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INVARIANTS OF CONFORMAL LAPLACIANS

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The conformal Laplacian $\square = d^*d + (n-2)s/4(n-1)$, acting on functions on a Riemannian manifold M^n with scalar curvature s, is a conformally invariant operator. In this paper we will use $\square$ to construct new conformal invariants: one of these is a pointwise invariant, one is the integral of a local expression, and one is a nonlocal spectral invariant derived from functional determinants.

We begin in §1 by describing the Laplacian $\square$ and its Green function in the context of conformal geometry. We then derive a basic formula giving the variation in the heat kernel of $\square$. This formula is strikingly simpler than the corresponding formula for the ordinary Laplacian given by Ray and Singer [15].

The heat kernel of $\square$ has an asymptotic expansion $k(t,x,x) \sim (4\pi t)^{-n/2}\sum a_k(x)t^k$. In §2 we prove that $a_{(n-2)/2}$ is a pointwise conformal invariant of weight -2, i.e. it satisfies $a_{(n-2)/2}(x;\lambda^2 g) = \lambda^2 a_{(n-2)/2}(x;g)$, where g is the metric and λ is any smooth positive function. In particular, this shows the existence of a nontrivial locally computable conformally invariant density naturally associated to the conformal structure of an even dimensional manifold. The key to the proof is to consider the parametrix of the Green's function, which is obtained from the heat kernel by an integral transform. One finds that $a_{(n-2)/2}$ occurs as the coefficient of the first log term in this parametrix, and its conformal invariance then follows directly from the conformal invariance of the Green's function.

In §3 we show that $\int a_{n/2}$ is a global conformal invariant (the calculations in §4 show that it is not a pointwise invariant). The proof is a direct calculation of the invariant of $\int a_{n/2}$ using equation (1.10).

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In §4 we explicitly compute the conformal invariants $\int a_{n/2}$ for $n = 2, 4, 6$ and $a_{(n-2)/2}$ for $n = 2, 4, 6, 8$. These computations are quite complicated when $n = 6$ and 8; they depend on the work of Gilkey [10] and connect nicely with the recent work of Fefferman and Graham [9]. We begin by reviewing the short list of explicitly known conformal tensors. For the conformal Laplacian acting on scalars, the invariant $a_{(n-2)/2}$, $n \leqslant 8$, is a sum of these conformal tensors. For the bundle Laplacian, $a_{(n-2)/2}$ contains additional terms; these have led us to discover a new conformal tensor (Theorem 4.1). It is jointly associated to the conformal structure of M and the hermitian structure of the bundle, and is natural and regular in the sense of [1]. The invariants $\int a_{n/2}$, $n \leqslant 6$, can then be expressed in terms of the Euler class of M and integrals of conformal tensors. In dimension 4, $\int a_2$ is particularly interesting: it has a topological lower bound which is realized if and only if M is self-dual.

We were led to consider $a_{(n-2)/2}$ and $\int a_{n/2}$ by the work of Branson and Ørsted [6]. Specifically, they (and Schimming [17] and Wünsch [19] independently) proved that $a_{(n-2)/2}$ is a conformal invariant for the wave operator. They then argued that the same result holds for the heat operators, and checked this for $n = 2, 4$. They also conjectured Theorem 3.1 which they have now proved by different methods [7].

In §5 we consider conformal deformations of the functional determinant of $\Box$. We prove that the determinant is a conformal invariant in odd dimensions (Theorem 5.3). This gives a new *global* conformal invariant, constructed from the spectrum of $\Box$. In even dimensions the determinant is not conformally invariant, but we obtain interesting invariants by considering the ratio of determinants. Specifically, we use the solution of the Yamabe problem on a spin manifold to construct conformal Laplacians $\Box_L$, $\Box_R$ on left and right spinors. The ratio $\det \Box_L / \det \Box_R$ is then a locally smooth function on the space of all metrics which we explicitly compute (equation (5.6)). Furthermore, the "conformal anomaly" $d \log(\det \Box_L / \det \Box_R)$ is a conformally invariant 1-form on each conformal class (given by integration against the $\hat{A}$-polynomial as a differential form in the curvature). The invariance of this conformal anomaly reflects a dichotomy: on each conformal class either $\det \Box_L / \det \Box_R$ is invariant (we give examples of this), or this ratio has no critical points in the conformal orbit.

The unifying feature to the invariants of this paper is the zeta function associated to the conformal Laplacian. Roughly speaking, the invariants in §2 is the residue at $s = 1$ of the local zeta function $\zeta(s, x)$, the invariant of §3 is $\zeta(0)$, and the conformal anomaly is $d(\zeta_L'(0) - \zeta_R'(0))$.

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1. The conformal Laplacian

The *conformal Laplacian* on an n-dimensional Riemannian manifold (M, g) is

$$\Box_g = d^*d + \alpha s, \tag{1.1}$$

where s is the scalar curvature of the metric g and $\alpha = (n-2)/4(n-1)$. This operator arises in the well-known formula describing the conformal behavior of scalar curvature: changing the metric conformally to $g_1 = e^{2f}g$ transforms αs to

$$\alpha s_1 = e^{-(n+2)f/2}\Box_g e^{(n-2)f/2}$$

(cf. [3]). Hence $\Box_{g_1} \cdot 1 = e^{-(w+2)f}\Box_g e^{wf}$, where $w = n/2 - 1$. Using this to express $\Box_{g_2} \cdot 1$, $g_2 = e^{2h}g_1$, in terms of $\Box_g$ and $\Box_{g_1}$ leads to

$$e^{-(w+2)f} \circ \Box_g \circ e^{wf} = \Box_{g_1}. \tag{1.2}$$

This formula is best understood in the context of conformal geometry.

Recall that a conformal structure on M is an equivalence class of metrics with $g_0 \sim g_1$ if $g_1 = e^{2f}g_0$ for some $f \in C^\infty(M)$. Equivalently, it is a reduction of the oriented GL(n) frame bundle of T^*M to a bundle F_c whose structure group is the conformal group $CO(n) = \mathbf{R}^* \times SO(n)$. For each $w \in \mathbf{R}$ we have a representation w: $A \mapsto (\det A)^{w/n}$ of $CO(n)$, and hence a trivial real line bundle $L^w = F_c x_w \mathbf{R}$ associated to the conformal structure. Each metric g_0 within the conformal class determines an $SO(n)$ subbundle $F \subset F_c$ and an identification $i_{g,w}$: $L^w \to L^0 = M \times \mathbf{R}$. Changing the metric to $e^{2f}g$ changes F to $e^{2f}F$ (multiplication within the fibers of F_c) and changes $i_{g,w}$ to $e^{-wf}i_{g,w}$. The number w is called the *weight* of L^w and sections of L^w are functions of weight w.

Equation (1.2) shows that when $w = n/2 - 1$ the operator $(i_{g,w+2})^{-1} \circ \Box_g \circ i_{g,w}$ is invariant under conformal changes of metric. Thus setting $w = n/2 - 1$ we obtain a Laplacian $\Box$: $\Gamma(L^w) \to \Gamma(L^{w+2})$ which depends only on the conformal class, and a commutative diagram

$$\begin{array}{ccc} \Gamma(L^w) & \xrightarrow{\ \square\ } & \Gamma(L^{w+2}) \\ \simeq \big\downarrow i_{g,w} & & \simeq \big\downarrow i_{g,w+2} \\ C^\infty(M) & \xrightarrow{\ \square_g\ } & C^\infty(M) \end{array} \tag{1.3}$$

for each metric g in the conformal class.

More generally, if (E, h) is a hermitian vector bundle over M, then a hermitian connection ∇ on E gives a conformal Laplacian

$$\square = (i_{g,w+2})^{-1}(\nabla^*\nabla + \alpha s)i_{g,w}\colon \Gamma(E \otimes L^w) \to \Gamma(E \otimes L^{w+2})$$

and a diagram corresponding to (1.3). Here $\nabla^*\nabla$ is the trace Laplacian of ∇ and g, given in local coordinates by $-(\det g)^{-1/2}\nabla_i g^{ij}(\det g)^{1/2}\nabla_j$. Note that ∇ is taken independent of g; conformal invariance fails if one tries to identify ∇ with the Levi-Civita connection (which is defined when E is associated to the orthogonal frame bundle).

