EQUIVARIANT MORSE THEORY
AND CLOSED GEODESICS

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In this paper we use the equivariant Morse theory to investigate the existence of closed geodesics on simply connected compact Riemannian manifolds, particularly the simply connected compact rank 1 symmetric spaces (CROSS's) S^n, $\mathbf{C}P^n$, $\mathbf{H}P^n$, CaP^2.

In §5 we show the existence of at least $g(\lambda, n)$ "short" closed geodesics without self-intersection on a homotopy CROSS sufficiently close to the standard metric, where $g(\lambda, n)$ is the cuplength of the space of unparameterized geodesics on the standard CROSS. In the nondegenerate case there will be $\lambda(\lambda + 1)n(n + 1)/4$ ($\lambda = 1, 2, 4, 8$).

In §6 we prove that if M is a simply connected compact Riemannian manifold with the rational homotopy type of a CROSS and if all closed geodesics on M are hyperbolic, then the number of distinct closed geodesics of length $\leqslant l$ on M grows at least like the prime numbers.

Birkhoff gave a proof of the existence of at least one closed geodesic on an n-sphere in 1927. In 1929 Lusternik and Schnirelmann claimed the existence of three closed geodesics without self-intersection for any metric on a 2-sphere. The proof however was only completed recently by Ballmann. The geodesics of these theorems are obtained by shortening deformations of the great and small "circles" on a sphere; such geodesics can be considered "short". The generalization to higher dimensional spheres was attempted by Alber, who gave a proof of the existence of $g(n)$ closed geodesics on an n-sphere, where $g(n)$ is the cuplength of the Grassmannian $G(2, n + 1)$. But Ballmann found a mistake in Alber's proof. Ballmann, Thorbergsson and Ziller ([11], [12]), Anosov [2] and the author [28] independently gave complete proofs of versions of Alber's theorem. In the nondegenerate case one obtains $n((n + 1)/2)$ closed geodesics on a sphere. Morse [37] showed that the latter number is optimal with the example of an ellipsoid with unequal axes. While such an ellipsoid will

always have an infinite number of closed geodesics, for axes sufficiently close to 1 only $n(n + 1)/2$ will have length less than any given number greater than 2π.

One of the most important methods in the theory of closed geodesics has been the Morse theory on some version of the space Λ of closed curves on a manifold M. The critical points of the energy function on Λ are the closed geodesics. However the "iterates" of a closed geodesic, which should be considered as geometrically the same, appear as separate critical points of the energy function. The theorem of Bott [12] on the index of these iterates paved the way for the theorem of Gromoll and Meyer [26]: If M is a simply connected compact Riemannian manifold and if the rational Betti numbers of Λ are unbounded, then M has infinitely many (geometrically distinct) closed geodesics. Klingenberg showed that the same is true if the Betti numbers for any prime field are unbounded. The question of which manifolds satisfy the conditions of the theory of Gromoll and Meyer was settled by Sullivan and Vigué-Poirrier: They showed that if M is compact simply connected the rational Betti numbers of Λ are unbounded if and only if $H^*(M; \mathbf{Q})$ is not a truncated polynomial ring in one generator. Among the manifolds not covered by the theorem are the CROSS's and certain symmetric spaces of rank > 1. Ziller [48] has proved that for rank > 1 the $\mathbf{Z}_2$ Betti numbers of Λ are unbounded.

On the other hand a form of the "Birkhoff-Lewis fixed point theorem", proved by Moser [38], implies the existence of infinitely many closed geodesics near a generic nonhyperbolic geodesic. While we know of no example of a metric on any compact simply connected Riemannian manifold with all geodesics hyperbolic, the theorem of §6 does complete the proof of the generic existence of infinitely many closed geodesics on manifolds with these homotopy types. (See also [24] which shows that hyperbolic geodesics are more prevalent than had previously been suspected.) Note that a version of the theorem of Sullivan and Vigué-Poirrier for prime fields would complete the proof of the generic existence of infinitely many closed geodesics for simply connected compact Riemannian manifolds.

Klingenberg has given proofs of the generic existence of infinitely many closed geodesics and of the existence of infinitely many closed geodesics on a compact Riemannian manifold. However these proofs depend upon the "divisibility lemma" which is controversial.

Many of the mistakes in the history of closed geodesics stem from the fact that at some point it seems impossible to gain more information from the Morse theory on Λ without dividing out by the natural $O(2)$-action: one would like to deal with the "unparameterized" closed curves. But the group action is

not free and the quotient space of Λ by the group action has singularities.

In his attempt to calculate the "circular connectivities" of a sphere, i.e. the $\mathbf{Z}_2$ Betti numbers of the quotient of ΛS^n by the $O(2)$-action, Morse made a mistake which was later repeated by Bott and others and finally caught by A. S. Svarc. The problem was the fact that the group action is not free. The $\mathbf{Z}_2$ cohomology of the quotient has still not to our knowledge been computed despite many attempts; however it appears to have infinitely many generators in most dimensions! In §4.3 we calculate for the s.c. CROSS's what we would like to propose as a new candidate for the title "circular connectives": the mod 2 equivariant cohomology of Λ for the group $O(2)$.

The equivariant Morse theory has been used by Atiyah and Bott [3] in their work on the Yang Mills equations. The idea is this: Given a group action on a manifold X which leaves a differentiable function invariant one would like to "divide out" by the action. But if the group action is not free the quotient will have singularities. To resolve these singularities one looks instead at the *homotopy quotient* of Borel. The equivariant Morse theory is then the Morse theory on the homotopy quotient or, equivalently, the Morse theory on the space X itself with the equivariant cohomology replacing ordinary cohomology. All the standard theorems and (recent) proofs of Morse theory carry over to the equivariant case.

§1 deals with some well-known facts about equivariant cohomology and methods of computation. In §2 the equivariant Morse theory is introduced; topological implications are discussed. In §3 we outline the usual facts about Morse theory on the Sobolev space $H^1[S^1, M]$ and give an elementary proof of an equivariant version of Palais' theorem to the effect that $H^1[S^1, M]$ has the weak homotopy type of the space of continuous closed curves. §4 contains computations for CROSS's. This paper was written with a reader in mind who is more familiar with the geometry involved than the topology.

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1. Equivariant cohomology

1.1 Preliminaries. Let G be a locally compact Lie group. A G-space is a topological space X with a continuous G-action. We shall also assume that the action of G is proper, i.e., $G \times X \to X \times X$ by $(g, x) \to (gx, x)$ is a proper

map. This ensures that, if X is Hausdorff [16]:

(i) The stability subgroup H_x of a point $x \in X$ is compact.

(ii) The orbit $G \circ x$ of x in X is closed.

(iii) The orbit space G/X is Hausdorff in the quotient topology.

Proper group actions have many of the properties of compact group actions since we are guaranteed the existence of a slice; see [39].

A G-map from a G-space X to a G-space Y is a continuous map ϕ with $\phi g = g\phi$ for all g in G. If A is a category of topological spaces whose morphisms are continuous maps, we can form the category A_G: An object in A_G is a space X in A together with a continuous G-action so that $g: X \to X$ is a morphism of A for all g in G. The morphisms of A_G are the G-maps of A. A G-object or morphism will mean "in the category A_G", e.g. G-manifold, G-diffeomorphism, G-homotopy equivalence.

A G-manifold will be a differentiable (possibly infinite dimensional) paracompact Hausdorff manifold M with a continuous G-action so that $g: M \to M$ is differentiable for all $g \in G$. (Differentiable will mean sufficiently smooth, e.g. C^∞.) Note that if M is a locally compact G-manifold then the G-action $G \times M \to M$ is differentiable; see [9], [39].

If M is such a manifold with a free G-action, i.e. $gx \neq x$ for $g \neq 1$, then the projection $M \to G\backslash M$ is a smooth fibration and $G\backslash M$ is also a manifold. However if the action is not free, $G\backslash M$ will in general have singularities. In order to "resolve" these singularities we will use the homotopy quotient of Borel.

If X and Y are G-spaces we denote by $X \times_G Y$ the quotient of $X \times Y$ by the diagonal G-action, with the quotient topology. There are natural projections from $X \times_G Y$ to $G\backslash X$ and $G\backslash Y$; the inverse image of the orbit $G \circ x \in G\backslash X$ is identified with $H_x\backslash Y$, with $H_x \subset G$ the stability group of x.

1.2 Classifying spaces (see [10], [30]). A *universal G-bundle* is a principal G-bundle whose total space EG is contractible. The base space $BG = G\backslash EG$ is a *classifying space* for G. Such bundles always exist; BG is unique up to homotopy type in the category of CW complexes.

For example, if G is discrete an Eilenberg-Maclane space $K(G; 1)$ is a BG. If G is embedded in $GL(n; k)$ with $k = \mathbf{R}$ or $\mathbf{C}$ then

$$F_k(n, \infty) = \{\text{ordered } n\text{-frames in } k^\infty\}$$

is the total space of a universal G-bundle. If $\hat{F}_\mathbf{R}(n, \infty)$ is the space of orthonormal n-frames then the Grassmannian $G_\mathbf{R}(n, \infty) = O(n)\backslash \hat{F}_\mathbf{R}(n, \infty)$ of n-planes in $\mathbf{R}^\infty$ is a classifying space for the orthogonal group $O(n)$.

The classifying property of BG is the following: If E is a principal G-bundle over a paracompact base B, then there is a map $\alpha: B \to BG$ so that $\alpha^*(EG) = E$

and thus a correspondence

$$\left\{ \begin{array}{c} \text{Equivalence classes of} \\ G\text{-bundles over } B \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Homotopy classes of} \\ \text{maps } B \to BG \end{array} \right\}.$$

For example if B^n is a manifold embedded in $\mathbf{R}^\infty$ and if $\alpha: B \to G_{\mathbf{R}}(n, \infty)$ gives the usual identification of the tangent space to B at a point with a linear subspace of $\mathbf{R}^\infty$, then α is a classifying map for the orthonormal frame bundle associated to the tangent bundle TB of B, and

$$TB \cong \alpha^* \big(\hat{F}_{\mathbf{R}}(n, \infty) \times_{O(n)} \mathbf{R}^n \big).$$

If $\gamma: G \to G'$ is a continuous homomorphism, then we can form the principal G'-bundle

$$EG \times_G G' \to BG,$$

where G acts on G' by γ. By the above this bundle is the pull-back of EG' under a map $\Gamma: BG \to BG'$. Thus a map $\gamma: G \to G'$ induces a map G (up to homotopy) so that

$$
\begin{array}{ccc}
EG \times_G G' & \longrightarrow & EG' \\
\downarrow & & \downarrow \\
BG & \xrightarrow{\quad \Gamma \quad} & BG'
\end{array}
$$

is a pull-back diagram. In fact Milnor has shown [34] that any principal G'-bundle over BG is induced by a map $G \to G'$.

