J. DIFFERENTIAL GEOMETRY
23 (1986) 97-107

THE SPECTRAL GEOMETRY OF A TOWER OF COVERINGS

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Let M be a compact Riemannian manifold and let $\{M_i\}$ be a family of finite-sheeted covering spaces of M with the induced Riemannian metric.

In this paper, we wish to study the behavior of the first eigenvalue $\lambda_1(M_i)$ as i tends to infinity. Here λ_1 is given by the variational formula

$$\lambda_1(N_i) = \inf_f \frac{\int_N \|df\|^2}{\int_N |f|^2}, \tag{1}$$

where f ranges over functions which are perpendicular to the constant function ~ $\int_N f = 0$.

It might appear, particularly from the perpendicularity condition, that the behavior of λ_1 depends rather delicately on the metric properties of M. However, motivated by our work on λ_0 of coverings [1], we were led to the point of view that the asymptotic properties of λ_1 as i tends to infinity should be governed only by combinatorial properties of the fundamental group of M.

Our main result, Theorem 1, confirms that this is indeed the case. To state the combinatorial property which emerges, let us first recall that a finite-sheeted covering space M_i of M is described by a subgroup $\pi_1(M_i)$ of finite index in $\pi_1(M)$. We now fix, once and for all, generators $g_1, \cdots, g_k$, and for each i we consider the combinatorial graph Γ_i described as follows: The vertices of Γ_i are the finite number of cosets of $\pi_1(M)/\pi_1(M_i)$. Two vertices of Γ_i are joined by an edge if the corresponding cosets differ by left multiplication by one of the g_i's.

For each i, we let h_i denote the following number: Let $E = \{E_j\}$ be a collection of edges of Γ_i such that $\Gamma_i - E$ disconnects into two pieces, A and B. Let $\#(E)$ denote the number of edges in E, and $\#(A)$ and $\#(B)$ the

Received April 7, 1985 and, in revised form, October 7, 1985. This work was partially supported by National Science Foundation grant DMS-83-15552. The author is an Alfred P. Sloan Fellow.

number of vertices in A and B. We then let h_i denote the infimum

(2) $$h_i = \inf_E \frac{\#(E)}{\min(\#(A), \#(B))}.$$

We then have:

Theorem 1. *$\lambda_1(M_i) \to 0$ as $i \to \infty$ if and only if $h_i \to 0$ as $i \to \infty$.*

The proof of Theorem 1 proceeds much as the proof of the main theorem of [1]. The central point is to make use of Cheeger's inequality [8]

(3a) $$\lambda_1(N) \geqslant \tfrac{1}{4}h^2,$$

where h is the isoperimetric constant of N,

(3b) $$h = \inf_S \frac{\operatorname{area}(S)}{\min(\operatorname{vol}(A), \operatorname{vol}(B))},$$

and S runs over hypersurfaces of N which divide N into two pieces, A and B.

The analogy of (2) with (3b) should be fairly clear. The content of Theorem 1 is then to say that the combinatorial isoperimetric constants h_i and the geometric isoperimetric constants $h(M_i)$ stay within a bounded ratio of each other, independent of i.

The proof of Theorem 1 occupies §1.

The existence of a combinatorial condition as in Theorem 1 then implies a number of consequences, some of which we mention in §3. The idea of estimating λ_1 by graph-theoretic techniques was considered for the case of surfaces by P. Buser [4], [5], where he used this technique in studying the question: Is there a family of compact Riemann surfaces, with area tending to infinity, with λ_1 bounded from below? He later settled this question in [3], as did Vigneras in [15]. Both of their solutions use delicate number-theoretic machinery, including Jacquet-Langlands theory. One of our results in §3 gives a general solution to this problem:

Corollary 6 (§3). *Let M be a compact manifold with a surjection $\pi_1(M) \to \mathbf{Z} * \mathbf{Z}$. Then there are families of arbitrarily large coverings $\{M_i\}$ of M and a constant $c > 0$ such that $\lambda_1(M_i) > c$ for all i.*

Note that the class of manifolds M with a surjective map $\pi_1(M) \to \mathbf{Z} * \mathbf{Z}$ is quite large—it includes all surfaces of genus $\geqslant 2$, and also examples of hyperbolic manifolds of all dimensions (see [3], [12] for a general discussion of this point).