Now suppose that (M, g) is compact. Let $G_g(x, y)$ be the Greens function for $\square_g$. The identification $i_{g,w}\colon L^w \to L^0 = \mathbf{R}$ determines a constant section $|dx|^w = (i_{g,w})^{-1}1 \in \Gamma(L^1)$ with the property that $|dx|^n = dv_g$ is the volume form determined by the metric. The conformal Green's function is the section of the exterior tensor product bundle $L^w \boxtimes L^w$ over $M \times M$ defined by

$$G(x, y) = G_g(x, y)\,|dx|^w \boxtimes |dy|^w.$$

For each $h \in \Gamma(L^{w+2})$ in the image of $\square$ the convolution taking h to $f(x) = \int G(x, y)h(y)$ is a well-defined operator $\Gamma(L^{w+2}) \to \Gamma(L^w)$ which solves the equation $\square f = h$. The conformal invariance of $G(x, y)$ implies that

$$G_{e^{2f}g}(x, y) = G_g(x, y) \cdot \exp[-w(f(x) + f(y))]. \tag{1.4}$$

Most of our theorems will involve the heat kernel of $\square_g$. This is the distribution $k_g(t, x, y)$ on $[0, \infty) \times M \times M$ which satisfies

$$\begin{aligned} &(\partial_t + \square_g)k_g(t, x, y) = 0, \qquad t > 0, \\ &k_g(0, x, y) = \delta(x, y). \end{aligned} \tag{1.5}$$

As is well known (cf. [13], [15]) k_g exists, is smooth for $t > 0$, satisfies $k(t, x, y) = k(t, y, x)$, and is a semigroup in t under convolution. For $\phi \in C^\infty((0, t] \times M)$

$$\begin{aligned} \phi(t, x) = {} & \int ds \int_M k(t-s, x, z)(\partial_s + \square)\phi(s, z)\, dv(z) \\ & + \lim_{\delta \to 0} \int k(t-\delta, x, z)\phi(\delta, z)\, dv(z). \end{aligned} \tag{1.6}$$

This is seen by writing the first integral on the right as a limit of integrals over $[\delta, t-\delta] \times M$, integrating by parts and using (1.5).

The following variational formula will be important later.

Lemma 1.1. *Let $g_\varepsilon = e^{2\varepsilon f} g$ be a smooth 1-parameter family of metrics, and $k_\varepsilon(t,x,y)$ the heat kernels for $\partial_t + \square_{g_\varepsilon}$. Then*

$$\frac{d}{d\varepsilon} k_\varepsilon(t,x,y)|_{\varepsilon=0} = -\int_0^t ds \int_M f(z)[k_0(t-s,x,z)\partial_s k(s,z,y) + \leftrightarrow]\, dv(z), \tag{1.7}$$

where $\leftrightarrow$ denotes the same term with x and y reversed.

Proof. Set $\square'_\varepsilon = e^{-w\varepsilon f}\square e^{w\varepsilon f}$. Then $\square_\varepsilon = e^{-2\varepsilon f}\square'_\varepsilon$ by (1.2). Differentiating the expression $(\partial_t + \square_\varepsilon)k_\varepsilon = 0$ gives

$$(\partial_t + \square)\dot{k} = 2f\square k_0 + \dot{\square}'_\varepsilon k_0 = -2f\partial_t k_0 - w(f\square - \square f)k_0,$$

where the dot denotes $d/d\varepsilon$ at $\varepsilon = 0$. For $\phi = \dot{k}$, the last term in (1.6) vanishes, as can be seen by using the product rule for $d/d\varepsilon$ and the convolution property of k. We can then symmetrize (1.6) in x and y, noting that $\dot{k}(t,x,y) = \frac{1}{2}[\dot{k}(t,x,y) + \dot{k}(t,y,x)]$. This gives (1.7) plus the additional term

$$-w\int_0^t ds \int_M \{k(t-s,x,z)[f(z)\square - \square f(z)]k(s,z,y) + \leftrightarrow\}\, dv(z),$$

which vanishes since $\square$ is selfadjoint.

2. A local invariant

Near the diagonal of $M \times M$ the heat kernel k of $\square_g$ on a hermitian vector bundle E with compatible connection ∇ has a parametrix

$$p_L(t,x,y) = (4\pi t)^{-n/2} e^{-r^2/4t} \sum_{k=0}^{L} a_k(x,y) t^k, \tag{2.1}$$

where $r = \text{dist}(x,y)$, and for each L and $t \leqslant 1$

$$\begin{aligned} |(k-p_L)(t,x,y)| &\leqslant c_1 t^{L-n/2+1} \exp(r^2/t), \\ |\partial_t(k-p_L)(t,x,y)| &\leqslant c_1 t^{L-n/2} \exp(r^2/t) \end{aligned} \tag{2.2}$$

for some $c_1 > 0$ (cf. [13], [15]). On the diagonal, the coefficients $a_k(x,x) \in \Gamma(\text{End}(E))$ are regular local invariants of weights $-2k$ as in [1]. This means that: (i) $a_k(x,x)$ is given by a universal expression in the metrics of M and E, the connection coefficients of ∇, and their derivatives, (ii) a_k is invariant under the action of the group of all bundle automorphisms, (iii) when the metric g is scaled to $\lambda^2 g$, $\lambda \in \mathbf{R}$, then a_k changes to $\lambda^{-2k} a_k$.

Now the $a_k(x, x)$ are invariants of the *Riemannian* structure of M constructed from the conformal Laplacian, and one might ask whether they are *conformal* invariants, i.e. does (iii) hold when λ is a smooth function? Unfortunately this is not the case because the distance function r—which is crucial to the expansion (2.1)—is not well behaved under conformal changes of metric. Indeed, while $a_0(x, x) = 1$ is invariant, $a_1(x, x)$ is a multiple of the scalar curvature so is not conformally invariant. There is, however, one notable exception:

Theorem 2.1. *Suppose that* $n = \dim M$ *is even and let* $w = n/2 - 1$. *Then the* End(E)*-valued density* $a_w(x) = a_w(x, x)\, dv_g(x)$ *is conformally invariant of weight* -2, *i.e.*,

$$(2.3) \qquad \bar{a}_w(x) = e^{2f(x)} a_w(x),$$

where the left-hand side is computed from the metric $\bar{g} = e^{2f} g$.

Remark. In contrast, the heat kernel expansion (2.1) for the usual Laplacian d^*d contains *no* conformal invariants: in the conformal normal coordinate systems used in [12] the heat kernel of d^*d agrees with the euclidean heat kernel to arbitrarily high order.

Before proving Theorem 2.1 we note that there is a close relationship between the heat kernel parametrix (2.1) and the parametrix of the Green's operator (cf. [8]). We shall give an elementary proof below. Consider a general second order selfadjoint elliptic operator $D = \nabla^*\nabla$ + (lower order terms) on $\Gamma(E)$ over a compact Riemannian manifold M. Standard elliptic theory shows that there is a complete basis of $L^2(E)$ consisting of smooth eigenfunctions $\{\phi_i\}$ of D whose eigenvalues are real, discrete, and bounded below. Hence $L^2(E)$ decomposes as the direct sum of the finite dimensional spaces $L^2_-(E)$, $L^2_0(E)$ spanned by the eigenfunctions with negative and zero eigenvalues, and the infinite dimensional space $L^2_+(E)$ of eigenfunctions with positive eigenvalues. The inverse of D: $L^2_+(E) \oplus L^2_-(E) \to L^2_+(E) \oplus L^2_-(E)$ is then given by convolution with the Green's function $G(x, y) = G^+(x, y) + G^-(x, y)$, where

$$(2.4) \qquad G^+(x, y) = \sum_{\lambda_k > 0} \lambda_k^{-1} \phi_k(x) \otimes \phi_k^*(y)$$

and G^- is the corresponding sum over $\{\lambda_k < 0\}$. Similarly the heat kernel $k(t, x, y)$ of D is $k = k^+ + k^0 + k^-$, where

$$(2.5) \qquad k^+(t, x, y) = \sum_{\lambda_k > 0} e^{-t\lambda_k} \phi_k(x) \otimes \phi_k^*(y)$$

and k^-, k^0 are the corresponding sums over $\{\lambda_k < 0\}$ and $\{\lambda_k = 0\}$.

Theorem 2.2. *In a neighborhood of the diagonal of $M \times M$, the Green's function of D has an expansion*

$$G(x, y) = \begin{cases} P(x, y) + \Phi_0(x, y), & n \text{ odd}, \\ P(x, y) - (4\pi)^{-n/2} a_w(x, y) \log r^2 + \Phi_1(x, y), & n \text{ even}, \end{cases}$$

where Φ_0 and Φ_1 are bounded, $w = n/2 - 1$, and

$$(2.6)\quad P(x, y) = (4\pi)^{-n/2} \sum_{k=0}^{[(n-3)/2]} a_k(x, y)\left(\frac{r}{2}\right)^{2-n+2k} \Gamma\left(\frac{n}{2} - k - 1\right).$$

Proof. First note that k^- and k^0 have asymptotic expansions as $t \to 0$ which involve only nonnegative powers of t; hence $k^+(t, x, y)$ has a parametrix which agrees with (2.1) for $L < n/2$. Now (2.4) and (2.5) imply that

$$(2.7)\qquad G^+(x, y) = \int_0^\infty k^+(t, x, y)\, dt.$$

For $L = [n/2 - 1]$:

(a) The function $\phi_1(x, y) = \int_0^{1/4} (k^+ - p_L)(t, x, y)\, dt$ is bounded by (2.2).

(b) Setting $\lambda = r^2/4t$ we have

$$\int_0^{1/4} t^{-1} \exp(-r^2/4t)\, dt = -\log r^2 + \phi_2(x, y),$$

where

$$\phi_2(x, y) = \int_1^\infty \lambda^{-1} \exp(-\lambda)\, d\lambda + \int_{r^2}^1 \lambda^{-1}(\exp(-\lambda) - 1)\, d\lambda$$

is a smooth function.