1.3 The Homotopy quotient (see [10], [29]). If X is a G-space the homotopy quotient of X is defined by

$$X_G = X \times_G EG.$$

X_G is the "universal G-bundle with fiber X". Associated to X_G is the mixing diagram:

$$
\begin{array}{ccccc}
X & \longleftarrow & X \times EG & \longrightarrow & EG \\
\downarrow & & \downarrow & & \downarrow \\
G \backslash X & \longleftarrow & X_G & \longrightarrow & BG \\
& \pi_1 & & \pi_2 &
\end{array}
$$

The projection $\pi_1: X_G \to G \backslash X$ can be thought of as sort of "blow-up" in the sense of homotopy of the singularities which occur in $G \backslash X$ when the action of G is not free. If X is a G-manifold with $G \times X \to X$ differentiable and we have a G-manifold for EG, then $\pi_2: X_G \to BG$ is a smooth fibration with fiber X and X_G is a manifold. Moreover π_1 is a weak homotopy equivalence whenever the action is free. Note that if $x \in X$, $\pi_1^{-1}(G \circ x) \approx H_x \backslash EG$ with H_x the stability group of x. Thus $\pi_1^{-1}(G \circ x)$ *has the homotopy type of the classifying space* BH_x. If H_x is trivial, $\pi_1^{-1}(G \circ x)$ has the homotopy type of a point.

As an example consider the group $G = \mathbf{Z}_2$ acting on $\mathbf{R}^2$ by reflection about the origin. Let $EG = S^\infty$ with G acting by the antipodal map. The inverse image $\pi_1^{-1}(G \circ x)$ of a general point in $G \backslash X$ is EG (a point in homotopy!) The inverse image under π_1 of the orbit of the origin is the projective space $\mathbf{R}P^\infty$.

Another way to view the homotopy quotient is this: $X \times EG$ is a space with the homotopy type of X on which we have a free G-action reflecting the G-action on X. Without changing the homotopy type of G, we have "made G act freely". The homotopy quotient is the quotient of $X \times EG$ by this free action.

If X and Y are G-spaces and $\phi \colon X \to Y$ a G-map, then $\phi \times \mathrm{id} \colon X \times EG \to Y \times EG$ descends to a continuous map

$$\phi_G \colon X_G \to Y_G.$$

Thus $X \to X_G$ is a functor from the category of G-spaces and G-maps to that of topological spaces and continuous maps. If ϕ is a G-homotopy equivalence, then ϕ_G is a homotopy equivalence.

If X is a G'-space and $\gamma \colon G \to G'$ a homomorphism, then X is also, by γ, a G-space. It is easy to see that the homotopy quotient X_G is the bundle over BG given by the pullback of the bundle $X_{G'} \to BG'$ over the map Γ induced by γ:

$$
\begin{array}{ccc}
X_G & \longrightarrow & X_{G'} \\
\downarrow & & \downarrow \\
BG & \xrightarrow{\ \Gamma\ } & BG'
\end{array}
$$

1.4 Equivariant cohomology (see [10], [29]).   If X is a G-space the equivariant cohomology of X is defined by

$$(*) \qquad\qquad H_G^*(X) = H^*(X_G).$$

H_G^* satisfies by definition the exactness, homotopy and excision axioms for cohomology theories but not the dimension axiom: If we consider a point as a trivial G-space then

$$H_G^*(\mathrm{pt.}) = H^*(BG).$$

H_G^* is a functor from G-spaces to $H^*(BG)$-modules. The structure of $H_G^*(X)$ as an $H^*(BG)$-module is given from the cohomology theory point of view (left hand side of $(*)$) by the G-map

$$X \to \mathrm{pt.}$$

and from the homotopy quotient point of view (right hand side of $(*)$) by the map

$$\pi_2 \colon X_G \to BG.$$

A homomorphism γ: $G \to G'$ gives naturally, as we have seen, a functor from G'-spaces to G-spaces and maps $X_G \to X_{G'}$ and $BG \to BG'$. The induced maps $H^*_{G'} \to H^*_G$ and $H^*(BG') \to H^*(BG)$ are compatible with the module structure.

1.5 G-vector bundles. A G-vector bundle over a G-manifold M is a G-space V which is a (differentiable) vector bundle over M and so that g: $V \to V$ is a vector bundle map; i.e. g preserves and is linear on the fibers. The G-map $V \to M$ induces the vector bundle $V_G \to M_G$. For example the tangent bundle TM of a G-manifold M is naturally a G-vector bundle; if $gx = y$ ($g \in G$; $x, y \in M$) then g: $T_x M \to T_y M$ is the derivative of the map g: $M \to M$. If G acts by isometries then V is a *G-Riemannian vector bundle*.

If $V \to M$ is a G-vector bundle the restriction of V to an orbit $G \circ x$ is a G-vector bundle over $G \circ x \approx G/H_x$. Now let H be a compact subgroup of G and consider a G-vector bundle $V \to G/H$. The action of H on V gives a representation of H on the fiber F over the identity coset in G/H. Since the G-action gives a trivialization of the pullback of V under the map $G \to G/H$ this representation determines the bundle $V \to G/H$ uniquely. Thus we have a 1-1 correspondence

$$\left\{ \begin{array}{l} G\text{-vector bundles} \\ \text{over } G/H \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Representations} \\ \text{of } H \end{array} \right\}.$$

The inclusions $H \to G$ and $F \to V$ induce isomorphisms:

$$\begin{array}{ccc} F_H & \xrightarrow{\ \sim\ } & V_G \\ \downarrow & & \downarrow \\ BH = (\text{pt.})_H & \xrightarrow{\ \sim\ } & (G/H)_G \end{array}$$

The bundle $F_H \to BH$ is the *canonical bundle over BH associated to the representation of H on F*.

If $V \to M$ is a real G-orientable Riemannian vector bundle (if G is connected this simply means that V is orientable; otherwise we must insist also that G preserve the orientation) we have the *Thom isomorphism*

$$H^*_G(DV, SV) \cong H^{*-\lambda}_G(M),$$

where D and S denote the disk and sphere bundles and λ is the dimension of V. This follows from the standard Thom isomorphism since the bundle $V_G \to M_G$ will be orientable precisely when V is G-orientable.

1.6 Examples and techniques. The Serre spectral sequence for the fibration $M_G \to BG$ with fiber M allows one to compute $H^*_G(M)$ in many cases; often the spectral sequence also gives information about the structure of $H^*_G(M)$ as an $H^*(BG)$-module. We mention some standard results.

If $S^1 = SO(2)$ is the circle group then $S^1 \backslash S^\infty = \mathbf{C}P^\infty$ is a BS^1 and

$$H^*(BS^1) \cong \mathbf{Z}[w],$$

the polynomial ring on a generator in dimension 2. The map $\gamma_m \colon S^1 \to S^1$ by $\gamma_m(g) = g^m$ induces $\Gamma_m^* \colon H^*(BS^1) \to H^*(BS^1)$ by $w \to mw$.

Let $\mathbf{Z}_n$ be the cyclic group of order n. Embedding $\mathbf{Z}_n$ in S^1 we see that the lens space $\mathbf{Z}_n S^\infty$ is a $B\mathbf{Z}_n$. We have

$$H^*(B\mathbf{Z}_n) \cong \mathbf{Z}[w]/nw = 0.$$

In particular if n divides m the inclusions $\mathbf{Z}_n \to \mathbf{Z}_m \to S^1$ induce surjections $H^*(BS^1) \to H^*(B\mathbf{Z}_n) \to H^*(B\mathbf{Z}_n)$.

The Euler class of the canonical bundle over $B\mathbf{Z}_n$ corresponding to the 2-dimensional representation with eigenvalues $e^{\pm \pi i m/n}$ is m-times a generator of $H^2(B\mathbf{Z}_n) \cong \mathbf{Z}_n$.

$H^*(B\mathbf{Z}_n)$ can be computed from a free resolution of $\mathbf{Z}[\mathbf{Z}_n]$. The computation works as well for any local coefficient system on $B\mathbf{Z}_n$ given as a $\mathbf{Z}[\mathbf{Z}_n]$-module. If X is a $\mathbf{Z}_n$-space the $\mathbf{Z}[\mathbf{Z}_n]$-module $C^*(X)$ gives the local coefficient system for the fibration $X_{Z_n} \to B\mathbf{Z}_n$. In particular there is a spectral sequence converging to $H^*(X_{\mathbf{Z}_n})$ with $E_0^{pq} = C^q(X)$ and differentials

$$d \colon E_0^{pq} \to E_0^{p,q+1} \quad \text{the ordinary differential } C^q(X) \to C^{q+1}(X),$$

$$\delta \colon E_0^{pq} \to E_0^{p+1,q} \quad \text{the map } C^q(X) \to C^q(X) \text{ induced by}$$

$$\begin{cases} (1 - t) & (p \text{ even}), \\ (1 + t + \cdots + t^{n-1}) & (p \text{ odd}), \end{cases}$$

where t is a generator of $\mathbf{Z}_n$.

For example let σ be an orthononal representation of $\mathbf{Z}_n$ on $\mathbf{R}^\lambda$ whose associated $\mathbf{Z}_n$-bundle $V \to \text{pt.}$ is *not* orientable, i.e. $\det \sigma(t) = -1$. Then n is even; taking the d-cohomology $H_{\mathbf{Z}_n}^*(DV, SV)$ which has E_1 term $\equiv 0$ except in the row $q = \lambda$. In this row we have (setting $t = -1$)

$$\mathbf{Z} \underset{2}{\to} \mathbf{Z} \underset{0}{\to} \mathbf{Z} \underset{2}{\to} \mathbf{Z} \to \cdots.$$

Thus

$$H_{\mathbf{Z}_n}^j(DV, SV) = \begin{cases} \mathbf{Z}_2, & \lambda < j \text{ and } \lambda + j \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

If $O(n)$ is the orthogonal group then

$$H^*(BO(n); \mathbf{Z}_2) = \mathbf{Z}_2[w_1, \ldots, w_n],$$

where w_i in dimension i is the ith Steifel-Whitney class.