In general, bounding the combinatorial constant of Theorem 1 can be rather subtle. To that end, we prove the following general result in §2.

Theorem 3. *Suppose $\pi = \pi_1(M)$ has Kazhdan's Property* T. *Then there exists a constant $C > 0$ such that $\lambda_1(M') \geqslant C$ for all finite coverings M' of M.*

Actually, somewhat more is true, but we will leave a more detailed discussion of Kazhdan's Property T to §2.

It is a pleasure to thank John Millson and Peter Sarnak for many helpful conversations about this material.

1. Proof of Theorem 1

We now fix a compact manifold M, and consider finite coverings M_i of M. We would like to model the spectral geometry of M_i in terms of the combinatorial data given by the inclusion $\pi_1(M_i) \to \pi_1(M)$.

To that end, let us pick a fundamental domain F for M in the universal cover $\tilde{M}$ of M. Then the deck transformations $g_1, \cdots, g_k$ pairing F with an adjacent fundamental domain clearly generate $\pi_1(M)$. We may now pick a point x in the interior of F, and consider the graph in $\tilde{M}$ whose vertices consist of the orbit of x under deck transformations, and where two vertices are joined by an edge when the corresponding copies of F meet.

The resulting graph is visibly invariant under deck transformations, and so projects onto any covering M_i of M. When M_i is a finite covering space, this is a finite graph, the number of vertices being the degree of the covering, and it is easily seen to be the graph Γ_i described in the introduction.

For any connected graph Γ, we may consider the combinatorial isoperimetric constant

$$h(\Gamma) = \inf_E \frac{\#(E)}{\min(\#(A), \#(B))}$$

as E ranges over sets of edges of Γ such that $\Gamma - E$ disconnects into two pieces, A and B. Here, $\#(A)$ (resp. $\#(B)$) denotes the number of vertices in the piece A (resp. B).

Set $h_i = h(\Gamma_i)$, where Γ_i is the graph defined above.

The main result of this section is:

Theorem 1. *Given a family of coverings M_i of M, $\lambda_1(M_i) \to 0$ as $i \to \infty$ if and only if $h_i \to 0$ as $i \to \infty$.*

The proof of Theorem 1 closely follows the proof of Theorem 1 of [1].

We first show:

Lemma 1. *There exists a constant c_1 independent of i such that $\lambda_1(M_i) \leqslant c_1 \cdot h_i$.*

Proof. Suppose we are given a set of edges E_1 in Γ_i separating Γ_i into A and B. Let us choose $\varepsilon > 0$, fixed for the discussion. We then consider the function f_E on M_i defined as follows: Suppose $\#(A) \leqslant \#(B)$. Then we choose f_E to be 1 on the interior of the union of fundamental domains belonging to A, $-c$ on the interior of fundamental domains belonging to B, and changing linearly with distance in an ε-neighborhood of where a face of a fundamental domain belonging to A meets a fundamental domain belonging to B. The constant $c \leqslant 1$ is chosen so that $\int_{M_i} f_E = 0$.

We now calculate the Rayleigh quotient of f_E. But $\int_{M_i} \|df_E\|^2 \leqslant d_1 \cdot \#(E)$, since $df \equiv 0$ away from where the domains from A meet those of B, and we get only a bounded contribution from each common face, corresponding to an edge in E. On the other hand

$$\int_{M_i} |f_E|^2 \geqslant d_2(\#(A) + c^2\#(B)) \geqslant d_2\#(A),$$

since f_E is constant on some fixed amount of each fundamental domain. Hence

$$\frac{\int_{M_i} \|df_E\|^2}{\int_{M_i} |f_E|^2} \leqslant \frac{d_1}{d_2} \cdot \frac{\#(E)}{\#(A)}$$

and Lemma 1 is proved.