(c) The function $\phi_3(x, y) = \int_{1/4}^\infty k^+(t, x, y)\, dt$ satisfies $(D_x + D_y)\phi_3 = 2k^+(1/4, x, y)$. Since $k^+(1/4, x, y)$ is a smooth function on $M \times M$, elliptic theory implies that $\phi_3(x, y) \in C^\infty(M \times M)$.

(d) Set $N = [(n - 3)/2]$. It is straightforward to check that $\phi_4(x, y) = \int_{1/4}^\infty P_N(t, x, y)\, dt$ is bounded and that $\int_0^\infty P_N(t, x, y)\, dt$ is the function $P(x, y)$ of (2.6).

Now when n is odd, $N = L$ and (2.7) and (a)–(d) above give $G(x, y) = P(x, y) + \Phi_0(x, y)$, where $\Phi_0 = G^- + \phi_1 + \phi_3 - \phi_4$ is a bounded function. When n is even, $L = N + 1$ and we similarly obtain

$$G(x, y) = P(x, y) - (4\pi)^{-n/2} a_w(x, y) \log r^2 + \Phi_1(x, y),$$

where $\Phi_1 = G^- + \phi_1 + \phi_2(4\pi)^{-n/2} a_w(x, y) + \phi_3 - \phi_4$ is bounded.

Proof of Theorem 2.1. Fix $x \in M$. We know (2.3) holds when f is constant, so we may assume that $f(x) = 0$. The metrics g and $\bar{g}$ are then equal at x. Fixing an orthonormal frame of T_xM and applying the exponential maps of g

and $\bar{g}$ gives normal coordinate charts $\varepsilon, \bar{\varepsilon}$: $\mathbf{R}^n = T_xM \to U\,M$ with $\bar{\varepsilon}\varepsilon^{-1} = \mathrm{Id} + O(r^2)$. From (1.4) we have

$$0 = \varepsilon^*\left[G_g(x, y)\exp[wf(y)] - (\bar{\varepsilon}^{-1})^*\bar{\varepsilon}^*G_{\bar{g}}(x, y)\right]$$

while Theorem 2.2 gives

$$\varepsilon^*G_g(x, y) = \varepsilon^*P(x, y) - (4\pi)^{-n/2}\varepsilon^*a_w(x, y)\log r^2 + O(1)$$

and a similar formula for $\bar{\varepsilon}^*G_{\bar{g}}(x, y)$. Now combine these, expanding $\exp(wf)$, $\varepsilon^*a_k(x, y)$, and $\varepsilon^*\bar{a}_k(x, y)$ in their Taylor series at x. The result is

$$r^{2-n}Q - (4\pi)^{-n/2}[a_w(x, x) - \bar{a}_w(x, x)]\log r^2 = O(1),$$

where $r^2 = |y|^2$ and Q is a *polynomial* in y. This implies that $a_w(x, x) = \bar{a}_w(x, x)$.

3. A global invariant

Let $\{a_k\}$ be the heat kernel coefficients of the conformal Laplacian on a bundle E. In this section we will show that on an even dimensional manifold, $\int \mathrm{tr}\, a_{n/2}(x, x)\, dv_g$ is a conformal invariant (here tr denotes the trace in the fiber of E at x).

To fix notation, let (M^n, g) be a compact Riemannian manifold and let $\mathcal{O}_g = \{e^{2f}g | f \in C^\infty(M)\}$ be the set of metrics conformal to g. For each k, the formula $J_k(g) = \int_M \mathrm{tr}\, a_k(x, x)\, dv_g$ defines a function J_k: $\mathcal{O}_g \to \mathbf{R}$.

Theorem 3.1. *The differential of* J_k: $\mathcal{O}_g \to \mathbf{R}$ *is* $(n - 2k)\mathrm{tr}\, a_k$, *i.e.*

$$\delta_f J_k = \frac{d}{d\varepsilon} J_k\left(e^{2\varepsilon f}g\right)\Big|_{\varepsilon=0} = (n - 2k)\int_M f \cdot \mathrm{tr}\, a_k.$$

In particular, if n is even, then $J_{n/2} = \int_M \mathrm{tr}\, a_{n/2}$ *is a conformal invariant.*

This theorem is proved by calculating the term-by-term variation in the heat kernel asymptotics. A direct approach requires careful analysis (cf. [5], [7]). We will instead use the zeta function which is needed in §5. It is defined by

$$\zeta(s) = \sum_{\lambda_i \neq 0} |\lambda_i|^{-s},$$

where $\{\lambda_i\}$ are the eigenvalues of $\square$ counted with multiplicity. Taking the Mellin transform gives

$$\zeta(s) = \sum_{\lambda_i < 0} |\lambda_i|^{-2} + \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\int_M \mathrm{tr}\, k^+(t, x, x)\, dx\, dt, \tag{3.1}$$

where k^+ is the heat kernel (2.5). It follows that $\zeta(s)$ is analytic for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to $\mathbf{C}$ with only simple poles (cf. [15]).

Now write $k^+ = k - k^0 - k^-$ as in (2.5). Noting that

$$\int \operatorname{tr}(k^0 + k^-)(t,x,x)\,dx = \sum_{\lambda_i \leqslant 0} e^{-\lambda_i t} = \#\{\lambda_i \leqslant 0\} + O(t),$$

and that $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1, we have

$$\begin{aligned} \zeta(0) &= \#\{\lambda < 0\} \\ &\quad + \operatorname{Res}_0 \int_0^1 t^{s-1}\,dt \left[\int_M \operatorname{tr} k(t,x,x)\,dx - \#\{\lambda_i \leqslant 0\} + O(t) \right] \\ &= \operatorname{Res}_0 \int_0^1 t^{s-1} \int_M \operatorname{tr} k(t,x,x)\,dx\,dt - \#\{\lambda_i = 0\}. \end{aligned} \tag{3.2}$$

Substituting in the parametrix (2.1) gives

$$\zeta(0) = (4\pi)^{-\frac{n}{2}} \int_M \operatorname{tr} a_{n/2} - \#\{\lambda_i = 0\}. \tag{3.3}$$

Similarly, one finds (cf. [16]) that when $n = \dim M$ is even

$$\begin{aligned} \operatorname{Res}_k \zeta(s) &= (4\pi)^{-\frac{n}{2}} \frac{1}{(k-1)!} \int_M \operatorname{tr} a_{\frac{n}{2}-k}, \qquad k = 1, 2, \cdots, \frac{n}{2}, \\ \zeta(-k) &= (-1)^k k! (4\pi)^{-\frac{n}{2}} \int_M \operatorname{tr} a_{\frac{n}{2}+k} + \left(1 - \frac{1}{k!}\right) \sum_{\lambda_i < 0} |\lambda_i|^k, \qquad k \in \mathbf{Z}^+, \end{aligned} \tag{3.4}$$

and when n is odd

$$\operatorname{Res}_{n/2-k} \zeta(s) = (4\pi)^{-\frac{n}{2}} \left[\Gamma\left(\frac{n}{2} - k\right)\right]^{-1} \int_M \operatorname{tr} a_k, \qquad k = 0, 1, \cdots. \tag{3.5}$$

We can now prove Theorem 3.1. First suppose n is even. Because $\dim \ker \square$ is conformally invariant (by 1.2) equations (3.3) and then (3.2) give

$$\delta_f J_{n/2} = (4\pi)^{n/2} \delta_f \zeta(0) = (4\pi)^{n/2} \cdot \operatorname{Res}_0 \int_0^1 t^{s-1} \int_M \operatorname{tr} \delta_f k(t,x,x)\,dx\,dt.$$

From (1.7)

$$\delta_f \int_M \operatorname{tr} k(t,x,x)\,dx = \int_0^t ds \int_M -2f(z) \int_M \langle k(t-s,x,z), \partial_s k(s,z,x) \rangle_x,$$

where $\langle\ ,\ \rangle$ is the inner product in the fibers of E. The innermost integral on the right is

$$\begin{aligned} \operatorname{tr} e^{-(t-s)\square} \partial_s k(s,z,x) &= -\operatorname{tr} e^{-(t-s)\square} \square k(s,z,x) \\ &= -\operatorname{tr} \square e^{-(t-s)\square} k(s,z,x) = \operatorname{tr} \partial_t k(t,z,z). \end{aligned}$$

Hence

$$\delta_f J_{n/2} = (4\pi)^{n/2} \operatorname{Res}_0 \int_0^1 t^s \int_M -2f(z) \operatorname{tr} \partial_t k(t,z,z)\,dz\,dt.$$

Finally, plugging in the parametrix (2.1) yields

$$\text{(3.6)}\quad \delta_f J_{n/2} = \operatorname{Res}_0 \int_M -2f(x) \sum_{k=0}^{N} \frac{n-2k}{s+k-n/2} \operatorname{tr} a_k(x,x)\, dx = 0.$$

This computation is easily modified to give

$$\delta \operatorname{Res}_k \zeta(s) = (4\pi)^{-n/2} \frac{2k}{(k-1)!} \int_M f \operatorname{tr} a_{n/2-k}, \qquad k = 1, 2, \cdots, \frac{n}{2},$$

$$\text{(3.7)}\quad \delta\zeta(-k) = (-1)^k k! (4\pi)^{-n/2} (-2k) \cdot \int_M f \operatorname{tr} a_{n/2+k} + \left(1 - \frac{1}{k!}\right) \delta \sum_{\lambda_i < 0} |\lambda_i|^k, \qquad k \in \mathbf{Z}^+,$$

if dim M is even, and

$$\text{(3.8)}\quad \delta \operatorname{Res}_{n/2-k} \zeta(s) = (4\pi)^{-\frac{n}{2}} \left[\Gamma\left(\frac{n}{2} - k\right)\right]^{-1} (n-2k) \int_M f \cdot \operatorname{tr} a_k, \qquad k = 0, 1, 2, \cdots,$$

if dim M is odd. The theorem follows from equations (3.4)–(3.8).

