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TOPOLOGICAL INVARIANTS AND EQUIDESINGULARIZATION FOR HOLOMORPHIC VECTOR FIELDS

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The integrals of a holomorphic vector field Z defined in an open subset $\mathcal{U}$ of $\mathbf{C}^n$ are complex curves parametrized locally as the solutions of the differential equation

$$\frac{dz}{dT} = Z(z), \qquad T \in \mathbf{C}, z \in \mathcal{U}.$$

They define a complex one-dimensional foliation $\mathcal{F}_Z$ of $\mathcal{U}$ with singularities at the zeros of Z. The purpose of this paper is to exhibit several topological invariants of these foliations near a singular point.

Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions defined in some neighborhood of $0 \in \mathbf{C}^n$ and let $I(Z_1,\cdots,Z_n)$ be the ideal generated by the germs at $0 \in \mathbf{C}^n$ of the coordinate functions of Z. We define the *Milnor number* μ of the vector field Z at $0 \in \mathbf{C}^n$ as

$$\mu = \dim_{\mathbf{C}} \mathcal{O}_n / I(Z_1,\cdots,Z_n).$$

This number is finite if and only if $0 \in \mathbf{C}^n$ is an isolated singularity of Z, a hypothesis which we will assume from now on. In this case μ coincides with the topological degree of the Gauss mapping induced by Z, considered as a real vector field, in a small $(2n-1)$-sphere around $0 \in \mathbf{C}^n$. In Theorem A we show that: *the Milnor number of Z is a topological invariant of $\mathcal{F}_Z$ provided that $n \geqslant 2$.*

Consider now a polydisc $B \subset \mathcal{U}$ centered at $0 \in \mathbf{C}^n$ and let $f\colon B \to \mathbf{C}^k$, $f(0) = 0$, be an irreducible analytic function. Then $V = f^{-1}(0)$ is an analytic subvariety and we say *V is invariant by Z if for any $p \in V$ we have $df(p) \cdot Z(p) = 0$.* Suppose $k = n-1$. Then $\dim_{\mathbf{C}} V = 1$ and $V - \{0\}$ is a leaf of $\mathcal{F}_Z$. Moreover, if B is small enough, then $D = B \cap V$ is homeomorphic to a 2-disc via a Puiseaux's parametrization. Then the restriction of Z to D can be considered as a real vector field X defined in a 2-disc. *The multiplicity of Z*

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along V is defined as the topological degree of the Gauss mapping induced by X. Then Theorem B of Chapter I asserts that *the multiplicity of Z along V is a topological invariant of $\mathcal{F}_Z$.*

In Chapter II we concentrate in dimension two. In this dimension the integrals of the vector field $Z = Z_1\partial/\partial z_1 + Z_2\partial/\partial z_2$, i.e. the leaves of $\mathcal{F}_Z$, are the integrals of the differential form $\omega = -Z_2\,dz_1 + Z_1\,dz_2$. The main result of this chapter is an equidesingularization theorem for vector fields or 1-forms generalizing the equidesingularization theorem for analytic curves ($\omega = df$). In 1932 W. Burau [2] and O. Zariski [17] proved independently that homeomorphic singular curves in $\mathbf{C}^2$ have isomorphic desingularizations at singular points. The desingularization theorem for vector fields (see [1], [8], [16]) is achieved by a process described in II (2.1). Essentially it says that after finitely many successive quadratic transformations (blow ups) at singular points the foliation $\mathcal{F}_Z$ is transformed into a foliation $\tilde{\mathcal{F}}_Z$ with a finite number of singularities, all of them *simple* and lying in the divisor. This means that around any singular point p there is a local coordinate chart (z_1, z_2), $z_1(p) = z_2(p) = 0$, where $\tilde{\mathcal{F}}_Z$ is induced by the vector field $\tilde{Z}$:

$$\tilde{Z} = \tilde{Z}_1\frac{\partial}{\partial z_1} + \tilde{Z}_2\frac{\partial}{\partial z_2}, \qquad \tilde{Z}_1(0) = \tilde{Z}_2(0) = 0,$$

and the eigenvalues λ_1, λ_2 of $d\tilde{Z}(0)$ satisfy one of the following conditions:

(i) $\lambda_1 \neq 0$ and $\lambda_2 = 0$ or viceversa.

(ii) $\lambda_1 \neq 0 \neq \lambda_2$ and $\lambda_1/\lambda_2 \notin Q_+$.

The significance of simple singularities in this context is that they are persistent once they appear in the desingularization and no further explosions will simplify them. The local topological structure of these singularities has been studied by several authors [3, 10].

A *generalized curve* is, by definition, a vector field Z inducing a foliation $\mathcal{F}_Z$ whose desingularization admits only simple singularities with nonvanishing eigenvalues. We will show that the topological structure of these vector fields is strongly dependent upon the structure of their invariant one dimensional varieties (separatrices) passing through $0 \in \mathbf{C}^2$. Theorem C asserts: *If Z is a generalized curve and Z' is any vector field such that $\mathcal{F}_{Z'}$ and $\mathcal{F}_Z$ are topologically equivalent near $0 \in \mathbf{C}^2$, then Z' is also a generalized curve and the desingularizations of Z' and Z are isomorphic.* The algebraic multiplicity at $0 \in \mathbf{C}^2$ of a vector field or 1-form is, by definition, the degree of its first nonzero term in its Taylor development at $0 \in \mathbf{C}^2$. For curves ($\omega = df$) this multiplicity (k) has the following well-known geometric interpretation: $k + 1$ is the intersection number of ($f = 0$) with a generic complex line passing through $0 \in \mathbf{C}^2$. We prove in Chapter II that this interpretation survives for generalized curves: *If k*

is the algebraic multiplicity of a generalized curve, then $k+1$ *is the sum of intersection numbers of all separatrices with a generic complex line passing through* $0 \in \mathbf{C}^2$. Indeed, a more general formula is shown relating the algebraic multiplicity of any vector field with the multiplicities along the divisor of its desingularization. As a consequence of this and Theorem C we obtain that: *the algebraic multiplicity of a generalized curve is a topological invariant in the space of all holomorphic vector fields with singularity at* $0 \in \mathbf{C}^2$, which extends the corresponding result on curves.

Let us remark that foliations given by the level curves of a function f: $(\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ are generalized curves as well and the above statements specialize to the Burau-Zariski Theorem in this case, as it follows from King's Theorem [7] which asserts that topologically equivalent curves in the sense of [17] have associated foliations by level curves also equivalent.

A final word must be said on the reason why Theorem C has been proved only for generalized curves. The main difficulty is that it is unknown whether topological equivalences can be extended to the divisors in the desingularizations. If this were the case a proof would follow quickly from Theorem B of Chapter I. We overcome this difficulty in the case of generalized curves because there exists a one-to-one correspondence between singularities in the divisors (out of intersections of projective lines) and separatrices.

CHAPTER I
THE MILNOR NUMBER INVARIANCE AND APPLICATIONS

1. Index of an isolated singularity

Let $Z = (Z_1, \cdots, Z_n)$ be a holomorphic vector field defined in an open set $\mathcal{U} \subset \mathbf{C}^n$, $n \geqslant 2$. Given a C^r, $r \geqslant 0$, real vector field X defined in $\mathcal{U}$, we say that *X is tangent to Z* if for each $z \in \mathcal{U}$ we have

$$X(z) \in \mathbf{C}Z(z) = \{\lambda Z(z); \lambda \in \mathbf{C}\}.$$

The index of X at $p \in \mathcal{U}$, denoted by $\operatorname{ind}_p(X)$, is the topological degree of the map $X/\|X\|$: $S_r^{2n-1}(p) \to S^{2n-1}$, where $\|X\|^2 = \sum_{j=1}^{2n} x_j^2$, $S_r^{2n-1}(p) = \{z; \|z-p\| = r\}$ and $r > 0$ is small enough. In $S_r^{2n-1}(p)$ and S^{2n-1} we take the orientations defined by a normal vector field pointing outside the balls bounded by the spheres.

Proposition 1. *Let X and Y be continuous real vector fields tangent to Z such that $p \in \mathscr{U}$ is an isolated singularity for Z, X and Y. Then* $\operatorname{ind}_p(X) = \operatorname{ind}_p(Y)$.

Proof. It is sufficient to prove that $\operatorname{ind}_p(X) = \operatorname{ind}_p(Z)$ considering Z as a real vector field. Since X is tangent to Z and $\{Z(z), iZ(z)\}$ is a base for the real vector space $\mathbf{C}Z(z)$, if $Z(z) \neq 0$, it follows that for any $z \in S_r^{2n-1}(p)$, r small, we can write

$$X(z) = \alpha(z)Z(z) + \beta(z)iZ(z) = (\alpha(z) + i\beta(z))Z(z),$$

where $f = (\alpha, \beta)\colon S_r^{2n-1}(p) \to \mathbf{R}^2 - \{0\}$ is continuous. Since $\pi_{2n-1}(\mathbf{R}^2 - \{0\}) = \{0\}$ for $n \geqslant 2$, there exists a homotopy

$$f_t = (\alpha_t, \beta_t)\colon [0,1] \times S_r^{2n-1} \to \mathbf{R}^n - \{0\}; \qquad f_0 = f, f_1 = (1,0).$$

Let $X_t = (\alpha_t + i\beta_t)Z$ and $F_t = X_t/\|X_t\|$. Then F_t is a homotopy between $X/\|X\|$ and $Z/\|Z\|$. Since the degree is a homotopy invariant, the proposition is proved.

2. The Milnor number of a holomorphic vector field

Let $\mathscr{O}_{n,p}$ be the ring of germs at $p \in \mathscr{U}$ of holomorphic functions and $[Z_1, \cdots, Z_n] \subset \mathscr{O}_{n,p}$ the ideal generated by the components of Z. The number

$$\mu = \mu(Z, p) = \dim_{\mathbf{C}} \mathscr{O}_{n,p}/[Z_1, \cdots, Z_n]$$

is called the Milnor number of Z at p.

The following facts are well known (cf. [9]):

(1) $\mu = 0$ if and only if $Z(p) \neq 0$.

(2) $0 < \mu < \infty$ if and only if p is an isolated singularity for Z.

(3) $\mu = 1$ if and only if $\det(\partial Z_i(p)/\partial z_j)_{1 \leqslant i, j \leqslant n} \neq 0$.

(4) Let $0 < \mu < \infty$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $c \in \mathbf{C}^n$ with $\|c\| < \delta$ then the number of solutions of the equation $Z(z) = c$ in the ball $B_p(\varepsilon) = \{z \mid \|z - p\| < \varepsilon\}$ is at most μ. Furthermore, if $p_1, \cdots, p_m$, $m \leqslant \mu$, are such solutions, then

$$\sum_{i=1}^{m} \mu(Z - c, p_i) = \mu.$$

(5) $\mu = \operatorname{ind}_p(Z)$.

Property (5) is an easy consequence of the others. In fact, let $\varepsilon > 0$ be such that p is the unique singularity of Z in $B = \{z \mid \|z - p\| \leqslant \varepsilon\}$. Let $k = \inf\{\|Z(z)\|;\ z \in \partial B\} > 0$. If c is a regular value of Z with $\|c\| < \min\{k, \delta\}$, where $\delta > 0$ is as in (4), then the number of solutions of $Z(z) = c$ in B is μ.

Let $Z^t = Z - tc$. Then $Z^t(z) \neq 0$ for $\|z\| = \varepsilon$, which implies that F: $[0,1] \times \partial\overline{B} \to S^{2n-1}$, given by $F(t, z) = Z^t(z)/\|Z^t(z)\|$ is a homotopy between $F_0 = Z/\|Z\|$ and $F_1 = Z - c/\|Z - c\|$. Hence F_0 and F_1 have the same degree. Let $p_1, \cdots, p_\mu$ be the singularities of $Z - c$. For each $j = 1, \cdots, \mu$ let $B_j \subset B$ be a small ball around p_j such that $B_j \cap B_k = \varnothing$ for $j \neq k$. Let G_j: $\partial B_j \to S^{2n-1}$ be defined by $G_j(z) = Z_j(z) - c/\|Z_j(z) - c\|$. By degree theory (cf. [11]) we have

$$\text{degree } F_1 = \sum_{j=1}^{\mu} \text{degree } G_j.$$

Now, since c is a regular value of Z, it follows that

$$\text{degree } G_j = \mu\big(Z - c, p_j\big) = 1, \qquad j = 1, \cdots, \mu.$$

Therefore $\operatorname{ind}_p(Z) = \mu$. q.e.d.

We proceed now to give an alternative definition for the index of a holomorphic vector field which will be useful in the proof of Theorem A. Let

$$\varphi \colon \overline{D}_\varepsilon \times \overline{B}_r \to B_\rho \subset \mathbf{C}^n$$

be the local complex flow of Z, where $D_\varepsilon = \{T \in \mathbf{C};\ |T| < \varepsilon\}$, $B_r = \{z \in \mathbf{C}^n;\ |z| < r\}$ and $0 < r < \rho$.