Lemma 1 shows that if $h_i \to 0$ as $i \to \infty$, then $\lambda_1(M_i) \to 0$ as $i \to \infty$. To complete the proof of Theorem 1, we must now show:

Lemma 2. *If $\lambda_1(M_i) \to 0$ as $i \to \infty$, then $h_i \to 0$ as $i \to \infty$.*

The idea used to establish Lemma 2 is to use Cheeger's inequality

$$\lambda_1(N) \geqslant \tfrac{1}{4}h^2,$$

where $h = h(N)$ is the geometric isoperimetric constant

$$h(N) = \inf_S \frac{\operatorname{area}(S)}{\min(\operatorname{vol}(N_1), \operatorname{vol}(N_2))},$$

where S ranges over hypersurfaces of N which divides N into two pieces $N - S = N_1 \cup N_2$.

Just as in [1], the idea is to compare the geometric isoperimetric constant $h(M_i)$ with the combinatorial isoperimetric constant $h(\Gamma_i)$. If we can prove an inequality of the form $h(\Gamma_i) \leqslant (\text{const})h(M_i)$, then Lemma 2 follows, and hence the theorem.

We run into two obstacles not present in [1]. First of all, the main technical estimate of [1], Theorem 3, is no longer available to us. Secondly, it may happen that the "geometrically smallest piece" may be "combinatorially larger". Both of these difficulties arise from the fact that the denominator in the definition of h is the minimum of two terms.

To prove the lemma, we consider the problem of minimizing the geometric isoperimetric constant $h(M_i)$ among all hypersurfaces S. As in [1] and [6], the minimum is realized by an integral current T_i, which is regular outside a set of high codimension, and which has constant mean curvature η_i on the regular part of T_i.

As in [1], we must now argue that the mean curvature η_i is bounded independent of i, but the argument of [1] does not apply. This problem was considered by Buser in [6], where he showed that if the Ricci curvature of a manifold N is scaled to be $\geqslant -(n-1)$, then we have the inequality

$$|\eta| \leqslant 1 + \frac{h(N)}{n}, \qquad n = \dim(N).$$

In particular, since $h(M_i) \leqslant h(M)$, we see that $|\eta_i| \leqslant (\text{const})$ for some (const) independent of i.

We now tentatively divide Γ_i into three sets $\mathscr{A}_i$, $\mathscr{B}_i$ and $\mathscr{C}_i$, where $\mathscr{C}_i$ consists of those domains which meet T_i, and $\mathscr{A}_i$ and $\mathscr{B}_i$ are those fundamental domains contained in M_{i1} and M_{i2}, respectively.

It is easily seen that

$$\operatorname{vol}(F)\cdot\#(\mathscr{A}_i) \leqslant \operatorname{vol}(M_{i1}) \leqslant \operatorname{vol}(F)(\#(\mathscr{A}_i)+\#(\mathscr{C}_i))$$

and similarly for $\mathscr{B}_i$.

It further follows from the bounded mean curvature of T_i, as in [1, Proposition 3], that there is a constant d such that

$$(\dagger) \qquad \operatorname{area}(T_i) \geqslant (d)\#(\mathscr{C}_i).$$

Let us first assume that $\#(\mathscr{C}_i) \geqslant (\#(\Gamma_i))/10$. It then follows from (†) that

$$\frac{\operatorname{area}(T_i)}{\operatorname{vol}(M_i)} \geqslant \frac{(d)\#(\mathscr{C}_i)}{\operatorname{vol}(F)\#(\Gamma_i)} \geqslant \frac{(d)}{(10)(\operatorname{vol}(F))}$$

so that $h(M_i) \geqslant d/5\operatorname{vol}(F)$.