4. Conformal invariants: Examples

In this section we will explicitly calculate the conformal invariants of Theorems 2.1 and 3.1 for dimensions 2, 4, 6, and 8. These invariants are polynomials in the curvature tensor R_{ijkl} $(= \langle \partial/\partial x_k, (\nabla_i \nabla_j - \nabla_j \nabla_i) \partial/\partial x_l \rangle$ in our conventions) and its covariant derivatives. Few such conformally invariant curvature polynomials are known. However the results of §2 give new invariants of this type. We will write down the simplest of these.

For $n > 4$ the space of curvature tensors decomposes under the action of $O(n)$ into three irreducible pieces corresponding to the scalar curvature $s = R^{ij}{}_{ij}$, the traceless Ricci curvature $B_{ij} = R^k{}_{ikj} - (s/n) g_{ij}$, and the Weyl curvature

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2} (g_{jl} B_{ik} - g_{jk} B_{il} + g_{ik} B_{jl} - g_{il} B_{jk}) - \frac{s}{n(n-1)} (g_{jl} g_{ik} - g_{jk} g_{il}).$$

The Weyl curvature is conformally invariant, and we can construct conformally invariant scalars by completely contracting powers of W_{ijkl}; examples include $|W|^2 = W^{ijkl} W_{ijkl}$ and the invariants

$$B_{16} = W_{ijkl} W_{klpq} W_{pqij}, \qquad B_{17} = W_{ijkl} W_{ipkq} W_{jplq}$$

which will appear frequently (here, and below, repeated lower indices have been contracted with the metric). Recently Fefferman and Graham [9] have shown the existence of a series of increasingly complicated conformally invariant scalars. They explicitly computed the first of these; in dimension n it is

$$\Omega_n = |\nabla W|^2 + \frac{8}{n-2}\Big[4B_{17} + B_{16} - \frac{1}{n-1}s|W|^2 + \langle W, \Delta W\rangle$$
$$+\frac{n-10}{n-2}\left(\nabla_k B_{ij}\nabla_i B_{kj} - |\nabla B|^2\right)\Big] + \frac{2(n-10)}{n^2(n-1)}|\nabla s|^2, \tag{4.1}$$

where ΔW is the tensor with components $W_{ijkl,pp}$ (this also appears in [11]).

Theorem 2.1 also gives a series of conformally invariant scalars $a_{n/2-1}(x,x)$. Moreover, when applied to the conformal Laplacian on a hermitian vector bundle E, Theorem 2.1 shows the existence of a series of conformally invariant scalars $\operatorname{tr} A_{n/2-1}(x,x)$ naturally associated to the conformal structure of M and the hermitian connection ∇ on E. These are polynomials in the Riemannian curvature, the curvature $F_{ij} \in \operatorname{End}(E)$ of ∇, and their covariant derivatives. Since $\operatorname{tr} A_{n/2-1}$ is a multiple of $a_{n/2-1}$ plus terms involving F_{ij}, the sum of the terms involving F_{ij} is itself conformally invariant (cf. (4.5)). The first such example is rather obvious (it is essentially $|F|^2$ when $n=6$). As we will see below, the next example ($n=8$) has the form

$$a_1\langle F, \Delta F\rangle + a_2|\nabla F|^2 + a_3 \operatorname{tr}\nabla_j F_{ij}\nabla_k F_{ik} + a_4 s|F|^2 \tag{4.2}$$

for some constants a_i. It is natural to guess that this is the $n=8$ case of a conformal invariant $\hat{\Omega}_n$ which, like the Fefferman-Graham invariant, exists in all dimensions. Assuming this, we can find the a_i as follows.

Under a conformal change of metric $g \mapsto \bar{g} = e^{2f}g$ the bundle curvature F is unchanged and the Christoffel symbols become

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\partial_k f - \delta^i_k\partial_j f - g_{jk}g^{il}\partial_l f.$$

Computations then show that

$$\langle F, \Delta F\rangle \mapsto e^{-6f}[\langle F, \Delta F\rangle - (2n-6)A + (n-4)B - (2n-8)C - 4D + 2E],$$
$$|\nabla F|^2 \mapsto e^{-6f}\left[|\nabla F|^2 + 6A - 6B + (2n-8)C + 4D\right],$$
$$\nabla_j F_{ij}\nabla_k F_{ik} \mapsto e^{-6f}\left[\nabla_j F_{ij}\nabla_k F_{ik} + (n-4)^2 C + (2n-8)D\right],$$
$$s|F|^2 \mapsto e^{-6f}\left[s|F|^2 - (n-1)(n-2)A + 2(n-1)E\right],$$

where $A = |df|^2|F|^2$, $B = \langle df \cdot F, \nabla F\rangle$, $C = |\partial_i f F_{ik}|$, $D = \partial_i f F_{ij} \nabla_k F_{kj}$, and $E = d{*}df \cdot |F|^2$. Requiring that (4.2) be conformally invariant gives an overdetermined system of equations for the a_i. These have a solution, giving us a new conformal invariant:

Theorem 4.1. *In dimension* $n \geqslant 2$

$$\hat{\Omega}_n = (n-4)\langle F, \Delta F\rangle + \frac{(n-4)^2}{6}|\nabla F|^2 + \frac{10-n}{3}\operatorname{tr}\nabla_j F_{ij}\nabla_k F_{ik} - \frac{n-4}{n-1}s|F|^2 \tag{4.3}$$

is a conformally invariant scalar of weight 6.

In dimension 4, $\hat{\Omega}_4 = 2\nabla_j F_{ij} \nabla_k F_{ik} = 8|D^*F|^2$ is a multiple of the square of the norm of the gradient of the (conformally invariant) Yang-Mills action $\int |F|^2$. In dimension 10, $|F|^2$ is conformally invariant of weight 4 and $\hat{\Omega}_{10} = -3\Box|F|^2$.

We can now calculate the heat kernel coefficients $a_i(x,x)$ for the conformal Laplacian (1.1). According to Gilkey [10]

$$\begin{aligned} a_0(x,x) &= 1, \\ a_1(x,x) &= \frac{4-n}{12(n-1)}s, \\ a_2(x,x) &= \frac{1}{180}\left[|W|^2 - \frac{n-6}{n-2}|B|^2 + \frac{3(n-6)}{2(n-1)}\Delta s \right. \\ &\qquad \left. + \frac{(n-6)(5n^2-18n+4)}{8n(n-1)^2}s^2\right]. \end{aligned} \tag{4.4}$$

There is a similar expression for $a_3(x,x)$ involving 17 curvature invariants; it is computed in the appendix. The formulas become even longer when one turns to the conformal Laplacian on a vector bundle of rank k. As above we denote the heat kernel coefficients for the bundle Laplacian by $A_i(x,y)$ and those of the scalar Laplacian by $a_i(x,y)$. Gilkey's results show that the traces of the $A_i(x,x)$ are:

$$\begin{aligned} &\operatorname{tr} A_0(x,x) = k, \qquad \operatorname{tr} A_1(x,x) = k a_1(x,x), \\ &\operatorname{tr} A_2(x,x) = k a_2(x,x) + \frac{1}{12}|F|^2, \\ &\operatorname{tr} A_3(x,x) = k a_3(x,x) + \frac{1}{180}\left[6\langle F, \Delta F\rangle + 4|\nabla F|^2 \right. \\ &\qquad + \operatorname{tr}\nabla_j F_{ij}\nabla_k F_{ik} - 6\operatorname{tr} F_{ij}F_{jk}F_{kl} + 2\operatorname{tr} W_{ijkl}F_{ij}F_{kl} \\ &\qquad \left. + \frac{16-2n}{n-2}\operatorname{tr} B_{ij}F_{ik}F_{jk} - \frac{(n-4)(5n+8)}{4n(n-1)}s|F|^2\right]. \end{aligned} \tag{4.5}$$

Comparing (4.1), (4.3)–(4.5), and (A.1) yields the following surprisingly simple formulas.