Haefliger [27] has shown that one can use *Sullivan theory* to compute $H_G^*(\Gamma; \mathbf{Q})$, where Γ is the space of continuous sections of a G-fiber bundle.

Sullivan and Vigue-Poirrier [45] investigated the following example: Let $G = S^1$ and consider the S^1-bundle $S^1 \times M \to S^1$, where M is a manifold and S^1 acts trivially on M. Then Γ is the space of continuous maps from S^1 to M with S^1 acting by a change of parameter. If $M = S^{2m+1}$, $m > 0$, then $H^*_G(\Gamma)$ has minimal model $Q[c] \otimes \Lambda(x, \bar{x})$ with $\deg c = 2$, $2m + 1 = \deg x = \deg \bar{x} + 1$ and $dc = 0$, $dx = -c\bar{x}$, $d\bar{x} = 0$. Thus

$$H^*_G(\Gamma; Q) \cong Q[c, \bar{x}]/c\bar{x}.$$

When G is discrete an Eilenberg-Maclane space $K(G, 1)$ is a BG and $H^*(BG)$ is the Eilenberg-Maclane cohomology of G which is computed algebraically from a free resolution [17]. The analog for a topological group is a *semi-simplicial construction*. A semi-simplicial space (which corresponds to a free resolution when G is discrete) is constructed whose geometric realization is a BG (or X_G for a G-space X); the cohomology of this semi-simplicial space is that of BG (or X_G), [34], [35], [14].

2. Equivariant Morse theory

Using the Morse theory one gains information about the topology of a manifold by studying the critical points of an appropriate function or, working backward, deduces information about the critical set of a function on a manifold from the topology. Often such problems come equipped with a "built-in" symmetry in the form of a group action on the manifold which leaves the function invariant; one would like to divide out by the group action. There is ample evidence that this is not always a good idea in the long history of mistaken proofs in the subject of closed geodesics. An alternative is the equivariant Morse theory. This can be thought of as the usual Morse theory on the homotopy quotient of the given manifold, or as the Morse theory on the manifold itself using the functor H^*_G rather than H^*. While we always have the first point of view in mind, the second has technical advantages. As many authors have noted, cf. [25], [46], the usual proofs of the standard theorems of Morse theory are also valid in the equivariant category. The first part of this section is devoted to statements of these theorems.

2.1 Statement of the theorems in the equivariant context. The customary set-up for the Morse theory on Hilbert manifolds starts with a complete Hilbert manifold M with a differentiable function satisfying condition (C) of Palais and Smale [40]. In the equivariant setting for a Lie group G this means:

M is a complete G-Riemannian manifold equipped with a G-invariant differentiable function f. The appropriate form of condition (C) is

(C_G) Let $\{x_n\}$ be a sequence of points in M so that $f(x_n)$ is bounded, and $|\nabla f(x_n)|$ converges to 0. Then $\{x_n\}$ contains a subsequence whose image in $G \backslash M$ converges.

If G is the trivial group this is condition (C).

We shall also assume that f satisfies a regularity condition: If $x \in M$ is a critical point then $G \to G \circ x$ is differentiable. The orbit of x will then be an embedded submanifold.

Condition (C_G) is used in two ways: First it implies that if $K \subset M$ is the set of critical points of f and $a, b \in \mathbf{R}$, then the image of $f^{-1}[a, b] \cap K$ in $G \backslash M$ is compact; second it ensures that $|\nabla f|$ will be bounded away from 0 on the complement of any neighborhood of K in $f^{-1}[a, b]$.

Let M be a complete G Riemannian manifold and let f be a differentiable G-function satisfying (C_G).

Theorem A. *If $[a,b]$ contains no critical values then M^b is G-diffeomorphic to M^a.*

The proof is by pushing M^b down to M^a along $-\nabla f$. (See Palais [40]; the proof is still valid if one puts a G- front of every third word as are the proofs cited below.)

Let $A \subset M$ be a critical manifold of f, i.e. a submanifold of M consisting of critical points. The symmetric bilinear Hessian form Hf is defined on the normal bundle $N(A)$ to A in M; see Meyer [36]. A is *nondegenerate* if the selfadjoint operator S_0 given by

$$\langle S_0 u_x, v_x \rangle = \tfrac{1}{2} Hf(u_x, v_x)$$

is invertible (as bounded operator) on $N_x(A)$ for each $x \in A$. If A is nondegenerate and χ is the characteristic function of $[0, \infty)$ then $P = \chi(S_0)$ is a G-orthogonal bundle projection. We define the negative and positive bundles of A by

$$N^-(A) = (1 - P)N(A); \quad N^+(A) = PN(A).$$

The *index* of A is the dimension of $N^-(A)$. It follows from (C_G) that nondegenerate critical manifolds are isolated and that if the critical manifolds are isolated and that if the critical set K of f consists entirely of such manifolds, then the critical manifolds of f are isolated.

Let E, E' be G-Riemannian vector bundles over a manifold B. Then $DE \oplus DE'$ is a *handle-bundle* of type (E, E').

Theorem B. *Let M be a complete G-Riemannian manifold and let f be a differentiable G-invariant function satisfying (C_G). If $[a,b]$ contains one critical*

value $c \in (a, b)$, *and if* $K \cap f^{-1}(c)$ *is a union of nondegenerate critical G-manifolds* A_i, *then* M^b *is G-diffeomorphic to* M^a *with the handle-bundles* (N^+(A_i), $N^-(A_i)$) *disjointly attached along* $DN^+(A_i) \oplus SN^-(A_i)$.

For the proof see Palais [40] and Wassermann [46].

Next we turn to the case of isolated degenerate critical manifolds. That is, we consider submanifolds A_i of M having neighborhoods U_i with $U_i \cap K = A_i$, but now with no restriction on the Hessian form.

In the equivariant theory we have the analog of the local invariants of Gromoll-Meyer and an equivariant version of the splitting theorem; see Gromoll and Meyer [25]. For our purposes we need only the following version of the principle of subordinated homology classes:

Theorem C. *If* $[a,b]$ *contains one critical value* $c \in (a, b)$, *if* $K \cap f^{-1}(c)$ *is a union of isolated critical G-manifolds* A_i, *and if* $u \in H_{G*}(M)$ *satisfies* $u_{|A_i} = 0$ *for each* i, *then*

$$\cap u\colon H_{G*}(M^b, M^a) \to H_{G*}(M^b, M^a)$$

is the zero map.

Proof. By the usual arguments, for any G-neighborhood U of $\cup A_i$ there is an $\varepsilon > 0$ so that $H_{G*}(U, U^{c-\varepsilon}) \to H_{G*}(M^b, M^{c-\varepsilon})$ is surjective. The latter is naturally isomorphic to $H_{G*}(M^b, M^a)$. Thus the above cap product factors through

$$\cap (u_{|U})\colon H_{G*}(U, U^{c-\varepsilon}) \to H_{G*}(U, U^{c-\varepsilon}).$$

2.2 Topological implications; nondegenerate case. Using Theorems A, B and C one can hope to calculate $H_G^*(M)$ once the structure of the critical point set of an appropriate function is known. Suppose M is a complete G-Riemannian manifold and that f is a G-function satisfying (C_G). From Theorem A it follows that if $[a, b]$ contains no critical values, then

$$H_G^*(M^b, M^a) = 0.$$

Now suppose the hypotheses of Theorem B are met: $[a, b]$ contains one critical value $c \in (a, b)$ and $K \cap f^{-1}(c)$ is a union of nondegenerate critical manifolds A_i. Let $DN^-(A_i)$ be the closed disk bundle of the negative bundle over A_i. Then

$$H_G^*(M^b, M^a) \cong \bigoplus_i H_G^*(DN^-(A_i), SN^-(A_i)).$$

If all the bundles $N^-(A_i)$ are G-orientable and $G \backslash A_i$ is connected, then the Thom isomorphism yields

$$H_G^*(M^b, M^a) = \oplus H_G^{*-\lambda_i}(A_i),$$

where λ_i is the index of A_i.

Now when G is the trivial group one can look for nondegenerate critical points. In general the best that one can hope to find is nondegenerate critical orbits. Suppose $x \in M$ and the orbit $G \circ x$ is a nondegenerate critical manifold. We call this the *nondegenerate case* as the Hessian form has the minimum degeneracy required by the G-structure. Let $H = H_x$ be the stability group of x. Then

$$H_G^*(DN^-(G \circ x), SN^-(G \circ x)) = H^*(DV_G, SV_G),$$

where $V_G = (N^-(G \circ x))_G$ is the canonical bundle over BH determined by the representation of H on the fiber of $N^-(G \circ x)$. If N^- is G-orientable then

$$H_G^*(DN^-(G \circ x), SN^-(G \circ x)) = H^{*-\lambda}(BH).$$

If the critical set K consists entirely of nondegenerate critical manifolds, then by (C_G) we can find an increasing sequence $\{a_i\} \in \mathbf{R}$ with no limit points so that $[a_i, a_{i+1}]$ contains one (interior) critical value. If we know the structure of each negative bundle $N^-(A)$ and thus $H_G^*(M^{a_{i+1}}, M^{a_i})$, we can hope to recover the equivariant cohomology of M via the long exact sequences

$$(*) \qquad \cdots \to H_G^j(M^{a_{i+1}}, M^{a_i}) \to H_G^j(M^{a_{i+1}})$$

$$\to H_G^j(M^{a_i}) \xrightarrow{\partial} H_G^{j+1}(M^{a_{i+1}}, M^{a_i}) \to \cdots.$$

Thus inductively if we know $H_G^*(M^{a_i})$ (e.g. if f is bounded below) and can recover the *boundary maps* ∂ then, at least for coefficients in a field k, we can determine the k-module $H_G^*(M^{a_{i+1}})$.

