(\omega_0, \xi) \neq 0$ for $\xi \in \Sigma_{\omega_0}$, and a, b smooth functions vanishing at ω_0. Then if $\sigma([L^{(\alpha)}, L^{(\beta)}])(\omega_0, \xi) \neq 0$ either $Mb \neq 0$ or $M'a \neq 0$, thus either

$\sigma([M', L^{(\alpha)}])(\omega_0, \xi) \neq 0$ or $\sigma([M, L^{(\beta)}])(\omega_0, \xi) \neq 0$ contradicting the definition of m_1.

It is immediate that (ii) must satisfy (2.5). For (iii) we observe that if $n = 1$, there is essentially one nontrivial commutator of length 2 (up to multiplication by a function) which of course commutes with itself. Finally, for (iv) we first write as in (i)

$$L^{(\alpha)} = M + \sum_{j=1}^{l} a_j T_j, \qquad L^{(\beta)} = M' + \sum_{j=1}^{l} b_j T_j,$$

where $M, M' \in \mathbf{L} \oplus \overline{\mathbf{L}}$ and the T_j span the missing directions at ω_0. If $\alpha + \beta = m_j$, then (iv) implies $\alpha \leqslant m_1 - 1$ and $\beta \leqslant m_1 - 1$, hence all the a_j and the b_j must vanish at ω_0. The rest of the argument is the same as for (i). q.e.d.

Let $\omega_0 \in \Omega$ be fixed. We shall make the additional hypothesis that $\mathscr{V}$ is *integrable at* ω_0, i.e. that there exist $n + l$ complex valued functions ζ_i on Ω, in a neighborhood of ω_0, such that the matrix $D\zeta(\omega_0)$, $\zeta = (\zeta_1, \cdots, \zeta_{n+l})$, is of rank $n + l$ and such that

$$L\zeta_i = 0, \qquad i = 1, \cdots, n + l, \tag{2.8}$$

for all $L \in \mathbf{L}$. We shall always assume that $\zeta(\omega_0) = 0$. By shrinking Ω about ω_0 if needed, $M = \zeta(\Omega)$, the image of Ω under the mapping ζ, is a submanifold of $\mathbb{C}^{n+l}$ of real codimension l. We shall often identify Ω with M. Under this identification $\mathscr{V}$ is the subbundle of the antiholomorphic tangent vectors to M. We shall say that M is a *generic CR manifold in* $\mathbb{C}^{n+l}$.

Assume that Ω (or M) is of finite type at ω_0 (or at the origin) and let m_j be the Hörmander numbers at ω_0 with multiplicity l_j. Recall that $\sum_{j=1}^{r} l_j = l$. We shall define local coordinates (x, y, s) on Ω vanishing at ω_0, with $x, y \in \mathbb{R}^n$, $s = (s_1, \cdots, s_r)$ with $s_k \in \mathbb{R}^{l_k}$, $1 \leqslant k \leqslant r$, and dilations

$$\delta_t(x, y, s) = (tx, ty, t^{m_1}s_1, \cdots, t^{m_r}s_r) \tag{2.9}$$

for $t > 0$. If $p(x, y, s)$ is a polynomial we shall say that p is *homogeneous of weight m* if

$$p(\delta_t(x, y, s)) = t^m p(x, y, s) \quad \forall (x, y, s) \in \mathbb{R}^{2n+l}, \forall t > 0.$$

With such coordinates, if $f(x, y, s)$ is smooth near 0 we shall say that f is of weight $\geqslant m$, and write $f = \mathcal{O}(m)$ if the Taylor expansion of f at the origin is a sum of homogeneous polynomials of weight $\geqslant m$.

If $p(x, y, s)$ is a real valued homogeneous polynomial of weight $m > 1$, we shall say that p is *M-pluriharmonic of weight m* if there exists a holomorphic function $F(\zeta)$ with $F(0) = 0$, $F'(0) = 0$, in a neighborhood of $\zeta(\omega_0) = 0$ in

$\mathbb{C}^{n+l}$ such that

$$p(x, y, s) = \Im m\ F(\zeta)|_M + \mathcal{O}(m+1).$$

Here $\zeta = (\zeta_1, \cdots, \zeta_{n+l})$ where the $\zeta_i|_M$ satisfy (2.8) and therefore are functions of (x, y, s).

Our first result gives normal coordinates on M in the general finite type case, reproving a result of Bloom and Graham [7]; for the semi-rigid case a special form of the coordinates is given. We shall also give results for the case where only certain directions are of finite type.

Theorem 1. *Let M be a generic CR manifold in $\mathbb{C}^{n+l}$, of real codimension l, and of finite type at the origin. One can find local coordinates (x, y, s) on M with $x, y \in \mathbb{R}^n$, $s \in \mathbb{R}^l$, and local holomorphic coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^l$, such that M is locally represented by*

$$(2.10)\qquad \begin{aligned} &z_j = x_j + iy_j, \qquad j = 1, 2, \cdots, n, \\ &w_k = s_k + i\left[p_{m_k}(z, \bar{z}, s_1, \cdots, s_{k-1}) + \mathcal{O}(m_k + 1)\right], \qquad k = 1, \cdots, r, \end{aligned}$$

where $w_k \in \mathbb{C}^{l_k}$, the m_k's are Hörmander numbers of M at the origin, l_k is the multiplicity of m_k, and p_{m_k} is a homogeneous polynomial of weight m_k (with the dilations (2.9)) valued in $\mathbb{R}^{l_k}$. Also for any $\eta \in \mathbb{R}^{l_k} \setminus \{0\}$, $\eta \cdot p_{m_k}$ is not M-pluriharmonic of weight m_k.

Furthermore, the p_{m_k}, $1 \leqslant k \leqslant r$, may be chosen to be independent of all the s_j if and only if M is semi-rigid at the origin.

More generally if $1 \leqslant j < r$, the p_{m_k}, $1 \leqslant k \leqslant j$, may be chosen independent of s if and only if M is semi-rigid at the origin up to the jth Hörmander number m_j.

Finally if M is real analytic then all the remainder terms $\mathcal{O}(m_k + 1)$ in (2.10) are real analytic functions in (x, y, s).

We shall prove Theorem 1 in §4, using canonical coordinates for the associated antiholomorphic vector fields. These coordinates will be defined in §3. The methods used here are partly based on a theorem of Helffer and Nourrigat [8].

If M is *not of finite type* at the origin, i.e. $m_r = \infty$ and $l_r = 0$, we shall also define local coordinates $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}^l$ with $s = (s_1, \cdots, s_r)$, $s_k \in \mathbb{R}^{l_k}$ for $1 \leqslant k < r$, and $s_r \in \mathbb{R}^{l'_r}$ with $l'_r = l - \sum_{j=1}^{r-1} l_j$. We define dilations

$$\delta_t(x, y, s_1, \cdots, s_{r-1}) = \left(tx, ty, t^{m_1}s_1, \cdots, t^{m_{r-1}}s_{r-1}\right), \qquad t > 0,$$

and, we say that a polynomial $p(x, y, s_1, \cdots, s_{r-1})$ is homogeneous of weight m if

$$p \circ \delta_t = t^m p \quad \forall t > 0.$$

Any monomial in the x, y, s variables will be of weight ∞ if at least one of its factor is a component of s_r. Similarly to the finite type case, a smooth funtion $f(x, y, s)$ is of weight $\geqslant m$, $f = \mathcal{O}(m)$, if each monomial of its Taylor series at the origin is of weight $\geqslant m$ (possibly of weight ∞).

Theorem 2. *Let M be a generic CR manifold in $\mathbb{C}^{n+l}$, of real codimension l, and not of finite type at the origin. Let $m_1 < \cdots < m_{r-1}$ be the finite Hörmander numbers at the origin with multiplicities $l_1, \cdots, l_{r-1}$. For any $N > m_{r-1}$, one can find local coordinates x, y, s on M with $x, y \in \mathbb{R}^n$, $s \in \mathbb{R}^l$ as above, and local holomorphic coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^l$ such that M is locally represented by*

$$\begin{aligned} &z_j = x_j + iy_j, \qquad 1 \leqslant j \leqslant n, \\ &w_k = s_k + i\left[p_{m_k}(x, y, s_1, \cdots, s_{m_{k-1}}) + \mathcal{O}(m_k + 1)\right], \qquad 1 \leqslant k < r; \end{aligned} \tag{2.11}$$

$$w_r = s_r + i\mathcal{O}(N), \tag{2.12}$$

where $w_k \in \mathbb{C}^{l_k}$, $1 \leqslant k < r$, $w_r \in \mathbb{C}^{l'_r}$ with $l'_r = l - \sum_{j=1}^{r-1} l_j$, and p_{m_k} is a homogeneous polynomial with weight m_k valued in $\mathbb{R}^{l_k}$. Also for any $\eta \in \mathbb{R}^{l_k} \setminus \{0\}$, $\eta \cdot p_{m_k}$ is not M-pluriharmonic of weight m_k.

Furthermore, if in addition M is real analytic, then all the remainder terms in (2.11) *are real analytic functions of x, y, s, and* (2.12) *can be replaced by*

$$w_r = s_r + if(x, y, s) \cdot s_r, \tag{2.13}$$

where f is an $l'_r \times l'_r$ real matrix with real analytic coefficients vanishing at the origin of $\mathbb{R}^{2n+l}$.

Remark. In fact (2.12) can be made more precise as shown in the proof of Theorem 2. For every N we can find smooth functions f_0 valued in $\mathbb{C}^{l'_r}$ and f_1 valued in $\mathbb{C}^{l'_r \times l'_r}$ such that on M

$$w_r = s_r + f_0(z, \bar{z}, s_1, \cdots, s_{r-1}) + f_1(z, \bar{z}, s) \cdot s_r, \tag{2.14}$$

and in addition,

$$f_0 = \mathcal{O}(N), \qquad L_j f_0 = \mathcal{O}(\infty), \quad 1 \leqslant j \leqslant n, \qquad f_1(0) = 0.$$

By making the change of variable $s'_r = \Re e\, w_r$, it is clear that (2.14) implies (2.13).

Theorem 2 will be proved in §4. An interesting consequence is the following.

Theorem 3. *Let M be a generic CR manifold of $\mathbb{C}^{n+l}$ of real codimensional l and assume $0 \in M$. If,*

(i) *there exist $2 \leqslant m_1 < \cdots < m_{r-1} < \infty$, $l_j \geqslant 1$, $1 \leqslant j \leqslant r-1$, $\sum_{j=1}^{r-1} l_j < l$, and a holomorphic submanifold S of $\mathbb{C}^{n+l}$, $0 \in S$, $\dim S = n + \sum_{j=1}^{r-1} l_j$, such that $M \cap S$ is a generic CR submanifold of S of real codimension $\sum_{j=1}^{r-1} l_j$ and of* *finite type at the origin, with Hörmander numbers* m_j *and multiplicities* l_j, $1 \leqslant j \leqslant r-1$, *then*

(ii) *M is not of finite type at the origin with finite Hörmander numbers* $m_1, \cdots, m_{r-1}$, *and multiplicities* $l_1, \cdots, l_{r-1}$, *and furthermore there exist coordinates as in Theorem* 2 *with* (2.12) *replaced by*

$$w_r = s_r + if(x, y, s) \cdot s_r, \tag{2.15}$$

where f is an $l_r' \times l_r'$ *real smooth matrix*, $f(0) = 0$, *and* $l_r' = l - \sum_{j=1}^{r-1} l_j$.

In addition, if M is real analytic and is not of finite type, then (i) *holds*; *in particular when M is real analytic* (i) *and* (ii) *are equivalent.*

From Theorem 3 we obtain the following sufficient condition for nonextendability of CR functions.

(2.16) Corollary. *If M satisfies condition* (i) *of Theorem* 3 (*in particular if M is real analytic and not of finite type at* 0), *then for any* $k \geqslant 0$, *there exists a CR function on M of class* C^k *which does not extend to be holomorphic in any wedge of edge M.*

Proof. Using Theorem 3 we can find holomorphic coordinates (z, w) in $\mathbb{C}^{n+l}$ such that (2.15) holds. Therefore we have on M

$$w_r \cdot w_r = {}^tAAs_r \cdot s_r,$$

where A is the $l_r' \times l_r'$ matrix $I + if(x, y, s)$. Since ${}^tAA = I - {}^tff + i(f + {}^tf)$ and since $f(0) = 0$, it follows that near the origin $\Re e(w_r \cdot w_r) \geqslant 0$. For every integer $k \geqslant 0$, $(w_r \cdot w_r)^{k+1/3}$ is a CR function of class C^k. One can check that its hypoanalytic wave front set at the origin (see §6) contains a line and therefore, by Theorem 7 of §6, it cannot extend holomorphically to any wedge of edge M.

Remarks. (1) If $r = 1$ in condition (i) of Theorem 3, then $S \cap M = S$; therefore condition (i) states that M contains a holomorphic manifold of complex dimension n. In particular this is the case when $l = 1$ and (i) holds.

(2) In the case when M is a hypersurface ($l = 1$) Trépreau [16] has recently shown that condition (i) of Theorem 3 is necessary and sufficient for the existence of a CR function on M which does not extend holomorphically to either side of M. We conjecture that for $l > 1$ condition (i) of Theorem 3 is also necessary for the conclusion of Corollary (2.16).