Proposition 4.2. *The heat kernel coefficient* $A_{n/2-1}(x,x)$ *for the conformal Laplacian on a bundle of rank k satisfies*

$$
\begin{aligned}
&\operatorname{tr} A_1(x,x) = 0, && n=4,\\
&\operatorname{tr} A_2(x,x) = \frac{k}{180}|W|^2 + \frac{1}{12}|F|^2, && n=6,\\
&\operatorname{tr} A_3(x,x) = \frac{k}{9\cdot 7!}(81\Omega_8 - 352B_{17} - 64B_{16})\\
&\qquad + \frac{1}{120}\left(\hat{\Omega}_8 - 4\operatorname{tr} F_{ij}F_{jk}F_{kl} + 2W_{ijkl}F_{ij}F_{kl}\right), && n=8.
\end{aligned} \tag{4.6}
$$

Each term in these expressions is a conformally invariant scalar. The heat coefficients for the conformal Laplacian on functions are obtained by setting $k=1$ and $F=0$ in (4.6).

The integral invariant of Theorem 3.1 involves the Euler characteristic of M, which is

$$
\begin{aligned}
\chi(M) &= (4\pi)^{-2}\cdot\frac{1}{2}\int_M |W|^2 - 2|B|^2 + \frac{1}{6}s^2,\\
\chi(M) &= (4\pi)^{-3}\cdot\frac{1}{12}\int_M -16B_{17} + 4B_{16} - 12B_{ij}W_{iklm}W_{jklm}\\
&\qquad + 6B_{ij}B_{kl}W_{ikjl}q + 3B_{ij}B_{jk}B_{ki} + \frac{2}{5}s|W|^2 - \frac{6}{5}s|B|^2 + \frac{4}{75}s^3
\end{aligned} \tag{4.7}
$$

in dimensions 4 and 6 respectively. From (4.4), (4.5), (4.7), (A.2), and (A.3) we obtain

Proposition 4.3. *The invariant* $I_{n/2} = (4\pi)^{-n/2}\int_M \operatorname{tr} A_{n/2}(x,x)\,dv_g$ *of Theorem* 3.1 *is*

$$
\begin{aligned}
I_1 &= \frac{1}{6}\chi(M),\\
I_2 &= \left[\frac{(4\pi)^{-2}}{120}\int_M k|W|^2 + 10|F|^2\right] - k\frac{\chi(M)}{180},\\
I_3 &= \frac{-k}{9\cdot 7!}\left[30\chi(M) - (4\pi)^{-3}\int_M 54\Omega_6 + 204B_{17} + 47B_{16}\right]\\
&\qquad + \frac{1}{720}(4\pi)^{-3}\int_M 9\hat{\Omega}_6 - 32\operatorname{tr} F_{ij}F_{jk}F_{kl} + 2\operatorname{tr} W_{ijkl}F_{ij}F_{kl}.
\end{aligned} \tag{4.8}
$$

Again, each term is conformally invariant.

The result of Proposition 4.3 is particularly interesting when $n = 4$. On an oriented 4-manifold the curvature tensors W_{ijkl} and F_{ij} decompose into self-dual and anti-self-dual pieces $W^{\pm}_{ijkl}$ and $F^{\pm}_{ij}$. The Pontrjagin numbers of M and E are then given by

$$p_1(M) = \frac{1}{4\pi^2}\int_M |W^+|^2 - |W^-|^2\, dv_g, \quad p_1(E) = \frac{1}{8\pi^2}\int_M |F^+|^2 - |F^-|^2\, dv_g.$$

We can always orient M and E so that $p_1(M) \geqslant 0$ and $p_1(E) \geqslant 0$. From (4.8) we obtain:

Theorem 4.4. *The heat kernel for the conformal Laplacian on a rank k vector bundle E over a compact orientable Riemannian 4-manifold has an asymptotic expansion satisfying* $\int k(t, x, x) = (4\pi t)^{-2}\operatorname{vol}(M) + I_2 + O(t)$, *where*

$$\begin{aligned} I_2 &= \frac{1}{16\pi^2}\int_M \operatorname{tr} A_2(x, x) \\ &= \frac{1}{960\pi^2}\int_M k|W^-|^2 + 10|F^-|^2 \\ &\quad + \frac{k}{480}\left[|p_1(M)| - \frac{8}{3}\chi(M)\right] + \frac{1}{6}|p_1(E)|. \end{aligned}$$

Hence $480\, I_2 \geqslant k[|p_1(M)| - \frac{8}{3}\chi(M)] + 80|p_1(E)|$ *for any metric and connection on $E \to M$, and equality holds if and only if $E \to M$ (oriented appropriately) is a self-dual bundle over a self-dual* 4*-manifold.*

Self-duality is, of course, a conformally invariant condition on 4-manifolds.

5. Functional determinants and nonlocal invariants

Consider a general bundle conformal Laplacian $\square$ with ν negative eigenvalues (counted with multiplicity). Motivated by the observation that $-\zeta'(0)$ is formally $\sum \log \lambda_i$, one defines the determinant of $\square$ by

$$\det\square = \begin{cases} (-1)^{\nu} e^{-\zeta'(0)} & \text{if } \ker\square = \{0\}, \\ 0 & \text{if } \ker\square \neq \{0\}. \end{cases}$$

The determinant is a smooth function of the metric (cf. [14]). This is seen by choosing an $a > 0$ not in the spectrum and writing

$$\zeta(s) = \sum_{\lambda_i < a} |\lambda_i|^{-s} + \sum_{\lambda_i > a} \lambda_i^{-s};$$

both summands are then locally smooth functions even near metrics with nontrivial kernel.

In this section we will show that $\det\Box$ is a conformal invariant on odd dimensional manifolds. For even dimensional spin manifolds, we will introduce conformal Laplacians $\Box_L$ and $\Box_R$ on left and right spinors, and show that the variation of $\det\Box_L/\det\Box_R$ is a conformally invariant one-form on each conformal class of metrics. For special metrics, this ratio is itself a conformal invariant.

We first calculate the variation of $\zeta'(0)$ for the conformal Laplacian $\Box$ on a vector bundle E. Let

$$\operatorname{tr} k(t,x,x) \sim (4\pi)^{-n/2}\sum_k \operatorname{tr} a_k(x) t^{k-n/2}$$

as top dimensional forms. For an orthonormal basis $\{\psi_i\}$ of $\ker\Box$, define the "local Betti number" $\beta(x)$ to be $\beta(x) = \Sigma_i|\psi_i(x)|^2\,dv_g(x)$.

Proposition 5.1.

$$\delta_f\zeta'(0) = \begin{cases} \displaystyle\int_M 2f(x)\left[(4\pi)^{-n/2}\operatorname{tr} a_{n/2}(x) - \beta(x)\right], & n \text{ even}, \\ \displaystyle\int_M -2f(x)\beta(x), & n \text{ odd}. \end{cases}$$

Proof. Let F.P. denote the finite part at $s = 0$. Then

$$F.P.[\Gamma(s)\zeta(s)] = F.P.\left[\Gamma(s)(\zeta(0) + s\zeta'(0) + O(s^2))\right].$$

Since $\zeta(s)$ depends smoothly upon the metric within the conformal class and $\delta\zeta(0) = 0$ by Theorem 3.1, we have $\delta\zeta'(0) = \delta(\Gamma(s)\zeta(s))$ at $s = 0$. By (3.1),

$$\Gamma(s)\zeta(s) = \int_0^\infty t^{s-1}\left[\sum_{\lambda_i<0} e^{\lambda_i t} + \int \operatorname{tr} k^+(t,x,x)\right] dt$$

for $\operatorname{Re}(s) \gg 0$, so by Lemma 5.2 below

$$\delta_f\zeta'(0) = \int_0^\infty t^s\left[\int_M -2f\left(\sum_{\lambda_i<0}\partial_t e^{\lambda_i t}|\phi_i(x)|^2 + \partial_t \operatorname{tr} k^+(t,x,x)\right)\right] dt\Bigg|_{s=0}.$$

The integral over M is of exponential decay in t, so we may switch the order of integration and integrate by parts to obtain

$$\begin{aligned}\delta_f\zeta'(0) = &-\int_M 2f\left[t^s\left(\sum_{\lambda_i<0} e^{\lambda_i t}|\phi_i(x)|^2 + \operatorname{tr} k^+(t,x,x)\right)\right]_{t=0}^{t=\infty}\Bigg|_{s=0} \\ &+ \int_M 2f\left[s\int_0^\infty t^{s-1}\left(\sum_{\lambda_i<0} e^{\lambda_i t}|\phi_i(x)|^2 + \operatorname{tr} k^+(t,x,x)\right)\right]\Bigg|_{s=0}.\end{aligned}$$

The integrand of the first integral is zero at $t = \infty$ for all s, and is $O(t^{s-n/2})$ for t finite and hence zero at $t = 0$ for $\mathrm{Re}(s) \gg 0$. The second integral is $\int_M 2f(x)s\Gamma(s)\zeta(s,x)|_{s=0}$, where the local zeta function $\zeta(s,x)$ is defined by

$$\zeta(s,x) = \sum_{\lambda_n \neq 0} |\phi_n(x)|^2 |\lambda_n|^{-s}$$

for $\{\phi_n\}$ an orthonormal basis of the λ_n-eigenspace [16]. The proposition follows as in (3.2) and (3.3).