Now suppose that $\#(\mathscr{C}_i) \leqslant \#(\Gamma_i)/10$, which by the above must be the case if $h(M_i) \to 0$. Then

$$\frac{\operatorname{area}(T_i)}{\operatorname{vol}(M_i)} \geqslant \frac{d}{\operatorname{vol}(F)}\left[\frac{\#(\mathscr{C}_i)}{\#(\mathscr{A}_i)+\#(\mathscr{C}_i)}\right].$$

If $\operatorname{vol}(M_{i1}) \leqslant \operatorname{vol}(M_{i2})$, then we construct our sets A_i, B_i, and E_i as follows: First, place all the edges from all the vertices in $\mathscr{C}_i$ into E_i, so that $\#(E_i) = (\text{const})\#(\mathscr{C}_i)$. We now replace edges from E_i back into the graph so that all the points of $\mathscr{C}_i$ are joined to $\mathscr{A}_i$, and we set $A_i = \mathscr{A}_i \cup \mathscr{C}_i$, $B_i = \mathscr{B}_i$.

We now have that

$$\frac{\#(E_i)}{\#(A_i)} \leqslant (\text{const})\frac{\operatorname{area}(T_i)}{\operatorname{vol}(M_{i1})},$$

but it may now happen that $\#(A_i) \geqslant \#(B_i)$. But from $\operatorname{vol}(M_{i1}) \leqslant \operatorname{vol}(M_{i2})$ we see that

$$\#(\mathscr{A}_i) \leqslant \#(\mathscr{B}_i) + \#(\mathscr{C}_i)$$

and from our assumption we have

$$\#(\mathscr{C}_i) \leqslant (\#(\mathscr{A}_i) + \#(\mathscr{B}_i) + \#(\mathscr{C}_i))/10$$

from which we readily conclude that

$$\#(\mathscr{B}_i) \geqslant 4\#(\mathscr{A}_i)/5$$

so that

$$\frac{\#(E_i)}{\min(\#(A_i), \#(B_i))} \leqslant \frac{4}{5}(\text{const})\frac{\text{area}(T_i)}{\text{vol}(M_i)}$$

and so $h(\Gamma_i) \leqslant (\text{const})h(M_i)$, as desired.

2. Property T

In this section, we study groups for which the behavior of the isoperimetric constants h_i, and therefore the first eigenvalues of coverings, can be controlled.

Let us then fix a discrete group π and generators $g_1, \cdots, g_k$ for π. Let us denote by Γ the graph of π, whose vertices are the elements of π and whose edges correspond to multiplication by the generators of π.

Let us denote by $h_0(\Gamma)$ the isoperimetric constant of Γ defined as follows:

$$h_0(\Gamma) = \inf_E \frac{\#(E)}{\#(A)},$$

where E runs over a collection of edges separating Γ into a finite part A and a possibly infinite part. Note that $h_0(\Gamma) = 0$ when π is finite.

Recall from [1] that π is amenable if and only if $h_0(\Gamma) = 0$. The equivalence of this with the standard definition of amenability in terms of left-invariant means depends on a delicate combinatorial theorem due to Folner [9].

Recall also that π is said to be residually finite if, given any element $g \in \pi$ other than the identity, there is a homomorphism $\phi\colon \pi \to F$, where F is a finite group, such that $\phi(g) \neq \text{id}$. Residual finiteness is the usual state of affairs among groups arising in geometry.

Theorem 2. *Suppose that $\pi = \pi_1(M)$ is a group which is infinite, amenable, and residually finite. Then there are finite coverings M_i of M such that $\lambda_1(M_i) \to 0$ as $i \to \infty$.*

Proof. Geometrically, residual finiteness says that, given any compact set $C \subset \tilde{M}$, there is a finite covering M' of M and a covering map $f\colon \tilde{M} \to M'$, such that f is 1-to-1 on C.