3. Canonical coordinates for a generic CR structure

As in §2 let $(\Omega, \mathcal{V})$ be a generic CR structure satisfying (2.1) and (2.2), and $\omega_0 \in \Omega$. We begin by putting the vector fields L_j, $1 \leqslant j \leqslant n$, a given local basis of $\mathbf{L}$ near ω_0, into a convenient form by the use of canonical coordinates. As in

Rothschild-Stein [13] and Helffer-Nourrigat [8] for coordinates as in (2.9) we extend the dilations (2.9) to vector fields by setting the weights of ∂_{x_j} and ∂_{y_j} to be -1 and the weight of ∂_{s_k} to be $-m_k$. If p_j is homogeneous of weight j and ∂_t is of weight $-m$, then $p_j\partial_t$ is said to be homogeneous of weight $j-m$. Similarly, a sum of such terms is of weight $\geqslant k$, $k \in \mathbb{Z}$, if the lowest weight of a homogeneous summand is $\geqslant k$.

The following is partly based on a method due to Helffer and Nourrigat [8] (see [13] for a more general context).

Theorem 4. *Let $\mathscr{V}$ be a generic CR bundle in Ω of finite type at $\omega_0 \in \Omega$ with Hörmander numbers $m_1 < m_2 < \cdots < m_r$, and let $L_1, L_2, \cdots, L_n$ be a basis for the sections of $\mathscr{V}$ near ω_0. Then there exist local coordinates (x, y, s) in Ω, $x, y \in \mathbb{R}^n$, $s = (s_1, \cdots, s_r)$, $s_k \in \mathbb{R}^{l_k}$, such that*

$$L_j = \partial_{\bar{z}_j} + \sum_{k=1}^{r} q^j_{m_k-1}\partial_{s_k} + \mathscr{O}(0), \tag{3.1}$$

where $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$, $q^j_{m_k-1} = q^j_{m_k-1}(z, \bar{z}, s_1, s_2, \cdots, s_{k-1})$ is a homogeneous polynomial of weight $m_k - 1$, and $\mathscr{O}(0)$ is a vector field of weight $\geqslant 0$. If, in addition, $\mathscr{V}$ is semi-rigid at ω_0, then the coordinates may be chosen so that the $q^j_{m_k-1}$ are independent of s.

Proof. We define the coordinates (x, y, s) as follows. Let $X_1, X_2, \cdots, X_n$, and $Y_1, Y_2, \cdots, Y_n$ be respectively the real and imaginary parts of the L_j, i.e.

$$L_j = X_j + iY_j, \qquad j = 1, 2, \cdots, n.$$

Let $\{S_{jp}\}$ be a set of real vector fields such that the following hold.

(3.2) The set $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, \{S_{jp}\}\}$ is a basis for the tangent space to Ω at ω_0.

(3.3) Each S_{jp} is a commutator of length m_j of the $\{X_k\}$ and $\{Y_k\}$, and m_j is the smallest integer for which S_{jp} is in the span of the commutators of length $\leqslant m_j$ at ω_0.

The existence of such S_{jp} is guaranteed by (2.4). In addition we have $1 \leqslant j \leqslant r$ and $1 \leqslant p \leqslant l_j$, where l_j is the multiplicity of m_j.

Now let $S_j = (S_{jp})$ and $s_j \cdot S_j = \sum_p s_{jp}S_{jp}$, $s_j \in \mathbb{R}^{l_j}$. Then the local coordinates (x, y, s) are defined by

$$(x, y, s) \leftrightarrow \exp 2(x \cdot X + y \cdot Y)\exp(s_1 \cdot S_1) \cdots \exp(s_{r-1} \cdot S_{r-1})\exp(s_r \cdot S_r) \cdot \omega_0. \tag{3.4}$$

We shall apply [8, Theorem 4.1] which implies the existence of vector fields $\hat{X}_j$ and $\hat{Y}_j$ homogeneous of weight -1 such that

$$X_j = \hat{X}_j + \mathscr{O}(0) \quad \text{and} \quad Y_j = \hat{Y}_j + \mathscr{O}(0), \tag{3.5}$$

$j = 1, 2, \cdots, n$.

To prove (3.1) we need to calculate $\hat{X}_j$ and $\hat{Y}_j$ explicitly. For this let $U_1, U_2, \cdots, U_{2n}$ be generators for the free nilpotent Lie algebra $\mathscr{G} = \mathscr{G}_1 + \mathscr{G}_2 + \cdots + \mathscr{G}_{m_r}$ of step m_r, and define the linear mapping $\lambda\colon \mathscr{G} \to C^\infty(\Omega, T\Omega)$, where $T\Omega$ is the tangent space to Ω, by

$$\lambda(U_i) = X_i, \quad i = 1, \cdots, n \tag{3.6}$$
$$\lambda(U_i) = Y_i, \quad i = n+1, \cdots, 2n,$$

and extend to the brackets of length $\leqslant m_r$ by putting

$$\lambda\left(\left[U_{i_1}\left[U_{i_2}, \cdots, \left[U_{i_{k-1}}, U_{i_k}\right] \cdots\right]\right]\right) = \left[\lambda(U_{i_1})\left[\lambda(U_{i_2}), \cdots \left[\lambda(U_{i_{k-1}}), \lambda(U_{i_k})\right] \cdots\right]\right].$$

For each $k \leqslant r$ let $\mathscr{H}_k(\omega_0)$ be the subspace of $\mathscr{G}$,

$$\mathscr{H}_k(\omega_0) = \mathscr{G}_k \cap \lambda_{\omega_0}^{-1}(V_{k-1}(\omega_0)),$$

where $V_{k-1}(\omega_0) = \lambda_{\omega_0}(\mathscr{G}_1 + \mathscr{G}_2 + \cdots + \mathscr{G}_{k-1})$, and let $\mathscr{H}(\omega_0)$ be the graded subalgebra

$$\mathscr{H} = \mathscr{H}(\omega_0) = \sum_{k=1}^{r} \mathscr{H}_k(\omega_0).$$

Then $\hat{X}_i = \pi_{0,\mathscr{H}}(X_i)$ and $\hat{Y}_i = \pi_{0,\mathscr{H}}(Y_i)$, where $\pi_{0,\mathscr{H}}$ is a particular realization of the realization induced from the trivial representation on $\mathscr{H}$ to $\mathscr{G}$. We shall construct that realization.

We let T_{jk} be the commutators of the $\{U_l\}$ corresponding to the S_{jk}; i.e., $\lambda(T_{jk}) = S_{jk}$, and let $\{H_{jk}\}$ be a basis of $\mathscr{H}_j$, $j = 1, \cdots, r$. Then any $g \in G = \operatorname{Exp} \mathscr{G}$ may be written uniquely as

$$(x, y, s, h) \leftrightarrow g = (\operatorname{Exp} h_r H_r) \cdots (\operatorname{Exp} h_1 H_1)(\operatorname{Exp} s_r T_r) \cdots (\operatorname{Exp} s_1 T_1)\left(\operatorname{Exp} 2\left(\sum_j x_j U_j + y_j U_{n+j}\right)\right). \tag{3.7}$$

Now, using the fact that $\mathscr{H}$ is a subalgebra, and using the Baker-Campbell-Hausdorff formula (see e.g. Varadarajan [18]), we find that there exist unique functions $\sigma_j^k(x, y, s_1, \cdots, s_{j-1}, t)$ and $\nu_j^k(x, y, s, h, t)$ such that for $1 \leqslant k \leqslant 2n$

$$g \operatorname{Exp} tU_k = \left(\operatorname{Exp} \nu_r^k \cdot H_r\right) \cdots \left(\operatorname{Exp} \nu_1^k \cdot H_1\right)\left(\operatorname{Exp} \sigma_r^k \cdot T_r\right) \cdots \left(\operatorname{Exp} \sigma_1^k \cdot T_1\right)\left(\operatorname{Exp} 2\left[\sum_{j=1}^{n} (x_j U_j + y_j U_{n+j}) + \frac{t}{2} U_k\right]\right). \tag{3.8}$$

Then $\pi_{0,\mathscr{H}}(U_k)$ is defined as

$$\pi_{0,\mathscr{H}}(U_j) = \tfrac{1}{2}\partial_{x_j} + \sum r_{m_k}^j(x, y, s_1, \cdots, x_{k-1})\partial_{s_k}, \tag{3.9}$$
$$\pi_{0,\mathscr{H}}(U_{j+n}) = \tfrac{1}{2}\partial_{y_j} + \sum r_{m_k}^{j+n}(x, y, s_1, \cdots, s_{k-1})\partial_{s_k},$$

where $r_{m_k}^j = \frac{d}{dt}\sigma_k^j|_{t=0}$.

It is easy to check that the $r^j_{m_k}$ are homogeneous of weight $m_k - 1$ by using the Baker-Campbell-Hausdorff formula to calculate the general form of the coefficients. Hence $\hat{X} + i\hat{Y}$ has the desired form. This proves the first part of the theorem.

For the second part we note first that $\mathscr{V}$ is semi-rigid at ω_0 if and only if

$$[\mathscr{G}^2, \mathscr{G}^2] \subset \mathscr{H}, \tag{3.10}$$

where $\mathscr{G}^2 = \mathscr{G}_2 + \mathscr{G}_3 + \cdots + \mathscr{G}_r$. Hence

$$\left[\left[U_j, U_k\right], T_{pq}\right] \in \mathscr{H} \tag{3.11}$$

for any $1 \leqslant j,\ k \leqslant 2n$, and $1 \leqslant p \leqslant r$, and similarly for higher brackets. Hence each σ^k_j is independent of s. This completes the proof of Theorem 4.

In case $\mathscr{V}$ is not of finite type we shall prove a modification of Theorem 4.

Theorem 5. *Let $\mathscr{V}$ be a generic CR bundle in Ω not of finite type at $\omega_0 \in \Omega$ with Hörmander numbers $m_1 < m_2 < \cdots < m_{r-1} < \infty$ and $m_r = \infty$, and multiplicities $l_1, \cdots, l_{r-1}, l_r = 0$. Define $l'_r = l - \sum_{k=1}^{r-1} l_k$. Let $L_1, \cdots, L_n$ be a local basis of $\mathbf{L}$ near ω_0. Then there exist local coordinates (x, y, s) in Ω, $x, y \in \mathbb{R}^n$, $s = (s_1, \cdots, s_r)$, $s_k \in \mathbb{R}^{l_k}$ for $k = 1, \cdots, r-1$, and $s_r \in \mathbb{R}^{l'_r}$, such that*

$$L_j = \partial_{\bar{z}_j} + \sum_{k=1}^{r-1} q^j_{m_k-1}(z, \bar{z}, s_1, \cdots, s_{k-1})\partial_{s_k} + q_\infty(z, \bar{z}, s)\partial_{s_r} + \mathcal{O}(0), \tag{3.12}$$

where the $q^j_{m_k-1}$ are homogeneous polynomials with weight $m_k - 1$, q_∞ is of weight ∞ as defined in §2 (s_r having weight ∞), and $\mathcal{O}(0)$ is a vector field in $\partial/\partial z_j$, $\partial/\partial \bar{z}_j$, $\partial/\partial s_k$, $1 \leqslant k \leqslant r-1$, of weight $\geqslant 0$.

Furthermore all the coefficients of L_j are real analytic if the CR structure $\mathscr{V}$ is real analytic.

Proof. We define coordinates (x, y, s) similarly to those of Theorem 4. We define X_j, Y_j, $1 \leqslant j \leqslant n$, and $\{S_{jp}\}$, $1 \leqslant j \leqslant r-1$, $1 \leqslant p \leqslant l_j$, as in (3.2) and (3.3), except that here $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, \{S_{jp}\}\}$ spans only those directions reached as commutators of the form (2.3). Let S_{rp}, $1 \leqslant p \leqslant l'_r$, be a set of vector fields which, together with the preceding ones, define at ω_0 a basis of $T_{\omega_0}(\Omega)$.

Then the coordinates (x, y, s) are defined by

$$\begin{aligned}(x, y, s) \leftrightarrow {} & \exp 2(x \cdot X + y \cdot Y + s_r \cdot S_r)\exp(s_1 \cdot S_1) \\ & \cdots \exp(s_{r-1}S_{r-1}) \cdot \omega_0.\end{aligned} \tag{3.13}$$

Now we may apply again [8, Théorème 4.1] to obtain

$$(3.14)\qquad X_j = \frac{1}{2}\partial_{x_j} + \sum_{k=1}^{r-1} r^j_{m_k-1}(s_r, x, y, s_1, \cdots, s_{k-1})\partial_{s_k} + \mathcal{O}(0),$$

$$(3.15)\qquad Y_j = \frac{1}{2}\partial_{y_j} + \sum_{k=1}^{r-1} r^{j+n}_{m_k-1}(s_r, x, y, s_1, \cdots, s_{k-1})\partial_{s_k} + \mathcal{O}(0),$$

where s_r is regarded as a weight 1 variable (so that ∂_{s_r} has weight -1), $r^j_{m_k-1}$ is homogeneous of weight $m_k - 1$, and $\mathcal{O}(0)$ is a vector field of weight $\geqslant 0$. Note that since the coefficients of ∂_{s_r} must vanish at ω_0, these differentiations are included in the error term $\mathcal{O}(0)$. Now we may rewrite (3.14) as

$$(3.16)\quad X_j = \frac{1}{2}\partial_{x_j} + \sum_{k=1}^{r-1} \tilde{r}^j_{m_k-1}(x, y, s_1, \cdots, s_{k-1})\partial_{s_k} + \frac{1}{2}f_j(x, y, s)\partial_{s_r} + \mathcal{O}',$$

where $f_j(0,0,0) = 0$, $\mathcal{O}'$ is a vector field in the directions ∂_{x_j}, ∂_{y_k}, and ∂_{s_k}, $1 \leqslant k \leqslant r-1$, of weight $\geqslant 0$ when s_r is regarded to have weight ∞, and similarly for Y_j in (3.15). To prove Theorem 5 we must show that f_j is of weight ∞ when s_r is given weight ∞, i.e. every term in the Taylor series of f_j has a component of s_r as a factor.