Lemma 5.2. (a) $\delta\int_M \mathrm{tr}\, k^{\pm}(t,x,x)\,dx = -2t\int_M f(x)\partial_t \mathrm{tr}\, k^{\pm}(t,x,x)\,dx$.

(b) $\delta\sum_{\lambda_i<0} e^{\lambda_i t} = -2t\partial_t\int_M \sum_{\lambda_i<0} f(x)e^{\lambda_i t}|\phi_i(x)|^2\,dx$.

Proof. Let $\{\phi_i(\varepsilon,x)\}$ be a basis of the negative eigenspace of $\Box_{e^{2\varepsilon f}g}$ which is orthonormal with respect to the L^2 inner product $\langle\ ,\ \rangle$ of the metric g. Then $0 = \langle \phi_i, \dot\phi_i\rangle$, where the dot denotes $d/d\varepsilon$ at $\varepsilon = 0$. Hence

$$\begin{aligned}\delta\int_M \mathrm{tr}\, k^-(t,x,x)\,dx &= \delta\sum_{\lambda_i<0}\langle \phi_i, k_t\phi_i\rangle \\ &= \sum_{\lambda_i<0} 2\langle \phi_i, k_t\dot\phi_i\rangle + \langle \phi_i, \dot k_t\phi_i\rangle.\end{aligned}$$

For each i the first term is $2e^{-\lambda_i t}\langle \phi_i, \dot\phi_i\rangle = 0$, and by Lemma 1.1 the second term is

$$-2\int_0^t ds\sum_{\lambda_i<0}\left\langle fe^{-(t-s)\lambda_i}\phi_i, \partial_s e^{-s\lambda_i}\phi_i\right\rangle = -2t\partial_t\sum_{\lambda_i<0}\left\langle f\phi_i, k_t\phi_i\right\rangle.$$

This gives (a) for k^-, and hence for $k^- + k^0$ (since $\delta(\dim\ker\Box) = 0$). We showed in §3 that (a) holds for k, so it holds for k^+ by subtraction. Part (b) follows easily after noting that the above calculation is valid on each negative eigenspace.

Since ν is a conformal invariant, the variation of $\zeta'(0)$ is the variation of $-\log\det\Box$ if $\det\Box \neq 0$. Thus Proposition 5.1 gives $\delta_f\det\Box$ and produces our first invariant: $\det\Box$ on an odd dimensional manifold. (Note that this is nonzero on a conformal class admitting a metric of positive scalar curvature).

Theorem 5.3. (a) *If* $\dim M$ *is odd*, $\det\Box$ *is a conformal invariant.*

(b) *If* $\dim M = n$ *is even*, $\delta_f\det\Box = -(4\pi)^{-n/2}\det\Box\int_M 2f(x)\mathrm{tr}\, a_{n/2}(x)$.

Let $\mathscr{M}$ denote the space of metrics and consider a conformal orbit $\mathscr{O}_g = \{e^{2f}g\colon f \in C^\infty(M)\} \subset \mathscr{M}$. Choose a vector field $X = (Xg, g) \in \Gamma(T\mathscr{O}_g) \simeq C^\infty(M)\times\mathscr{O}_g$. If $\det\Box \neq 0$, (b) above defines a 1-form $w = d\log\det\Box$ on $\mathscr{O}_g$ by

$$\omega(X)(g) = -2(4\pi)^{-\frac{n}{2}}\int_M X_g\cdot \mathrm{tr}\, a_{n/2}. \tag{5.1}$$

Physicists call ω a *conformal anomaly*. What is anomolous is that, naively, one might expect that the determinant of an invariant operator to be invariant. However, $\Box$: $\Gamma(E \otimes L^w) \to \Gamma(E \otimes L^{w+2})$ is an operator between two *different* bundles. Hence the eigenvalues and the determinant depend on an identification $L^w = L^{w+2}$ (which is equivalent to a choice of metric). The 1-form ω measures the variation in this identification.

It has recently been shown that the existence of other similar anomalies (e.g. the chiral anomaly discussed by Atiyah and Singer [2]) is forced by topological considerations. In contrast, the conformal anomaly has no such topological content ($\mathcal{O}_g$ and $\mathcal{M}$ are contractible).

To introduce the Laplacians which produce an invariant in even dimensions, we have to discuss the Yamabe problem. The Yamabe invariant μ of a conformal class $\mathcal{O}$ is

$$\mu = \inf_{g \in \mathcal{O}} \frac{\int S_g \, dv_g}{V_g^{1-2/n}},$$

where V_g is the volume of (M, g). By the solution of the Yamabe problem [3], [18], [12] in each conformal class there is a metric g_0 which realizes this infimum; it has constant scalar curvature $s_0 = \mu V_{g_0}^{2/n}$. Moreover, this metric is unique if $\mu \leqslant 0$ and we normalize by requiring $V_{g_0} = 1$.

Now $C^\infty(M)$ acts on $\mathcal{M}$ via conformal transformations. Since $\mathcal{M}$ and $C^\infty(M)$ are contractible, the fibration $\mathcal{M} \to \mathcal{M}/C^\infty(M)$ admits global sections. Indeed, the solution of the Yamabe problem provides a smooth section over the set of conformal classes with $\mu \leqslant 0$. When $\mu > 0$ the lack of uniqueness in the Yamabe problem complicates matters, but we can still define local sections as follows.

Fix a metric g_0 of constant scalar curvature μ and volume 1 and set $p = 4/n - 2$. For each nearby metric g, the Yamabe problem is equivalent to solving the nonlinear elliptic eigenvalue problem $\Box_\varepsilon f = \alpha\lambda f^{p+1}$ (then $f^p g_\varepsilon$ has constant scalar curvature λ). By linearizing this equation and applying the implicit function theorem one can show that the Yamabe problem can be uniquely solved for volume 1 metrics in a neighborhood of g_0 provided $\frac{1}{4}(n+2)\mu/(n-1)$ is not an eigenvalue of $\Box_{g_0}$. Thus for generic g_0 we obtain a smooth local section σ: $U \to \mathcal{M}$ whose image consists of metrics of volume 1 with constant scalar curvature $s = \mu$ depending smoothly on g.

Now let (M, g) be a compact oriented spin manifold. We will use this neighborhood U to parametrize two smooth families $\Box_L$, $\Box_R$ of conformal Laplacians on the bundles S_L, S_R of left and right spinors. Let $[h]$ denote the conformal class of a metric h. Define a map from U to connections on the frame bundle by

$$[h] \mapsto \text{the Levi-Civita connection for } \sigma[h]$$

This connection lifts to connections ∇_L, ∇_R on S_L, S_R (the spin bundles are constructed with respect to $\sigma[h]$). Define

$$\square_L\colon \Gamma(S_L \otimes L^w) \to \Gamma(S_L \otimes L^{w+2})$$

by

$$\square_L(g) = \nabla_L^* \nabla_L + \frac{n-2}{4(n-1)} s(g)$$

for any $g \in [h]$. Define $\square_R$ similarly.

Thus defined, $\square_L(g)$ and $\square_R(g)$ are nonnegative operators. Indeed, this property is conformally invariant, and holds for metrics of constant scalar curvature because the Lichnerowicz formula $\partial\!\!\!/^2 = \nabla^*\nabla + s/4$ for the square of the Dirac operator implies that

$$\square = \nabla^*\nabla + \frac{n-2}{4(n-1)} s = \partial\!\!\!/^2 - \frac{s}{4(n-1)} \tag{5.2}$$

on S_L and S_R. Furthermore, if $\mu \neq 0$, then $s \neq 0$ in (5.2) and $\square_L$, $\square_R$ are positive operators.

Corollary 5.4. *Let $\mathcal{O}_g$ be a conformal orbit with $\mu \neq 0$ and let $\hat{a}_g(x)$ be the $\hat{A}$-polynomial as a differential form in the curvature. Then in the notation of* (5.1)

$$\left[d \log\left(\frac{\det \square_L}{\det \square_R} \right)(X) \right]_g = -2 \int_M X_g \cdot \hat{a}_g. \tag{5.3}$$

Proof. Apply Proposition 5.1, noting that by Gilkey's Theorem [1] $\eta = \operatorname{tr}[a^L_{n/2}(x) - a^R_{n/2}(x)]$ is a Pontrjagin form and hence a pointwise conformal invariant [4]. We may thus assume that the scalar curvature s is constant. Then (5.2) gives $e^{-t\square} = e^{-t\partial\!\!\!/^2} e^{t\gamma s}$, where γ is a constant, so

$$\eta = \sum_{k+l=n/2} \operatorname{tr}\left(a_k^{\partial\!\!\!/^2_L} - a_k^{\partial\!\!\!/^2_R} \right) \frac{(\gamma s)^l}{l!}.$$

By Gilkey's Theorem again, only the term with $k = n/2$ is nonzero. The heat equation proof of the Atiyah-Singer Index Theorem shows that $(4\pi)^{-n/2}\eta = \hat{a}(x)$ and completes the proof.