Lemma 1. *Let τ: $\overline{B}_r - \{0\} \to D_\varepsilon - \{0\}$ be a continuous function such that for any $z \in \overline{B}_r - \{0\}$ and $t \in (0,1]$ we have $\varphi(t\tau(z), z) \neq z$. Let g: $\partial B_r \to S^{2n-1}$ be defined by*

$$g(z) = \frac{\varphi(\tau(z), z) - z}{\|\varphi(\tau(z), z) - z\|}.$$

Then $\operatorname{ind}_0(Z) = \operatorname{degree}(g)$.

Proof. Let G: $[0,1] \times \partial B_r \to S^{2n-1}$ be defined by

$$G(t, z) = \frac{\varphi(t\tau(z), z) - z}{\|\varphi(t\tau(z), z) - z\|}, \qquad t \neq 0,$$

and

$$G(0, z) = \frac{\tau(z)}{|\tau(z)|} \cdot \frac{Z(z)}{\|Z(z)\|}.$$

Then $G(1, z) = g(z)$ and

$$\begin{aligned}
\lim_{t\to 0} G(t, z) &= \frac{\tau(z)}{|\tau(z)|} \lim_{t\to 0} \left\| \frac{\varphi(t\tau(z), z) - z}{t\tau(z)} \right\|^{-1} \cdot \lim_{t\to 0} \frac{\varphi(t\tau(z), z) - z}{t\tau(z)} \\
&= \frac{\tau(z)}{|\tau(z)|} \lim_{s\to 0} \left\| \frac{\varphi(s, z) - z}{s} \right\|^{-1} \cdot \lim_{s\to 0} \frac{\varphi(s, z) - z}{s} \\
&= \frac{\tau(z)}{|\tau(z)|} \cdot \frac{Z(z)}{\|Z(z)\|}.
\end{aligned}$$

It follows that G is continuous and therefore is a homotopy between $g(z)$ and $G_0(z) = \tau(z)/|\tau(z)| \cdot Z(z)/\|Z(z)\|$. Now, since $n \geqslant 2$, $\pi_{2n-1}(S^1) = \{0\}$, hence $\tau/|\tau|$: $\partial B_r \to S^1$ is homotopic to the constant $1 \in S^1$ and g is homotopic to $Z/\|Z\|$. Therefore $\operatorname{ind}_0(Z) =$ degree g.

Lemma 2. *Let φ: $D_\varepsilon \times \overline{B}_{r'} \to B_\rho$ be the local complex flow of Z. Suppose that $Z(z) \neq 0$ for $z \in \overline{B}_r - \{0\}$, where $0 < r \leqslant r'$. Then there exists $\delta > 0$ such that for any $T \in D_\delta - \{0\}$ and $z \in \overline{B}_r - \{0\}$, we have $\varphi(T, z) \neq z$.*

Proof. Suppose by contradiction that there exist sequences $T_n \to 0$, $0 < |T_n| < \varepsilon$, and $z_n \in \overline{B}_r - \{0\}$ such that $\varphi(T_n, z_n) = z_n$ for all $n \geqslant 1$. Let L_n be the leaf of $Z/\overline{B}_r$ passing through z_n. The lemma is a consequence of the following assertions:

(i) $L_n \cap \partial \overline{B}_r \neq \varnothing$.

(ii) For all $n \geqslant 1$ we have $\varphi(T_n, z) = z$ for all $z \in L_n \cap \overline{B}$.

In fact, suppose that (i)–(ii) are true. From (i) we take $w_n \in L_n \cap \partial \overline{B}_r$. Then $\|w_n\| = r$ and by taking a subsequence if necessary we can suppose that $w_n \to w_0$, $\|w_0\| = r$. By (ii) we have $\varphi(T_n, w_n) = w_n$ and so

$$Z(w_0) = \lim_{n \to \infty} \frac{\varphi(T_n, w_n) - w_n}{T_n} = 0,$$

which is a contradiction.

Proof of (i). Let L be a nonsingular leaf of Z and suppose by contradiction that $L \subset B_r = \operatorname{int}(\overline{B}_r)$. Let $\tilde{r} = \sup\{\|z\|;\ z \in L\}$ and $q_n \in L$ such that $q_n \to q_0$ as $n \to \infty$, where $\|q_0\| = \tilde{r} \leqslant r$. Since $Z(q_0) \neq 0$, by the complex flow box theorem there exist $\alpha > 0$, a neighborhood Q of q_0 and a holomorphic diffeomorphism ψ,

$$\psi\colon D_\alpha \times B_\alpha \to Q, \qquad B_\alpha = \{z \in \mathbf{C}^{n-1};\ \|z\| < \alpha\},$$

such that

$$\psi_*\left(\frac{\partial}{\partial t}\right) = Z \quad \text{and} \quad \psi(0,0) = q_0, \quad \text{where } \frac{\partial}{\partial t}(T, z) = (1,0).$$

In terms of flows we have $\psi(T_0 + T, z) = \varphi(T, \psi(T_0, z))$ provided that both members are defined.

For $q \in Q$ let $P_q = \psi(D_\alpha \times p_2\psi^{-1}(q))$ be the plaque of Z/Q, where p_2: $D_\alpha \times B_\alpha \to B_\alpha$ is the natural projection. Then P_q is a small disk contained in the leaf of Z through q and ψ_q: $D_\alpha \to P_q$ defined by $\psi/D_\alpha \times p_2\psi^{-1}(q)$ is an analytic immersion. Now ψ_{q_0} is not constant, therefore by the maximum principle $0 \in D_\alpha$ is not a local maximum of $z \in D_\alpha \to \|\psi_{q_0}(z)\|$, hence there exists $p_0 \in P_{q_0}$ such that $\|p_0\| > \|q_0\| = \tilde{r}$. By continuity, it follows that there exists $\delta > 0$ such that if $\|q - q_0\| < \delta$, then the plaque P_q contains a point p

with $\|p\| > \tilde{r}$. Therefore if n is sufficiently big, there is $p \in P_{q_n}$ such that $\|p\| > \tilde{r}$, which is a contradiction since $p \in L$.

Proof of (ii). Let L be a leaf of Z/B_r, $r' > r$. We consider L with the intrinsic topology, that is, the topology generated by plaques contained in L (cf. [14]). Suppose that $\varphi(T_0, z_0) = z_0$ for some $z_0 \in L$ and $0 < |T_0| < \varepsilon$. We shall prove that $\varphi(T_0, z) = z$ for any $z \in L$. Let $A = \{z \in L \mid \varphi(T_0, z) = z\}$. Then A is closed and $A \neq \varnothing$. Since L is connected it is sufficient to prove that A is open in L. Let $q \in A$. Since $Z(q) \neq 0$ we can parametrize a neighborhood of q in L by the flow φ_q: $D_\varepsilon \to L$, $\varphi_q(T) = \varphi(T, q)$. By the properties of the flow, if $|T| < \varepsilon - |T_0| = \rho$, we have

$$\varphi(T_0, \varphi(T, q)) = \varphi(T + T_0, q) = \varphi(T, \varphi(T_0, q)) = \varphi(T, q),$$

because $q \in A$. This implies that for $p \in \varphi_q(D_\rho)$ we have $\varphi(T_0, p) = p$ and so $\varphi_q(D_\rho) \subset A$ and A is open. This finishes the proof of Lemma 2.

Corollary. *Let Z be as before, $\delta > 0$ as in Lemma 2 and τ: $B_r - \{0\} \to D_\delta - \{0\}$ a continuous function. Define f: $B_r - \{0\} \to \mathbf{C}^n - \{0\}$ by $f(z) = \varphi(\tau(z), z)$. Then $f(z) \neq z$ for any $z \in B_r - \{0\}$. Furthermore, if $(f - \mathrm{id})_*$: $H_{2n-1}(\mathbf{C}^n - \{0\}) \to H_{2n-1}(\mathbf{C}^n - \{0\})$ is the map induced by $f - \mathrm{id}$ in the homology level, then $(f - \mathrm{id})_*$ is the multiplication by $\mu = \mathrm{ind}_0(Z)$.*

Observe that $H_{2n-1}(B_r - \{0\}) \subset H_{2n-1}(\mathbf{C}^n - \{0\})$ and this inclusion is an isomorphism between the two groups.

Proof. Since $0 < |\tau(z)| < \delta$ we have by Lemma 2 that $\varphi(\tau(z), z) \neq z$, $z \in B_r - \{0\}$. On the other hand by Lemma 1, $\mu = \mathrm{ind}_0(Z) = \mathrm{degree}(g)$, where $g(z) = (\varphi(\tau(z), z) - z)/\|\varphi(\tau(z), z) - z\|$, and so g_*: $H_{2n-1}(\partial B_r) \to H_{2n-1}(S^{2n-1})$ is given by $g_*(\sigma) = \mu\sigma$, where σ is a generator of $H_{2n-1}(\partial B_r)$. Now i_s: $\mathbf{C}^n - \{0\} \to \partial B_s$, given by $i_s(z) = sz/\|z\|$, is a homotopy equivalence, therefore the commutativity of the diagram

$$\begin{tikzcd}
H_{2n-1}(B_r - \{0\}) \arrow[r, "i_{r*}"] \arrow[d, "(f - \mathrm{id})_*"'] & H_{2n-1}(\partial B_r) \arrow[d, "g_*"] \\
H_{2n-1}(\mathbf{C}^n - \{0\}) \arrow[r, "i_{1*}"] & H_{2n-1}(S^{2n-1})
\end{tikzcd}$$

implies the corollary.

3. Topological invariance of the Milnor number

This section is devoted to the proof of

Theorem A. *The Milnor number of a holomorphic vector field is a topological invariant in $\mathbf{C}^n$, $n \geqslant 2$. In other words, if Z and $\tilde{Z}$ are holomorphic vector fields in $\mathbf{C}^n$, $n \geqslant 2$, locally topologicaly equivalent at p and $\tilde{p}$, then $\mu(Z, p) = \mu(\tilde{Z}, \tilde{p})$.*

Suppose that Z and $\tilde{Z}$ are locally equivalent at $0 \in \mathbf{C}^n$ by a homeomorphism $h\colon B_{r'} \to h(B_{r'}) = U$. Let $\varphi\colon D_\varepsilon \times \overline{B}_r \to \mathbf{C}^n$ and $\tilde{\varphi}\colon D_\varepsilon \times \overline{B}_{\tilde{r}} \to \mathbf{C}^n$ be the local complex flows of Z and $\tilde{Z}$ respectively. If we take r' small we can suppose that $B_{r'} \subset B_r$ and $U \subset B_{\tilde{r}}$. Let $0 < \rho < r'$ be such that $\varphi(D_\varepsilon \times B_\rho) \subset B_{r'}$ and $\tilde{\varphi}(D_\varepsilon \times h(B_\rho)) \subset U$. Let $\delta = \delta(Z)$ and $\tilde{\delta} = \delta(\tilde{Z})$ be as in Lemma 2, that is, for all $(T, z) \in (D_\delta - \{0\}) \times (B_r - \{0\})$ and $(\tilde{T}, \tilde{z}) \in (D_{\tilde{\delta}} - \{0\}) \times (B_{\tilde{r}} - \{0\})$ we have $\varphi(T, z) \neq z$ and $\tilde{\varphi}(\tilde{T}, \tilde{z}) \neq \tilde{z}$.

Lemma 3. *There exist continuous functions $\tau\colon B_\rho - \{0\} \to (0, \delta)$ and $\tilde{\tau}\colon h(B_\rho) - \{0\} \to D_{\tilde{\delta}} - \{0\}$ such that for all $z \in B_\rho - \{0\}$ we have*

$$h(\varphi(\tau(z), z)) = \tilde{\varphi}(\tilde{\tau}(h(z)), h(z)).$$

Proof. We need a notation. Let $z_0 \in B_r - \{0\}$. Since $Z(z_0) \neq 0$, by the complex flow box theorem, there exist $\alpha > 0$, a neighborhood $Q(z_0) = Q$ of z_0 and a holomorphic diffeomorphism $g\colon D_\alpha \times B_\alpha \to Q$, $B_\alpha = \{z \in \mathbf{C}^{n-1} \mid \|z\| < \alpha\}$, such that $g(0,0) = z_0$ and $g(T + T_0, w) = \varphi(T, g(T_0, w))$, provided the two members are defined. Let $V(z_0) = V = g(D_{\alpha/2} \times B_\alpha) \subset Q$. We call the triple (V, Q, α) a distinguished flow box of Z. For $q \in Q$ we call $P_q = g(D_\alpha \times p_2(g^{-1}(q)))$ the plaque of q in Q, where $p_2\colon D_\alpha \times B_\alpha \to B_\alpha$ is the natural projection. We remark that the leaves of Z/Q are the plaques of Q and the foliation of Z/Q is trivial. Moreover if $|T| < \alpha/2$ and $q \in V$, then $\varphi(T, q) \in P_q \subset Q$. We shall use the notation $(\tilde{V}, \tilde{Q}, \tilde{\alpha})$ for a distinguished flow box of $\tilde{Z}$, and $\tilde{P}_{\tilde{q}}$ for a plaque of $\tilde{Q}$.