For this we calculate f_j directly from the coordinates (3.13). Indeed,

$$f_j(x, y, s) = \frac{d}{dt} g\big(\exp tX_j(\exp 2(x \cdot X + y \cdot Y + s_r \cdot S_r))(\exp s_1 \cdot S_1) \cdots (\exp s_{r-1} \cdot S_{r-1}) \cdot \omega_0\big)\big|_{t=0},$$

where $g(x, y, s) = s_r$ (under the identification (3.13)). Since the given vector fields span the tangent space at ω_0, there exist functions α, β, γ of (t, x, y, s) (but independent of the original vector field variables) such that

$$(3.17)\quad \begin{aligned} &\exp tX_j(\exp 2(xX + yY + s_rS_r))(\exp s_1 \cdot S_1) \cdots \exp(s_{r-1}S_{r-1}) \cdot \omega_0 \\ &\quad = \exp 2(\alpha X + \beta Y + \gamma_r S_r)(\exp \gamma_1 \cdot S_1) \cdots \exp(\gamma_{r-1}S_{r-1}) \cdot \omega_0. \end{aligned}$$

Then

$$f_j(x, y, s) = \frac{d}{dt}\,\gamma_r(t, x, y, s)\big|_{t=0}.$$

It suffices to show that the Taylor series of γ_r around 0 vanishes identically when $s_r = 0$. Setting $s_r = 0$ in (3.17) and inverting the term on the right, we obtain

$$(3.18)\quad \begin{aligned} &(\exp - \gamma_{r-1}S_{r-1}) \cdots (\exp - \gamma_1 S_1) \exp - 2(\alpha X + \beta Y + \gamma_r S_r) \\ &\quad \cdot (\exp tX_j) \exp 2(xX + yY)(\exp s_1 S_1) \cdots \exp(s_{r-1}S_{r-1}) \cdot \omega_0 = \omega_0. \end{aligned}$$

Using the Baker-Campbell-Hausdorff formula, we may expand the product of exponentials on the left-hand side of (3.18) to obtain (as an identity in the real analytic case and as an asymptotic expansion in the C^∞ case) an exponential of the form

$$(3.19) \qquad \exp(\tilde{\gamma}_1 \cdot S_1 + \cdots + \tilde{\gamma}_{r-1} S_{r-1} + \tilde{\gamma}_r \cdot S_r + \tilde{\alpha} \cdot X + \tilde{\beta} \cdot Y),$$

where the $\tilde{\gamma}_j$, $\tilde{\alpha}$, and $\tilde{\beta}$ are functions of t, x, y, s and $u = (u_1, \cdots, u_{2n+l})$, where u is a set of variables in which the vector fields are acting. Since the exponential of a nonvanishing vector field at ω_0 cannot fix ω_0, we must have $\tilde{\gamma}_j$, $\tilde{\alpha}$, and $\tilde{\beta}$ all identically zero when $u = \omega_0$.

Suppose now, by contradiction, that the Taylor series of γ_r at $s_r = 0$ is not identically zero. Let k be the lowest degree (in t, x, y, $s_1, \cdots, s_{r-1}$) of the nonvanishing terms. (Here we take degree in the ordinary sense, with all the variables of degree 1.) Then k is the lowest degree of the nonzero coefficient of S_r obtained from summing the terms in the exponentials in (3.18). Since $\tilde{\gamma}_r = 0$ at $u = \omega_0$, there must be a term in the expansion, coming from commutators, which cancels the kth degree terms of γ_r. However, by the hypothesis the vector fields in S_r are not obtained as commutators, of any length, of the X, Y, and S_j, $1 \leqslant j \leqslant r - 1$. Hence the commutator must include S_r. In that case, its coefficient must be of degree at least $k + 1$ in $(t, x, y, s_1, \cdots, s_{r-1})$ and hence cannot cancel the degree k terms in γ_r. This contradiction shows that γ_r is flat when $s_r = 0$, which completes the proof of Theorem 5.

The following proposition will also be useful in the proof of Theorem 1.

(3.20) Proposition. *With the assumptions and notation of Theorem* 4, *let* $L_j = \hat{L}_j + \mathcal{O}(0)$, *where* $\hat{L}_j$ *is the homogeneous part of weight* -1 *in the right-hand side of* (3.1). *Then the following holds*:

(i)

$$(3.21) \qquad [\hat{L}_j, \hat{L}_k] = 0, \qquad j, k = 1, \cdots, n,$$

(ii) *For every* k, $1 \leqslant k \leqslant m_r$, *and every* $\xi \in T^*_{\omega_0}\Omega$,

$$(3.22) \qquad \sigma(L^{(k)})(\omega_0, \xi) = \sigma(\hat{L}^{(k)})(\omega_0, \xi),$$

where $L^{(k)}$ *is a commutator of the form* (2.3), *with each* M_j *being one of the* L_p *or the* $\overline{L}_p$, *and with a similar definition for* $\hat{L}^{(k)}$ *where* L_p *is replaced by* $\hat{L}_p$.

(iii) *Also if* $\mathscr{V}$ *is semi-rigid then the coordinates* (x, y, s) *in Theorem* 4 *may be chosen so that for* $j = 1, \cdots, n$, $k = 1, \ldots, r$,

$$(3.23) \qquad q^j_{m_k - 1}(z, \bar{z}) = -i \frac{\partial}{\partial \bar{z}_j} \hat{p}_{m_k}(z, \bar{z}),$$

where $\hat{p}_{m_k}$ *is homogeneous of degree* m_k *valued in* $\mathbb{R}^{l_k}$.

Proof. For the first statement note that $[L_j, L_k]$ is of weight $\geqslant -1$, since $[L_j, L_k]$ is a linear combination of the L_p. Since $[\hat{L}_j, \hat{L}_k]$ is of weight -2, (3.21) follows. Similar homogeneity arguments easily yield (ii). To prove (iii) note that in the semi-rigid case, the coefficients $q^j_{m_k - 1}$ are independent of s, and so (3.21) implies for $j, p = 1, \cdots, n$, $k = 1, \cdots, r$,

$$\frac{\partial}{\partial \bar{z}_j} q^p_{m_k - 1}(z, \bar{z}) = \frac{\partial}{\partial \bar{z}_p} q^j_{m_k - 1}(z, \bar{z}).$$

Dolbeault's lemma implies the existence of homogeneous polynomial $r_{m_k}(z, \bar{z})$ such that

$$q^j_{m_k - 1}(z, \bar{z}) = -i \frac{\partial}{\partial \bar{z}_j} r_{m_k}(z, \bar{z}).$$

Taking $\hat{p}_{m_k} = \Re e\, r_{m_k}$ and making the change of coordinates $s'_k = s_k - \Im m\, r_{m_k}(z, \bar{z})$, $1 \leqslant k \leqslant r$, yield the desired result (3.23).

4. Complex coordinates for M and the proofs of Theorems 1, 2, and 3

The key step in the proof of Theorem 1 is the following:

(4.1) Proposition. *Let M be a generic CR manifold in $\mathbb{C}^{n+l}$ of codimension l, and of finite type at the origin. If $L_1, \cdots, L_n$ is a local basis of $\mathbf{L}$ near the origin, there exist holomorphic coordinates (z', w') in $\mathbb{C}^{n+l}$ such, in the coordinate system of Theorem 4 we have on M*

$$(4.2) \qquad \begin{aligned} z'_j &= x_j + iy_j + \mathcal{O}(2), \qquad 1 \leqslant j \leqslant n, \\ w'_k &= s_k + \hat{p}_{m_k}(z, \bar{z}, s_1, \cdots, s_{k-1}) + \mathcal{O}(m_k + 1), \qquad 1 \leqslant k \leqslant r, \end{aligned}$$

where $\hat{p}_{m_k}$ is homogeneous of weight m_k. Furthermore if M is semi-rigid at the origin the holomorphic coordinates (z', w') may be chosen so that the $\hat{p}_{m_k}$ are independent of the s variables.

Proof. We start with any holomorphic coordinates $\zeta = (\zeta_1, \cdots, \zeta_{n+l})$ in $\mathbb{C}^{n+l}$. Of course the restriction of each ζ_j to M satisfies (2.8). Using the coordinates z, $\bar{z}$, s of Theorem 4, we can write

$$(4.3) \qquad \zeta|_M = Az + B\bar{z} + Cs + Q(z, \bar{z}, s),$$

where A, B are $n \times n$ complex matrices, C is an $l \times n$ matrix, and the Taylor expansion of Q starts with quadratic terms in z, $\bar{z}$, s.

Applying the vector fields L_j given by (3.1)–(4.3) we obtain $B \equiv 0$. Since $d(\zeta_1 \mid_M), \cdots, d(\zeta_{n+l} \mid_M)$ are linearly independent at the origin, after a linear complex change of coordinates in $\mathbb{C}^{n+l}$, and relabeling the coordinates

$\hat{z}_1, \cdots, \hat{z}_n, \hat{w}_1, \cdots, \hat{w}_r$ ($\hat{w}_j \in \mathbf{C}^{l_j}$), we can assume that on M

$$\begin{aligned} \hat{z}_j &= z_j + q_j(z, \bar{z}, s), \quad 1 \leqslant j \leqslant n, \\ \hat{w}_j &= s_j + \hat{q}_j(z, \bar{z}, s), \quad 1 \leqslant j \leqslant r, \end{aligned} \tag{4.4}$$

where q_j and $\hat{q}_j$ start with quadratic terms.

We can take $z_j' = \hat{z}_j$, $1 \leqslant j \leqslant n$, in order to satisfy the first part of (4.2). We need to make other holomorphic changes of coordinates to define the w_j'. From (4.4) we can write

$$\hat{w}_k = s_k + r_2^k + \cdots + r_{m_k}^k + \mathcal{O}(m_k + 1), \tag{4.5}$$

where each $r_j^k = r_j^k(z, \bar{z}, s)$ is homogeneous of weight j, and the r_j^k and $\mathcal{O}(m_k + 1)$ have no linear terms.

If the L_j are given by (3.1) we can write

$$L_j = \hat{L}_j + L_j^0 + L_j^1 + \cdots, \tag{4.6}$$

where $\hat{L}_j$ is the principal term of weight -1, and L_j^p is homogeneous of weight p. If the L_j are not analytic, the equality in (4.6) is taken in the sense of Taylor series around 0.

Now we proceed inductively. Suppose that for all $j < j_0$ we may find w_j' in the form given by (4.2). We shall construct w_{j_0}' using $\hat{w}_{j_0}$ of (4.5). Indeed, if all $r_j^{j_0}$, $j < m_{j_0}$ are zero, we take $w_{j_0}' = \hat{w}_{j_0}$ and we are done. If not, let k_0 be the smallest integer for which $r_{k_0}^{j_0} \neq 0$. Since $r_{k_0}^{j_0}$ is homogeneous of weight $k_0 < m_{j_0}$, we may write

$$r_{k_0}^{j_0} = r_{k_0}^{j_0}\left(z, \bar{z}, s_1, \cdots, s_{j_0 - 1}\right).$$

By homogeneity we have $\hat{L}_j r_{k_0}^{j_0} = 0$ for all j. We claim that there is a holomorphic polynomial $p(z, w) = p(z, w_1, \cdots, w_{j_0 - 1})$ such that

$$p\left(z, s_1 + i\hat{p}_{m_1}, \cdots, s_{j_0 - 1} + i\hat{p}_{m_{j_0} - 1}\right) = r_{k_0}^{j_0}\left(z, \bar{z}, s_1, \cdots, s_{j_0 - 1}\right). \tag{4.7}$$

Indeed, the left-hand side is clearly a solution of the system $\hat{L}_j f = 0$, $j = 1, \cdots, n$, and since the coefficients of $\hat{L}_j$ are analytic, it suffices to show that p can be chosen so that the Cauchy data on the noncharacteristic manifold $\{y = 0\}$ agree with that of $r_{k_0}^{j_0}$. For this, note that $r_{k_0}^{j_0}(x, x, s)$ can be written as a polynomial p in x and $s_j' = s_j + ip_{m_j}(x, x, s_1, \cdots, s_{j-1})$, i.e.,

$$p(x, s') = r_{k_0}^{j_0}(x, x, s),$$

which proves the claim (4.7).