We define the *Poincare series* for a field k by

$$P_G(M; k, t) = \sum_j \text{Rank } H_G^j(M; k) t^j$$

and the *Morse series* by

$$\mathfrak{M}_G(M; f, k, t) = \sum_i P_G(DN^-(A_i), SN^-(A_i)),$$

where A_i ranges over the (nondegenerate) critical manifolds of M. If each $N^-(A_i)$ is G-orientable then

$$\mathfrak{M}_G(M; f, k, t) = \sum_i t^{\lambda_i} P_G(A_i; k, t).$$

In particular in the nondegenerate case the contribution of a critical orbit to the Morse series is the Poincare series of the classifying space of the stability group "jacked up" by the index t^λ. The Morse inequalities, which are a direct result

of the long exact sequences (∗), can be stated as follows [11]:

> Let f be bounded below. Then there is a series $Q(t)$ with nonnegative integer coefficients so that

$$\mathfrak{M}_G(M; f, k, t) = P_G(M; k, t) + (1 + t)Q(t).$$

If $Q(t) \equiv 0$, then f is called a *perfect Morse function* (for the field k). This means that the boundary maps of (∗) are all trivial; for each i, j,

$$H_G^j(M^{a_{i+1}}; k) = H_G^j(M^{a_i}; k) \oplus H_G^j(M^{a_i+1}, M^{a_i}; k)$$

and there is no "cancelling out" of the cohomology of one critical set with that of another. In this case f has the "minimal critical set" (i.e. the smallest Morse series) prescribed by the topology of M (i.e. by the Poincare series).

Now the Morse inequalities reflect the information contained in the long exact sequences (∗) when H_G^* is considered as a functor to k-modules. In the equivariant case we actually have much more information, since the boundary maps

$$\partial\colon H_G^*(M^{a_i}) \to H_G^{*+1}(M^{a_{i+1}}, M^{a_i})$$

are $H^*(BG)$-module maps. This is important in practice as $H_G^*(M^{a_i})$ is likely to have infinitely many generators as a k-module, but will be a finitely generated $H^*(BG)$-module if $H^*(M^{a_i})$ has finite rank.

2.3 Completing manifolds and self-completing bundles. If one can find G-completing manifolds for the critical manifolds A then the boundary maps must vanish. A G-completing manifold for A is a G-submanifold B of M satisfying:

(i) $A \subset B$ as a submanifold of codim $\lambda = $ index A,

(ii) $f_{|B-A} < f(A)$,

(iii) $H_G^*(B, B^{f(A)-\varepsilon}) \to H_G^*(B)$ is injective for all $\varepsilon > 0$.

(The third condition will be satisfied e.g., if there is a map $B \to A$ consistent with the inclusion $A \to B$ giving B as a G-fibration over A whose fiber is a G- and $\pi_1(A)$-orientable manifold.)

Proof. Suppose for simplicity that $f(A) = 0$ and $K \cap f^{-1}(0) = A$. Commutativity of the diagram

$$
\begin{array}{ccccc}
H_G^{*-1}(M^{-\varepsilon}) & \xrightarrow{\ \partial\ } & H_G^*(M^\varepsilon, M^{-\varepsilon}) & \longrightarrow & H_G^*(M^\varepsilon) \\
& & \downarrow & & \downarrow \\
& & H_G^*(B, B^{-\varepsilon}) & \hookrightarrow & H_G^*(B)
\end{array}
$$

implies that the boundary map ∂ is zero.

The notion of completing manifolds goes back to Morse [37]. In the equivariant situation one encounters the phenomenon of *self-completing bundles* (see Atiyah-Bott [3]). First note that the above argument also works for a manifold B with boundary if $\partial B \cap A = \varnothing$; however in this case (iii) will never hold if A is finite dimensional in the ordinary Morse theory (trivial G). Let $B = DN^-(A)$. Then $B^{-\varepsilon} \simeq SN^-(A)$; if $N^-(A)$ is G-orientable the Thom isomorphism identifies the map (iii) with the map

$$\cup \chi \colon H_G^{*-\lambda}(A) \to H_G^*(A)$$

by multiplication with the Euler class $\chi \in H_G^\lambda(A)$ of the bundle $(N^-(A))_G \to A_G$. Thus if multiplication with χ is injective on $H_G^*(A)$, the boundary maps ∂ must be trivial, no matter how the negative bundle $N^-(A)$ is "attached"! We call $N^-(A)$ a G-self-completing bundle.

For example put $G = S^1$; let $A = S^1/\mathbf{Z}_n$. If $V \to A$ is the bundle given by a representation with eigenvalues $e^{\pm 2\pi i m/n}$ for m prime to n, then V is S^1-self-completing for $\mathbf{Z}_n$ coefficients.

2.4 Topological implications; degenerate case. In the degenerate case Lusternik-Schnirelmann theory allows one to use the homology of M to predict critical points. If G is trivial and the critical set K of f consists entirely of nondegenerate critical points, then the Morse inequalities imply that the number of critical points of f is bounded below by the sum of the Betti numbers $P(M; k, 1)$ for any field k. On the other hand an isolated degenerate critical point (e.g. a monkey's saddle) can contribute more than 1 to $P(M; k, 1)$. Theorem C is the analog in the equivariant theory of the principle of subordinated homology classes, which in the ordinary Morse theory gives a correspondence between subordinated homology classes and *distinct* (possibly degenerate) critical points.

For our purposes the Birkhoff minimax principle can be stated as follows (Klingenberg [32]).

Let f be a G-differentiable function on a complete G-Riemannian manifold M, satisfying (C_G). Let $a, b \in \mathbf{R}$ and suppose that f has no critical values in $(a, a + \varepsilon]$ for some $\varepsilon > 0$. Let $w_0 \in H_{G*}(M^b, M^a)$ be a nontrivial homology class. Put

$$\kappa(w_0) = \underset{w \in w_0}{\text{Inf}} \ \underset{p \in \text{Image}(w)}{\text{Sup}} \ f(p),$$

where w ranges over the repesentatives of the class w_0. Then $\kappa(w_0) > a$, and $\kappa(w_0)$ is a critical value of f.

Suppose f is bounded below on M and that the critical set K of f consists of a union of isolated critical orbits. It follows from Theorem C that if $w_1, w_2 \in H_{G*}(M; k)$ with $w_1 = w_2 \cap x$, $\dim x > 0$, (i.e. w_1 is *subordinate* to w_2) *and if*

$x|_{K \cap f^{-1}(\kappa(w_2))} \equiv 0$, then $\kappa(w_1) < \kappa(w_2)$. Under these conditions w_1 and w_2 produce distinct critical orbits by the minimax construction. Note however that if $f^{-1}(\kappa(w_2))$ contains a critical point with nontrivial stability group, then $K \cap f^{-1}(\kappa(w_2))$ is likely to have a large cuplength.

2.5 The fixed point set of a normal subgroup. Let M be a complete G-Riemannian manifold. Let $H \subset G$ be a normal subgroup and let $M^H \subset M$ be the fixed point set of H. Since H is normal, M^H is a G-space. If $x \in M^H$ then the action of G induces an orthogonal representation of H on the tangent space $T_x(M)$. Put

$$T_x^H(M) = \left\{ v \in T_x(M) \mid hv = v \quad \forall h \in H \right\}.$$

M^H is a closed geodesic G-submanifold of M. Moreover it is clear that if ϕ: $M \to M'$ is a G-map (resp. diffeomorphism, homotopy equivalence, vector bundle map) then

$$\phi|_{M^H} : M^H \to M'^H$$

is a G-map (resp. etc.).

Now suppose that f is a differentiable G-function on M satisfying (C_G). Then $\nabla f|_{M^H} \subset T^H(M^H)$ and the critical set of the restriction of f to M^H is equal to $M^H \cap K$. Since M^H is closed the restriction of f to M^H also satisfies (C_G), and the diffeomorphisms of Theorems A and B restrict to diffeomorphisms of the same type on M^H. If A is a nondegenerate critical G-submanifold of M, then A^H is a submanifold of A and $N^{H-}(A^H)$ is a G-subbundle of $N^-(A^H)$ (the restriction of $N^-(A)$ to A^H).

Suppose now that $A = A^H$ and that A is connected mod G. Then $N^-(A)$ splits:

$$N^-(A) = N^{H-}(A) \oplus N^{H-}(A)^\perp.$$

Let λ and λ_H be the index of A in M (dimension of N^-) and the index of A in M^H (dimension of N^{H-}). If N^{H-} and $N^{H-\perp}$ are G-orientable then Thom isomorphism identifies the map

$$(*) \qquad H_G^*(DN^-, SN^-) \to H_G^*(DN^{H-}, SN^{H-})$$

induced by the bundle inclusion with the map

$$H_G^{*-\lambda}(A) \to H_G^{*-\lambda_H}(A)$$

given by multiplication with the Euler class $\chi \in H_G^*(A)$ of the bundle $(N^{H-}(A)^\perp)_G \to A_G$. If the latter bundle is self-completing then $(*)$ is an injection.

3. Morse theorem for closed curves

3.1 Preliminaries. Let M be a compact Riemannian manifold. $\Lambda = H^1[S^1, M]$ will be the space of H^1 maps from $S^1 = \mathbf{R}/\mathbf{Z}$ to M. The C^∞ vector bundle $H^1[S^1, TM] \to H^1[S^1, M]$ is naturally identified with the tangent bundle $T\Lambda \to \Lambda$. If $c \in \Lambda$ is C^∞, $T_c(\Lambda)$ can be viewed as the space of H^1 vector fields along c (i.e. $V(t) \in T_{c(t)}M$). We define a metric on $T_c(\Lambda)$ by

$$\langle V, W \rangle = \int_0^1 \langle V(t), W(t) \rangle \, dt + \int_0^1 \left\langle \frac{dv}{dt}, \frac{dw}{dt} \right\rangle \, dt.$$

This metric has a unique extension to a Riemannian metric on $T\Lambda$. The energy function is defined by

$$f(c) = \frac{1}{2} \int_0^1 \left| \frac{dc}{dt} \right|^2 \, dt.$$

The group $O(2)$ of isometries of the circle acts naturally on Λ leaving f invariant. (For general facts about Λ and the proofs of the following theorems see Anosov [2], Eliasson [21], Eells [19], Klingenberg [32].)

1. $\Lambda = H^1[S^1, M]$ is a complete separable C^∞ $O(2)$-Riemannian manifold. $M \to H^1[S^1, M]$ is a functor from the category of compact differentiable maps to that of $O(2)$-manifolds, $O(2)$-invariant differentiable maps.

2. The energy function f is a C^∞ $O(2)$-invariant map $\Lambda \to \mathbf{R}$. The critical points of f are the closed geodesics on M. The energy function satisfies condition (C) of Palais and Smale.

Note that $O(2) \times \Lambda \to \Lambda$ is continuous but not differentiable; however if $c \in \Lambda$ is C^∞ then $O(2) \to O(2) \circ c$ is differentiable. Thus if c is a geodesic, $O(2) \circ c$ is a submanifold of Λ.