Let (V, Q, α) and $(\tilde{V}, \tilde{Q}, \tilde{\alpha})$ be distinguished flow boxes for Z and $\tilde{Z}$ respectively such that $h(Q) \subset \tilde{V}$. Since h is an equivalence between Z and $\tilde{Z}$, it is clear that $h(P_q) \subset \tilde{P}_{h(q)}$. It follows that there exists a continuous function $S\colon D_{\alpha/2} \times V \to D_{2\tilde{\alpha}}$ such that $h(\varphi(T, q)) = \tilde{\varphi}(S(T, q), h(q))$ and $S(T, q) = 0$ if and only if $T = 0$.

Let $\{(V_j, Q_j, \alpha_j)\}_{j=0}^\infty$ and $\{(\tilde{V}_j, \tilde{Q}_j, \tilde{\alpha}_j)\}_{j=0}^\infty$ be countable sets of distinguished flow boxes for Z and $\tilde{Z}$ respectively satisfying the following properties:

(a) $\{V_j\}_{j=0}^\infty$ and $\{\tilde{V}_j\}_{j=0}^\infty$ are locally finite coverings of $B_\rho - \{0\}$ and $h(B_\rho) - \{0\}$ respectively.

(b) For any $j \in \mathbf{N}$ there exists $i(j) = i$ such that $h(Q_j) \subset \tilde{V}_i$.

(c) $\alpha_j < \delta/2$ and $\tilde{\alpha}_j < \tilde{\delta}/4$ for any $j \in \mathbf{N}$.

Let $S_j\colon D_{\alpha_j/2} \times V_j \to D_{2\tilde{\alpha}_i}$ be so that $h(\varphi(T, q)) = \tilde{\varphi}(S_j(T, q), h(q))$ for $(T, q) \in D_{\alpha_j/2} \times V_j$. Observe that if $q \in V_j \cap V_{j'} \neq \varnothing$ and $|T| < \frac{1}{2}\min\{\alpha_j, \alpha_{j'}\}$, then $\varphi(T, q) \in Q_j \cap Q_{j'}$ and the equality

$$h(\varphi(T, q)) = \tilde{\varphi}\big(S_j(T, q), h(q)\big) = \tilde{\varphi}\big(S_{j'}(T, q), h(q)\big)$$

implies that $S_j(T, q) = S_{j'}(T, q)$. Let $\tau\colon B_\rho - \{0\} \to (0, \delta)$ be a continuous function such that $\tau(q) \leqslant \alpha_j/2$ if $q \in V_j$. Define $\tilde{\tau}\colon h(B_\rho) - \{0\} \to D_{\tilde{\delta}} - \{0\}$

by $\tilde{\tau}(\tilde{q}) = S_j(\tau(h^{-1}(\tilde{q})), h^{-1}(\tilde{q}))$ if $\tilde{q} \in H(V_j)$. By the preceding remark, it is clear that $\tilde{\tau}(\tilde{q})$ does not depend on j such that $\tilde{q} \in h(V_j)$ and so $\tilde{\tau}$ is well defined. Moreover

$$h(\varphi(T(q), q)) = \tilde{\varphi}(S_j(\tau(q), q), h(q)) = \tilde{\varphi}(\tilde{\tau}(h(q)), h(q)).$$

q.e.d.

We write $f(z) = \varphi(\tau(z), z)$, $z \in B_\rho - \{0\}$, and $\tilde{f}(w) = \tilde{\varphi}(\tilde{\tau}(w), w)$, $w \in h(B_\rho) - \{0\}$. By Lemma 3 we have $h \circ f = \tilde{f} \circ h$. On the other hand, by the corollary, $(f - \mathrm{id})_*$ and $(\tilde{f} - \mathrm{id})_*$, at the homology level, are the multiplications by μ and $\tilde{\mu}$ respectively. Let

$$h_*: H_{2n-1}(\mathbf{C}^n - \{0\}) \to H_{2n-1}(\mathbf{C}^n - \{0\})$$

be the isomorphism induced by h. Clearly the following lemma implies Theorem A.

Lemma 4. *The following diagram commutes*:

$$\begin{array}{ccc} H_{2n-1}(\mathbf{C}^n - \{0\}) & \xrightarrow{(f-\mathrm{id})_*} & H_{2n-1}(\mathbf{C}^n - \{0\}) \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ H_{2n-1}(\mathbf{C}^n - \{0\}) & \xrightarrow{(\tilde{f}-\mathrm{id})_*} & H_{2n-1}(\mathbf{C}^n - \{0\}) \end{array}$$

Proof. Since $h \circ f = \tilde{f} \circ h$ we have $(\tilde{f} - \mathrm{id}) \circ h = \tilde{f} \circ h - h = h \circ f - h$. It is sufficient to prove that $h \circ f - h$ and $h \circ (f - \mathrm{id})$: $B_\rho - \{0\} \to C^n - \{0\}$ are homotopic. We define a homotopy F: $[0,1] \times (B_\rho - \{0\}) \to C^n - \{0\}$ by $F(t, z) = h(f(z) - (1 - t)z) - h(tz)$. Then F is continuous and $F(t, z) \neq 0$ for all $(t, z) \in [0,1] \times (B_\rho - \{0\})$, because $F(t, z) = 0$ implies $h(f(z) - (1 - t)z) = h(tz)$ and since h is a homeomorphism $f(z) - (1 - t)z = tz$, hence $f(z) = z$, which contradicts $\varphi(\tau(z), z) \neq z$.

4. Invariance of the multiplicity along a subvariety

We observe that Theorem A is false for $n = 1$. For instance, let $Z(z) = z^m$ and $\tilde{Z}(z) = z^n$, $m \neq n$. Then $\mathrm{ind}_0(Z) = m \neq \mathrm{ind}_0(\tilde{Z}) = n$. On the other hand, $\mathbf{C} - \{0\}$ is the unique nonsingular leaf for Z and $\tilde{Z}$, which implies that the identity of $\mathbf{C}$ is an equivalence between Z and $\tilde{Z}$.

Nevertheless Theorem A remains true for $n = 1$ in the "restricted case", as we shall see below.

Definition. Let Z be a germ at $p \in \mathbf{C}^n$ of a holomorphic vector field and V a germ at p of an analytic subvariety, say given by $V = f^{-1}(0)$, where f: $(\mathbf{C}^n, p) \to (\mathbf{C}^k, 0)$ is a germ at p of an analytic mapping. We say that V is *invariant by* Z if for any $q \in V$ we have $df_q \cdot Z(q) = 0$.

It follows from the definition that if V is an invariant subvariety, then V is saturated by the foliation defined by Z, that is, for any $q \in V$ the leaf L_q of Z through q satisfies $L_q \subset V$. From the point of view of topological equivalence the situation is particularly interesting when p is an isolated singularity and $\dim_{\mathbf{C}} V = 1$. In this case if we restrict Z to a sufficiently small neighborhood of p, then $V - \{p\}$ is a leaf of Z and we have the following proposition.

Proposition 2. *Let p and $\tilde{p}$ be isolated singularities of Z and $\tilde{Z}$ respectively and V a germ at p of an analytic irreducible subvariety invariant by Z and with complex dimension* 1. *Suppose that h: $(\mathbf{C}^n, p) \to (\mathbf{C}^n, \tilde{p})$ is a local topological equivalence between (Z, p) and $(\tilde{Z}, \tilde{p})$. Then $\tilde{V} = h(V)$ is a germ at $\tilde{p}$ of an irreducible analytic subvariety invariant by $\tilde{Z}$.*

Proof. Let h: $U \to \tilde{U}$ be a local equivalence between Z and $\tilde{Z}$, where U, $\tilde{U}$ are open neighborhoods of p, $\tilde{p}$ respectively, such that $Z(q) \neq 0$ if $q \in U - \{p\}$ and $\tilde{Z}(\tilde{q}) \neq 0$ if $\tilde{q} \in \tilde{U} - \{\tilde{p}\}$. We can suppose also that $V \cap U$ is connected. From now on we consider Z and $\tilde{Z}$ restricted to U and $\tilde{U}$ respectively and $V \subset U$. Since $V - \{p\}$ is connected, invariant by Z and $\dim_C V = 1$ it follows that $V - \{p\}$ is a leaf of Z. This implies that $\tilde{V} - \{\tilde{p}\} = h(V - \{p\})$ is a leaf of $\tilde{Z}$. Therefore $\tilde{V} - \{\tilde{p}\}$ is an analytic submanifold of $\tilde{U}$. On the other hand the closure in U of $\tilde{V} - \{\tilde{p}\}$ is

$$\mathrm{cl}_{\tilde{U}}(\tilde{V} - \{\tilde{p}\}) = \mathrm{cl}_{\tilde{U}}(h(V - \{p\})) = h(\mathrm{cl}_U(V - \{p\})) = h(V) = \tilde{V}.$$

It follows from Remmer-Stein's theorem (cf. [6]) that $\tilde{V}$ is an analytic subvariety, clearly invariant by $\tilde{Z}$. q.e.d.

Now let p be a singularity of Z and V an invariant subvariety such that $p \in V$ and $\dim_{\mathbf{C}} V = 1$. If B is a small ball around p, then $B \cap V$ is homeomorphic to a 2-dimensional disc. Such homeomorphism could be realized for example by a Puiseaux's parametrization of $B \cap V$. We are in the following situation: We have a disk $D = B \cap V$ and a vector field $X = Z/D$ with an unique singularity $p \in D$. In this case if we consider X as a real vector field, then the index of X at p is well defined. This motivates the following definition.

Definition. Let $Z, V, B, D = B \cap V$ and $p \in D$ be as before. We define the multiplicity of Z along V at $p \in D$ as the topological index of $X = Z/D$ at p, considered as a real vector field in D. We shall use the notation $\mathrm{ind}_p(Z/V)$ for this index.

We have the following result.

Theorem B. *The multiplicity of Z along V is a topological invariant. More specifically, let Z and $\tilde{Z}$ be holomorphic vector fields locally topologically equivalent by a homeomorphism h: $(\mathbf{C}^n, p) \to (\mathbf{C}^n, \tilde{p})$, where p and $\tilde{p}$ are isolated singularities of Z and $\tilde{Z}$ respectively. Suppose that V is an invariant subvariety of Z with $p \in V$ and $\dim_{\mathbf{C}} V = 1$. Let $\tilde{V} = h(V)$. Then $\mathrm{ind}_p(Z/V) = \mathrm{ind}_{\tilde{p}}(\tilde{Z}/\tilde{V})$.*

Now let us compute this index in terms of a Puiseaux's parametrization of V.

Proposition 3. *Let V be an invariant subvariety of Z of dimension* 1 *and $p \in V$ an isolated singularity of Z. Let α: $(D,0) \to (\mathbf{C}^n, p)$ be a Puiseaux's parametrization of a neighborhood of p in V. Then there exists a unique holomorphic vector field X in D such that $d\alpha \cdot X = Z \circ \alpha$. Furthermore if $X(t) = \sum_{j \geqslant m} a_j t^j$, with $a_m \neq 0$, then $\operatorname{ind}_p(Z/V) = m$.*

Proof. We can suppose $p = 0$. Since α: $(D,0) \to (\mathbf{C}^n, 0)$ is a Puiseaux's parametrization of V, then α is one-to-one, holomorphic in D and $\alpha'(t) \neq 0$ if $t \neq 0$ (cf. [12]).