From (4.7) and the fact that $p(z, w)$ is again homogeneous of weight k_0 we have

$$p(z', w') = r_{k_0}^{j_0}\left(z, \bar{z}, s_1, \cdots, s_{j_0 - 1}\right) + \mathcal{O}(k_0 + 1); \tag{4.8}$$

here we recall that $z' = \hat{z}$ and $w_j' = \hat{w}_j$ for $1 \leqslant j < j_0$. Now if we put

$$\hat{w}_{j_0}' = \hat{w}_{j_0} - p(z', w'), \tag{4.9}$$

it follows from (4.8) that we have

$$\hat{w}_{j_0}' = s_{j_0} + r_{k_0+1}'^{j_0} + \cdots + r_{m_{j_0}}'^{j_0} + \mathcal{O}(m_{j_0} + 1),$$

where the $r_j'^{j_0}$ are of weight j, and we may proceed by induction.

We assume now that M is semi-rigid and of finite type at the origin. We must show that the holomorphic coordinates (z', w') may be chosen so that the $\hat{p}_{m_k}$ in (4.2) are independent of s. Recall that by Theorem 4,

$$\hat{L}_j = \partial_{\bar{z}_j} + \sum q_{m_k-1}^j(z, \bar{z})\partial_{s_k}, \tag{4.10}$$

where $\hat{L}_j$ is as in (4.6). By induction, we may assume that $\hat{p}_{m_j}$ is independent of the s variables for $j \leqslant k - 1$, i.e.,

$$w_j' = s_j + \hat{p}_{m_j}(z, \bar{z}) + \mathcal{O}(m_j + 1), \qquad j \leqslant k - 1. \tag{4.11}$$

Suppose

$$w_k' = s_k + \hat{p}_{m_k}(z, \bar{z}, s_1, s_2, \cdots, s_{k-1}) + \mathcal{O}(m_k + 1)$$

and that

$$\hat{p}_{m_k}(z, \bar{z}, s_1, s_2, \cdots, s_{k-1}) = \sum_{|\alpha| \leqslant N} c_\alpha(z, \bar{z}) s^\alpha,$$

$$s^\alpha = s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_{k-1}^{\alpha_{k-1}}, \quad |\alpha| = \sum |\alpha_j|,$$

where N is the largest integer for which some $c_\alpha(z, \bar{z})$, $|\alpha| = N$, is nonzero. By induction on N it will suffice to show that there is a holomorphism fixing $z', w_1', \cdots, w_{k-1}'$ so that the new $\hat{p}_{m_k}$ in w_k' has (actual) degree less than N in the s variables. Since by (4.10)

$$0 = \hat{L}_j\big(s_k + \hat{p}_{m_k}(z, \bar{z}, s_1, \cdots, s_{k-1})\big) = \frac{\partial}{\partial \bar{z}_j} \sum_{|\alpha|=N} c_\alpha s^\alpha + \sum_{|\alpha|<N} d_\alpha s^\alpha, \tag{4.12}$$

and the distinct s^α are linearly independent, we conclude that

$$\frac{\partial}{\partial \bar{z}_j} c_\alpha(z, \bar{z}) = 0, \qquad |\alpha| = N, 1 \leqslant j \leqslant n,$$

i.e. $c_\alpha = c_\alpha(z)$ is holomorphic. We take

$$\tilde{w}_k' = w_k' - \sum_{|\alpha|=N} c_\alpha(z) w'^\alpha.$$

We clearly have

$$\tilde{w}_k' = s_k + \hat{p}_{m_k}'(z, \bar{z}, s_1, \cdots, s_{k-1}) + \mathcal{O}(m_k + 1),$$

where $\hat{p}_{m_k}'$ is still homogeneous of weight m_k and its (actual) degree in the s variables is strictly less than N. This proves Proposition (4.1).

Proof of Theorem 1. Let $L_1, \cdots, L_n$ be a basis of $\mathbf{L}$ and (x, y, s) the coordinates of Theorem 4. By Proposition (4.1) we may find holomorphic coordinates z', w' in C^{n+l} satisfying (4.2). We choose new coordinates on M by letting

$$x' = \Re e\, z', \quad y' = \Im m\, z', \quad s' = \Re e\, w'.$$

Since

$$x' - x = \mathcal{O}(2), \qquad y' - y = \mathcal{O}(2),$$

$$s_k' - s_k - \Re e\, \hat{p}_{m_k}(z, \bar{z}, s_1, \cdots, s_{k-1}) = \mathcal{O}(m_k + 1),$$

we conclude that there are homogeneous polynomials p_{m_k} of weight m_k such that on M

$$\Im m\, w_k' = p_{m_k}(z', \bar{z}', s_1', \cdots, s_{k-1}') + \mathcal{O}(m_k + 1),$$

and p_{m_k} is independent of s' in the semi-rigid case, which proves (2.10) (if we drop the primes).

Now we prove by contradiction that for any $\eta \in \mathbb{R}^{l_k} \setminus \{0\}$, $\eta \cdot p_{m_k}$ is not M-pluriharmonic of weight m_k. Assume for some $\eta \in \mathbb{R}^{l_k} \setminus \{0\}$, $\eta \cdot p_{m_k}$ is M-pluriharmonic of weight m_k. After a linear change of variables in $\mathbb{R}^{l_k}$ we may assume $\eta = (1, 0, \cdots, 0)$. Therefore there exists a holomorphic polynomial homogeneous of weight m_k, $F(z, w_1, \cdots, w_{k-1})$, such that

$$p_{m_k}^1(z, \bar{z}, s_1, \cdots, s_{k-1}) = \Im m\, F(z, s_1 + ip_{m_1}, \cdots, s_{k-1} + ip_{m_{k-1}}),$$

where $p_{m_k}^1$ is the first component of p_{m_k}. After making the holomorphic change of coordinates

$$w_{k,1}' = w_{k,1} - F(z, w_1, \cdots, w_{k-1})$$

($w_{k,1}$ is the first component of w_k in $\mathbb{C}^{l_k}$), putting

$$s_{k,1}' = s_{k,1} - \Re e\, F(z, s_1 + ip_{m_1}, \cdots, s_{k-1} + ip_{m_{k-1}}),$$

and then dropping the primes, we conclude that (2.10) holds with

$$p_{m_k}^1 \equiv 0. \tag{4.13}$$

Since by homogeneity we have

$$\hat{L}_j(s_k + ip_{m_k}) = 0, \qquad 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant r,$$

where $\hat{L}_j$ is as in (4.6), we conclude that the coefficient of $\partial/\partial s_{1,1}$ in $\hat{L}_j$ is identically zero. On the other hand, by Proposition (3.20)(ii) we know that the structure defined by the $\hat{L}_j$ is also of finite type (in fact with the same Hörmander numbers and multiplicities as the original structure). We find a contradiction since $\partial/\partial s_{1,1}$ can never be obtained as a commutator of the $\hat{L}_j$ and their conjugates.

Proof of Theorem 2. Let $L_1, \cdots, L_n$ be a basis of $\mathbf{L}$ and (x, y, s) the coordinates given by Theorem 5. We start with holomorphic coordinates $\zeta = (\zeta_1, \cdots, \zeta_{n+l})$ in $\mathbb{C}^{n+l}$. Of course the restriction of each ζ_j to M satisfies (2.8). As in the proof of Proposition (4.1), after a linear complex change of coordinates and relabeling them $\hat{z}_1, \cdots, \hat{z}_n, \hat{w}_1, \cdots, \hat{w}_r$ ($\hat{w}_j \in \mathbb{C}^{l_j}$), (4.4) holds.

We introduce new coordinates on M given by

$$\hat{x} = \Re e\, \hat{z}, \quad \hat{y} = \Im m\, \hat{z}, \quad \hat{s} = s.$$

After dropping the "tildas," using (3.12) and changing the basis $L_1, \cdots, L_n$ we can assume

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{r-1} c_k^j(z, \bar{z}, s)\frac{\partial}{\partial s_k} + q_\infty(z, \bar{z}, s)\frac{\partial}{\partial s_r} \tag{4.14}$$

with

$$c_k^j(z, \bar{z}, s) = q_{m_k-1}^j(z, \bar{z}, s_1, \cdots, s_{k-1}) + \mathcal{O}(m_k - 1)$$

and $q_{m_k-1}^j$ and q_∞ are as in Theorem 5.

Again as in the proof of Proposition (4.1) and after a holomorphic change of coordinates in $\mathbb{C}^{n+l}$ we can assume that on M we have

$$\begin{aligned} &z_j' = z_j, \qquad 1 \leqslant j \leqslant n, \\ &w_k' = s_k + \hat{p}_{m_k}(z, \bar{z}, s_1, \cdots, s_{k-1}) + \mathcal{O}(m_k - 1), \qquad 1 \leqslant k \leqslant r-1; \end{aligned} \tag{4.15}$$

$$w_r = s_r + f_0(z, \bar{z}, s_1, \cdots, s_{r-1}) + f_1(z, \bar{z}, s) \cdot s_r, \tag{4.16}$$

where the $\hat{p}_{m_k}$ are homogeneous of weight m_k, and f_0, f_1 are smooth (and analytic when M is real analytic), with $f_0(0) = 0$, $df_0(0) = 0$, $f_1(0) = 0$.

In fact making the change of variables

$$s_k' = \Re e\, w_k', \qquad 1 \leqslant k \leqslant r-1,$$

we can assume that

$$w_k = s_k + i\left[p_{m_k}(z, \bar{z}, s_1, \cdots, s_{k-1}) + \mathcal{O}(m_k + 1)\right], \quad 1 \leqslant k \leqslant r-1, \tag{4.17}$$

where p_{m_k} is homogeneous of weight m_k and p_{m_k} and $\mathcal{O}(m_k + 1)$ are valued in $\mathbb{R}^{l_k}$.

Now we write the Taylor expansion of f_0 as a sum of homogeneous polynomial of weight j

$$f_0 \sim \sum_{j=2}^{\infty} r_j(z, \bar{z}, s_1, \cdots, s_{r-1}) \tag{4.18}$$

and

$$\hat{L}_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{r-1} q^j_{m_k}(z, \bar{z}, s_1, \cdots, s_{k-1}) \frac{\partial}{\partial s_k}.$$

As in the proof of Proposition (4.1) we have $\hat{L}_j r_{j_0} = 0$, $1 \leqslant j \leqslant n$, where r_{j_0} is the first nonvanishing term in the series (4.18). We can eliminate such a term by making a change of coordinates in $\mathbf{C}^{n+l}$ of the form

$$w'_r = w_r - H(z, w_1, \cdots, w_{r-1})$$

with an appropriate holomorphic function H. We proceed inductively and we can assume that the series (4.18) starts with $j = N$. Note that applying the vector fields (4.14) to (4.16) yields $L_j f_0 = \mathcal{O}(\infty)$, $1 \leqslant j \leqslant n$, which proves (2.14) and hence completes the proof of Theorem 2 in the C^∞ case.

If M is real analytic, applying the vector fields (4.14) to (4.15) yields

$$L_j|_{s_r=0} f_0 = 0, \qquad 1 \leqslant j \leqslant n.$$

Since the $L_j|_{s_r=0}$ define a real analytic CR structure of codimension $l - l'_r$ and since f_0 is real analytic, we can find a holomorphic function $H(z, w_1, \cdots, w_{r-1})$ whose restriction to $M \cap \{s_r = 0\}$ is f_0. Now let

$$w'_r = w_r - H(z, w_1, \cdots, w_{r-1}).$$

Finally, if we take $s'_r = \Re e\, w'_r$, we note that w'_r has the desired form. This completes the proof of Theorem 2.