The space Λ of closed curves, together with the energy function f, is appropriate for equivariant Morse theory with the group $O(2)$. Thus one can use the cohomology of Λ to predict critical points, i.e. closed geodeiscs. However each geometrically distinct closed geodesic $c \in \Lambda$ gives rise to an infinite number of critical $O(2)$-orbits, corresponding to its mth *iterates* $c^m(t) = c(mt)$. In order to use Morse theory to predict closed geodesics we need to get a hold on the index of these iterates. The main theorem in this direction is due to Bott.

Let M be a compact Riemannian manifold. If $c: S^1 \to M$ is a closed geodesic, then the *index* and *nullity* of c are the index and nullity of the critical manifold $S^1 \circ c \subset \Lambda$.

Theorem [12]. *To each geodesic c on M^n is associated a matrix $P \in \mathrm{Sp}(n - 1)$. P in turn determines, up to an additive constant, a nonnegative*

integer valued function $a(z)$ *defined for* $|z| = 1$. *Let* $b(z) = \text{Nullity}(P - zI)$. *Then*:

(i) *The nullity of* c^m *is* $\sum_{z^m = 1} b(z)$.

(ii) *If* $b(z) = 0$, *then* a *is constant at* z; *the absolute value of the jump of* a *at a point* z *is bounded by* $b(z)$; $a(z) = a(\bar{z})$.

(iii) *The index of* c^m *is given by*

$$\lambda(c^m) = \sum_{z^m = 1} a(z).$$

The matrix P is the (linear) *Poincaré map* given by the "normal part" of the derivative of the symplectic diffeomorphism $TM \to TM$ given by the geodesic flow at the fixed point $(c(0), dc(0)/dt) \in TM$. The geodesic c is called *elliptic* if all eigenvalues of P lie on the unit circle and *hyperbolic* if none of the eigenvalues has absolute value 1. Note that if c is hyperbolic, then c^m is nondegenerate (i.e. nullity $c^m = 0$) for all m, and $\lambda(c^m) = m\lambda(c)$.

A closed curve has *multiplicity m* if it has stability group $\mathbf{Z}_m \subset SO(2)$; a *prime curve* is a curve of multiplicity 1.

3.2 "G-weak homotopy type" of Λ. Let $C^0[S^1, M]$ be the continuous closed curves on M with the usual sup norm. It is easy to see that a homotopy equivalence $M \to M'$ induces an $O(2)$-homotopy equivalence $C^0[S^1, M] \to C^0[S^1, M']$. Let $C^\infty[S^1, M]$ carry the C^∞ topology. The inclusions

$$(*) \qquad C^\infty[S^1, M] \to H^1[S^1, M] \to C^0[S^1, M]$$

are continuous; it follows from a fundamental theorem of Palais [41], [42] that the maps $(*)$ are homotopy equivalences. One would like to show that they are in fact $O(2)$-homotopy equivalences. We will settle for the

Theorem. *Let* $X = C^\infty[S^1, M]$ *or* $H^1[S^1, M]$ *and* $Y = C^0[S^1, M]$. *Let* H *and* G *be subgroups of* $O(2)$, *with* H *normal. If* $i: X \to Y$ *is the inclusion then*

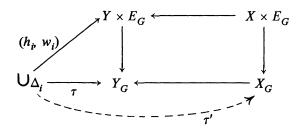$$i_G: \left(X_G, X_G^H\right) \to \left(Y_G, Y_G^H\right)$$

is a weak homotopy equivalence.

The theorem follows from the fact that, for each subgroup G, $i: X \to Y$ is a G-map with the property of the following lemma, which may be thought of as describing a "G-weak homotopy equivalence."

Lemma. *Let* $h: (K, A) \to (Y, i(X))$ *be a continuous map of a pair of compact sets into* Y *so that* $h|_A$ *lifts to a continuous map into* X. *Then there is a homotopy* $h_t: (X, A) \to (Y, i(X))$ *so that* $h_0 = h$, *and*:

(i) $h_{t|_A}$ *lifts to a (continuous) homotopy.*

(ii) h_1 *lifts to a continuous map* $K \to X$.

(iii) $h_0(x) = gh_0(y) \Rightarrow h_t(x) = gh_t(y)$ *for* $x, y \in K, g \in G$.

Assume the lemma, let $\tau\colon S^n \to Y_G$ be given. Triangulate S^n so that the restriction of τ to each simplex has a continuous lift to a map $(h_i, w_i)\colon \Delta_i \to Y \times E_G$. Putting $K = \bigcup_{\text{disjoint}} \Delta_i$ and $h\big|_{\Delta_i} = h_i$ by (iii), the homotopy $H_t \simeq (h_t, w)\colon \bigcup \Delta_i \to Y \times E_G$ descends to a homotopy from τ to a map with a continuous lift $\tau'\colon S^n \to X_G$.

Thus $\pi_n(X_G) \to \pi_n(Y_G)$ is surjective. The same argument with $\tau\colon (D^{n+1}, S^n) \to (Y_G, i(X_G))$ shows that $\pi_n(X_G) \to \pi_n(Y_G)$ is injective.

Proof of Lemma. We use a smoothing argument. Let $V \subset M^n$ be an open set with local chart $\psi\colon V \xrightarrow{\sim} B_{4\varepsilon} \subset \mathbf{R}^n$. Let (ϕ, ϕ') be a C^∞ partition of unity on $\mathbf{R}^n$ with $\phi \equiv 1$ on B_ε and $\phi' \equiv 1$ on $\mathbf{R}^n - B_{2\varepsilon}$. $\{e_j\}$ will denote an orthonormal basis of $\mathbf{R}^n$ and for $x \in \mathbf{R}^n$, x^j will be the jth coordinate. Let $u\colon \mathbf{R} \to \mathbf{R}$ be a nonnegative even C^∞ function with support in $[-1, 1]$ and $\int_{-\infty}^\infty u(t)\, dt = 1$. Put $u_s(t) = \frac{1}{s}u(\frac{t}{s})$. For $\gamma\colon \mathbf{R} \to \mathbf{R}^n$ we define $U_s\gamma\colon \mathbf{R} \to \mathbf{R}^n$ by $U_s\gamma = \sum e_j u_s^*[(\phi \circ \gamma)\gamma^j]$ (convolution product). Then there is a $\delta > 0$ so that for $c \in \operatorname{Im} h\colon K \to Y$ and $s \in (0, \delta]$,

    (i) $U_s(\psi \circ c)$ is C^∞ for $t \in (\psi \circ c)^{-1}(B_{4\varepsilon})$.

    (ii) $U_s(\psi \circ c) \equiv 0$ for $t \notin (\psi \circ c)^{-1}(B_{3\varepsilon})$.

    (iii)

$$\lim_{s \to 0} \ \operatorname*{Sup}_{\substack{t \in (\psi \circ c)^{-1}B_{4\varepsilon} \\ c \in \operatorname{Im} H}} \big| U_s(\psi \circ c)(t) - \phi(\psi \circ c)\psi \circ c(t) \big| = 0$$

(see e.g. [22]). Thus for $s \in (0, \delta]$ and $c \in \operatorname{Im} h$

$$(*) \qquad\qquad \psi^{-1}\big(U_s(\psi \circ c) + \phi'(\psi \circ c)(\psi \circ c)\big)$$

provides an approximation for the restriction of c to $(\psi \circ c)^{-1}(B_{4\varepsilon}))$ which is C^∞ for $t \in (\psi \circ c)^{-1}(B_\varepsilon)$. By (ii) the approximation extends (by the identity) to an approximation $W_s(c)$ for all t. (Note that (ii) depends on the fact that K is compact.) Moreover if $h\big|_A$ gives a continuous map $A \to X$, then the maps $K \times \{1\} \to X, A \times I \to X$ by $(x, s) \to W_{\delta s} \circ h(x)$ are continuous.

We can find a finite number of sets $V_i \subset M$ so that $M \subset \cup_{i=1}^{N} \psi_i^{-1}(B_{\varepsilon/2})$. Write $W_s^i(c)$ for the (extended) approximation (∗) corresponding to the set V_i. For sufficiently small $\delta > 0$ the homotopy $K \times I \to Y$ given by

$$h_s(x) = W_{\delta s}^1 \circ W_{\delta s}^2 \circ \cdots \circ W_{\delta s}^N h(x)$$

will have the desired properties. This homotopy is naturally G-invariant since all choices were made on the "source" manifold M.

4. Examples: Simply connected compact rank 1 symmetric spaces

4.1 Preliminaries.   A standard computation in Morse theory gives $H^*(\Lambda)$ if M is a S.C. CROSS, i.e., $M = S^n, CP^n, HP^n$ or CaP^2. We refer to Besse [7] for general facts about such spaces.

The standard metric on M is normalized so that the maximal sectional curvature is 1. Then all geodesics are closed and of length 2π; the Poincaré map is the identity and the critical set of the energy function on Λ consists of the critical manifolds A_m of geodesics of length $2\pi m$, $m \geqslant 0$. We have $A_0 \cong M$ and $A_m \cong STM$, the unit tangent bundle, for $m \geqslant 1$.

Let $a = n, 2, 4$ or 8 ($M = S^n, CP^n, HP^n, CaP^2$) and let $r = \dim M/a$ ($= 1, n, n, n$). Then

$$H^*(M) \cong \mathbf{Z}[u]/u^{r+1} = 0,$$

a truncated polynomial ring on one generator in dim a. If $x \in M$ and $V_{11} \in T_x M$ is a unit vector, then (see [7], [47]) there is an orthonormal basis $\{V_{ij} \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant a\}$ for $T_x M$ so that:

(i) the V_{1j} span (by exp) a *projective line* of M, i.e., a totally geodesic a-sphere of constant curvature 1,

(ii) V_{1j} and $V_{ij'}$, $i \neq 1$, span a totally geodesic $\mathbf{R}P^2$ of constant curvature $\frac{1}{4}$.

Note that the projective line of (i) is uniquely determined by V_{11}. It is easy to see by counting zeros of Jacobi fields that A_m ($m > 0$) is a nondegenerate critical manifold of index

$$\lambda(A_m) = (2m - 1)(a - 1) + (m - 1)(r - 1)a.$$

The negative bundle $N^-(A_m)$ is orientable.