Let $t \in D - \{0\}$. Then the tangent space of V at $\alpha(t)$ is generated by $\alpha'(t)$. Therefore, by invariance, we can write $Z(\alpha(t)) = X(t)\alpha'(t) = d\alpha(t) \cdot X(t)$, where $X(t) \in \mathbf{C}$. Clearly X: $D - \{0\} \to \mathbf{C}$ is holomorphic. Let us prove that X extends analytically to 0. We can write $\alpha(t) = (\alpha_1(t)), \cdots, \alpha_n(t))$, where $\alpha_j(t) = t^{m_j}\xi_j(t)$, $m_j \geqslant 1$, $\xi_j(0) \neq 0$ or $\xi_j \equiv 0$. Let $m_k = \min\{m_j \mid \xi_j \neq 0\}$. Since $0 \in \mathbf{C}^n$ is a singularity of Z, the Taylor series of Z_k at $0 \in \mathbf{C}^r$ can be written as $Z_k(z) = \sum_{|\sigma| \geqslant 1} a_\sigma z^\sigma$, where $\sigma = (\sigma_1, \cdots, \sigma_n)$, $z^\sigma = z_1^{\sigma_1} \cdots z_n^{\sigma_n}$ and $|\sigma| = \sigma_1 + \cdots + \sigma_n$. From this expression we have

$$Z_k(\alpha(t)) = \sum_{|\sigma| \geqslant 1} a_\sigma t^{\langle \sigma, m\rangle} \xi^\sigma, \quad \text{where } \langle \sigma, m \rangle = \sum_{i=1}^{n} \sigma_i m_i.$$

It follows that $Z_k(\alpha(t)) = t^{m_k}\lambda(t)$, where $\lambda(t)$ is analytic in a neighborhood of 0. On the other hand we have $\alpha_k'(t) = m_k t^{m_k - 1}\xi_k(t) + t^{m_k}\xi_k'(t) = t^{m_k-1}u(t)$, where $u(0) \neq 0$. This implies that

$$X(t) = \frac{1}{\alpha_k'(t)} Z_k(\alpha(t)) = t\frac{\lambda(t)}{u(t)}$$

is analytic in a neighborhood of 0.

Since α: $D \to V$ is a diffeomorphism outside $0 \in D$ it is clear that the indices of X at $0 \in D$ and Z/V at $0 \in V$ are equal.

Now let $X(t) = t^m\gamma(t)$, where $\gamma(0) \neq 0$. If $\rho > 0$ is small, then the variation of the argument of $X(\rho e^{i\theta}) = \rho^m e^{im\theta}\gamma(\rho e^{i\theta})$, as θ varies from 0 to 2π, is $2m\pi$. Hence $\operatorname{ind}_0(X) = m$. This ends the proof.

Proof of Theorem B. Except for Lemma 1 the proof is similar to the proof of Theorem A. Let Z and $\tilde{Z}$ be holomorphic vector fields with an isolated singularity at $0 \in \mathbf{C}^n$, $n \geqslant 2$, and suppose they are locally equivalent by a homeomorphism h: $B_r \to h(B_r) = U$. Let $V \subset B_r$ be an invariant variety of Z with $0 \in V$ and $\tilde{V} = h(V)$. We want to prove that $\operatorname{ind}_0(Z/V) = \operatorname{ind}_0(\tilde{Z}/\tilde{V})$.

Let φ and $\tilde{\varphi}$ be the local complex flows of Z and $\tilde{Z}$ respectively, let τ and $\tilde{\tau}$ be continuous functions as in Lemma 3 and let $f(z) = \varphi(\tau(z), z)$ and $\tilde{f}(w) = \tilde{\varphi}(\tilde{\tau}(w), w)$ be as before, so that we have $h \circ f = \tilde{f} \circ h$. Let α: $D \to V$ and $\tilde{\alpha}$: $D \to \tilde{V}$ be Puiseaux's parametrizations of V and $\tilde{V}$ and let X and $\tilde{X}$ be the vector fields in D given by Proposition 3, that is, such that $d\alpha \cdot X = Z \circ \alpha$ and $d\tilde{\alpha} \cdot \tilde{X} = \tilde{Z} \circ \tilde{\alpha}$. Define F: $\alpha^{-1}(B_\rho) \to D$ and $\tilde{F}$: $\tilde{\alpha}^{-1}(h(B_\rho)) \to D$ by $F = \alpha^{-1} \circ f \circ \alpha$ and $\tilde{F} = \tilde{\alpha}^{-1} \circ \tilde{f} \circ \tilde{\alpha}$. If $H = \tilde{\alpha} \circ h \circ \alpha^{-1}$, then clearly $H \circ F = \tilde{F} \circ H$. The same proof of Lemma 4 implies that the following diagram commutes:

$$\begin{tikzcd}
H_1(\mathbf{C} - \{0\}) \arrow[r, "(F-\mathrm{id})_*"] \arrow[d, "H_*"'] & H_1(\mathbf{C} - \{0\}) \arrow[d, "H_*"] \\
H_1(\mathbf{C} - \{0\}) \arrow[r, "(\tilde{F}-\mathrm{id})_*"] & H_1(\mathbf{C} - \{0\})
\end{tikzcd}$$

From this fact, it is sufficient to prove that $(F - \mathrm{id})_*$ and $(\tilde{F} - \mathrm{id})_*$ are the multiplications by $m = \mathrm{ind}_0(Z/V) = \mathrm{ind}_0(X)$ and $\tilde{m} = \mathrm{ind}_0(\tilde{X})$ respectively. We shall prove this fact for $(\tilde{F} - \mathrm{id})_*$.

Using the homotopy equivalence i_s: $\mathbf{C} - \{0\} \to S_s^1$, $i_s(z) = sz/|z|$, we see that it is sufficient to prove that the maps γ_0, γ_1: $S_r^1 \to S^1$ defined by $\gamma_0(z) = \tilde{X}(z)/|\tilde{X}(z)|$ and $\gamma_1(z) = (\tilde{F}(z) - z)/|\tilde{F}(z) - z|$ are homotopic. Let $\tilde{\psi}(T, z) = \tilde{\alpha}^{-1}(\tilde{\varphi}(T, \tilde{\alpha}(z)))$ be the local flow in D induced by $\tilde{\varphi}$. Define G: $[0,1] \times S_r^1 \to S^1$ by

$$G(t, z) = \frac{\tilde{\psi}(t\tilde{\tau} \circ \tilde{\alpha}(z), z) - z}{|\tilde{\psi}(t\tilde{\tau} \circ \tilde{\alpha}(z), z) - z|}.$$

Then $G(1, z) = \gamma_1(z)$ and

$$\lim_{t \to 0} G(t, z) = \frac{\tilde{\tau} \circ \tilde{\alpha}(z)}{|\tilde{\tau} \circ \tilde{\alpha}(z)|} \cdot \frac{\tilde{X}(z)}{|\tilde{X}(z)|}.$$

Now, the maps 1 and $\tilde{\tau}$: $U - \{0\} \to \mathbf{C} - \{0\}$ are homotopic, because $U - \{0\} = h(B_r) - \{0\}$ is homotopically equivalent to S^{2n-1} and $\pi_{2n-1}(C - \{0\})$ is trivial. If $\tilde{\tau}_t$ is a homotopy between 1 and $\tilde{\tau}$, then $\tilde{\tau}_t \circ \tilde{\alpha}/|\tilde{\tau}_t \circ \tilde{\alpha}|$ is a homotopy between 1 and $\tilde{\tau} \circ \tilde{\alpha}/|\tilde{\tau} \circ \tilde{\alpha}|$, which ends the proof of Theorem B.

5. Applications

(5.1) Vector fields for which the first nonzero jet is isolated. Here we consider two holomorphic vector fields Z^1 and Z^2 in $\mathbf{C}^n$, $n \geqslant 2$, whose Taylor's series at 0 are of the form

$$Z^j(z) = \sum_{i=k_j}^{\infty} Z_i^j(z), \qquad j = 1,2,$$

where Z_i^j are homogeneous vector fields of degree i ($j = 1,2$) and $Z_{k_1}^1$ and $Z_{k_2}^2$ (the first nonzero jets of Z^1 and Z^2) have 0 as an isolated singularity. We have the following result.

Proposition 4. *Let Z^1 and Z^2 be as before and suppose that they are locally topologically equivalent at* 0. *Then $k_1 = k_2$.*

Proof. We shall prove that $\mu(Z^j, 0) = k_j^n$ ($j = 1,2$). From Theorem A it follows that $k_1^n = k_2^n$ and so $k_1 = k_2$.

Let $Z = Z_k + Z_{k+1} + \cdots$ be the Taylor's series of Z in a neighborhood of 0, where 0 is an isolated singularity of Z_k. Let $m = \inf\{\|Z_k(z)\|;\ \|z\| = 1\}$ and $m_j = \sup\{\|Z_j(z)\|;\ \|z\| = 1\}$. We have $\|Z_j(z)\| \leqslant m_j\|z\|^j$, $j \geqslant k+1$. This implies that

$$\|Z(z)\| \geqslant \left(m - \sum_{j \geqslant k+1} m_j\|z\|^{j-k}\right)\|z\|^k.$$

By the convergence of the series, there exists $\varepsilon > 0$ such that for $\|z\| < \varepsilon$ we have $m - \sum_{j \geqslant k+1} m_j\|z\|^{j-k} \geqslant m/2$ and so $\|Z(z)\| \geqslant m\|z\|^k/2$ for $\|z\| < \varepsilon$. Let $0 < r < \varepsilon$ and consider the homotopy G: $[0,1] \times S_r^{2n-1} \to S^{2n-1}$ given by

$$\begin{aligned} G(t,z) &= \frac{Z(tz)}{\|Z(tz)\|} = \sum_{j \geqslant k} t^j Z_j(z) \Big/ \Big\| \sum_{j \geqslant k} t^j Z_j(z) \Big\| \\ &= \sum_{j \geqslant k} t^{j-k} Z_j(z) \Big/ \Big\| \sum_{j \geqslant k} t^{j-k} Z_j(z) \Big\|. \end{aligned}$$

Then $G(1,z) = Z(z)/\|Z(z)\|$ and $G(0,z) = Z_k(z)/\|Z_k(z)\|$. Hence $\mu(Z,0) = \mu(Z_k,0)$. Now, if $c \neq 0$ is a regular value of the map Z_k: $\mathbf{C}^n \to \mathbf{C}^n$, then by Bézout's theorem the number of solutions of the equation $Z_k(z) = c$ is exactly k^n. It follows from remark (4) in §2 that $\mu(Z_k,0) = k^n$. This proves the proposition.

Remark. In the case where $k_1 = 1$ we do not need to suppose that 0 is an isolated singularity for Z_{k_2}. This fact, together with $k_2 = 1$, follows from the local equivalence and the fact that $\mu(Z,0) = 1$ if and only if $\det(DZ(0)) \neq 0$.

(5.2) Complex saddle-nodes.

Definition. Let Z be a vector field in $\mathbf{C}^2$ with an isolated singularity at $p \in \mathbf{C}^2$. We say that p is *a saddle-node for Z* if the eigenvalues of the linear part of Z at p, $DZ(p)$, say λ_1 and λ_2, satisfy $\lambda_1 = 0$ and $\lambda_2 \neq 0$. If E_1 and E_2 are the eigenspaces of $DZ(p)$ corresponding to λ_1 and λ_2, we call E_1 and E_2 the weak direction and strong direction of Z respectively. The name saddle-node is used here in analogy with the real case.

Proposition 5. *Let Z and $\tilde{Z}$ be holomorphic vector fields in $\mathbf{C}^2$ with isolated singularities at p and $\tilde{p}$ respectively. Suppose that Z and $\tilde{Z}$ are locally topologically equivalent at p and $\tilde{p}$ and that p is a saddle-node for Z. Then $\tilde{p}$ is a saddle-node for $\tilde{Z}$.*

Proof. We need the following theorem (cf. [10]).

Theorem. *Let p be a saddle-node for Z and E the strong direction of Z at p. Then there exists a germ of analytic manifold $W \subset \mathbf{C}^2$, such that $p \in W$, W is tangent to E at p and W is invariant by Z.*

Since W is an analytic submanifold of $\mathbf{C}^2$, there exist analytical coordinates (x, y) in a neighborhood of p such that $x(p) = y(p) = 0$ and $W \subset \{(x, y) \mid y = 0\}$. Let $Z(x, y) = (P(x, y), Q(x, y))$ be the expression of Z in this coordinate system. The invariance of W implies that $Q(x, 0) \equiv 0$ and $P(x, 0) = \lambda x + a_2 x^2 + \cdots$, because W is tangent to E. It follows that $\operatorname{ind}_0(Z/W) = 1$. Now, let h: $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, \tilde{p})$ be a local equivalence between Z and $\tilde{Z}$. We can suppose that $\tilde{p} = 0$. Let $\tilde{W} = h(W)$. By Proposition 2, $\tilde{W}$ is an analytic subvariety of $\mathbf{C}^2$ and since $\pi_1(B_r - \tilde{W}) = Z$, $B_r = \{(u, v) \in \mathbf{C}^2 \mid |u|^2 + |v|^2 < r\}$, r small, we can conclude that $\tilde{W}$ is in fact a submanifold of $\mathbf{C}^2$ (cf. [14]). Let $\tilde{E}$ be the subspace of dimension 1 tangent to $\tilde{W}$. Then $\tilde{E}$ is invariant by $D\tilde{Z}(0)$ and by Theorem B $\operatorname{ind}_0(\tilde{Z}/\tilde{W}) = 1$. These facts imply that the eigenvalue of $D\tilde{Z}(0)/\tilde{E}$ is not zero. Now $\mu(Z, 0) > 1$ and so by Theorem A, $\mu(\tilde{Z}, 0) > 1$, which implies that $\det(D\tilde{Z}(0)) = 0$, hence the other eigenvalue of $D\tilde{Z}(0)$ is 0, therefore 0 is a saddle-node of $\tilde{Z}$.