Proof of Theorem 3. We first prove that (i) implies (ii). After a holomorphic change of coordinates in $\mathbf{C}^{n+l}$ we can assume that near the origin $S = \{(z, w) \in \mathbf{C}^{n+l}: w_r = 0\}$ with $w = (w_1, \cdots, w_r)$, $w_j \in \mathbf{C}^{l_j}$, $1 \leqslant j \leqslant r-1$, $w_r \in \mathbf{C}^{l'_r}$ with $l'_r = l - \sum_{j=1}^{r-1} l_j$, and $z \in \mathbf{C}^n$. Since $M \cap \{w_r = 0\}$ is a generic CR manifold of $\mathbf{C}^{n+\Sigma l_j}$, of finite type at the origin, we can assume that it is given by

$$\{(z, w') \in \mathbf{C}^{n+\Sigma l_j}: \Im m\, w' - \phi(z, \bar{z}, \Re e\, w') = 0\},$$

where ϕ has the normal form (2.10) of Theorem 1. Since M is generic, it is defined by $\rho_j(\zeta, \bar{\zeta}) = 0$, $j = 1, 2, \cdots, l$, $\zeta \in \mathbf{C}^{n+l}$, with $\partial\rho_1 \wedge \cdots \wedge \partial\rho_l \neq 0$ near the origin. Since ρ_j vanishes on $M \cap S$ we must have

$$\rho_j = A_j w_r + \bar{A}_j \bar{w}_r + C_j(\Im m\, w' - \phi(z, \bar{z}, \Re e\, w')), \qquad 1 \leqslant j \leqslant l, \tag{4.19}$$

where each A_j is a $1 \times l'_r$ matrix with complex smooth coefficients, and each C_j a $1 \times (l - l'_r)$ matrix with real smooth coefficients. The linear independence of $\partial\rho_j$ and the form of the ρ_j given by (4.19) imply that the $l \times l$ matrix $(\partial\rho_j/\partial w_k)$ is invertible. After applying an invertible $l \times l$ matrix with smooth real coefficients to the ρ_j we may assume that the $(l - l'_r) \times (l - l'_r)$ submatrix $C = (C_j)$, $1 \leqslant j \leqslant l - l'_r$, is the identity matrix, and $C_j = 0$ for $j > l - l'_r$. Therefore the $l'_r \times l'_r$ matrix $A = (A_k)_{l - l'_r + 1 \leqslant k \leqslant l}$ is also invertible. Hence we have, with $\rho' = (\rho_1, \cdots, \rho_{l - l'_r})$ and $\rho'' = (\rho'_{l - l_r + 1}, \cdots, \rho_l)$,

$$\rho' = \Im m\, w' - \phi(z, \bar{z}, \Re e\, w') + A' w_r + \overline{A'} \overline{w}_r, \tag{4.20}$$

$$\rho'' = A w_r + \overline{A} \overline{w}_r. \tag{4.21}$$

After a holomorphic change of variables in $\mathbb{C}^{l'_r}$ taking $\tilde{w}_r = 2iA(0)w_r$, and dropping the tildas we obtain

$$\rho'' = \Im m\, w_r + B w_r + \overline{B} \overline{w}_r, \tag{4.22}$$

where $B = B(z, \bar{z}, w', \overline{w}', w_r, \overline{w}_r)$ is an $l'_r \times l'_r$ complex matrix with smooth coefficients and $B(0) = 0$. Using the implicit function theorem and noting that $\rho'' = 0$ when $w_r = 0$, we may replace ρ'' by

$$\tilde{\rho}'' = \Im m\, w_r - \psi\big(z, \bar{z}, w', \overline{w}', \Re e\, w_r\big) \cdot \Re e\, w_r \tag{4.23}$$

with ψ an $l'_r \times l'_r$ real valued matrix with $\psi(0) = 0$. Replacing $\Im m\, w_r$ in (4.20) by $\psi \cdot \Re e\, w_r$, we obtain the desired conclusion that (i) implies (ii).

To prove that (ii) implies (i) in the case where M is real analytic, we use the coordinates of Theorem 2, and observe that the holomorphic manifold given by $\{w_r = 0\}$ satisfies the conditions of (i).

5. Uniqueness of the normal forms

The main result here is that the polynomials p_{m_k} of Theorem 1 in §2 are uniquely determined up to certain transformations.

Theorem 6. *Let M be a generic CR manifold in $\mathbb{C}^{n+l}$, and (z, w) a coordinate system satisfying* (2.10). *Suppose that there are new holomorphic coordinates (z', w') and homogeneous polynomials $p'_{m_k}(z', \bar{z}', s'_1, \cdots, s'_{k-1})$ such that M is defined by*

$$\begin{aligned} z'_j &= x'_j + i y'_j, \qquad j = 1, 2, \cdots, n, \\ w'_k &= s'_k + i\big[p'_{m_k}\big(z', \bar{z}', s'_1, \cdots, s'_{k-1}\big) + \mathcal{O}(m_k + 1)\big], \qquad 1 \leqslant k \leqslant r. \end{aligned} \tag{5.1}$$

Then there exist an invertible complex $n \times n$ matrix A, and real invertible $l_j \times l_j$ matrices B_j such that

$$\begin{aligned} &p'_{mk}\left(Az, \overline{Az}, B_1 s_1 + \Re e F_{m_1}(z), B_2 s_2 + \Re e F_{m_2}(z, \dot{w}_1), \cdots,\right. \\ &\qquad\qquad \left.B_{k-1}s_{k-1} + \Re e F_{m_{k-1}}(z, \dot{w}_1, \cdots, \dot{w}_{k-2})\right) \\ &\quad = B_k p_{m_k}(z, \bar{z}, s_1, \cdots, s_{k-1}) + \Im m F_{m_k}(z, \dot{w}_1, \cdots, \dot{w}_{k-1}), \end{aligned} \tag{5.2}$$

with

$$F_{m_j}(z, \dot{w}_1, \cdots, \dot{w}_{m_{j-1}}) = \sum_{|\alpha| + m\cdot\beta = m_j} c_{\alpha\beta} z^\alpha \dot{w}^\beta, \qquad c_{\alpha\beta} \in \mathbf{C}^{l_j}, \tag{5.3}$$

where $m \cdot \beta = \sum_{k=1}^{j-1} m_k |\beta_k|$, $\beta = (\beta_1, \cdots, \beta_{j-1})$, $\beta_k \in \mathbf{Z}_+^{l_k}$, and $\dot{w}_k$ is the homogeneous part of w_k, i.e. $\dot{w}_k = s_k + ip_{m_k}$.

Conversely, given matrices A, $\{B_j\}$ as above, and functions F_{m_j} as in (5.3), *let (z', w') be the holomorphism defined by*

$$\begin{aligned} z' &= Az, \\ w_1' &= B_1 w_1 + F_{m_1}(z), \\ &\cdots \\ w_l' &= B_l w_l + F_{m_l}(z, w_1, \cdots, w_{l-1}). \end{aligned} \tag{5.4}$$

Then there exist p'_{m_k} homogeneous of degree m_k such that (5.1) *and* (5.2) *hold.*

Proof. Assume first that (5.1) is given. Suppose that (z', w') is obtained from (z, w) by a holomorphic transformation. Then

$$z' = Az + Dw + Q(z,w), \qquad w' = Cz + Bw + \hat{Q}(z,w), \tag{5.5}$$

where

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}, \quad n \times n, \qquad D = \begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix}, \quad n \times l, \qquad C = \begin{bmatrix} C_1 \\ \vdots \\ C_r \end{bmatrix},$$

where C_j is an $l_j \times n$ matrix, and

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix},$$

where each B_j is an $l_j \times l$ matrix, and Q and $\hat{Q}$ are holomorphic functions beginning with quadratic terms.

To calculate p'_{m_1} we equate the imaginary parts of w_1' in (5.1) and (5.5). By considering terms of homogeneity $< m_1$ we obtain first $\Im m(C_1 z) = 0$, which implies $C_1 = 0$. Similarly, $\Im m\, B_1 = 0$, since s_1 appears in no other real linear terms. Equating the terms of weight m_1 we obtain

$$p'_{m_1}(Az, \overline{Az}) = B_1^1 p_{m_1}(z, \bar{z}) + \Im m\, F_{m_1}(z),$$

where $F_{m_1}(z)$ is the holomorphic polynomial of weight m_1 in the expansion of $\hat{Q}(z, w)$, and $B_1 = (B_1^1, \cdots, B_1^r)$, real matrices.

For the other w'_j we shall need the following.

(5.6) Lemma. *Let C, D, and $\hat{Q}$ be as in* (5.5), *we write $\hat{Q} = (\hat{Q}_1, \cdots, \hat{Q}_r)$. Then for each j*

(i) $C_j = 0$,

(ii) $\Im m\, B_j = 0$, *and*

(iii) *if $\hat{Q}_j = \Sigma c^j_{\alpha\beta} z^\alpha (w_1)^{\beta_1} \cdots (w_r)^{\beta_r}$, then $c^j_{\alpha\beta} = 0$ for $|\alpha| + \Sigma^r_{k=1} m_k |\beta_k| < m_j$.*

Proof. From (5.1) and (5.5) we have for each j,

$$s'_j + i\left[p'_{m_j}(z', \bar{z}', s'_1, \cdots, s'_{j-1}) + \mathcal{O}(m_j + 1)\right] = C_j z + B_j w + \hat{Q}_j(z, w). \tag{5.7}$$

For (i) we note that since $\Im m(C_j z)$ is the only imaginary term of degree 1, it must be zero, from which (i) follows. For (ii), we observe that $(\Im m\, B_j)w$ is the only imaginary term containing s_j as a linear term and hence must be 0.

For (iii) we conclude by equating the homogeneous parts of degrees $\leqslant m_j - 1$ in (5.7) that the sum of the terms on the right in (iii) with indices α, β, where $|\alpha| + \Sigma^r_{k=1} m_k |\beta_k| < m_j$, must be 0. Since the position of the s_k in this summand precludes any possibility of cancellation, the conclusion of (iii) follows. This proves the lemma.

The proof of the first part of Theorem 6 may now be completed by considering separately the imaginary parts of both sides of (5.7) as a sum of homogeneous terms and applying Lemma 5.6.

To prove the second claim we consider (5.4) and solve inductively (z, w) in terms of (z', w'). We get from (5.4)

$$\begin{aligned} z &= A^{-1} z', \\ w_1 &= B_1^{-1} w'_1 + q_{m_1}(z'), \\ &\cdots \\ w_l &= B_l^{-1} w'_l + q_{m_l}(z', w'_1, \cdots, w'_{l-1}), \end{aligned} \tag{5.8}$$

where $q_{m_j}(z', w'_1, \cdots, w'_{j-1})$ is homogeneous of weight m_j.

Now we take

$$p'_{m_1}(z', \bar{z}') = B_1 p_{m_1}(A^{-1}z', \overline{A^{-1}z'}) + \Im m\, F_{m_1}(A^{-1}z'), \tag{5.9}$$

and inductively for $2 \leqslant j \leqslant l$,

$$(5.10)\quad \begin{aligned} p'_{m_j}\left(z', \bar{z}', s'_1, \cdots, s'_{j-1}\right) = B_j p_{m_j}\Big(&A^{-1}z', \overline{A^{-1}z'}, B_1^{-1}s'_1 + \Re e\, q_{m_1}(z'), \cdots, \\ &B_{j-1}^{-1}s'_{j-1} + \Re e\, q_{m_{j-1}}\left(z', \dot{w}'_1, \cdots, \dot{w}'_{j-2}\right)\Big) \\ &+ \Im m\, F_{m_j}\Big(A^{-1}z', B_1^{-1}\dot{w}'_1 + q_{m_1}(z'), \cdots, \\ &B_{j-1}^{-1}\dot{w}'_{j-1} + q_{m_{j-1}}\left(z', \dot{w}'_1, \cdots, \dot{w}'_{j-2}\right)\Big) \end{aligned}$$

with

$$\dot{w}'_k = s'_k + ip'_{m_k}\left(z', \bar{z}', s'_1, \cdots, s'_{k-1}\right).$$

The reader can easily check that (5.1) and (5.2) follows from (5.8)–(5.10) which completes the proof of Theorem 6.

In the semi-rigid case the following corollary is an immediate consequence of Theorem 6.

(5.11) Corollary. *If in addition to the assumptions of Theorem* 6, *M is semi-rigid at the origin and if the p_{m_k} and the p'_{m_k} depend only on z and z' respectively, then we have*

$$p'_{m_k}\left(Az, \overline{Az}\right) = B_k p_{m_k}(z, \bar{z}) + \Im m\, F_{m_k}(z),$$

where the A and B_k are as in Theorem 6, *and F_{m_k} is a homogeneous holomorphic polynomial of degree m_k.*

6. Hypoanalytic wave front set and the mini-FBI transform

We assume in this section that M is a generic CR manifold in $\mathbb{C}^{n+l}$ of codimension l given by (1.1), i.e.,

$$(6.1)\qquad \Im m\, w = \phi(z, \bar{z}, s).$$

In addition to (1.2), and after a holomorphic change of coordinates in $\mathbb{C}^{n+l}$, we can also assume

$$(6.2)\qquad \phi''_{ss}(0) = 0.$$

We shall use the notion of hypoanalyticity introduced in [2]. As a criterion for hypoanalyticity we use the exponential decay of an FBI transform. However, instead of integrating on a maximally real manifold of $\mathbb{C}^{n+l}$ as in [2], we integrate on a maximally real manifold in $\mathbb{C}^l$. Such an integral was used in [4] in a simpler case (rigid structures). We shall refer to this transform as a *mini-FBI*. We give now a precise definition.

Let U be an open neighborhood of 0 in $\mathbb{C}^n$, and V an open neighborhood of 0 in $\mathbb{R}^l$. If $u \in C^\infty(U, \mathscr{E}'(V))$ ($\mathscr{E}'(V)$ is the space of distributions with compact support in V), its mini-FBI transform is defined by

$$(6.3) \quad F(u;\, z, w, \sigma) = \int_{M_s} e^{i(w-\tilde{w})\sigma - \langle\sigma\rangle(w-\tilde{w})^2} \Delta(w - \tilde{w}, \sigma)\tilde{u}(z, \bar{z}, \tilde{w})\, d\tilde{w},$$

where $M_z = \{w \in \mathbb{C}^l;\, (z, w) \in M\}$, i.e. M_z is parametrized by

$$V \ni s \mapsto s + i\phi(z, \bar{z}, s) \in M_z,$$

$w, \sigma \in \mathbb{C}^l$, $|\Im m\,\sigma| < |\Re e\,\sigma|$, $\langle\sigma\rangle = (\sum_{j=1}^l \sigma_j^2)^{1/2}$, $\Delta(w, \sigma) = \det \frac{\partial\theta}{\partial\sigma}(w, \sigma)$ with $\theta = \sigma + i\langle\sigma\rangle w$. For $w = s + i\phi(z, \bar{z}, s) \in M_z$ we set

$$\tilde{u}(z, \bar{z}, w) = u(z, \bar{z}, s);$$

finally $d\tilde{w} = d\tilde{w}_1 \wedge \cdots \wedge d\tilde{w}_l$.