Equation (5.3) shows that $d\log(\det\square_L/\det\square_R)$ is a *conformally invariant* 1-form on $\mathcal{O}_g$, dual to $-2\hat{a}(x)$ under the integration pairing. Furthermore, it shows that the $\hat{A}$-polynomial *as a differential form* (not just a cohomology class) arises naturally from the spectral analysis of the operator $\square_L$ and $\square_R$.

Whenever $\mu \neq 0$, $\square_L$, $\square_R$ are invertible and $\det\square_L/\det\square_R$ is a smooth, locally well-defined function of the metric. We will next give an explicit formula which shows that this ratio extends smoothly to the metrics over U with $\mu = 0$.

Theorem 5.5. *Let M be oriented so $\hat{A}(M) \geqslant 0$. The ratio $F(g) = \det\square_L/\det\square_R$ is a smooth locally defined function on the space of metrics. If $g_0 = \sigma([g]) = e^{-2f}g$ is the conformal metric of constant scalar curvature and volume* 1 *then*

$$\frac{\det\square_L}{\det\square_R} = \left|\frac{-\mu}{4(n-1)}\right|^{\hat{A}(M)} \exp\int_M -2f\cdot\hat{a}_g. \tag{5.4}$$

Proof. First suppose that $\mu \neq 0$ (by a theorem of Kazdan and Warner [3, Theorem 6.20] there is an open set of metrics with $\mu < 0$ when $n \geqslant 3$). Integrating (5.3) along the line $g_t = e^{2tf}g_0$, $0 \leqslant t \leqslant 1$, and noting that $\hat{a}(g_t) = \hat{a}(g)$ yields

$$F(g) = F(g_0)\exp\int_M -2f\hat{a}_g. \tag{5.5}$$

By (5.2) the eigenvalues of $(\square_{g_0})_{L,R}$ are those of $\partial\!\!\!/^2$ shifted by $-\mu/4(n-1)$. Since the nonzero eigenvalues of $\partial\!\!\!/^2$ match up ($\partial\!\!\!/$ gives an isomorphism of these eigenspaces),

$$(\zeta_L - \zeta_R)(z) = (N_L - N_R)\left|-\tfrac{1}{4}\mu/(n-1)\right|^{-z},$$

where N_L and N_R are the multiplicities of $\ker\partial\!\!\!/^2$ on S_L and S_R. Hence

$$F(g_0) = \det\square_L/\det\square_R = \left|-\tfrac{1}{4}\mu/(n-1)\right|^{\hat{A}(M)}. \tag{5.6}$$

Finally, when $\mu = 0$ we set $F(g_0) = 0$ if $\hat{A}(M) > 0$ and $F(g_0) = 1$ if $\hat{A}(M) = 0$, and define $F(g)$ by (5.5). This agrees with the limit as $\mu \to 0$ of (5.6) and smoothly extends (5.4) across metrics with $\mu = 0$ (recall that μ depends smoothly on g near g_0).

Remarks. 1. Lichnerowicz's formula implies that $\mu > 0$ can occur only if $\hat{A}(M) = 0$.

2. In dimensions $n \equiv 2 \bmod 4$, the two spin representations of $\mathrm{Spin}(n)$ are conjugate. Hence there is a conjugate linear isomorphsm c: $S_L \to S_R$ between the spin bundles of the constant scalar curvature metrics with $c^2 = 1$ and $\nabla c \equiv 0$. This implies that $\square_L$ and $\square_R$ have the same spectrum, so $\det\square_L/\det\square_R$ is always one for these metrics. Note that $\hat{a}(x) \equiv 0$ in this case.

The conformal invariance of the 1-form (5.3) means that either the function $\det\Box_L/\det\Box_R$ has no critical point on a given conformal class, or else this ratio is itself a conformal invariant. We now give some families of metrics for which the latter holds.

Proposition 5.6. (a) $\det\Box_L/\det\Box_R = 1$ *for a locally conformally flat metric.*

(b) *Let* $M = \Gamma\setminus G/H$ *be a locally homogeneous spin manifold of volume* 1 *whose metric is induced from a G-invariant metric on* G/H. *Then* $\det\Box_L/\det\Box_R$ *is invariant under volume preserving conformal variations, and is equal to* $|-\frac{1}{4}\mu/(n-1)|^{\hat{A}(M)}$.

Proof. (a) If the Weyl tensor vanishes, we have $\hat{a}(x)\equiv 0$ and $\hat{A}(M)=0$, so the right-hand side of (5.4) is 1.

(b) The G-invariance of the metric implies that $\hat{a}(x)$ is constant, and when g_t is a volume-preserving family of metrics the integral of $X=\dot{g}_t$ is zero. Hence (5.3) is zero and (5.5) becomes $F(g)=F(g_0)$, and the proposition follows.

We may also consider conformal Laplacians acting on spinors with values in a bundle E. Set

$$\Box = \nabla^*\nabla + F_{ij}e^i\cdot e^j\cdot + \frac{(n-2)}{4(n-1)}s,$$

where $\{e^i\cdot\}$ is Clifford multiplication by an orthonormal basis of one-forms of E with respect to a bundle metric h, ∇ is a fixed connection compatible with h and a metric g of constant scalar curvature on M, F_{ij} is the curvature of ∇, and s and the adjoint ∇^* are computed with respect to g. This Laplacian is conformally invariant and satisfies $\Box = \partial\!\!\!/_E^2 - \frac{1}{4}s/(n-1)$, so the results of this paper remain valid for $\Box$ (except that $\Box$ may have nonzero kernel if $\mu>0$) by replacing $\hat{a}(x)$ with $\hat{a}(x)\mathrm{ch}(x)$, where ch is the Chern character of E.

Finally if $n=2p$ the signature operator $D=d+d^*$ on p-forms has the Bochner formula

$$D^2 = \nabla^*\nabla - W_{ijkl}a_i^*a_ja_k^*a_l + \frac{n}{4(n-1)}s.$$

Here W is the Weyl tensor, and for an orthonormal frame of one-forms $\{\theta_i\}$ with dual vector fields $\{X_i\}$, $a_i^* = \theta_i\wedge$ and $a_j = \iota_{X_j}$. The space of complex p-forms decomposes into the ± 1 eigenspaces $\Omega^p_\pm$ of $i^{p^2}*$. If we consider

$$\Box_\pm = \nabla^*\nabla - W_{ijkl}a_i^*a_ja_k^*a_l + \frac{(n-2)}{4(n-1)}s$$

acting on $\Omega^p_\pm$ (weighted appropriately), then $\square_\pm = (D_\pm)^2 - \frac{1}{2}s/(n-1)$. Since the operator $L = dd^* - d^*d$ matches up the nonzero eigenspaces of D^2 on $\Omega^p_\pm$, the results of this paper also carry over to $\square_\pm$ (except for the $\mu > 0$ case as above) with $\hat{a}(x)$ replaced by Hirzebruch's L-polynomial $L(x)$. This gives a nontrivial conformal anomaly on nonspin manifolds of dimension $4k$.

Appendix

The results of §4 require an explicit expression for $a_3(x,x)$ in terms of the Weyl, traceless Ricci, and scalar curvatures. This is obtained by a long calculation, summarized here.

Applying Gilkey's results [10] to the conformal Laplacian (1.1), $n > 2$, we obtain $a_3(x,x) = (4\pi)^{-n/2}\Sigma c_i A_i$, where the 17 curvature invariants $\{A_i\}$ and their coefficients $\{c_i\}$ are listed in the first two columns of Table I. (Note that the curvature R_{ijkl} used in [10] differs from ours by a minus sign, and that the heat kernel coefficients tabulated in [10] are for the operator $\nabla^*\nabla - E$, so in our notation $E = -\alpha s$.)