CHAPTER II
THE EQUIDESINGULARIZATION THEOREM

It is a well-known theorem in the theory of singularities of curves in $\mathbf{C}^2$ that locally homeomorphic singularities have isomorphic desingularizations. We discuss here the corresponding problem for isolated singularities of holomorphic vector fields in $\mathbf{C}^2$.

1. Generalized curves

The blowing-up method (Cf. [10]). Let Z be a holomorphic vector field defined in an open subset $\mathscr{U} \subset \mathbf{C}^2$, with $0 \in \mathscr{U}$ and $Z(0) = 0$, i.e., $0 \in \mathbf{C}^2$ is a *singularity* of Z. We assume that $0 \in \mathbf{C}^2$ is the unique singularity of Z in $\mathscr{U}$. Most of the time we will be interested in the foliation $\mathscr{F}_Z$ induced by the 1-form ω,

$$\omega = Z_2(x, y)\, dx - Z_1(x, y)\, dy = 0,$$

where

$$Z = Z_1(x, y)\frac{\partial}{\partial x} + Z_2(x, y)\frac{\partial}{\partial y}.$$

Clearly $\mathscr{F}_Z$ is regular in $\mathscr{U}$ except at $0 \in \mathbf{C}^2$. The singularity is said to be *simple* if the eigenvalues λ_1, λ_2 of $dZ(0)$ satisfy one of the conditions:

(i) $\lambda_1 \cdot \lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbf{Q}_+$;

(ii) $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

The blow up (or explosion) of $0 \in \mathbf{C}^2$ consists of replacing $0 \in \mathbf{C}^2$ by a one-dimensional projective line P considered as the set of limit directions at $0 \in \mathbf{C}^2$. We introduce complex coordinates in $\mathscr{U}^{(1)} = \mathscr{U} \setminus \{0\} \cup P$ as follows: any open subset of $\mathscr{U} \setminus \{0\}$ keeps its coordinates; in order to cover P we use two charts $\varphi\colon V_1 \times \mathbf{C} \to \mathscr{U} \setminus (y = 0) \cup P \setminus \{0\}$ and $\psi\colon \mathbf{C} \times V_2 \to U \setminus (x = 0) \cup P \setminus \{\infty\}$ related by $\psi^{-1} \circ \varphi(t, x) = (t^{-1}, tx)$, $t \neq 0$; that is, we select points 0 and ∞ in P (which is the same as choosing two independent complex lines in $\mathbf{C}^2$); these charts cover neighborhoods of $P \setminus \{\infty\}$ and $P \setminus \{0\}$. The projection $\pi^{(1)}\colon \mathscr{U}^{(1)} \to \mathscr{U}$ is given by $\pi(p) = p$ for $p \in P$ and $\pi(p) = 0$ for $p \in P$ and is written in these coordinates as $(x, t) \mapsto (x, tx)$ and $(u, y) \mapsto (uy, y)$ respectively. We now lift the foliation $\mathscr{F}_Z$ to $\mathscr{U}^{(1)}$. Suppose

$$\dot{x} = a_\nu(x, y) + R_1(x, y),$$
$$\dot{y} = b_\nu(x, y) + R_2(x, y),$$

where $(a_\nu(x, y), b_\nu(x, y))$ is the first nonzero jet of Z at $0 \in \mathbf{C}^2$; $\nu = \nu_Z \in \mathbf{N}$ is the *algebraic multiplicity of* Z (or $\mathscr{F}_Z$) at the singularity. We have the following equations for $\pi^* Z$:

$$(*) \qquad \begin{aligned} \dot{x} &= x^\nu\big(a_\nu(1, t) + xR_1'(x, t)\big), \\ \dot{t} &= x^{\nu-1}\big(b_\nu(1, t) - ta_\nu(1, t)\big) + x^\nu\big(R_2'(x, t) - tR_1'(x, t)\big); \end{aligned}$$

$$\dot{u} = y^{\nu-1}\big(a_\nu(u,1) - ub_\nu(u,1)\big) + y^\nu\big(R_1''(u, y) - uR_2''(u, y)\big),$$
$$\dot{y} = y^\nu\big(b_\nu(u,1) + yR_2''(u, y)\big).$$

Now, all points of P are singularities of $\pi^* Z$. We have two ways of desingularizing it, according to whether $b_\nu(1, t) - ta_\nu(1, t)$ is identically zero or not.

(i) *Nondicritical singularity.* $b_\nu(1, t) - ta_\nu(1, t) \neq 0$. Dividing $(*)$ by $x^{\nu-1}$ we get

$$(**) \qquad \begin{aligned} \dot{x} &= x\big(a_\nu(1, t) + xR_1'(x, t)\big), \\ \dot{t} &= b_\nu(1, t) - ta_\nu(1, t) + x\big(R_2'(x, t) - tR_1'(x, t)\big). \end{aligned}$$

The expression found in the other coordinate system (after dividing by $y^{\nu-1}$) fits with $(**)$ to define a foliation $\mathscr{F}_Z^{(1)}$ in $\mathscr{U}^{(1)}$ having P as an invariant set. More precisely, up to a certain number of isolated singularities given by the roots of $b_\nu(1, t) - ta_\nu(1, t) = 0$, P is a leaf of $\mathscr{F}_Z^{(1)}$. Observe that $\mathscr{F}_Z^{(1)}$ and $\pi^* \mathscr{F}_Z$ coincide outside $P^{(1)}$.

(ii) *Dicritical singularity*. $b_\nu(1, t) - ta_\nu(1, t) \equiv 0$. After dividing $(*)$ by x^ν we find

$$\dot{x} = a_\nu(1, t) + xR_1'(x, t),$$
$$\dot{t} = R_2'(x, t) - tR_1'(x, t).$$

Combining this with the correspondent expression in the other coordinate system, we get equations of a foliation $\mathscr{F}_Z^{(1)}$ which coincides with $\pi^*\mathscr{F}_Z$ outside P but this time the projective line P is no longer an invariant set. The foliation $\mathscr{F}_Z^{(1)}$ is transverse to P except at a finite number of points (the roots of $a_\nu(1, t) = 0$) which may or may not be singularities.

It is important to notice in both cases that the foliations are locally given by analytic expressions. Therefore, we can repeat the process at any of the singularities of $\mathscr{F}_Z^{(1)}$. A new foliation $\mathscr{F}_Z^{(2)}$ is found in a neighborhood of a union of projective lines having normal crossings and again exhibiting a finite number of singularities. The process can be repeated as long as we want. After k blow ups we have a foliation $\mathscr{F}_Z^{(k)}$ defined in a neighborhood $\mathscr{U}_Z^{(k)}$ of a union $\mathscr{P}_Z^{(k)}$ of projective lines with normal crossings and a proper analytic projection $\pi_Z^{(k)}\colon \mathscr{U}_Z^{(k)} \to \mathscr{U}$ which sends $\mathscr{P}_Z^{(k)}$ to $0 \in \mathbf{C}^2$ and such that $\pi_Z^{(k)}\colon \mathscr{U}_Z^{(k)} \setminus \mathscr{P}_Z^{(k)} \to \mathscr{U} \setminus \{0\}$ is an isomorphism between the foliations $\mathscr{F}_Z^{(k)}$ and $\mathscr{F}_Z$. We will write $(\mathscr{U}_Z^{(k)}, \pi_Z^{(k)}, \mathscr{P}_Z^{(k)}, \mathscr{F}_Z^{(k)})$ to denote a kth blow up of Z at $0 \in \mathbf{C}^2$; $\pi^{(k)}$ will be called its projection and $\mathscr{P}_Z^{(k)}$ its *divisor*. The divisor is a union of embedded projective lines intersecting transversely at points called *corners*, each corner being the intersection of two projective lines. The Desingularization Theorem for vector fields ([1], [8], [16]) asserts that all singularities become simple after a finite number of blow ups and these just repeat themselves under new explosions. Therefore if we start exploding $0 \in \mathbf{C}^2$ and if we agree that simple singularities, once they appear, are not submitted to further explosions, we definitely arrive to a situation where after l blow ups all singularities become simple. This defines uniquely the blow up $(\mathscr{U}_Z^{(l)}, \pi_Z^{(l)}, \mathscr{P}_Z^{(l)}, \mathscr{F}_Z^{(l)})$ as *the desingularization of* Z *at* $0 \in \mathbf{C}^2$. An alternative notation we will use for the desingularization of Z is $(\tilde{\mathscr{U}}_Z, \pi_Z, \mathscr{P}_Z, \tilde{\mathscr{F}}_Z)$. Sometimes when no confusion arises we will drop the subindices in either notation. In fact, we will often say that the divisor $\mathscr{P}$ with marked points, which are the singularities of $\tilde{\mathscr{F}}$, is the desingularization of Z.

2. The nondicritical case (finitely many separatrices)

In order to make the exposition more understandable we will assume first that the dicritical case does not appear in the blow up process. The remaining case will be considered in §5.

A formula for the computation of the algebraic multiplicity. In this section we describe how to compute the algebraic multiplicity of a singularity from data read off from its desingularization.

Definition. The *weight* $\rho(P)$ of a projective line P which appears in the process of desingularization is:

(i) 1, if the projective line appears immediately after exploding $0 \in C^2$,

(ii) the sum of the weights of the projective lines meeting at the singularity which was blown up to originate P.

Observe that once P is created we assign to it an integer $\rho(P)$ which will remain untouched under further explosions. This number coincides with the algebraic multiplicity of an irreducible curve whose desingularization is transversal to P. This fact will be proved in §3.

Suppose now that a smooth invariant analytic line S contains the singularity of a foliation $\mathscr{F}$ (given in a neighborhood of the singularity by an analytic vector field). Choosing convenient coordinates we may write

$$\dot{x} = x^n P(x) + yQ(x, y),$$
$$\dot{y} = yR(x, y),$$

where $y = 0$ stands for S and $P(0) \neq 0$.

Definition. The integer $n \in \mathbf{N}$ is the *multiplicity* of $\mathscr{F}$ in $q \in S$ along S. We denote it by $\mu_{\mathscr{F}}(q, S)$; it is easy to check the invariance of the definition under analytic changes of coordinates.

We consider now the desingularization $(\tilde{\mathscr{U}}, \pi, \tilde{P}, \tilde{\mathscr{F}})$ of Z. Let $p \in \tilde{P}$ be a singularity of $\tilde{\mathscr{F}}$. We take

$$\varphi(p, P) = \begin{cases} \mu_{\tilde{\mathscr{F}}}(p, P) & \text{if } p \in P \text{ is not a corner,} \\ \mu_{\tilde{\mathscr{F}}}(p, P) - 1 & \text{if } p \in P \text{ is a corner.} \end{cases}$$

Notice that two integers $\varphi(p, P_1)$ and $\varphi(p, P_2)$ are associated to a corner $\{p\} = P_1 \cap P_2$.

Theorem 1. *The algebraic multiplicity $\nu = \nu_Z$ of Z is given by*

$$\nu + 1 = \sum_{p \in \tilde{P}} \rho(P)\varphi(p, P),$$

where the summation runs through the singularities of $\tilde{\mathscr{F}}$.