We shall write $F(z, w, \sigma)$ instead of $F(u;\, z, w, \sigma)$ when there is no possible confusion. We can write the mini-FBI in a more explicit form:

$$(6.4) \quad \begin{aligned} F(z, w, \sigma) = \int_{\mathbb{R}^l} & e^{i(w-\tilde{s}-i\phi(z,\bar{z},\tilde{s}))\sigma - \langle\sigma\rangle(w-\tilde{s}-i\phi(z,\bar{z},\tilde{s}))^2} \\ & \cdot\Delta(w - \tilde{s} - i\phi(z, \bar{z}, \tilde{s}), \sigma)u(z, \bar{z}, \tilde{s}) \det\big(I + i\phi'_{\tilde{s}}(z, \bar{z}, \tilde{s})\big)\, d\tilde{s}. \end{aligned}$$

We shall use an inversion formula for the mini-FBI similar to the one used in [2] (see also [14], [4]). For $z \in U$ and $s \in V$ denote by $\gamma_{z,s}$ the manifold of $\mathbb{C}^l$ parametrized by

$$(6.5) \qquad \mathbb{R}^l \ni \eta \mapsto \sigma = {}^t\big(I + i\phi'_s(z, \bar{z}, s)\big)^{-1}\eta \in \gamma_{z,s}.$$

The reader can easily check that if U and V are small enough, there exists $C > 0$ such that

$$(6.6) \qquad \Re e\big[-i(w - \tilde{w})\sigma + \langle\sigma\rangle(w - \tilde{w})^2\big] \geqslant C|\sigma|\,|w - \tilde{w}|^2$$

for $w, \tilde{w} \in M_z$, $\sigma \in \gamma_{z,s}$, and $z \in U$, $s \in V$.

The inversion formula now reads

$$(6.7) \qquad u(z, \bar{z}, s) = \frac{1}{(2\pi)^l}\int_{\gamma_{z,s}} F(z, s + i\phi(z, \bar{z}, s), \sigma)\, d\sigma,$$

or equivalently by using (6.3):

$$(6.8) \quad \begin{aligned} & u(z, \bar{z}, s) \\ & = \frac{1}{(2\pi)^l}\iint_{M_z\times\gamma_{z,s}} e^{i(w-\tilde{w})\sigma - \langle\sigma\rangle(w-\tilde{w})^2}\Delta(w - \tilde{w}, \sigma)\tilde{u}(z, \bar{z}, \tilde{w})\, d\tilde{w}\, d\sigma. \end{aligned}$$

In (6.7) and (6.8) we have used the notation $d\sigma = d\sigma_1 \wedge \cdots \wedge d\sigma_l$ and $w = s + i\phi(z, \bar{z}, s)$.

In order to prove (6.8) we observe that the right-hand side of (6.8) can be written

$$\tag{6.9} \frac{1}{(2\pi)^l} \iint_{M_z \times \Gamma_{z,s}} e^{i(w-\tilde{w})\theta} \tilde{u}(z, \bar{z}, \tilde{w})\, d\tilde{w}\, d\theta,$$

where $\Gamma_{z,s}$ is the image of $\mathbb{R}^l$ under the mapping

$$\mathbb{R}^l \ni \eta \mapsto \theta = A\eta + i\langle A\eta\rangle(w - \tilde{w})$$

with $A = {}^t(I + i\phi'_s(z, \bar{z}, s))^{-1}$.

In fact the integral defined by (6.9) must be considered as

$$\tag{6.10} \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^l} \iint_{M_z \times \Gamma_{z,s}} e^{i(w-\tilde{w})\theta - \varepsilon\theta^2} \tilde{u}(z, \bar{z}, \tilde{w})\, d\tilde{w}\, d\theta.$$

Deforming the domain of integration in $d\theta$ from $\Gamma_{z,s}$ to $\mathbb{R}^l$ and using Corollary 4.3 in [2] completes the proof of (6.8).

Remark. Note that (6.6) is not needed in order to prove (6.7) or (6.8). However for the proof of Theorem 7 and in §9 we need to consider integrals similar to the one in the right-hand side of (6.7), in which the integration is carried over a subset of $\gamma_{z,s}$. Condition (6.6) is then crucial to define such integrals.

It is convenient to write a basis $L_1, \cdots, L_n$ of $\mathbf{L}$ in the form

$$\tag{6.11} L_j = \frac{\partial}{\partial \bar{z}_j} - i \sum_{k=1}^{l} \phi_{k,\bar{z}_j} M_k, \qquad 1 \leqslant j \leqslant n,$$

with $M_k = \sum_{p=1}^{l} a_{k,p}(z, \bar{z}, s)\partial/\partial s_p$, the matrix $(a_{k,p})$ being the inverse of the matrix $(I + i\phi'_s(z, \bar{z}, s))$. Note that we have

$$M_k\big(s_j + i\phi_j(z, \bar{z}, s)\big) = \delta_{k,j}, \quad \big[M_k, M_j\big] = 0, \qquad 1 \leqslant k,\ j \leqslant l,$$
$$\big[L_p, M_k\big] = 0, \qquad 1 \leqslant p \leqslant n, 1 \leqslant k \leqslant l.$$

Using the identity

$$\int_{M_z} uM_j v\, dw = -\int_{M_z} \big(M_j u\big) v\, dw, \qquad 1 \leqslant j \leqslant l,$$

for $u \in C^\infty(U, \mathscr{E}'(V))$ and $v \in C^\infty(U \times V)$ (see similar proof in [2]), and (4.4) in [2], it is easy to check the following:

$$\tag{6.12} F\big(L_j u;\ z, w, \sigma\big) = \frac{\partial}{\partial \bar{z}_j} F(u;\ z, w, \sigma), \qquad 1 \leqslant j \leqslant n.$$

We shall take $u(z,\bar{z},s) = \chi(s)h(z,\bar{z},s)$, where h is a CR distribution defined in $\Omega = U \times V$ (i.e. $L_j h = 0$, $1 \leqslant j \leqslant n$), and $\chi \in C_0^\infty(V)$, $\chi \equiv 1$, in a neighborhood of 0. Its mini-FBI satisfies the following useful property:

(6.13) Lemma. *If U and V are small enough, there exist open sets U' and $\tilde{V}$, $0 \in U' \subset U$, $0 \in \tilde{V} \subset \mathbb{C}^l$, and a holomorphic function $G(z,w,\sigma)$ defined in the domain*

$$(6.14) \qquad z \in U', \quad w \in \tilde{V}, \quad \sigma \in \mathbb{C}^l, \quad |\Im m\, \sigma| < \frac{1}{2}|\Re e\, \sigma|,$$

such that

$$(6.15) \qquad |F(z,w,\sigma) - G(z,w,\sigma)| \leqslant Ce^{-|\sigma|/C}$$

uniformly for z, w, σ in the domain (6.14).

Proof. This is similar to the proof of Lemma II.1 in [4]. If U and V are small enough, it follows from (6.12) and the definition (6.3) that

$$|\partial_{\bar{z}} F(z,w,\sigma)| \leqslant Ce^{-|\sigma|/C}$$

for z, w, σ in a domain of the form (6.14).

Solving the equations $\partial_{\bar{z}} Q(z,w,\sigma) = \partial_{\bar{z}} F(z,w,\sigma)$ with $|Q(z,w,\sigma)| \leqslant C'e^{-|\sigma|/C'}$ (which can be done by the argument in [4]) and taking $G(z,w,\sigma) = F(z,w,\sigma) - Q(z,w,\sigma)$, we complete the proof of the lemma.

It follows, as in [4], from the inversion formula (6.7) that if the mini-FBI of $u = \chi h$ (where h is CR) satisfies

$$(6.16) \qquad |F(z,w,\sigma)| \leqslant Ce^{-|\sigma|/C},$$

uniformly for z, w, σ in a domain of the form (6.14), then h is the restriction to M of a holomorphic function in a neighborhood of the origin in $\mathbb{C}^{n+l}$, i.e. h is hypoanalytic at 0 in the terminology of [2]. Conversely if h is hypoanalytic at 0, it is easy to see, by deforming the domain of integration in (6.3), that (6.16) holds.

We will say that a CR distribution h is *hypoanalytic* at $(0,\sigma^0)$, $\sigma^0 \in \mathbb{R}^l \setminus 0$, if (6.16) holds uniformly for (z,w) in a neighborhood of 0 in $\mathbb{C}^{n+l}$ and σ in a conic neighborhood of σ^0 in $\mathbb{C}^l$. This notion of microlocal hypoanalyticity is equivalent to the one introduced in [2] (for a detailed proof see Treves [17]). However this fact is not essential in this present paper. If h is not hypoanalytic at $(0,\sigma)$ we say that $(0,\sigma)$ is in the *hypoanalytic wave front set* of h, and write

$$(0,\sigma) \in WFh \quad \text{or} \quad \sigma \in WF_0 h.$$

If Γ is a closed strictly convex cone $\subset \mathbb{R}^l$, we denote by $\mathring{\Gamma}$ its polar or dual cone,

$$\mathring{\Gamma} = \{v \in \mathbb{R}^l;\ v \cdot \sigma > 0\ \forall \sigma \in \Gamma \setminus 0\}.$$

If $\mathscr{C}$ is an open cone of $\mathbb{R}^l$, and $\mathscr{O}$ an open neighborhood of 0 in $\mathbb{C}^{n+l}$ we say that the open set

$$\mathscr{W} = \mathscr{W}_{\mathscr{C}} = \mathscr{W}(\mathscr{O}, \mathscr{C}) = \{(z, w) \in \mathscr{O};\ \Im m\, w - \phi(z, \bar{z}, s) \in \mathscr{C}\}$$

is a *wedge* with *edge* M.

If H is a tempered holomorphic function defined in $\mathscr{W}$ (i.e. $|H(z,w)| \leqslant Cd((z,w), M)^{-N}$ where $d((z,w), M)$ is the distance from (z,w) to M), then its boundary value, $h = bH$, is a CR distribution on M. We say that h extends holomorphically to $\mathscr{W}$.

We have the following result:

Theorem 7. *Let Γ be a strictly convex closed cone contained in $\mathbb{R}^l$, and h a CR distribution defined on M. The following properties are equivalent*:

(a) $WF_0 h \subset \Gamma$.

(b) *For every open cone $\mathscr{C} \subset \mathbb{R}^l$, with $\mathscr{C} \subset\subset \mathring{\Gamma}$, there exists an open neighborhood $\mathscr{O}$ of the origin in $\mathbb{C}^{n+l}$ such that h extends holomorphically to the wedge $\mathscr{W}(\mathscr{O}, \mathscr{C})$.*

Proof. The statement of this theorem is the same as Theorem II.2 of [4] (rigid case). Its proof is based on a deformation of the domain of integration in (6.3), and on the inversion formula (6.7), (6.8). We leave the details to the reader.

Remark. Throughout this paper we consider CR distributions defined in Ω. However for the proofs in the following sections it suffices to consider only CR functions of class C^1. Indeed using arguments similar to those in [4] and [5] it can be shown that any CR distribution h can be locally written

$$h = \left(\sum_{k=1}^{l} M_k^2 \right)^N h_1,$$

where h_1 is a CR function of class C^1, the M_k are as in (6.11), and $N \in \mathbb{Z}_+$.

7. Extendability of CR distributions from generic semi-rigid CR manifolds

The main result of this section is the following theorem announced in the introduction.

Theorem 8. *Any CR distribution on a generic semi-rigid CR manifold of finite type extends holomorphically to a wedge of edge M.*

Theorem 8 is an immediate consequence of Theorem 7 and the following result.

Theorem 9. *Let h be a CR distribution defined on a generic semi-rigid manifold of finite type defined by* (6.1). *There exists a strictly convex closed cone $\Gamma \subset \mathbb{R}^l$ such that $WF_0 h \subset \Gamma$.*

Before proving Theorem 9, we need to state a microlocal result.

Since M is semi-rigid we can make use of Theorem 1 and find holomorphic coordinates z, w in $\mathbb{C}^{n+l}$ such that on M we have

$$(7.1)\qquad \begin{aligned} w_1 &= s_1 + i\left[p_{m_1}(z,\bar z) + \mathcal{O}(m_1+1)\right],\\ &\ \vdots\\ w_r &= s_r + i\left[p_{m_r}(z,\bar z) + \mathcal{O}(m_r+1)\right], \end{aligned}$$

where m_j are the Hörmander numbers at the origin, $2 \leqslant m_1 < m_2 < \cdots < m_r < \infty$, and p_{m_j} is valued in $\mathbb{R}^{l_j}$ and homogeneous of degree m_j. Since M is of finite type we know that for all $\eta \in \mathbb{R}^{l_j} \setminus \{0\}$, $p_{m_j}(z,\bar z)\cdot\eta$ is nonpluriharmonic.

It is convenient to write (7.1) in the form

$$(7.2)\qquad w = s + i\left[p(z,\bar z) + R(z,\bar z,s)\right],$$

where $p(z,\bar z) = (p_{m_1}(z,\bar z),\cdots, p_{m_r}(z,\bar z))$ and R is the remainder term.