The relative cycles of $H_*(\Lambda)$ can be completed by the method of Bott and Samelson [15]: To complete the negative bundle over A_m we let $B_m \subset \Lambda$ be the set of curves c having

$$\begin{cases} c\big|_{[j/2m,(j+2)/2m]} & \text{a geodesic of length } \pi, \\ c\big|_{[j/m,(j+1)/m]} & \text{lies in a projective line as in (i)}. \end{cases}$$

Let ϕ_t: $\Lambda \to \Lambda$ be the flow along $-\nabla f$. Then for $t > 0$, $\tilde{B}_m = \phi_t B_m$ is a completing manifold for the bundle $N^-(A)$. (Note a point in $B_m - A_m$ has "corners".) Thus the energy function is perfect for ordinary cohomology on Λ. We have

$$H^j(\Lambda) \cong H^j(M) \oplus \left[\bigoplus_{m=1}^{\infty} H^{j-\lambda(A_m)} STM \right].$$

Remark. The completing manifold B_m is *not* $O(2)$-invariant. This is reflected in the fact that the energy function is not perfect for $H^*_{SO(2)}$; see the remark at the end of the next section. B_m *is* invariant under the dihedral subgroup $H \subset O(2)$, although it is not an H-completing manifold. This will be used in §4.3.

4.2 $H^*_{SO(2)}(\Lambda, \Lambda^0; \mathbf{Q})$ for M a s.c. CROSS. Now let $G = SO(2)$. We compute the rational Poincaré polynomial $P_G(\Lambda, \Lambda^0; \mathbf{Q}, t)$. Over $\mathbf{Q}$, $H^*_G(A_m) \cong H^*_G(A)$ where $A = A_1$ (see Lemma 1, §6.1). A spectral sequence calculation for the fibrations $A \to M$, $A \to G\backslash A$ yields the Poincaré polynomials of the critical manifolds:

M	$P_0(A; \mathbf{Q}, t)$	$P_G(A; \mathbf{Q}, t)$	$\lambda(A_m)$
S^n, n even	$1 + t^{2n-1}$	$\dfrac{1 - t^{2n}}{1 - t^2}$	$(2m-1)(n-1)$
S^n, n odd	$(1 + t^{n-1})(1 + t^n)$	$\dfrac{(1 - t^{n-1})(1 + t^{n-1})}{1 - t^2}$	$(2m-1)(n-1)$
$r > 1$	$\dfrac{(1 - t^{an})(1 + t^{an+a-1})}{1 - t^a}$	$\dfrac{(1 - t^{an})(1 - t^{an+a})}{(1 - t^a)(1 - t^2)}$	$a - 1 + (m-1)(na + a - 2)$

The energy function is G-perfect for rational coefficients on (Λ, Λ^0) since the Morse polynomial is lacunary, i.e. all cohomology is in either even or odd dimensions. Thus the Poincaré polynomial is given by $\sum_m t^{\lambda(A_m)} P_G(A; \mathbf{Q}, t)$:

M	$P_G(\Lambda, \Lambda^0; \mathbf{Q}, t)$
S^n, n even	$t^{n-1}\left[\dfrac{1}{1 - t^2} + \dfrac{t^{2n-2}}{1 - t^{2n-2}} \right]$
S^n, n odd	$t^{n-1}\left[\dfrac{1}{1 - t^2} + \dfrac{t^{n-1}}{1 - t^{n-1}} \right]$
$\mathbf{C}P^n$	$\dfrac{t}{1 - t^2}\left(1 + t^2 + \cdots + t^{2n}\right)$
$\mathbf{H}P^n$	$t^3\left[\dfrac{1}{1 - t^2} + \dfrac{t^{4n+2}}{1 - t^{4n+2}} \right]\left(1 + t^4 + \cdots + t^{4n-4}\right)$
CaP^2	$t^7\left[\dfrac{1}{1 - t^2} + \dfrac{t^{22}}{1 - t^{22}} \right]\left(1 + t^8\right)$

Remark. The energy function is not perfect for G-cohomology on Λ for the standard metric on S^{2n+1} if we do not mod out by the point curves Λ^0. One can detect some nonzero boundary maps by computing the rational cohomology of Λ by Haefliger's method (§1.6). We have

$$P_G(\Lambda; \mathbf{Q}, t) = \frac{1}{1 - t^2} + \frac{t^{2n}}{1 - t^{2n}}.$$

But the Morse polynomial is given by $P_G(\Lambda^0; \mathbf{Q}, t) + P_G(\Lambda, \Lambda^0; \mathbf{Q}, t)$ so that

$$\mathfrak{M}_G(M; f, \mathbf{Q}, t) = \frac{1 + t^{2n+1}}{1 - t^2} + \frac{t^{2n}}{1 - t^2} + \frac{t^{4n}}{1 - t^{2n}}.$$

The difference $(\mathfrak{M} - P)$ is $t^{2n+1}(1 + t)/(1 - t^2)$. It follows that for $x = yw^i \in H_G^*(\Lambda^0)$ with $y \in H^{2n+1}(S^{2n+1})$ and $w \in H^2(BG)$ nonzero and $j \geqslant 0$, $\partial x \in H_G^{*+1}(\Lambda, \Lambda^0)$ is nonzero.

In particular $H_G^*(\Lambda, \Lambda^0; \mathbf{Q})$ is *not* a torsion $H^*(BG)$-module, giving an example of the failure of the localization theorem [10] in the noncompact case.

4.3 $H_{0(2)}^*(\Lambda, \Lambda^0; \mathbf{Z}_2)$ for M a s.c. CROSS. Next we compute the mod 2 series $P_G(\Lambda, \Lambda^0; \mathbf{Z}_2, t)$ for G the full orthogonal group $O(2)$. We find the polynomials $P_G(A_m; \mathbf{Z}_2, t)$ and show that the energy function is again perfect. While the negative bundles $N^-(A_m)$ are not in general orientable they are, of course, orientable mod 2. All coefficients will be mod 2.

The computation of $P_G(A_m; t)$ breaks down to two cases, depending on the parity of m. For the applications we will be particularly interested in the first part of the

Lemma. (i) *If m is odd then $H_G^*(A_m; \mathbf{Z}_2)$ is generated over $H^*(BG)$ by powers u_2^k, $2k < r$, of a class u_2 which is the restriction to $H_G^*(A_m)$ of a global class $u_2 \in H_G^{2a}(\Lambda)$.*

(ii) *For m odd*

$$P_G(A_m, t) = \frac{(1 - t^{a+1})(1 - t^{ar})(1 - t^{ar+a})}{(1 - t)(1 - t^2)(1 - t^{2a})}.$$

(iii) *When m is even $H_G^*(A_m)$ is generated over $H_G^*(\Lambda)$ by a class $u_1 \in H_G^a(A_m)$.*

(iv) *For m even*

$$P_G(A_m, t) = \frac{(1 - t^{ar})(1 - t^{ar+a})}{(1 - t)(1 - t^a)(1 - t^2)}.$$

Proof. Put $A = A_m$. First suppose that m is odd but r is even, so that M has even Euler characteristic. Then

$$H^*(A) = \mathbf{Z}_2[u] \otimes \Lambda(x)/u^r = 0,$$

with dim $u = a$ and dim $x = ar + a - 1$. When m is odd the map $(A)_G \to G \backslash A$ induces an isomorphism in mod 2 cohomology. Consider the spectral sequence for $A_G \to BG$. Since $G \backslash A$ is a $(2ar - 2)$-dimensional manifold, and since Rank $H^k(BG)$ grows linearly, we must have $0 \neq d^{a+1}u \in H^{a+1}(BG)$. Let $I \subset H^*(BG)$ be the ideal generated by $d^{a+1}u$. The E^{a+2}-term of the spectral sequence looks like:

$$
\begin{array}{|l}
\hline
\\
\quad B \\
\hline
\quad A \\
\\
\hline
\end{array}
$$

Section A consists of rows 0 through $ar - 1$ and is generated over $\mathbf{Z}_2$ by $u^{2k}H^*(BG)/I$ with $0 \leqslant k \leqslant (r - 2)/2$; section B is rows ar through $2ar - 1$ and is generated by the $xu^{2k}H^*(BG)/I$. Now all differentials of the term u^2 in A must vanish since the transgression of a square is zero. To show that all terms in B must have nonzero differentials it is sufficient to see that *if* so, then the polynomial $P_G(A, T)$ we obtain is consistent with Poincaré duality, i.e. $t^{2ar-2}P(1/t) = P(t)$. But this polynomial is easily seen to be the product of three Poincaré duality polynomials. Thus $H_G^*(A)$ is generated over $H^*(BG)$ by classes u_2^k which restrict in $H^*(A)$ to the u^{2k}.

When r is odd we have

$$H^*(A) = \mathbf{Z}_2[u] \otimes \Lambda(x)/u^{r+1} = 0$$

with dim $u = a$ and dim $x = ar - 1$. The spectral sequence is more complicated since it may be that $d^a x \neq 0$. However the previous argument still goes through.

Note that for spheres S^n, by uniqueness of the classifying map the canonical map $\pi_2 : (A_1)_G \to BG$ is homotopy equivalent to the inclusion

$$G(2, n + 1) \to G(2, \infty).$$

To complete the proof of the first half of the lemma we must show that the classes $u_2^k \in H_G^*(A)$ are restrictions of classes in $H_G^*(\Lambda)$. First note that u_2 is characterized by the fact that its image in $H^*(A)$ is the square of the pullback under the "evaluation map" of a generator u of $H^*(M)$. Next take a class in $H_G^a(\Lambda^0) \cong H^a(M \times BG)$ whose image in $H^*(\Lambda^0) \cong H^*(M)$ is u, and square it:

$$
\begin{array}{ccccc}
M \cong \Lambda^0 & \longrightarrow & \Lambda & \longleftarrow & A \\
\downarrow & & \downarrow & & \downarrow \\
\Lambda_G^0 & \longrightarrow & \Lambda_G & \longleftarrow & A_G
\end{array}
$$

Since this class is a square it is the restriction to $H_G^{2a}(\Lambda^0)$ of a class $y \in H_G^{2a}(\Lambda)$. The map $H_G^*(\Lambda) \to H^*(\Lambda^0)$ factors through $H^*(\Lambda)$. Since $H^{2a}(\Lambda)$ has only one generator over $\mathbf{Z}_2$, the image of y in $H^*(\Lambda)$ must be equal to $e^*(u^2)$, where

$e: \Lambda \to \Lambda^0$ is the evaluation map. But then the restriction of y to A is equal to e^*u^2, since $e: A \to \Lambda^0$ is the composition of $e: \Lambda \to \Lambda^0$ with the inclusion $A \to \Lambda$.