Proof. Let us prove the formula for the foliation $\mathscr{F}^{(1)}$. Call P the projective line created from the explosion of Z, and $p_1, \cdots, p_s$ the singularities of $\mathscr{F}^{(1)}$. Then

$$\nu + 1 = \sum_{i=1}^{s} \varphi(p_i, P).$$

In fact, from

$$\dot{x} = x\big(a_\nu(1,t) + xR_1'(x,t)\big),$$
$$\dot{t} = b_\nu(1,t) - ta_\nu(1,t) + x\big(R_2'(x,t) - tR_1'(x,t)\big),$$

we find that the singularities of $\mathscr{F}^{(1)}$ are the roots of $b_\nu(1,t) - ta_\nu(1,t) = 0$ (we may arrange the coordinates so that ∞ is not a singularity of $\mathscr{F}^{(1)}$; in this case, degree $a_\nu(1,t) = \nu$). Now

$$b_\nu(1,t) - ta_\nu(1,t) = (t-t_1)^{a_1}, \cdots, (t-t_s)^{a_s}.$$

We conclude that $\varphi(p_i, P) = a_i$ and

$$\sum_{\mathscr{F}^{(1)}(p)=0} \varphi(p_i, P) = \sum_{i=1}^{s} a_s = \nu + 1.$$

In order to prove the formula in general, assume that we have

$$(*) \qquad \nu + 1 = \sum_{\mathscr{F}^{(k)}(p)=0} \rho(P)\varphi(p,P)$$

at the kth stage of the desingularization process. If $p \in P$ is to be exploded to a projective line P', two cases arise:

(i) p is not a corner. Introduce coordinates (x, y) at $p \in P$, y along P and x normal to it and write $\mathscr{F}_X^{(k)}$ near $p \in P$ as a vector field having an algebraic multiplicity ν'. We know that $\nu' + 1 = \sum \mu(q, P')$, where $q \in P'$ are singularities of $\mathscr{F}^{(k+1)}$. Let us still write $\{p\} = P \cap P'$ and denote for the moment by $\varphi'(p, P)$ the number $\mu_{\mathscr{F}_X^{(k+1)}}(p, P) - 1$. It is easy to see that $\varphi'(p, P) = \varphi(p, P) - (\nu' - 1)$. Observe that in $(*)$ $\varphi(p, P)$ appears multiplied by a factor $\rho(P)$; so we can replace $\rho(P) \cdot \varphi(p, P)$ by

$$\begin{aligned}
\rho(P)\big(\varphi'(p,P) + \nu' - 1\big) &= \rho(P)\left(\varphi'(p,P) - 1 + \sum_{q\in P'} \mu(q,P') - 1\right) \\
&= \rho(P)\big(\varphi'(p,P) - 1\big) + \sum_{q\neq p} \rho(P)\varphi(q,P') \\
&\quad + \rho(P)\big(\varphi(p,P') - 1\big).
\end{aligned}$$

We have $\rho(P) = \rho(P')$. Again changing the notation we find that $\rho(P)\varphi(p, P)$ has been replaced by

$$\rho(P)\big(\varphi(p,P) - 1\big) + \sum_{q\neq p} \rho(P')\varphi(q,P') + \rho(P')\big(\varphi(p,P') - 1\big).$$

(ii) p is a corner $P_1 \cap P_2$. Let $\{p_1\} = P_1 \cap P'$ and $\{p_2\} = P_2 \cap P'$. The idea is to find a substitute in $(*)$ for

$$\rho(P_1)\varphi(p,P_1) + \rho(P_2)\varphi(p,P_2).$$

As before, call ν' the algebraic multiplicity of a local analytic expression for $\mathscr{F}^{(k)}$ at $p \in P_1 \cap P_2$. We have

$$\nu' + 1 = \big(\varphi(p_1, P') + 1\big) + \big(\varphi(p_2, P') + 1\big) + \sum_{q \notin \{p_1, p_2\}} \varphi(q, P'),$$

$$\varphi(p_1, P_1) = \varphi(p, P_1) - (\nu' - 1),$$

$$\varphi(p_2, P_2) = \varphi(p, P_2) - (\nu' - 1).$$

so that

$$\begin{aligned}
&\rho(P_1)\varphi(p, P_1) + \rho(P_2)\varphi(p, P_2) \\
&\quad = \rho(P_1)\big(\varphi(p_1, P_1) + \nu' - 1\big) + \rho(P_2)\big(\varphi(p_2, P_2) + \nu' - 1\big) \\
&\quad = \rho(P_1)\varphi(p_1, P_1) + \rho(P_2)\varphi(p_2, P_2) + (\nu' - 1)(\rho(p_1) + \rho(P_2)) \\
&\quad = \rho(P_1)\varphi(p_1, P_1) + \rho(P_2)\varphi(p_2, P_2) \\
&\qquad + \left(\varphi(p_1, P') + \varphi(p_2, P') + \sum_{q \neq p_1, p_2} \varphi(q, P')\right)\rho(P').
\end{aligned}$$

Therefore the $(k + 1)$th stage of the desingularization also produces a formula $(*)$ for the computation of $\nu + 1$.

3. Generalized curves and their desingularizations

We want to consider now vector fields whose singularities in the desingularization can be foreseen before starting the blow up process (remark: we don't include the corners). The idea is the following: to detect the singularity in the desingularization picture by simply looking at the invariant curve which passes through it transversely to the projective line; this curve has a projection which is also an invariant curve but now containing $0 \in \mathbf{C}^2$. So we must guarantee that a normal invariant curve issues from every singularity located at the desingularization picture of the vector field. Let us adopt the following definition.

Definition. The vector field Z is a *generalized curve* if $\tilde{\mathscr{F}}$ in the desingularization of Z has no singularities with zero eigenvalues.

Definition. A *separatrix* of Z is a connected integral curve V of Z such that $\overline{V} = V \cup \{0\}$.

By the discussion above there exists a well-defined one-to-one correspondence between the separatrices of Z and the singularities of $\tilde{\mathscr{F}}$ which are not corners. We shall prove that the desingularization of a generalized curve is intimately linked to that of its separatrices.

As an example we may consider $Z = Z_f$ given by

$$\dot{x} = -\partial_2 f(x, y), \qquad \dot{y} = \partial_1 f(x, y),$$

where $f\colon \mathcal{U} \subset \mathbf{C}^2 \to \mathbf{C}$ is an analytic function having a singularity at $0 \in \mathbf{C}^2$. The integral curves of Z_f are the connected components of $f(x, y) = c$ for $c \in \mathbf{C}^*$ small and its separatrices the connected components of $f^{-1}(0)$ in $\mathcal{U} - \{0\}$. All integral curves are closed in $\mathcal{U}$ and do not approach $0 \in \mathbf{C}^2$, exception made to $f(x, y) = 0$. This property is not compatible with the existence of singularities of $\tilde{\mathcal{F}}_{Z_f}$ which have zero eigenvalue; in fact, an invariant curve along the nonzero eigenvalue direction would have a holonomy of the type $z \mapsto z + az^{k+1} + \cdots$, where $a \neq 0$ and $k \in \mathbf{N}$, in which case the leaves would no longer be closed [3].

More generally, let Z be a vector field in $\mathcal{U}$ with $Z(0) = 0$ and let $(S_j)_{j=1}^{k}$ be the set of separatrices of Z at $0 \in \mathbf{C}^2$. Then to each separatrix S_j there corresponds an irreducible function $f_j\colon \mathcal{U} \to \mathbf{C}$, $f_j(0) = 0$, such that $S_j = f_j^{-1}(0) \cap (\mathcal{U} \setminus \{0\})$. Then $f = \prod_{j=1}^{k} f_j$ is a decomposition of f in irreducible factors and if $S = \bigcup_{j=1}^{k} S_j$, then $S = f^{-1}(0) \cap (\mathcal{U} \setminus \{0\})$.

Theorem 2. *Let Z be a generalized curve with an isolated singularity at $0 \in \mathbf{C}^2$. If S is the union of separatrices of Z, then S and Z have the same desingularizations.*

Having the same desingularization means that the same sequence of explosions will lead to the desingularizations of Z and f above. As a consequence the final structure of projective lines and marked points in the divisor are the same. Notice that the statement is not completely true when Z is already a simple singularity; in order to unify all the possibilities we prefer to submit it to an explosion before saying it is desingularized.

Remark. $S \neq \varnothing$ by [4].

The proof of Theorem 2 is based upon the following lemma.

Lemma 1. *Any singularity of a generalized curve which possesses exactly two transversal smooth separatrices is simple.*

Proof. A curve which becomes transverse to a projective line of weight greater than one cannot be smooth. Therefore if ν is the algebraic multiplicity of Z we obtain from Theorem 1 that $\nu + 1 = 2$, i.e. $\nu = 1$. Let $A = DZ(0)$. Then $A \neq 0$ and we have the following possibilities:

(i) $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$,

(ii) $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$; $\lambda_1/\lambda_2 \in \mathbf{Q}_+$;

(iii) $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$; $\lambda_1/\lambda_2 \notin \mathbf{Q}_+$.

Case (i) will be excluded from our considerations because A has two independent eigenspaces (namely, the tangent spaces of the separatrices), so it is diagonalizable. In case (ii) if $\lambda_1 = 0$, Z is not a generalized curve; if $\lambda_1/\lambda_2 = n \in \mathbf{Z}_+$, the normal form of Dulac [5] allows us to write $Z = (\lambda_1 z_1 + a \cdot z_2^n, \lambda_2 z_2)$ and if $a \neq 0$, Z has only one separatrix; if $a = 0$, Z has dicritical components in its desingularization. Similarly if $\lambda_2/\lambda_1 \in \mathbf{Z}_+$. Finally if neither $\lambda_1/\lambda_2 \in \mathbf{Z}_+$ nor $\lambda_2/\lambda_1 \in \mathbf{Z}_+$, then Z is linearizable [13] and its desingularization has dicritical components. Only (iii) remains, but this means that the singularity is simple. q.e.d.

Proof of Theorem 2. Let us follow Z through the sequence of explosions which is demanded to desingularize S. It follows from the definition that $\mathscr{F}^{(k)}$ is a generalized curve near its singularities in $\mathscr{P}^{(k)}$. Through the corners pass two invariant projective lines. Thus from Lemma 1 we obtain they are simple singularities. The points where the transformed irreducible curves of S touch the divisor must also be singularities of $\mathscr{F}^{(k)}$; again by Lemma 1 they are simple singularities. Suppose $\mathscr{F}^{(k)}$ has another singularity; it will have only one separatrix contained in the divisor. From Theorem 1 we find a local algebraic multiplicity for $\mathscr{F}^{(k)}$ equal to zero which is not possible. q.e.d.

Remark. Let us consider again the example $Z = Z_f$; suppose f is irreducible. By Theorem 2, the desingularizations of Z_f and f are the same. Apply now the formula of Theorem 1: we get $\nu_Z + 1 = \rho(P)$, where P is the projective line in $\mathscr{P}$ transverse to the transformed curve of $f(x, y) = 0$. But $\nu_Z + 1$ is the algebraic multiplicity of f at $0 \in \mathbf{C}^2$, so that $\nu_f = \rho(P)$. Observe that further explosions do not change the weight of the projective line transversal to the transformed curve of $f(x, y) = 0$. Now suppose $f(x, y) = 0$ is reducible. Once we desingularize one of its components, we obtain a projective line transversal of weight equal to the algebraic multiplicity of that component.

Theorem 3. *Let Z be a generalized curve and $f_1(x, y) = 0, \cdots, f_k(x, y) = 0$ the equations of its separatrices. Then*

$$\nu_Z + 1 = \nu_{Z_f} + 1 = \sum_{i=1}^{k} \nu_{f_i}.$$

Proof. If $f = f_1 \cdots f_k$, then, by Theorem 2, Z and Z_f have the same desingularizations. Now apply the remark discussed above.

4. The Milnor number of a generalized curve

The reader must have noticed that the converse to Theorem 2 is missing. In fact, the converse is untrue. We have to introduce an algebraic condition in order to assure that a vector field which has the same desingularization as its separatrices is a generalized curve. This condition is to be discussed now.

We turn our attention to the Milnor number of a singularity of a vector field in $\mathbf{C}^2$. As before, we denote by $f(x, y) = 0$ the equation for the union of the separatrices of Z.

Theorem 4. *Given a vector field* Z *one has* $\mu(Z,0) \geqslant \mu(Z_f,0)$, *and equality holds if and only if* Z *is a generalized curve.*

Proof. Let us observe first that the theorem is true for simple singularities. In general we proceed by induction on the number of explosions Z_f needs to become desingularized. We will make use of a formula which relates the Milnor number of a vector field to the Milnor numbers of the singularities appearing in the first explosion, namely

$$(*) \qquad \mu(Z,0) = \nu_Z^2 - (\nu_Z + 1) + \sum_{i=1}^{s} \mu\left(\mathscr{F}_Z^{(1)}, p_i\right),$$

where $p_1, \cdots, p_s$ are the singularities of $\mathscr{F}_Z^{(1)}$ [10]. Suppose now Z_f needs just one explosion to become desingularized. In this case $\mathscr{F}_{Z_f}^{(1)}$ has singularities $p_1, \cdots, p_s$, all of them simple, thus with Milnor number equal to one; through each of them passes a smooth curve transversal to the projective line and invariant for $\mathscr{F}_{Z_f}^{(1)}$. This also holds for $\mathscr{F}_Z^{(1)}$, except that perhaps $p_1, \cdots, p_s$ are no longer simple and other singularities $p_{s+1}, \cdots, p_k$ of $\mathscr{F}_Z^{(1)}$ may exist. Accordingly we have two cases.

(1) Z is a generalized curve. Then $s = k$ and all singularities $p_1, \cdots, p_s$ are simple; so, from Theorem 1, $\nu_Z = \nu_{Z_f}$ and, by $(*)$, $\mu(Z,0) = \mu(Z_f,0)$.

(2) Z is not a generalized curve. Then

$$\sum_{i=1}^{k} \mu\left(\mathscr{F}_Z^{(1)}, p_i\right) > \sum_{j=1}^{s} \mu\left(\mathscr{F}_{Z_f}^{(1)}, p_j\right).$$

In fact, if $k > s$ this is obvious. On the other hand if $k = s$ and if all the singularities $p_1, \cdots, p_k$ have Milnor number one, it follows that $\mathscr{F}_Z^{(1)}$ at p_i has algebraic multiplicity one. Now a case by case argument like in the proof of Lemma 1 leads to a contradiction.