Similarly to [6] we say that $\sigma^0 \in \mathbb{R}^l \setminus \{0\}$ satisfies the *line sector property* if there exists a vector $V \in \mathbb{C}^n \setminus \{0\}$ such that for all $\zeta \in \mathbb{C}$

$$(7.3)\qquad \sigma^0 \cdot p(\zeta V, \overline{\zeta V}) = q_m(\zeta,\bar\zeta) + \mathcal{O}\left(|\zeta|^{m+1}\right)$$

with q_m nonpluriharmonic real homogeneous polynomial of degree m, and moreover there exist a sector $\mathcal{S}$ in the complex plane and $\mu \in \mathbb{C}$ satisfying

$$(7.4)\qquad \begin{cases} q_m(\zeta,\bar\zeta) + \Re e\,\mu\zeta^m|_{\mathcal{S}} < 0,\\ \text{angle}\,\mathcal{S} > \pi/m. \end{cases}$$

Remark. Note that if (7.3) holds then necessarily m is one of the m_j's. Also note that it follows from Corollary (5.11) that for the characteristic vector σ_0 to satisfy the line sector property is independent of the choice of the holomorphic coordinates (z,w).

We can now state and prove a microlocal result which was given in [6] in the case of a hypersurface ($l = 1$).

Theorem 10. *With the assumptions of Theorem* 9, *if* $\sigma^0 \in \mathbb{R}^l \setminus 0$ *satisfies the line sector property*, *then any CR distribution on M is hypoanalytic at* $(0,\sigma^0)$.

Proof of Theorem 10. Consider the complex coordinates (z,w) for which (7.1) (and (7.2)) are valid. For $\varepsilon > 0$, we define new coordinates (z',w') by the dilations corresponding to the homogeneity

$$z = \varepsilon z',\qquad w_j = \varepsilon^{m_j} w_j',\quad s_j = \varepsilon^{m_j} s_j',\qquad 1 \leqslant j \leqslant r.$$

After dropping the primes we see that the new coordinates on M satisfy

$$(7.5)\qquad w = s + i\left[p(z,\bar z) + \mathcal{O}(\varepsilon)\right],$$

where $p(z,\bar z)$ is the same as in (7.2).

After a real change of coordinates in $\mathbb{C}^l$, and a complex one in $\mathbb{C}^n$ we may assume

$$\sigma^0 = (1, 0, \cdots, 0), \qquad V = (1, 0, \cdots, 0), \tag{7.6}$$

where V is the vector in (7.3).

In these new coordinates we write $w = (w_1, \cdots, w_l)$, where the w_j are the scalar components of w (and not vectors as in (7.1)). We have on M, by using (7.3), (7.5), (7.6),

$$\begin{aligned} w_1 &= s_1 + i\left[q_m(z_1, \bar{z}_1) + \mathcal{O}\left(|z_1|^{m+1} + |z_2|^2 + \cdots + |z_n|^2 + \varepsilon\right)\right], \\ w_j &= s_j + i\left[\mathcal{O}\left(|z|^2 + \varepsilon\right)\right], \qquad 2 \leqslant j \leqslant l, \end{aligned} \tag{7.7}$$

where q_m is the same as in (7.3).

Now we make a second dilation. For $\delta > 0$ put

$$\begin{aligned} z_1 &= \delta z_1', \qquad z_j = \delta^m z_j, \quad 2 \leqslant j \leqslant n, \\ w_1 &= \delta^m w_1', \qquad w_j = \delta w_j', \quad 2 \leqslant j \leqslant l \ (s' = \Re e\, w'). \end{aligned}$$

After again dropping the primes, the new coordinates satisfy on M

$$\begin{aligned} w_1 &= s_1 + i\left[q_m(z_1, \bar{z}_1) + \mathcal{O}(\delta) + \mathcal{O}(\varepsilon/\delta^m)\right], \\ w_j &= s_j + i\left[\mathcal{O}(\delta) + \mathcal{O}(\varepsilon/\delta)\right], \qquad 2 \leqslant j \leqslant l. \end{aligned}$$

Now we choose $\varepsilon = \delta^{m+1}$, and we have

$$w_1 = s_1 + i\left[q_m(z_1, \bar{z}_1) + \mathcal{O}(\delta)\right], \qquad w_j = s_j + i\mathcal{O}(\delta), \quad 2 \leqslant j \leqslant l.$$

From now on the rest of proof follows closely the one of Theorem III.1 in [4]. We must show that the mini-FBI of $u = \chi h$, where χ is a cutoff function chosen independent of δ, is exponentially decreasing in a conic neighborhood of σ_0. We will not repeat the arguments of the proof in [4], we only indicate the important steps.

Thanks to Lemma III.1 in [4] we can find a small domain $D \subset \mathbb{C}$, $0 \in D$, and a holomorphic function $f(z_1)$ in D such that $f(0) = 0$,

$$-q_m(z_1, \bar{z}_1) + \Re e f(z_1) - \left(q_m(z_1, \bar{z}_1)\right)^2\Big|_{\partial D} > 0$$

and $|f(z_1)| \leqslant 1/2C$ for $z_1 \in D$, where C is the constant in (6.15).

We then choose $\delta > 0$ small enough, and we use the maximum principle in the z_1-plane for the holomorphic function $e^{-\sigma_1 f(z_1)} G(z, w, \sigma)$, where G is given by Lemma (6.13), to prove the desired exponential decay.

Proof of Theorem 9. Define the following set:

$$S = \left\{\sigma \in \mathbb{R}^l \backslash \{0\}\, ;\ \sigma \text{ does not satisfy the line sector property}\right\}.$$

Theorem 10 states that for any CR distribution h on M,

$$(7.8) \qquad WF_0 h \subset S.$$

On the other hand, given any $\sigma \in \mathbb{R}^l \setminus \{0\}$, it is easy to check that at most one of the vectors σ and $-\sigma$ is in S, i.e., $S \cup \{0\}$ does not contain any line. A very slight and obvious modification of Lemma III.3 of [4] shows that the set S is a convex cone of $\mathbb{R}^l \setminus \{0\}$. Since $WF_0 h$ is closed we conclude from (7.8) that its convex hull is a strictly convex closed cone $\Gamma \subset S$, which completes the proof of Theorem 9.

8. Other microlocal hypoanalyticity results. Examples

In this section we give some microlocal hypoanalyticity results for CR distributions defined on generic manifolds which are not necessarily semi-rigid.

First let us recall a definition introduced in [4]. If $q_m(\zeta, \bar{\zeta})$ is a real homogeneous polynomial in ζ, $\bar{\zeta}$ ($\zeta \in \mathbb{C}$) of degree $m \geqslant 2$, we say that it has the *extension property* if any CR function defined near the origin on the hypersurface

$$\Sigma = \left\{(\zeta, \eta) \in \mathbb{C}^2;\ \Im m\, \eta = q_m(\zeta, \bar{\zeta})\right\}$$

extends holomorphically to the side of Σ defined by $\Im m\, \eta < q_m(\zeta, \bar{\zeta})$.

In particular if q_m satisfies (7.4), then it has the extension property (cf. [6], [4]).

We can now state the following result.

Theorem 11. *Let M be a generic CR manifold in $\mathbb{C}^{n+l}$ defined by* (1.1) *and $\sigma^0 \in \mathbb{R}^l \setminus \{0\}$. Assume there exist a holomorphic curve γ defined in D, an open neighborhood of* 0 *in $\mathbb{C}$, γ: $D \to \mathbb{C}^n$, $\gamma(0) = 0$, and a homogeneous polynomial $p_m(\zeta, \bar{\zeta})$, $\zeta \in \mathbb{C}$, $m \geqslant 2$, valued in $\mathbb{R}^l$ satisfying*:

(i) $\phi(\gamma(\zeta), \overline{\gamma(\zeta)}, 0) = p_m(\zeta, \bar{\zeta}) + \mathcal{O}(|\zeta|^{m+1})$,

(ii) *the polynomial $\zeta \mapsto \sigma^0 \cdot p_m(\zeta, \bar{\zeta})$ has the extension property.*

Then any CR distribution on M is hypoanalytic at $(0, \sigma^0)$.

Proof. As in the proof of Theorem III.4 in [4] it suffices to prove the theorem when $\gamma(\zeta) = (\zeta, 0, \cdots, 0)$. Also we may assume $\sigma^0 = (1, 0, \cdots, 0)$.

It follows from (i) that we have

$$(8.1) \quad \phi(z, \bar{z}, s) = p_m(z_1, \bar{z}_1) + \mathcal{O}\left(|z_1|^{m+1} + |s|\,|z| + |z'|\,|z_1| + |z'|^2 + |s|^2\right)$$

with $z = (z_1, z')$.

Condition (ii) implies that q_m, the first component of $p_m(z_1, \bar{z}_1)$, satisfies the extension property.

For $\delta > 0$ we consider the following dilation:

$$\begin{aligned} z_1 &= \delta \tilde{z}_1, \\ z' &= \delta^m \tilde{z}', \\ w_1 &= \delta^m \tilde{w}_1 \qquad (s_1 = \delta^m \tilde{s}_1), \\ w_j &= \delta^{m-1/2} \tilde{w}_j \qquad \left(s_j = \delta^{m-1/2}\tilde{s}_j\right), 2 \leqslant j \leqslant l. \end{aligned}$$

Using (8.1) we conclude that we have on M:

$$\begin{aligned} \tilde{w}_1 &= \tilde{s}_1 + i\left[q_m\left(\tilde{z}_1, \bar{\tilde{z}}_1\right) + \mathcal{O}(\delta^{1/2})\right], \\ \tilde{w}_j &= \tilde{s}_j + i\mathcal{O}(\delta^{1/2}), \qquad 2 \leqslant j \leqslant l. \end{aligned} \tag{8.2}$$

From this point the rest of the proof follows closely the one of Theorem III.4 in [4]. We leave the details to the reader.

(8.3) Corollary. *Let M be a generic CR manifold defined by* (1.1), *and assume that m_1 (the first Hörmander number at* 0) *is finite. Let $\sigma^0 \in \mathbb{R}^l \setminus \{0\}$. If there exists $V \in \mathbb{C}^n \setminus \{0\}$ such that for $\zeta \in \mathbb{C}$*

$$\sigma^0 \cdot \phi\left(\zeta V, \overline{\zeta V}, 0\right) = q_{m_1}(\zeta, \bar{\zeta}) + \mathcal{O}\left(|\zeta|^{m_1+1}\right), \tag{8.4}$$

where the homogeneous polynomial q_{m_1} has the extension property, then any CR distribution on M is hypoanalytic at $(0, \sigma^0)$.

The proof of Corollary (8.3) is based on Theorem 11 and the following lemma.

(8.5) Lemma. *Assume that M is a generic CR manifold defined by* (1.1). *If its first Hörmander number m_1 at the origin is finite then there exist a homogeneous polynomial of degree m_1, $p_{m_1}(z, \bar{z})$, valued in $\mathbb{R}^l$ and a holomorphic polynomial of degree m_1, $F(z)$, valued in $\mathbb{C}^l$ such that*

$$\phi(z, \bar{z}, 0) = p_{m_1}(z, \bar{z}) + \Im m\, F(z) + \mathcal{O}\left(|z|^{m_1+1}\right). \tag{8.6}$$

Proof. We choose a basis $L_1, \cdots, L_n$ of $\mathbf{L}$ of the form (6.11). If $L^{(k)}$ is a commutator of the form (2.3) where M_j is either one of the L_p or one of the $\overline{L}_p$, then there are $\alpha, \beta \in \mathbb{Z}_+^n$, $|\alpha| + |\beta| = k$, $|\beta| \geqslant 1$, such that

$$L^{(k)} = \pm i \sum_{p=1}^{l} \phi_{p, z^\alpha \bar{z}^\beta}\left(M_p + \overline{M}_p\right) + \sum_{p=1}^{l} \sum_{|\alpha'| + |\beta'| < k} \phi_{p, z^{\alpha'} \bar{z}^{\beta'}} N_{p, \alpha' \beta'},$$

where $N_{p,\alpha'\beta'}$ is a complex vector field. Then it follows from the definition of the first Hörmander number m_1 that

$$\phi_{z^{\alpha'}\bar{z}^{\beta'}}(0) = 0 \quad \forall \alpha', \beta', |\alpha'| + |\beta'| < m_1, |\beta'| \geqslant 1,$$

$$\phi_{z^{\alpha}\bar{z}^{\beta}}(0) \neq 0 \quad \text{for some } \alpha, \beta, |\alpha| + |\beta| = m_1, |\beta| \geqslant 1,$$

which yields (8.6).

Proof of Corollary (8.3). After the holomorphic change of coordinates $w' = w - F(z)$, we can assume that $F(z) = 0$ in (8.6). Then we apply Theorem 11 with $\gamma(\zeta) = \zeta V$ to reach the conclusion of the corollary.

(8.7) Corollary. *Assume that M is a generic CR manifold in $\mathbb{C}^{n+l}$ of codimension l, and that m_1, its first Hörmander number at the origin, is odd with multiplicity $l_1 = l - 1$. Then there exists a line $L \subset \Sigma_0 \cup \{0\}$, where Σ_0 is the set of characteristic covectors at the origin, such that for any CR distribution h on M*

$$WF_0 h \subset L. \tag{8.8}$$

Proof. Since $l_1 = l - 1$, we have only two Hörmander numbers $2 \leqslant m_1 < m_2 \leqslant \infty$. We make use of the coordinates in Theorem 1 if $m_2 < \infty$, or Theorem 2 if $m_2 = \infty$. Since m_1 is odd we can apply Corollary (8.3) for all $\sigma^0 \in \mathbb{R}^l \setminus \{0\}$, $\sigma^0 \neq (0, \cdots, 0, \lambda)$, $\lambda \in \mathbb{R} \setminus \{0\}$, *and reach the conclusion* (8.8) *where L is the line spanned by the vector* $(0, \cdots, 0, 1)$.