Since π is amenable, by the theorem of [1], there are test functions f_ε with compact support on $\tilde{M}$, with Rayleigh quotient $< \varepsilon$. Since π is infinite, we may translate f_ε by some element $g \in \pi$ so that f_ε and $g \cdot f_\varepsilon$ have disjoint support. In particular, they are perpendicular in L^2.

Residual finiteness now says that there is a finite covering M'_ε of M such that $\mathrm{supp}(f_\varepsilon) \cup \mathrm{supp}(g \cdot f_\varepsilon)$ maps injectively into M'_ε. It follows from the minimax characterization of λ_1 that $\lambda_1(M'_\varepsilon) < \varepsilon$.

We remark that this theorem was also shown by Sunada [14] by a similar argument. We will strengthen this theorem by some remarks in the next section.

We now turn to the problem of finding π for which one may bound h_i, and hence $\lambda_1(M_i)$, from below. This can be a delicate combinatorial problem, as we will see below.

We recall the following notion, due to Kazhdan:

Definition ([10] **and** [11]). A discrete group π with generators $g_1, \cdots, g_k$ has Property T if there exists a constant $\varepsilon > 0$ with the following property: For any unitary representation of π in a Hilbert space $\mathscr{H}$, if there is a vector X with $\|X\| = 1$ and $\|g \cdot X - X\| < \varepsilon$ for $g = g_1, \cdots, g_k$, then there is a vector Y with $\|Y\| = 1$ and $g \cdot Y = Y$ for all $g \in \pi$.

Intuitively, Property T says that the trivial representation is isolated among all unitary representations of π.

There is an analogous notion of Property T for Lie groups, and Kazhdan [11] shows that if π is a discrete subgroup of cofinite volume in G, and G has Property T, then the same is true for π. Furthermore, one can study the unitary representations for G to determine if G has Property T. In particular, if G is the group of isometries of a symmetric space of rank $\geqslant 2$, then G has Property T. Among the symmetric spaces of rank 1, some have Property T and some do not.

Theorem 3. *Suppose* $\pi = \pi_1(M)$ *has Property* T. *Then there exists a constant* $C > 0$ *with the following properties*:

(i) *If* M' *is a finite covering of* M, *then* $\lambda_1(M') > C$.

(ii) *If* M' *is an infinite covering of* M, *then* $\lambda_0(M') > C$.

We would like to thank S. T. Yau for pointing out to us that our method for proving (i) gives (ii) as well.

One may ask if the converse to Theorem 3, or even just Theorem 3(i), is true. There are two reasons why this will not be the case. The first reason is that a group may fail to have many subgroups of finite index, for reasons which have nothing to do with Property T. On a deeper level, it will be clear from the proof below that only certain unitary representations of π enter into the proof of Theorem 3. For instance, the group $\mathrm{SL}(2, \mathbf{Z}[1/p])$ is well-known to be a group which does not have Property T, but all of whose subgroups of finite index are congruence subgroups (private communication from Alex Lubotzky). $\mathrm{SL}(2, \mathbf{Z}[1/p])$ will thus satisfy the conclusion of Theorem 3(i).

One expects that a general group may well have unitary representations close to the trivial representation, so that π does not have Property T, but for which the conclusion of Theorem 3 is still valid. It is not hard to give a weakened version of Property T for which Theorem 3 and its converse are true. We refer the reader to [2] for a more detailed discussion of this.

To prove Theorem 3, suppose we are given a sequence of coverings M_i of M, that is to say a sequence of subgroups $\pi_i \subset \pi$. We separate into two cases:

Case (i). Suppose the M_i's are finite coverings of M with $\lambda_1(M_i) \to 0$.

By Theorem 1, it follows that the h_i's also tend to zero. We will show that this contradicts Property T.

Consider the Hilbert space $L^2(\pi/\pi^i)$ of functions on the coset space π/π_i, with inner product

$$\langle f, g\rangle = \frac{1}{\#(\Gamma_i)} \sum_{v\in\Gamma_i} f(v)\cdot g(v).$$

Let $\mathscr{H}_i$ be the subspace of $L^2(\pi/\pi^i)$ perpendicular to the constant function:

$$\sum_{v\in\Gamma_i} f(v) = 0.$$

Then clearly π acts unitarily on $\mathscr{H}_i$.