Now let m be even. Let $H \subset O(2)$ be the dihedral subgroup $\mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{Z}_2 \cong H_0 = H \cap SO(2)$, and let $H_1 = H/H_0$. Then $H^*(BH) \cong \mathbf{Z}_2[v_1, v_2]$ with dim $v_i = 1$, $H^*(BG) \to H^*(BH)$ by $w_1 \to v_1 + v_2$, $w_2 \to v_1 v_2$ and $H^*(BH) \to H(^*BH_0) \cong \mathbf{Z}[v]$ by $v_i \to v$. Mod 2 the subgroup H "captures" all the G-action in the sense that $H_G^*(A) \hookrightarrow H_H^*(A)$, since $A_H \to A_G$ is trivial mod 2.

Commutativity of

$$\begin{array}{ccc}
H^*(BH) & \longrightarrow & H_H^*(A) \\
\uparrow & & \uparrow \\
H^*(BG) & \longrightarrow & H_G^*(A)
\end{array}$$

gives $H_H^*(A) \cong H_G^*(A)[v_1]/v_1^2 + w_1 v_1 + w_2 = 0$, where w_1 and w_2 are identified with their images in $H_G^*(A)$. When m is even H_0 acts trivially on A and

$$H_H^*(A) \cong H_{H_1}^*(A) \otimes H^*(BH_0).$$

We view $H_G^*(A)$ as a subalgebra of the ring on the right.

We determine $P_{H_1}(A; \mathbf{Z}_2, t)$ using the fibration:

$$\begin{array}{c}
\mathbf{R}P^{ar-1} \longrightarrow H_1 \backslash A \cong A_{H_1} \\
\downarrow \\
M
\end{array}$$

The fibration is cohomologically trivial for $a > 2$ since $H^*(\mathbf{R}P^{ar-1})$ is generated by a class in dim 1; using commutativity of the fibration with the inclusions $S^2 \to S^3$, $S^2 \to \mathbf{C}P^n$ we see that it is in fact always trivial, and

$$P_{H_1}(A; \mathbf{Z}_2, t) = \frac{(1 - t^{ar})}{1 - t} \frac{(1 - t^{a(r+1)})}{1 - t^a}.$$

Next consider the fibration:

$$\begin{array}{c}
A \longrightarrow A_{H_1} \\
\downarrow \\
BH_1
\end{array}$$

We have $H^*(A) \cong \mathbf{Z}_2[u] \otimes \Lambda(x)/u^s = 0$ where $s = r(r+1)$ if M has even (odd) Euler characteristic, and dim $x = (2r - s + 1)a - 1$. Since $H^*(M) \hookrightarrow H^*(A_{H_1})$ all differentials of the entry u in the E^2-term of the spectral sequence must vanish. If we assume that all terms in the spectral sequence containing a factor of x have some nonzero differential we obtain the correct polynomial. Thus the assumption is justified; $H_G^*(A)$ is generated over $H^*(BG)$ by classes $u_1^k \in H_G^{ak}(A)$, $1 \leqslant k < s$. The image of u_1 in $H^*(A)$ is the pull back under the

evaluation map of a generator u of $H^*(M)$. As before u_1^2 is the image of a class in $H_G^{2a}(\Lambda)$.   q.e.d.

Finally we show that the energy function is G-perfect mod 2 for (Λ, Λ^0). Let $a_m = f(A_m) + \varepsilon$. Recall that the energy function is perfect for ordinary cohomology on Λ. Thus the right-hand vertical arrow in the following diagram is an injection:

$$
\begin{array}{ccc}
H_G^{\lambda(A_m)}\left(\Lambda^{a_m}, \Lambda^{a_{m-1}}\right) & \xrightarrow{\quad\sim\quad} & H^{\lambda(A_m)}\left(\Lambda^{a_m}, \Lambda^{a_{m-1}}\right) \\
\downarrow & & \downarrow \\
H_G^{\lambda(A_m)}\left(\Lambda^{a_m}\right) & \longrightarrow & H^{\lambda(A_m)}\left(\Lambda^{a_m}\right)
\end{array}
$$

(Each group in the upper row has one generator over $\mathbf{Z}_2$, corresponding to the Thom class of the negative bundle over A_m.) Commutativity implies that the left vertical arrow is an injection.

Let $x \in H_G^*(\Lambda^{a_m}, \Lambda^{a_{m-1}})$, $m \geqslant 1$. We want to show that the boundary $\partial x \in H_G^{*+1}(\Lambda, \Lambda^{a_m})$ is zero. But $H_G^*(\Lambda^{a_m}, \Lambda^{a_{m-1}}) \cong H_G^{*+\lambda(A_m)}(A_m)$ is generated over $H^*(BG)\{u_2^k\} \subset H_G^*(\Lambda)$ by classes in dim $\leqslant \lambda(A_m) + a$. It is sufficient to show that the boundaries of these generating classes are zero. But $H_G^*(\Lambda, \Lambda^{a_m}) = 0$ in dim $< \lambda(A_{m+1})$. Since by the above diagram the lowest dimension of $H_G^*(\Lambda, \Lambda^{a_m})$ cannot be killed we are done when $\lambda(A_{m+1}) > \lambda(A_m) + a$, i.e. $ra > 2$ or $M \neq S^2$.

For S^2 we work a little harder. Since $H_G^*(\Lambda^{a_m}, \Lambda^{a_{m-1}})$ is generated over $H_G^*(\Lambda)$ by a single class in dim $\lambda(A_m) = 2m - 1$ when m is odd, and by classes in dim $2m - 1, 2m + 1$ when m is even, it is sufficient to show that $H_G^*(\Lambda^{a_m}, \Lambda^{a_{m-1}}) \to H_G^*(\Lambda^{a_m}, \Lambda^0)$ is injective for m odd. Let m be odd. Since $H_G^*(\Lambda^{a_m}, \Lambda^{a_{m-1}}) \to H_H^*(\Lambda^{a_m}, \Lambda^{a_{m-1}})$ is injective for the dihedral subgroup $H \subset G$ it is enough to show $H_H^*(\Lambda^{a_m}, \Lambda^{a_{m-1}}) \to H_H^*(\Lambda^{a_m}, \Lambda^0)$ is injective.

Now the completing manifold constructed by the method of Bott-Samelson is invariant under H. But B_m does not fiber over A_m as an H-manifold; B_m contains points with stability group $H_1 \subset H$ while H acts freely on A_m. Thus it is not clear (in fact it is false for $m = 1$) that $H_H^*(B_m, B_m^{a_m - \varepsilon}) \to H_H^*(B_m)$ is injective. However, B_m does "complete" A_m "down to Λ^0": Let $B_m^0 \subset B_m$ be the fixed point set of H_1. Then H acts freely on $B_m - B_m^0$; $H_H^*(B_m, B_m^{a_m - \varepsilon}) \to H_H^*(B_m, B_m^0)$ is injective in the top dimension because $H \, B_m - B_m^0$ is a manifold. But

$$
H^*(A_m) \cong \mathbf{Z}_2[w_1, v_1]/w_1^3 = o, v_1^2 + v_1 w_1 + v_1 w_1^2 = 0
$$

is generated by classes from $H_H^*(B_m)$; by Poincare duality we have injectivity of $H_H^*(B_m, B_m^{a_m - \varepsilon}) \to H_H^*(B_m, B_m^0)$. Since no nontrivial geodesic is fixed by H_1, by pushing B_m down by $-\nabla f$ we can assume $f(B_m^0) \subset [0, \varepsilon]$, so that f is a G-perfect on (Λ, Λ^0).

Remark. The energy function is not G-perfect (or H-perfect) on Λ for $M = S^2$: Let x be a nonzero class in $H_3(\Lambda_G^{a_1}, \Lambda_G^{a_0})$. The image of x in $H_3(G \backslash \Lambda^{a_1}, G \backslash \Lambda^0)$ is represented by the set of "circles" on S^2, i.e. intersections of S^2 with 2-planes in $\mathbf{R}^3$. So $\partial x \in H_2(G \backslash \Lambda^0)$ is a generator of $H_2(S^2)$.

The Poincare series for (Λ, Λ^0) is given by $\Sigma t^{\lambda(A_m)} P_{O(2)}(A_m; \mathbf{Z}_2, t)$. We have

$$P_{O(2)}\left(\Lambda, \Lambda^0; \mathbf{Z}_2, t\right)$$
$$= \frac{t^{a-1}(1 - t^{ar})(1 - t^{ar+a})\left[1 - t^{a+1} + t^{ra+a-2}(1 + t^a)\right]}{(1 - t)(1 - t^2)(1 - t^{2a})(1 - t^{2(ra+a-2)})} \cdot$$

For $M = S^n$ this reduces to

$$\frac{t^{n-1}(1 - t^n)\left[1 - t^{n+1} + t^{2n-2}(1 + t^n)\right]}{(1 - t)(1 - t^2)(1 - t^{4n-4})} \cdot$$

5. Existence of short closed geodesics on s.c. CROSS's

A *circle* on S^n is the intersection of S^n with a 2-plane in $\mathbf{R}^{n+1}$. A circle on M_s for $M_s = CP^n, HP^n$ or CaP^2 is a circle on some isometrically embedded $S^a \subset M_s$. Put $\lambda = ar/n$. Let $g(\lambda, n)$ be the cuplength of the space of *great* circles on M_s.

Theorem. *Let M be a compact Riemannian manifold with the property that every geodesic loop on M has length $\geq 2\pi$. For $M_s = S^n, CP^n, HP^n$ or CaP^2 let $\alpha: M_s \to M$ be a differentiable homotopy equivalence with the property that the image under α of any circle on M_s has length $< 4\pi$. Then M has $g(\lambda, n)$ closed geodesics without self-intersection of length $< 4\pi$. If all geodesics on M are nondegenerate then there are $a(a + 1)r(r + 1)/4$ closed geodesics with the same property.*

Remarks. Alber [1] has computed the cuplength $g(1, n)$ of $G(2, n + 1)$; it is given by

$$g(1, n) = n + 2^k - 1, \quad \text{where } 2^k \leq n < 2^{k+1}.$$

Klingenberg [31] has computed $g(\lambda, n)$ for $n > 1$;

$$g(\lambda, n) = 2\lambda n - (2\lambda - 1)s - 1, \quad \text{where } 0 \leq s = n - 2^k < 2^k.$$

By a theorem of Klingenberg [18] the hypothesis on the length of loops will be satisfied if the sectional curvature satisfies $1 \geq K > \frac{1}{4}$ or, in the even dimensional case, $1 \geq K > 0$.