Each A_i is a polynomial in the scalar, Ricci, and full curvature tensors, and there is a corresponding polynomial B_i in the scalar, traceless Ricci, and Weyl tensors. These $\{B_i\}$ are listed in Table II. The formulas converting the $\{A_i\}$ to the $\{B_i\}$ appear in the last column of Table I. Using these, we find that $a_3(x,x) = (4\pi)^{-n/2}(7!)^{-1}\Sigma d_i B_i$, where the $\{d_i\}$ are listed in Table II for dimension n, then for dimensions 6 and 8. Thus for $n = 8$, $a_3(x,x)$ is

$$\text{(A.1)}\qquad \frac{1}{(4\pi)^3\cdot 7!}\left[\frac{-80}{9}B_{17} - \frac{44}{9}B_{16} + \frac{12}{7}B_{12} + 12B_9 + 9B_5 - 4B_4 + 4B_3 - \frac{9}{112}B_2\right].$$

The formula for $a_3(x,x)$ for $n = 6$ (3rd column of Table II) is more complicated, but its integral simplifies because the functions

$$C_1 = B_1,\quad C_2 = B_2 + B_6,\quad C_3 = B_3 + B_7,$$
$$C_4 = B_4 + B_8,\quad C_5 = B_5 + B_9,$$
$$C_6 = \frac{-(n-2)^2}{4n^2}B_2 + B_4 - \frac{1}{n-1}B_{11} - \frac{n}{n-2}B_{13} + B_{14},$$
$$C_7 = \frac{(n-2)(n-3)}{4n^2(n-1)}B_2 - \frac{n-3}{n-2}(B_3 - B_4) + \frac{1}{4}B_5 - \frac{1}{2n}B_{12} - \frac{1}{2}B_{15} + \frac{1}{4}B_{16} + B_{17}$$

TABLE I

$A_1 = \Delta\Delta s$†	$6(3 - 14\alpha)$	B_1
$A_2 = \lvert\nabla s\rvert^2$	$17 - 168\alpha + 420\alpha^2$	B_2
$A_3 = \lvert\nabla R_{ij}\rvert^2$	-2	$B_3 + \frac{1}{n}B_2$
$A_4 = \nabla_i R_{jk}\nabla_k R_{ij}$	-4	$B_4 + \frac{n-1}{n^2}B_2$
$A_5 = \lvert\nabla R\rvert^2$	9	$B_5 + \frac{4}{n-2}B_3 + \frac{2}{n(n-1)}B_2$
$A_6 = s\Delta s$	$28(1 - 11\alpha + 30\alpha^2)$	B_6
$A_7 = R_{ij}\Delta R_{ij}$	-8	$B_7 + \frac{1}{n}B_6$
$A_8 = R_{ij}\nabla_k\nabla_j R_{ik}$	$8(3 - 14\alpha)$	below
$A_9 = \langle R, \Delta R\rangle$	12	$B_9 + \frac{4}{n-2}B_7 + \frac{2}{n(n-1)}B_6$
$A_{10} = s^3$	$70(\frac{1}{18} - \alpha + 6\alpha^2 - 12\alpha^3)$	B_{10}
$A_{11} = s\lvert R_{ij}\rvert^2$	$-\frac{14}{3}(1 - 6\alpha)$	$B_{11} + \frac{1}{n}B_{10}$
$A_{12} = s\lvert R\rvert^2$	$\frac{14}{3}(1 - 6\alpha)$	$B_{12} + \frac{4}{n-2}B_{11} + \frac{2}{n(n-1)}B_{10}$
$A_{13} = R_{ij}R_{jk}R_{ki}$	$-\frac{208}{9} + 112\alpha$	$B_{13} + \frac{3}{n}B_{11} + \frac{1}{n^2}B_{10}$
$A_{14} = R_{ij}R_{kl}R_{ikjl}$	$\frac{64}{3} - 112\alpha$	below
$A_{15} = R_{ij}R_{iklm}R_{jklm}$	$-\frac{16}{3}$	below
$A_{16} = R_{ijkl}R_{klmn}R_{mnij}$	$\frac{44}{9}$	below
$A_{17} = R_{ijkl}R_{ipkq}R_{jplq}$	$\frac{80}{9}$	below

†$\Delta = \nabla_i\nabla_i$, repeated indices are contracted with the metric, and $R = R_{ijkl}$.

$$A_8 = \frac{n}{n-2}B_8 + \frac{1}{2n}B_6 + \frac{2}{n-2}B_{14} - \frac{2n}{(n-2)^2}B_{13} - \frac{2}{(n-1)(n-2)}B_{11},$$

$$A_{14} = B_{14} - \frac{2}{n-2}B_{13} + \frac{2n-3}{n(n-1)}B_{11} + \frac{1}{n^2}B_{10},$$

$$A_{15} = B_{15} + \frac{4}{n-2}B_{14} + \frac{2(n-4)}{(n-2)^2}B_{13} + \frac{1}{n}B_{12} + \frac{4(2n-3)}{n(n-1)(n-2)}B_{11} + \frac{2}{n^2(n-1)}B_{10},$$

$$A_{16} = B_{16} + \frac{12}{n-2}B_{15} + \frac{24}{(n-2)^2}B_{14} + \frac{8(n-4)}{(n-2)^2}B_{13}$$
$$+ \frac{6}{n(n-1)}B_{12} + \frac{24}{n(n-1)(n-2)}B_{11} + \frac{4}{n^2(n-1)^2}B_{10},$$

$$A_{17} = B_{17} - \frac{3}{n-2}B_{15} + \frac{3(n-4)}{(n-2)^2}B_{14} + \frac{2(8-3n)}{(n-2)^3}B_{13}$$
$$- \frac{3}{2n(n-1)}B_{12} + \frac{3(n-4)}{n(n-1)(n-2)}B_{11} + \frac{n-2}{n^2(n-1)^2}B_{10}.$$

TABLE II

Invariant	Coefficient	$n = 6$	$n = 8$
$B_1 = \Delta\Delta s$	$-3(\frac{n-8}{n-1})$	$\frac{6}{5}$	0
$B_2 = \|\nabla s\|^2$	$\frac{5n^4 - 76n^3 + 288n^2 - 128n + 16}{4n^2(n-1)^2}$	$\frac{-4}{45}$	$\frac{-9}{112}$
$B_3 = \|\nabla B_{ij}\|^2$	$-2(\frac{n-20}{n-2})$	7	4
$B_4 = \nabla_i B_{jk} \nabla_k B_{ij}$	-4	-4	-4
$B_5 = \|\nabla W\|^2$	9	9	9
$B_6 = s\Delta s$	$\frac{n-8}{2n(n-1)^2}$	$\frac{-2}{5}$	0
$B_7 = B_{ij}\Delta B_{ij}$	$-8(\frac{n-8}{n-2})$	4	0
$B_8 = B_{ij}\nabla_k\nabla_j B_{ik}$	$\frac{-4n(n-8)}{(n-1)(n-2)}$	$\frac{12}{5}$	0
$B_9 = \langle W, \Delta W\rangle$	12	12	12
$B_{10} = s^3$	$\frac{-(n-8)(35n^4 - 308n^3 + 688n^2 - 184n - 96)}{72n^2(n-1)^3}$	$\frac{2}{135}$	0
$B_{11} = s\|B\|^2$	$\frac{(n-8)(7n^3 - 17n^2 - 2n + 24)}{3n(n-1)^2(n-2)}$	$\frac{-76}{75}$	0
$B_{12} = s\|W\|^2$	$\frac{-(n-4)(7n+16)}{3n(n-1)}$	$\frac{-58}{45}$	$\frac{-12}{7}$
$B_{13} = B_{ij}B_{jk}B_{ki}$	$\frac{4(n-8)(11n^3 - 28n^2 + 32n - 24)}{27(n-1)(n-2)}$	$\frac{-64}{15}$	0
$B_{14} = B_{ij}B_{kl}W_{ijkl}$	$\frac{4(n-8)(5n^2 - 4n - 4)}{3(n-1)(n-2)^2}$	$\frac{76}{15}$	0
$B_{15} = B_{ij}W_{iklm}W_{jklm}$	$-\frac{16}{3}(\frac{n-8}{n-2})$	$\frac{8}{3}$	0
$B_{16} = W_{ijkl}W_{klmn}W_{mnij}$	$\frac{44}{9}$	$\frac{44}{9}$	$\frac{44}{9}$
$B_{17} = W_{ijkl}W_{ipkq}W_{jpkq}$	$\frac{80}{9}$	$\frac{80}{9}$	$\frac{80}{9}$

each satisfy $\int C_i \, dv_g = 0$. One then sees that

$$\text{(A.2)} \quad a_3(x,x) = \frac{1}{(4\pi)^3 \cdot 7!}\left[\frac{-5}{18}\bar{\chi} + 6\Omega_6 + \frac{68}{3}B_{17} + \frac{47}{9}B_{16} + \xi\right],$$

where Ω_n is given by (4.1), $\bar{\chi}$ is the integrand of (4.7), and $5\xi = -6C_1 - 2C_2 + 20C_3 + 12C_4 - 17C_6 + 60C_7$. Similarly, the integral of the heat kernel coefficient $\operatorname{tr} A_3(x,x)$ (equation (4.4)) simplifies because

$$\eta_n = \frac{1}{2}\nabla_i F_{jk}\nabla_i F_{jk} - \nabla_j F_{ij}\nabla_k F_{ik} + F_{ij}F_{jk}F_{ki} + \frac{1}{2}W_{ijkl}F_{ij}F_{kl}$$
$$+ \frac{n-4}{n-2}B_{ij}F_{ik}F_{jk} + \frac{n-2}{n(n-1)}sF_{ij}F_{ij}$$

satisfies $\int \operatorname{tr} \eta_n \, dv_g = 0$. For $n = 6$ we then have

$$\begin{aligned} \operatorname{tr} A_3(x, x) &= k a_3(x, x) \\ &= \frac{1}{720}\left[9\hat{\Omega}_6 - 32 \operatorname{tr} F_{ij} F_{jk} F_{ki} + 2 \operatorname{tr} W_{ijkl} F_{ij} F_{kl}\right] + 2 \operatorname{tr} \eta_6. \end{aligned} \tag{A.3}$$

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