Now suppose there are sets A_i, B_i, and E_i with

$$\frac{\#(E_i)}{\min(\#(A_i), \#(B_i))} \to 0 \quad \text{as } i \to \infty.$$

Consider the function f_i defined by

$$\begin{aligned} f_i(v) &= \sqrt{\frac{\#(B_i)}{\#(A_i)}} \quad \text{for } v \in A_i \\ &= -\sqrt{\frac{\#(A_i)}{\#(B_i)}} \quad \text{for } v \in B_i. \end{aligned}$$

Then clearly $\langle f_i, 1\rangle = 0$, $\|f_i\| = 1$.

Let us calculate $\|g_j\cdot f_i - f_i\|$. But $(g_jf_i - f_i)^2(v) = 0$ whenever the edge leading from v corresponding to g_j does not lie in E_i. Therefore

$$\begin{aligned} \|g_jf_i - f_i\|^2 &\leqslant \frac{\#(E_i)}{\#(A_i) + \#(B_i)}\left[\sqrt{\frac{\#(B_i)}{\#(A_i)}} + \sqrt{\frac{\#(A_i)}{\#(B_i)}}\right]^2 \\ &= \frac{\#(E_i)}{\#(A_i) + \#(B_i)}\left[2 + \frac{\#(B_i)}{\#(A_i)} + \frac{\#(A_i)}{\#(B_i)}\right]. \end{aligned}$$

Assuming that $\#(A_i) \leqslant \#(B_i)$, this is

$$\leqslant \frac{3}{2}\cdot\frac{\#(E_i)}{\#(A_i)} + \frac{\#(E)}{\#(A_i)}\cdot\frac{\#(B_i)}{\#(A_i) + \#(B_i)} \leqslant \frac{5}{2}\,\frac{\#(E_i)}{\#(A_i)}.$$

It follows that if $\#(E_i)/\#(A_i) \to 0$, then for all i sufficiently large, $\|g_j f_i - f_i\| < \varepsilon$.

Property T then implies that the constant function is in $\mathscr{H}_i$, but this is impossible, from the definition of $\mathscr{H}_i$.

Case (ii). Now suppose the M_i's are infinite coverings of M, with $\lambda_0(M_i) \to 0$. Then we use the theorem of [1] to show that the isoperimetric constants $h_0(M_i)$ of the graphs of these coverings are similarly going to 0.

We then consider the Hilbert space $L^2(\pi/\pi^i)$ of all L^2-functions on the vertices of the graphs Γ_i, with inner product

$$\langle f, g\rangle = \sum_{v\in\Gamma_i} f(v)\cdot g(v).$$

If $h_0(\Gamma_i) \to 0$ as $i \to \infty$, then we can find sets A_i, E_i such that $\#(E_i)/\#(A_i) \to 0$ as $i \to \infty$. We now take test functions which are 1 on A_i and 0 away from A_i. As above, we see that $\|g_i \cdot f_i - f_i\| \to 0$ as $i \to \infty$. Property T then implies that the constant function is an L^2-function on Γ_i. But this is impossible, since Γ_i is infinite.

3. Some applications

In this section, we give some simple applications of Theorem 1.

Our first result is:

Theorem 4. *Suppose there is a surjective homomorphism* $f\colon \pi_1(M) \to \pi_1(N)$. *Then*:

(a) *If* N *has finite coverings* N_i *with* $\lambda_1(N_i) \to 0$, *then the same is true of* M.

(b) *If there is a constant* C *and infinitely many coverings* N_i *of* N *with* $\lambda_1(N_i) \geqslant C$, *then there exists* C' *and infinitely many coverings* M_i *of* M *with* $\lambda_1(M_i) \geqslant C'$.