Let us assume now the theorem has been proved for all singularities whose set of separatrices needs $l \in \mathbf{N}$ explosions to be desingularized. Let Z be given such that it needs $l + 1$ explosions to reach the desingularized picture. Explode Z_f once and look to the $\mathscr{F}_{Z_f}^{(1)}$-singularities $p_1, \cdots, p_s$. All of these are also singularities of $\mathscr{F}_Z^{(1)}$. There appear the same alternatives as above.

(1) Z is a generalized curve. The points $p_1, \cdots, p_s$ are all the singularities of $\mathscr{F}_Z^{(1)}$. By the induction hypothesis, Theorem 1 and $(*)$ one finds $\mu(Z,0) = \mu(Z_f,0)$.

(2) Z is not a generalized curve. Therefore either $p_1, \cdots, p_s$ are all the singularities of $\mathscr{F}_Z^{(1)}$ so at least one of them is not a generalized curve or $\mathscr{F}_Z^{(1)}$ has additional singularities $p_{s+1}, \cdots, p_k$.

In the first situation

$$\sum_{i=1}^{s} \mu\left(\mathscr{F}_Z^{(1)}, p_i\right) > \sum_{i=1}^{s} \mu\left(\mathscr{F}_{Z_f}^{(1)}, p_i\right)$$

by the induction hypothesis; in the second one obviously

$$\sum_{i=1}^{k} \mu\left(\mathscr{F}_Z^{(1)}, p_i\right) > \sum_{j=1}^{s} \mu\left(\mathscr{F}_{Z_f}^{(1)}, p_j\right).$$

As $\nu_Z \geqslant \nu_{Z_f}$, we get from $(*)$ that $\mu(Z,0) > \mu(Z_f,0)$.

5. The case with infinitely many separatrices

We have not analysed up to now the appearance in the desingularization process of a dicritical singularity, i.e. of the type

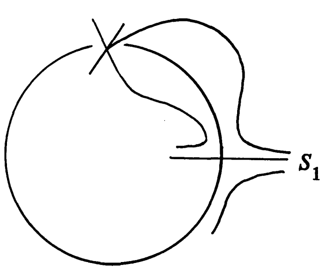
$$\dot{x} = a_\nu(x, y) + \cdots,$$
$$\dot{y} = b_\nu(x, y) + \cdots,$$

where $(a_\nu(x, y), b_\nu(x, y))$ is the homogeneous part of the vector field and $b_\nu(1, t) - ta_\nu(1, t) \equiv 0$. Remember that an explosion gives us a foliation

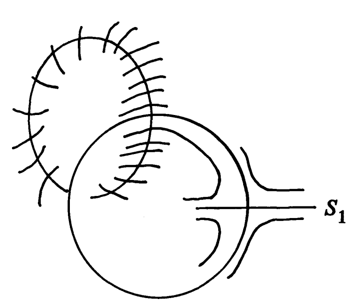
$$\dot{x} = a_\nu(1, t) + xR_1'(x, t),$$
$$\dot{t} = R_2'(x, t) - tR_1'(x, t)$$

in the first of the two coordinate systems. Now only at roots of $a_\nu(1, t)$ we may find singularities; outside these points the foliation is transverse to the projective line. This means that there exists a continuous family of separatrices of Z. Let us discuss some examples.

(i) $\dot{x} = 2x, \ \dot{y} = y.$



1st blowing up



2nd blowing up

After one blowing up we find two singularities, one of them dicritical. A further explosion applied to this singularity produces a desingularization with only one singular point.

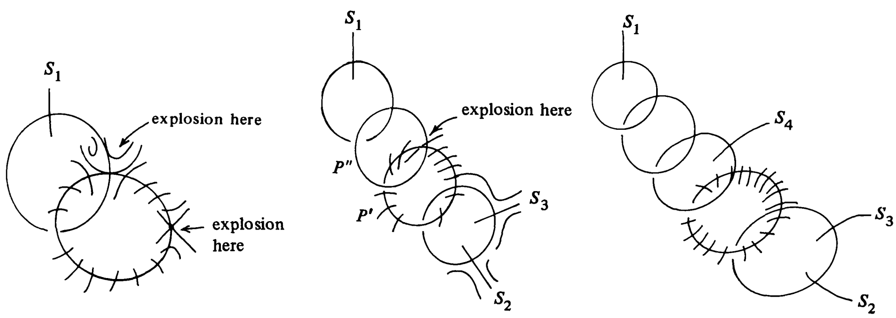
$$\text{(ii)}\qquad \begin{aligned} \dot{x} &= x^{2(k-2)} \cdot y, \\ \dot{y} &= x^{2k+1} + y^k + 2x^{2k-5} \cdot y^2, \qquad k \geqslant 4. \end{aligned}$$

The desingularization will be achieved in three steps. We consider first the case $k = 4$.

1*st step*. After two blow ups all three singularities of $\mathscr{F}^{(2)}$ have nonzero eigenvalues. The corner is a singularity and there is a singularity with both separatrices transverse to the divisor.

2*nd step*. An explosion is produced at each of these two singular points. At this stage the dicritical component is isolated in a projective line P' of the divisor, however through a corner $P' \cap P''$ passes a separatrix transverse to both P' and P''.

3*rd step*. A further blow up isolates the dicritical component in one projective line and now no separatrix passes through a corner. In the general case (any $k \geqslant 4$), it is necessary to proceed to k-blow ups in the corner of the 1st step in order to arrive to the 2nd step whose divisor will now have k projective lines. The passage to the 3rd step is the same as in $k = 4$.



1st step 2nd step 3rd step

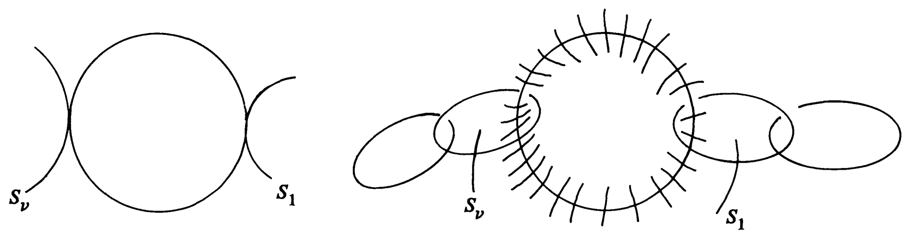
(iii) Let $P(t)$ be a polynomial of degree ν. Consider

$$\begin{aligned} \dot{x} &= x^{\nu+1} P\left(\frac{y}{x}\right), \\ \dot{y} &= x^{\nu} y P\left(\frac{y}{x}\right) + x^{\nu+2}. \end{aligned}$$

The first blow up yields:

$$\dot{x} = P(t), \ \dot{t} = 1.$$

There are no singularities; $\mathscr{F}^{(1)}$ is transverse to the projective line except at a finite member of points. Suppose for simplicity that all roots of $P(t)$ are simple. Then at each of these tangency points the desingularization needs two blowing ups.



The reader should notice that we are allowing additional blow ups; we look not only for simple singularities but also we insist on having all the separatrices desingularized. To be more precise, we adopt the following.

Definition. The set of separatrices of a vector field is said to be desingularized when (i) all separatrices have become smooth and disjoint; (ii) no separatrix passes through a corner; (iii) all separatrices are transverse to the divisor. If besides this, also the singularities appearing in the blow up are simple and lie in invariant projective lines, then the vector field is said to be desingularized.

Of course in the nondicritical case we just need to ask for all singularities being simple. We intend to show now that the Desingularization Theorem (which guarantees simple singularities after a number of blow ups) can be extended in the dicritical case to include also the conditions (i), (ii) and (iii) above. In order to do so let us introduce a "measure" of tangency between a vector field and a smooth curve.

Definition. Let Z be a vector field defined in a neighborhood of $p \in \mathbf{C}^2$ and S a smooth curve, $p \in S$, which is not invariant by Z. We consider local coordinates (x, y) at $p \in \mathbf{C}^2$ with $p = (0,0)$ and $S = (y = 0)$. Let

$$\dot{x} = a(x, y), \qquad \dot{y} = b(x, y)$$

be the equation for Z in this coordinate system. We then define *the order of tangency of Z with S* as the multiplicity of $0 \in C$ as a root of $b(x,0)$. We denote it by $\eta_Z(p, S)$ or $\eta_{\mathscr{F}}(p, S)$, where $\mathscr{F}$ is the foliation induced by Z. We leave to the reader the verification that this definition is independent of the coordinate system.

Lemma 2. *Let $\mathscr{F}$ be a foliation defined in a neighborhood of $p \in \mathbf{C}^2$, with algebraic multiplicity ν. Let S be a smooth curve containing p and not invariant by*

$\mathcal{F}$. Let $\mathcal{F}^{(1)}$ be the foliation obtained by blowing up at $p \in S$. Then

(i) *If $p \in S$ is a nondicritical singularity, then*

$$\eta_{\mathcal{F}^{(1)}}(p, S) = \eta_{\mathcal{F}}(p, S) - \nu.$$

(ii) *If $p \in S$ is a dicritical singularity, then*

$$\eta_{\mathcal{F}^{(1)}}(p, S) = \eta_{\mathcal{F}}(p, S) - (\nu + 1).$$

(iii) *If $p \in S$ is not a singularity and $\eta_{\mathcal{F}}(p, S) \neq 0$, then $p \in S$ is a* ***simple*** *nondicritical singularity for $\mathcal{F}^{(1)}$ with eigenvalues 1 and -1 and*

$$\eta_{\mathcal{F}^{(1)}}(p, S) = \eta_{\mathcal{F}}(p, S).$$

Proof. The blow up at $p \in S$ is represented in coordinates (x, t) by the change of coordinates $y = tx$. The curve S is now represented by $S = (t = 0)$ and $p = (0, 0)$. The foliation $\mathcal{F}^{(1)}$ near p is induced by the differential equations

$$\dot{x} = \frac{a(x, tx)}{x^{\sigma - 1}}, \qquad \dot{t} = \frac{b(x, tx) - ta(x, tx)}{x^{\sigma}},$$

where $\sigma = \nu$ in case p is nondicritical and $\sigma = \nu + 1$ in case p is a dicritical singularity. In both cases

$$\eta_{\mathcal{F}^{(1)}}(p, S) = \text{multiplicity of } \frac{b(x, 0)}{x^{\sigma}} = \eta_{\mathcal{F}}(p, S) - \sigma.$$

This proves (i) and (ii). Suppose now that $p \in S$ is not a singular point. Then $\mathcal{F}$ can be given by the vector field

$$\dot{x} = 1, \qquad \dot{y} = b(x, y)$$

and $\mathcal{F}^{(1)}$ is given locally around $p \in S$ by

$$\dot{x} = x, \qquad \dot{t} = b(x, tx) - t.$$

Therefore $p \in S$ is a simple singularity of $\mathcal{F}^{(1)}$ with eigenvalues 1 and -1 and

$$\eta_{\mathcal{F}^{(1)}}(p, S) = \text{multiplicity of } b(x, 0) = \eta_{\mathcal{F}}(p, S).$$

q.e.d.

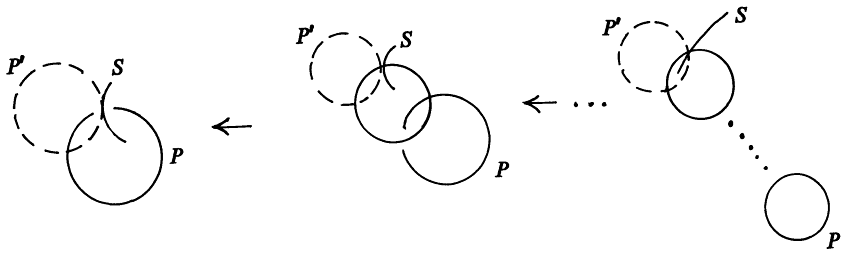
Now we can prove the extended Desingularization Theorem.

Theorem 5. *Let Z be a holomorphic vector field with infinitely many separatrices through $0 \in \mathbf{C}^2$. Then there exists a desingularization for Z.*

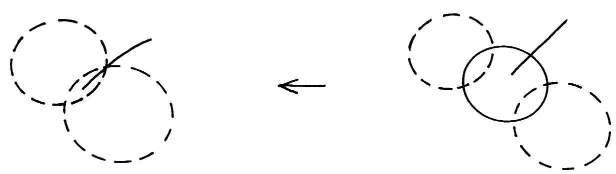
Proof. First of all we proceed as far as to make simple all the singularities which appear after blowing up Z. Let us list the possibilities for the separatrices of Z and handle each case.

(i) The separatrix is transverse to an invariant projective line, outside a corner. Clearly it is already desingularized.