(8.9) Examples. (a) Even for the case of a hypersurface, Theorem 11 gives results which cannot be obtained by using Theorem 10 (or [6]). As an example consider the hypersurface M in $\mathbb{C}^3$ defined by (1.1) with $z \in \mathbb{C}^2$, $w \in \mathbb{C}$, and

$$\phi(z, \bar{z}, s) = |z_1^3 - z_2^2|^2 + (\Re e\, z_2)|z_2|^4 + s|z_1|^2.$$

Here we have $m_1 = 4$. Applying Theorem 10 (or the result in [6]) we conclude that any CR distribution on M near the origin extends holomorphically to the side of M defined by $\Im m\, w - \phi > 0$. However using Theorem 11 with

$$\gamma(\zeta) = (\zeta^2, \zeta^3), \quad m = 15, \quad p_m(\zeta, \bar{\zeta}) = (\Re e\, \zeta^3)|\zeta|^{12},$$

we conclude, since m is odd, that any CR distribution near 0 is the restriction to M of a holomorphic function in a neighborhood of 0 in $\mathbb{C}^3$.

(b) Let $M \subset \mathbb{C}^4$ be defined by (1.1) with $n = 2$, $l = 2$, and

$$\phi_1(z, \bar{z}, s) = |z_1^2 - z_2^3|^2 (\Re e\, z_2), \qquad \phi_2(z, \bar{z}, s) = |z_1|^2 s_1 + |z_1|^{10} (\Re e\, z_1).$$

Here M is of finite type at the origin with Hörmander numbers $m_1 = 5$, $m_2 = 7$. M is not semi-rigid at the origin, therefore we cannot make use of Theorem 8. The homogeneous polynomials of Theorem 1 are here:

$$p_5 = |z_1|^4 (\Re e\, z_2), \qquad p_7 = |z_1|^2 s_1.$$

However, any CR distribution h on M near 0 extends holomorphically to a full neighborhood of 0 in $\mathbb{C}^4$. Indeed since $m_1 = 5$ is odd we can use Corollary (8.3) and conclude that h is hypoanalytic at $(0, \sigma^0)$ with $\sigma^0 = (\sigma_1, \sigma_2)$, $\sigma_1 \neq 0$. If $\sigma^0 = (0, \sigma_2)$ and $\sigma_2 \neq 0$, we can make use of Theorem 11 with

$$\gamma(\zeta) = (\zeta^3, \zeta^2), \quad m = 33, \quad p_m(\zeta, \bar{\zeta}) = \left(0, |\zeta|^{30}(\Re e\, \zeta^3)\right)$$

to reach the conclusion that h is hypoanalytic at $(0, \sigma^0)$.

9. Holomorphic decomposition of CR distributions

In [1] Andreotti and Hill proved the following decomposition result for a CR function h defined near a point $\omega_0 \in M$, where M is a smooth hypersurface in $\mathbb{C}^{n+1}$: There exists $\mathscr{O}$, a neighborhood of ω_0 in $\mathbb{C}^{n+1}$, with $\mathscr{O}^+$ and $\mathscr{O}^-$ the two sides of M in $\mathscr{O}$, and holomorphic functions H^+ and H^- defined in $\mathscr{O}^+$ and $\mathscr{O}^-$ respectively, such that

$$h = bH^+ + bH^-, \tag{9.1}$$

where bH^+ and bH^- denote respectively the boundary values of H^+ and H^- on M. More generally, following similar terminology of Henkin [9] for a generic CR manifold M in $\mathbb{C}^{n+l}$ of codimension l, we shall say that a CR distribution h defined near $\omega_0 \in M$ has a *holomorphic decomposition* if there exist convex open cones $\mathscr{C}_j \subset \mathbb{R}^l \setminus \{0\}$, $j = 1, \cdots, p$, and wedges $\mathscr{W}(\mathscr{O}, \mathscr{C}_j)$ of the form (1.3) such that

$$h = \sum_{j=1}^{p} bH_j \quad \text{near } \omega_0 \tag{9.2}$$

with H_j holomorphic in $\mathscr{W}(\mathscr{O}, \mathscr{C}_j)$. The existence of such a decomposition is known [9] for codimension 2 in the case where the first Hörmander number m_1 is 2, and the Levi form has either all eigenvalues positive or all negative, or two eigenvalues of different signs. In [4] the authors, together with F. Treves, prove that if M is rigid then any CR distribution has a holomorphic decomposition. In the case where M is semi-rigid and of finite type, our Theorem 8 in §7 shows that the decomposition (9.2) is always valid with $p = 1$.

In general, a recent example of Trépreau [15] in codimension 2 shows that there may exist CR functions for which there is no holomorphic decomposition. His example is a real analytic CR manifold M in $\mathbb{C}^3$ for which $l = 2$, having no finite Hörmander number. In the example, every CR distribution either extends to be holomorphic in a full neighborhood of the origin or has all of $\mathbb{R}^2 \setminus \{0\}$ in its hypoanalytic wavefront set at 0. Since the functions $h_j = bH_j$

in (9.2) must have $WF_0 h_j$ contained in a strictly convex cone (cf. Theorem 7), a holomorphic decomposition exists for a particular h on M if and only if h itself already extends to be holomorphic in a full neighborhood in $\mathbb{C}^3$.

Here we shall prove a positive result on holomorphic decomposition which contains the hypersurface theorem of Andreotti-Hill as a special case. The microlocal statement is the following.

Theorem 12. *Let M be a generic CR manifold of codimension l in $\mathbb{C}^{n+l}$, and assume $0 \in M$. Let h be a CR distribution on M near 0 with the property that there exist disjoint strictly convex closed cones $\Gamma_j \subset \mathbb{R}^l \setminus \{0\}$ with*

$$WF_0 h \subset \bigcup_{j=1}^{p} \Gamma_j. \tag{9.3}$$

Then h has a holomorphic decomposition of the form (9.2) *with the same p. In addition, the cones $\mathscr{C}_j$ in* (9.2) *can be chosen to be any open cones satisfying $\mathscr{C}_j \subset\subset \mathring{\Gamma}_j$.*

From Theorem 12 we can derive the following consequences.

(9.4) Corollary (*Andreotti-Hill* [1]). *If M is a hypersurface and h any CR distribution on M, then h has a holomorphic decomposition of the form* (9.1).

Indeed in the case of the hypersurface the space of characteristic covectors at the origin is one-dimensional, therefore (9.3) holds with $p = 2$, and Γ_j, $j = 1, 2$, being two disjoint half-lines.

The following corollary is a direct consequence of Theorem 12 and Corollary (8.7) in §8.

(9.5) Corollary. *Suppose that M is a generic CR manifold in $\mathbb{C}^{n+l}$ of codimension l, and $0 \in M$. Assume there exists a one-dimensional subspace $L \subset \Sigma_0 \cup \{0\}$, where Σ_0 is the set of all characteristic covectors at* 0, *such that for any CR distribution h on M*

$$WF_0 h \subset L.$$

Then any CR distribution decomposes

$$h = bH_1 + bH_2, \tag{9.6}$$

where H_j is holomorphic in a wedge $\mathscr{W}(\mathscr{O}, \mathscr{C}_j)$.

In particular, if the first Hörmander number m_1 of M at the origin is odd with multiplicity $l_1 = l - 1$, then all CR distributions have a holomorphic decomposition of the form (9.6).

Proof of Theorem 12. Let Γ_j' and Γ_j'', $j = 1, \cdots, p$, be strictly convex closed cones contained in $\mathbb{R}^l \setminus \{0\}$ such that

$$\Gamma_j \subset \operatorname{int} \Gamma_j' \subset\subset \operatorname{int} \Gamma_j'', \tag{9.7}$$

and the Γ_j'' are disjoint.

We use the inversion formula (6.7) (or (6.8)). If F is the mini-FBI transform of χh, $\chi \in C_0^\infty(V)$ and $\chi \equiv 1$ near 0, then if z and s are small enough we have

$$h(x,y,s) = \frac{1}{(2\pi)^l}\left(\sum_{j=1}^{p}\int_{\eta\in\Gamma_j'} F(z, s + i\phi(z,\bar{z},s),\sigma)\, d\sigma \right. \tag{9.8}$$
$$\left. + \int_{\eta\in\mathbb{R}^l\setminus\bigcup_j\Gamma_j'} F(z,s+i\phi(z,\bar{z},s)\sigma)\, d\sigma\right).$$

Here σ and η are related by (6.5).

We rewrite (9.8) in the form

$$h(x,y,s) = \sum_{j=1}^{p} h_j'(x,y,s) + h_0'(x,y,s). \tag{9.9}$$

Since F satisfies an estimate of the form (6.16) uniformly for (z,w) in a neighborhood of 0 in $\mathbb{C}^{n+l}$ and σ in the image of $\mathbb{R}^l\setminus\bigcup_j\Gamma_j'$ under the map (6.5) we conclude that there is a germ of smooth function $H_0(x,y,w)$ defined in a neighborhood of 0 in $\mathbb{C}^{n+l}$, holomorphic in w, such that

$$h_0'(x,y,s) = H_0(x,y,s+i\phi(x,y,s)).$$

On the other hand, if $\mathcal{O}$ is small enough there exist smooth functions $H_j(x,y,w)$, $j = 1,\cdots,p$, defined in the wedge $\mathscr{W}(\mathcal{O},\mathring{\Gamma}_j'')$, holomorphic in w (and with tempered growth as (z,w) approaches M) such that

$$h_j' = bH_j. \tag{9.10}$$

We conclude from (9.10) that for $k = 1,\cdots,n$,

$$L_k h_j' = b\frac{\partial H_j}{\partial \bar{z}_k}, \qquad j = 0,\cdots,p. \tag{9.11}$$

Applying L_k to (9.9) and using (9.11) yields

$$\sum_{j=1}^{r} b\left(\frac{\partial H_j}{\partial\bar{z}_k}\right) + \left.\left(\frac{\partial H_0}{\partial\bar{z}_k}\right)\right|_M = 0. \tag{9.12}$$

If we fix z near the origin, then $b\partial H_j/\partial\bar{z}_k$ is a distribution defined on M_z (maximally real manifold of $\mathbb{C}^l$). Its hypcanalytic wave front set at 0 (in the sense of [2]) is contained in Γ_j''. Since the Γ_j'' are disjoint, we conclude from (9.12) that $\partial H_j/\partial\bar{z}_k$ extends to be holomorphic in a full neighborhood of 0 as a function of w. [This is a hypoanalytic version of the edge of the wedge theorem (see e.g. Hörmander [10]).]

Now the rest of the proof is similar to that of [4, Theorem II.3]. Let $h_{jk} = L_k h'_j$ and $H_{j,k}(z, \bar{z}, w)$ be the holomorphic extension of h_{jk} in w. By the chain rule, for any q,

$$\partial_{\bar{z}_q} H_{jk}(z, \bar{z}, w)|_{w=s+i\pi} = L_q h_{jk}(z, \bar{z}, s). \tag{9.13}$$

Since the L_k commute with each others, (9.13) and a uniqueness argument imply that for all $j = 1, \cdots, p$,

$$\partial_{\bar{z}_k} H_{jq} = \partial_{\bar{z}_q} H_{jk}, \qquad 1 \leqslant k, q \leqslant n.$$

Hence we may find $U_j(z, \bar{z}, w)$ holomorphic in w such that

$$\partial_{\bar{z}_k} U_j = H_{jk}, \qquad 1 \leqslant k \leqslant n. \tag{9.14}$$

Now set

$$u_j(z, \bar{z}, s) = U_j(z, \bar{z}, s + i\phi(z, \bar{z}, s)), \qquad 0 \leqslant j \leqslant p,$$

$$h_j = h'_j - u_j + \frac{1}{p+1} \sum_{k=0}^{p} u_k, \qquad 0 \leqslant j \leqslant p. \tag{9.15}$$

Note that we can take $U_0 = H_0$ and therefore $u_0 = h'_0$. It follows from (9.9) and (9.15) that

$$h = \sum_{j=0}^{r} h_j. \tag{9.16}$$

On the other hand, since

$$L_k u_j = \frac{\partial}{\partial \bar{z}_k} U_j|_{w=s+i\phi},$$

we have

$$L_k u_j = L_k h'_j, \qquad 1 \leqslant k \leqslant n, 0 \leqslant j \leqslant p,$$

and we conclude that

$$L_k h_j = 0, \tag{9.17}$$

i.e., h_j is a CR distribution.

Finally, it is clear from (9.15) that h_j is the boundary value of

$$H_j - U_j + \frac{1}{p+1} \sum_{k=1}^{p} U_k,$$

which is defined in the wedge $\mathscr{W}(\mathscr{O}, \mathring{\Gamma}''_j)$, holomorphic with respect to w; (9.17) implies that it is also holomorphic in z, which completes the proof of Theorem 12.

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