We first observe the following:

Lemma. *Suppose we are given a group* π, *and two sets of generators* $G = \{g_1, \cdots, g_k\}$ *and* $H = \{h_1, \cdots, h_l\}$. *Then there are constants* C_1 *and* C_2 *such that* $C_1 h(\Gamma_i^H) \leqslant h(\Gamma_i^G) \leqslant C_2 h(\Gamma_i^H)$, *where* C_1, C_2 *are independent of* i *and* Γ_i^G (*resp.* Γ_i^H) *denotes the graph determined by the generators* G (*resp.* H).

Proof. Suppose we have a division $\Gamma_i^G - E_j = A_j \cup B_j$ of Γ_i^G into two pieces. Let $\mathscr{C}_j$ denote the collection of vertices of Γ_i^G adjoining an edge in E_j, and for each k, let $\mathscr{C}_j^k$ denote the set of vertices of Γ_i^G which may be joined to an edge in E_j by a path of k or fewer edges. It is evident that $\#(\mathscr{C}_j^k) \leqslant (l)^k \#(\mathscr{C}_j)$.

Choosing k so that any generator in G can be expressed as a word length $\leqslant k$ in the generators of H, and setting E_j' to be all the edges in Γ_i^H adjoining vertices in $(\mathscr{C}_j^k)$, it is evident that $A_j - (\mathscr{C}_j^k)$ is separated from $B_j - (\mathscr{C}_j^k)$ in $\Gamma_i^H - E_j'$. The lemma now follows routinely.

To prove the theorem, we now proceed as follows: Let us pick generators $g_1, \cdots, g_k$ for $\pi_1(M)$. Since $f\colon \pi_1(M) \to \pi_1(N)$ is surjective, the set $f(g_1), \cdots, f(g_k)$ now generates $\pi_1(N)$.

Suppose now that we are given subgroups π_i of $\pi_1(N)$. By the lemma, whether or not $h_i \to 0$ as $i \to \infty$ is independent of the choice of generators, so that we could have chosen $f(g_1), \cdots, f(g_k)$ in computing h_i. But now let $\pi_i' = f^{-1}(\pi_i) \subset \pi_1(M)$. Then calculating h_i is exactly the same for $\pi_i' \subset \pi_1(M)$ as for $\pi_1 \subset \pi_1(N)$, so that for the sequence of coverings M_i of M corresponding to π_i', $\lambda_1(M_i) \to 0$ if and only if $\lambda_1(N_I) \to 0$. This establishes Theorem 4.

Combining Theorem 2 with Theorem 4, we obtain a generalization of a theorem of Randol [13]:

Corollary 5. *Suppose $\pi_1(M)$ surjects onto a group which is infinite, amenable, and residually finite. Then there exists coverings M_i of M with $\lambda_1(M_i) \to 0$.*

Combining Theorem 2 with Theorem 4, we may show

Corollary 6. *Let M be a compact manifold such that $\pi_1(M)$ surjects onto $\mathbf{Z} * \mathbf{Z}$. Then there exists $c > 0$ and infinitely many coverings M_i of M with $\lambda_1(M_i) > c$ for all i.*

Proof. Let π be any finitely presented, infinite, residually finite group with Property T, for instance $\pi = \mathrm{SL}(3, \mathbf{Z})$, and let π_i be an infinite collection of subgroups of π of finite index. By Theorem 3, if N is a compact manifold with $\pi_1(N) = \pi$, then there is a constant $D > 0$ such that $\lambda_1(N_i) > D$ for all i, where N_i is the finite covering of N corresponding to π_i.

Suppose that π is generated by k elements. Then there is a finite covering M' of M whose fundamental group surjects onto $\mathbf{Z} * \mathbf{Z} * \cdots * \mathbf{Z}$ (k times). We then have a surjective homomorphism

$$\pi_1(M') \to \mathbf{Z} * \cdots * \mathbf{Z}\ (k \text{ times}) \to \pi$$

and Theorem 4 then completes the proof of the corollary.

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