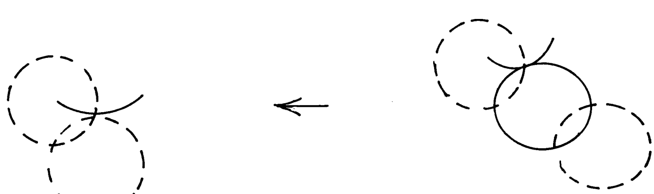
(ii) The separatrix is transverse to an invariant projective line at a corner. It follows that the other projective line at the corner is not invariant; the worse case appears when the separatrix is tangent to it. By Lemma 2, part (i), a sequence of blow ups at the point of tangency will eventually lead to the separatrix being transverse to both projective lines it touches. Therefore just an extra explosion is demanded to desingularize the separatrix.



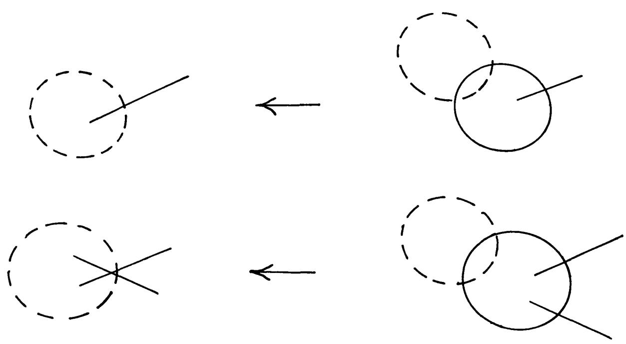
(iii) The separatrix passes through a corner where two noninvariant projective lines meet. If it is transverse to both projective lines we need to explode it just once.



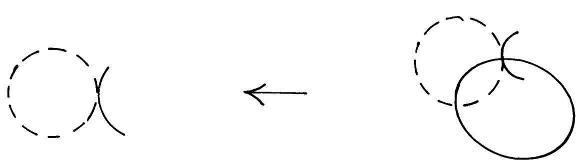
It could also happen that it is tangent to one of the projective lines; one explosion will bring us back to case (ii).



(iv) The separatrix is transverse to a noninvariant projective line and their intersection is a singularity (not a corner). Again just one explosion is needed.



(v) The separatrix is tangent to a noninvariant projective line (at a point which is not a corner). One explosion will produce case (ii) again:



We see that the possibilities above add up to a finite number of cases; in particular, all separatrices involved will eventually appear transversely to invariant projective lines, outside corners. As a consequence of the above constructions, the set of separatrices of Z is already desingularized. In order to avoid singularities in noninvariant projective lines, a further step must be taken. Suppose there exists a saddle-node at the intersection between two projective lines, with only one of them invariant. Furthermore, assume the singularity has only one separatrix (which necessarily will be contained in the invariant projective line). In this situation, we need an explosion at the singularity:

Now we can say that all noninvariant projective lines appearing in the desingularization are free from singularites; therefore they are fibered by separatrices (except at corners, where they necessarily meet invariant projective lines). The desingularization is completed now. q.e.d.

As we can see from Theorem 5, there exists a certain number of disjoint noninvariant projective lines fibered by disks which project, with two exceptions at most, on separatrices of Z. Each of such families of separatrices will be called a *dicritical component* of the vector field Z. Therefore the set of separatrices can be written as a union of a finite number of *isolated* separatrices (types (i)–(v) above) and a finite number of dicritical components of Z. We stress the point that although being infinite in number, the set of separatrices of a vector field is desingularized after a finite number of blow ups.

Definition. A vector field Z with a singularity at $0 \in \mathbf{C}^2$ is said to be a generalized curve when any singularity of its desingularization has nonvanishing eigenvalues.

This is the same definition as in the nondicritical case. The analogue of Theorem 2 also holds true. Call S the set of separatrices of Z.

Theorem 6. *If Z is a generalized curve, then Z and S have the same desingularizations.*

Proof. Desingularize S first; let us look for the singularities of $\mathscr{F}_Z^{(k)}$, where $\mathscr{F}_Z^{(k)}$ is the foliation which results from the application of the same explosions to Z. Given an isolated separatrix, the point where it crosses a projective line is a singularity of $\mathscr{F}_Z^{(k)}$; there are exactly two separatrices going through it, and transversal. By the proof of Lemma 1 it has algebraic multiplicity one; moreover, both eigenvalues must be nonzero. It is easy to see that the eigenvalues λ_1 and λ_2 of this singularity are not linked by any relation of the type $m\lambda_1 = n\lambda_2$ for $m, n \in \mathbf{N}$; in fact, either the singularity is equivalent to $\dot{x} = \lambda_1 x$, $\dot{y} = \lambda_2 y$ or it is equivalent to $\dot{x} = \lambda_1 x$, $\dot{y} = \lambda_2 y + \alpha x^p$ for $p \in \mathbf{N}$. In the first case, we would have infinitely many separatrices going through that singularity of $\mathscr{F}_Z^{(k)}$; in the second case, we would have obtained just one separatrix. None of these cases happens. Therefore this singularity is simple.

As for the corners, observe that there are no separatrices of Z going through them. There are two possible cases: either one of the projective lines meeting at that corner is fibered by a family of separatrices or not. In the first case, if the corner were a singularity of $\mathscr{F}_Z^{(k)}$ it would have at most one separatrix of $\mathscr{F}_Z^{(k)}$, contradicting Theorem 1. In the second case, both projective lines are invariant for $\mathscr{F}_Z^{(k)}$, and it is easy to see it must be a simple singularity.

Any other singularity of $\mathscr{F}_Z^{(k)}$ along an invariant projective line would imply the existence of separatrices of Z which are not in S; this is not possible. Finally, Theorem 1 implies that no singularities of $\mathscr{F}_Z^{(k)}$ may exist along noninvariant projective lines. q.e.d.

Unlike the situation where only a finite number of separatrices of Z exists, we have no analogue of Theorem 1 for the case with infinitely many separatrices. In fact, for each vector field we have a different formula relating its algebraic multiplicity to data obtained from the desingularization. We intend now to explain how this can be achieved.

Lemma 3. *Suppose that*

$$\dot{x} = a_\nu(x, y) + a_{\nu+1}(x, y) + \cdots,$$
$$\dot{y} = b_\nu(x, y) + b_{\nu+1}(x, y) + \cdots$$

are the differential equations for Z and $b_\nu(1,t) - ta_\nu(1,t) \equiv 0$. If P is the projective line and $\mathscr{F}^{(1)}$ the foliation which results from the explosion of $0 \in \mathbf{C}^2$, then

$$\nu - 1 = \sum_{p \in P} \eta_{\mathscr{F}^{(1)}}(p, P).$$

Proof. We can assume that $b_\nu(0,1) \neq 0$. Then all the nontransversal points of $\mathscr{F}^{(1)}$ relative to P will appear in the (x,t)-coordinate system:

$$\begin{aligned} \dot{x} &= a_\nu(1,t) + x(\cdots), \\ \dot{t} &= \left[b_{\nu+l}(1,t) - ta_{\nu+l}(1,t)\right] + x(\cdots). \end{aligned}$$

We conclude that the sum of the orders of tangency of $\mathscr{F}^{(1)}$ with P is exactly of degree $a_\nu(1,t)$. Now, we have $a_\nu(u,1) - ub_\nu(u,1) \equiv 0$, therefore $u/a_\nu(u,1)$ and if $a_\nu(x,y) = a_1(y)x + \cdots + a_\nu(y)x^\nu$, then degree $a_1(y) =$ degree $a_\nu(1,t) = \nu - 1$. q.e.d.

Let us imagine now that, while performing the desingularization of a vector field, we come across a dicritical singularity $p \in P \subset \mathscr{P}^{(k)}$ in some stage $\mathscr{F}^{(k)}$. Up to that point we may apply Theorem 1. What comes next? Two cases arise:

(a) $p \in P$ is not a corner. We have a term $\rho(P)\mu_{\mathscr{F}^{(k)}}(p, P)$ which appears in the formula given by Theorem 1. It is easy to see that

$$\mu_{\mathscr{F}^{(k+1)}}(p, P) = \mu_{\mathscr{F}^{(k)}}(p, P) - \nu.$$

If $p \in P$ is exploded to P' and $\{p_1, \cdots, p_s\}$ is the set of points, where $\mathscr{F}^{(k+1)}$ is not transversal to P', then $\nu - 1 = \sum_{i=1}^{s} \eta_{\mathscr{F}^{(k+1)}}(p_i, P')$, by Lemma 3. Hence $\rho(P)\mu_{\mathscr{F}^{(k)}}(p, P)$ will be replaced by

$$\begin{aligned} &\rho(P)\left[\mu_{\mathscr{F}^{(k+1)}}(p, P) + \nu\right] \\ &\quad = \rho(P)\left[\mu_{\mathscr{F}^{(k+1)}}(p, P) + \sum_{i=1}^{s} \eta_{\mathscr{F}^{(k+1)}}(p_i, P') + 1\right] \\ &\quad = \rho(P)\mu_{\mathscr{F}^{(k+1)}}(p, P) + \sum_{i=1}^{s} \rho(P')\eta_{\mathscr{F}^{(k+1)}}(p_i, P') + \rho(P'). \end{aligned}$$

(b) $p \in P_1 \cap P_2$ is a corner. We now have to replace a term like $\rho(P_1)\mu_{\mathscr{F}^{(k)}}(p, P_1) + \rho(P_2)\mu_{\mathscr{F}^{(k)}}(p, P_1)$. Assume p is blown up to P', and $p_1 \in P' \cap P_1, p_2 \in P' \cap P_2$. As before

$$\begin{cases} \mu_{\mathscr{F}^{(k+1)}}(p_1, P_1) = \mu_{\mathscr{F}^{(k)}}(p, P_1) - \nu, \\ \mu_{\mathscr{F}^{(k+1)}}(p_2, P_2) = \mu_{\mathscr{F}^{(k)}}(p, P_2) - \nu \end{cases}$$

and $\nu - 1 = \sum_{i=1}^{s} \eta_{\mathscr{F}^{(k+1)}}(p_i, P')$. It results in an expression like

$$\begin{aligned}
&\rho(P_1)\left[\mu_{\mathscr{F}^{(k+1)}}(p_1, P) + \nu\right] + \rho(P_2)\left[\mu_{\mathscr{F}^{(k+1)}}(p_2, P_2) + \nu\right] \\
&\quad = \rho(P_1)\mu_{\mathscr{F}^{(k+1)}}(p_1, P) + \rho(P_2)\mu_{\mathscr{F}^{(k+1)}}(p_2, P_2) \\
&\qquad + (\rho(P_1) + \rho(P_2))\left(1 + \sum_{i=1}^{s} \eta_{\mathscr{F}^{(k+1)}}(p_i, P')\right).
\end{aligned}$$

Therefore, we are able to produce a formula when a dicritical singularity appears. There is still a point to be studied: how to transform a singularity p which appears on a projective line P not invariant by $\mathscr{F}^{(k+1)}$. The resulting formula depends on the nature of this singularity. If it has just a finite number of separatrices, the order of tangency $\eta(p, P)$ will be changed to $\eta(p, P) - \nu$; if it is a dicritical singularity we will get $\eta(p, P) - (\nu + 1)$. In both cases we may write ν in terms of data arising from the explosion as we have been doing until now. This twofold situation is responsible for the absence of a universal formula.

What can be effectively said is the following: given the desingularization of the vector field, there exists a formula which involves: (1) a contribution from the singularities as in the nondicritical case; (2) an algebraic expression involving the weights of the projective lines; it reflects the presence of dicritical singularities in the process of desingularization, and depends only on the desingularization of the dicritical components.

Theorem 7. *Let Z, Z' be vector fields with $Z(0) = Z'(0) = 0$ and having S_Z, $S_{Z'}$ as sets of separatrices. Assume Z is a generalized curve and that S_Z and $S_{Z'}$ have isomorphic desingularizations. Then $\mu(Z', 0) \geqslant \mu(Z, 0)$, and equality holds if and only if Z' is a generalized curve.*

The proof is similar to that of Theorem 4, and it is left to the reader.

6. Proof of Theorem C

Theorem C. *Let Z be a generalized curve with a singular point $0 \in \mathbf{C}^2$. Let Z' be any vector field topologically equivalent to Z at $0 \in \mathbf{C}^2$. Then Z' is also a generalized curve and both Z and Z' have isomorphic desingularizations.*

Proof. Let S_Z and $S_{Z'}$ be the set of separatrices of Z and Z'. The topological equivalence between Z and Z' is also an equivalence between S_Z and $S_{Z'}$. Therefore S_Z and $S_{Z'}$ have isomorphic desingularizations. Since the Milnor number is an invariant up to topological equivalence, we obtain by Theorems 4 and 7 that Z' is also a generalized curve and by Theorems 3 and 6 they have isomorphic desingularizations.

Corollary. *The algebraic multiplicity of a generalized curve is a topological invariant.*

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