Using the space of "biangles" and a new proof of the sphere theorem, Ballman, Thorbergsson and Ziller [5] have proved the same result for spheres with *only* the assumption that $\frac{1}{4} \leq K \leq 1$.

Proof. All coefficients will be mod 2. The map $\alpha: M_s \to M$ induces an isomorphism $H_{O(2)}{}^*(\Lambda M_s, \Lambda^0 M_s) \xrightarrow{\sim} H_{O(2)}{}^*(\Lambda M, \Lambda^0 M)$. Now

$$H_{O(2)}{}^*\left(\Lambda^{2\pi^2} M_s, \Lambda^0 M_s\right) \cong H_{O(2)}{}^*{}_{-\lambda(A_1)}(A_1)$$

has $a(a + 1)r(r + 1)/4$ linearly independent classes, $g(\lambda, n)$ of which are pairwise subordinated by means of classes $w_1^q w_2^r u_2^k \in H_{O(2)}{}^*(\Lambda M_s)$. We will show that the images of these classes can be represented in $(\Lambda^{8\pi^2 - \varepsilon} M, \Lambda^0 M)$. These classes then correspond by the minimax construction to closed geodesics of energy $< 4\pi$. Such a geodesic c has trivial stability group by hypothesis, so that $H_{O(2)}{}^*(O(2) \circ c) = \mathbf{Z}_2$. The theorem then follows from §2.4.

Let $A \subset \Lambda M_s$ be the space of great circles, and let $\pi: C \to A$ be the $O(2)$-bundle of parametrized circles "parallel" to a great circle in A. If $c \in A$ then $(\pi^{-1}(c), \pi^{-1}(c) \cap \partial C) \to (\Lambda M_s, \Lambda^0 M_s)$ represents the Thom class of the negative bundle over A in $H_{\lambda(A_1)}(\Lambda M_s, \Lambda^0)$. (It is enough that C has the correct dimension and that $f < 2\pi^2$ on $C - A$.) Now let $x \in C_*(A_{O(2)})$ represent a class of $H_{O(2)}{}^*(A)$. The corresponding class in $H_{O(2)}{}^*{}_{+\lambda(A_1)}(\Lambda M_s, \Lambda^0)$ is represented by $\pi_{O(2)}^{-1}(x)$. By hypothesis $\alpha_*(\pi_{O(2)}^{-1} x) \in H_{O(2)}{}^*{}_{+\lambda(A_1)}(\Lambda M, \Lambda^0 M)$ lies in the homotopy quotient of the curves of length $\leqslant 4\pi - \varepsilon$ for some $\varepsilon > 0$.

By the Schwartz inequality the energy and length of a closed curve are related by $f(c) \geqslant \frac{1}{2}(l(c))^2$; we have equality if and only if c is parametrized proportional to arc length. To complete the proof we give an $O(2)$-deformation of the C^∞ map $\alpha: (C, \partial C) \to (\Lambda M, \Lambda^0)$ so that the image of C lies in $\Lambda^{8\pi^2 - \varepsilon} M$. This is done by pushing αc down along the gradient of the restriction of the energy function to the "reparametrizations" of αc.

A C^∞ map $\gamma: S^1 \to M$ induces a C^∞ function f_γ on $H^1[S^1, S^1]$ by $f_\gamma(\omega) = f(\gamma \circ \omega)$. Let $\phi(\gamma): \mathbf{R}^+ \to H^1[S^1, S^1]$ be the flow along $-\nabla f_\gamma$ with initial value the identity map $S^1 \to S^1$. We get a continuous map

$$\Phi: \left(C \times \mathbf{R}^+, \partial C \times \mathbf{R}^+\right) \to \left(\Lambda M, \Lambda^0\right) \quad \text{by } \Phi_t(c) = \alpha c \circ \phi_t(\alpha c).$$

Since C is compact we can find $\kappa > 0$ so that $l(\alpha c) < \kappa \Rightarrow f(\alpha c) < 8\pi^2 - \varepsilon$ for $c \in C$, and K so that $f(\alpha c) \leqslant K$ for $c \in C$. Consider the *immersions* γ: $S^1 \to M$ with $l(\gamma) \in [\kappa, 4\pi - \varepsilon]$ and $f(\gamma) \leqslant K$. Since the energy function on $H^1[S^1, S^1]$ satisfies condition (C) there is a $T \in \mathbf{R}^+$ so that for such γ, $f(\gamma \circ \phi_T(\gamma)) \leqslant 8\pi^2 - \varepsilon$. But the immersions are dense in ΛM (dim $M \geqslant 2$) so that Φ_T gives the desired deformation.

6. Hyperbolic case for s.c. rational CROSS's

6.1 "Localization" theorem for manifolds with all geodesics hyperbolic. Let M be a compact Riemannian manifold, all of whose closed geodesics are hyperbolic. Let $p > 2$ be prime; k a multiple of p.

Theorem. *Suppose* $H^k_{SO(2)}(\Lambda^\beta, \Lambda^\alpha; \mathbf{Z}_p)$ *contains a class which is annihilated by some nonzero element of* $H^*(BSO(2); \mathbf{Z}_p)$. *Then* Λ^β *contains a closed geodesic of index k whose multiplicity is prime to p.*

Corollary. *If M has only finitely many closed geodesics, then for p sufficiently large multiplication with* $H^*(BSO(2); \mathbf{Z}_p)$ *is injective on* $H^k_{SO(2)}(\Lambda, \Lambda^0; \mathbf{Z}_p)$.

The corollary follows from the theorem by Bott's formula (§3.1) if we take p greater than the index of each prime closed geodesic.

The idea is this: We put $G = SO(2)$. If we work mod p then a geodesic c whose multiplicity m is divisible by p has

$$H^*_G(G \circ c) \approx H^*(B\mathbf{Z}_m) \approx \mathbf{Z}_p[x_2] \otimes \Lambda(x_1),$$

a free $H^*(BG; \mathbf{Z}_p)$-module. If the multiplicity of c is prime to p then $G \circ c$ looks like a *point* in G-cohomology mod p. In the hyperbolic case with only finitely many closed geodesics, by Bott's formula for large primes p any G-cohomology which appears in a dimension divisible by p must "come from" a geodesic whose multiplicity is divisible by p and will generate a free $H^*(BG; \mathbf{Z}_p)$-module in $H^*_G(\Lambda, \Lambda^0)$. However, for example, for a sphere S^n with the standard metric closed geodesics with multiplicity divisible by p appear only with index $\geqslant (2p - 1)(n - 1)$ and the module structure of $H^*_G(\Lambda, \Lambda^0)$ provides a contradiction to the hypotheses of the Corollary.

Proof of Theorem. Let $G = SO(2)$ and let M be a compact Riemannian manifold. For $m \in \mathbf{Z}^+$ let $i_m: \Lambda_m \to \Lambda$ be the inclusion of the fixed point set of $\mathbf{Z}_m \subset G$ in Λ. By §2.5 Λ_m is a closed submanifold; the critical set of the restriction of the energy function f to Λ_m is the set of closed geodesics whose multiplicity is divisible by m.

Let $\Lambda \to \Lambda$ by $c \to c^m$, where $c^m(t) = c(mt)$. This is an embedding of Λ onto Λ_m; Λ_m is given as a G-manifold by Λ and by the homomorphism $\gamma_m: G \to G$ by $g \to g^m$. Since $f(c^m) = m^2 f(c)$, the index of c^m in Λ_m is equal to the index of c in Λ.

Let $\Gamma_m: BG \to BG$ be the map induced by γ_m and write X_m for the G-manifold given by X and γ_m. Recall that $H^*(BG; \mathbf{Z}) \approx \mathbf{Z}[w]$; $\Gamma^*_m: H^2(BG; \mathbf{Z}) \to H^2(BG; \mathbf{Z})$ is multiplication by m. Pulling back the spectral sequence for $X_G \to BG$ by Γ^*_m we have the first part of

Lemma 1. *Let p be prime.*

(i) $\Gamma^*_m: H^*_G(X, \mathbf{Z}_p) \to H^*_G(X_m; \mathbf{Z}_p)$ *is an isomorphism if* $(p, m) = 1$.

(ii) *If p divides m then* $H^*_G(X_m; \mathbf{Z}_p) \cong H^*(BG; \mathbf{Z}_p)\{H^*(X; \mathbf{Z}_p)\}$; *in particular* $H^*_G(X_m; \mathbf{Z}_p)$ *is a free* $H^*(BG; \mathbf{Z}_p)$-*module.*

The second part comes from looking at the pullback of spectral sequences induced by the inclusion $\mathbf{Z}_m \to G$ since $(X_m)_{\mathbf{Z}_m} \cong X \times B\mathbf{Z}_m$.

Note that we have two maps $i_m, \Gamma_m\colon \Lambda_{m_G} \to \Lambda_G$. The first is the identity on the base BG and "multiplication by m" on the fiber, i.e. $i_m\colon \Lambda_m \to \Lambda$. The second is the identity $\Lambda \to \Lambda$ on the fiber and "multiplication by m", i.e., Γ_m, on the base.

Let $p > 2$ be prime and let $c \in \Lambda_p$ be a nondegenerate geodesic of multiplicity m, index λ. The action of $\mathbf{Z}_m$ on the negative bundle over the orbit $G \circ c$ of c gives a splitting

$$N^-(G \circ c) = N_p^- \oplus N_0^-,$$

where N_{pc}^- is the negative bundle at c in Λ_p, i.e. the subspace on which $\mathbf{Z}_p \subset \mathbf{Z}_m$ acts as the identity and N_{oc}^- is a sum of 2-dimensional subspaces corresponding to representations with eigenvalues $e^{\pm i\alpha}$ where $e^{i\alpha}$ is an mth root of unity with $e^{i\alpha m/p} \neq 1$. Note that N_p^- is orientable if and only if N^- is. Let DN^- and SN^- be the unit disk and sphere bundles; let λ_p be the index of c in Λ_p.

Lemma 2. *If N^- is orientable then $H_G^*(DN^-, SN^-; \mathbf{Z}_p) \cong H_G^{*+\lambda}(B\mathbf{Z}_m; \mathbf{Z}_p)$ is a free $H^*(BG; \mathbf{Z}_p)$-module on generators in* dim $\lambda, \lambda + 1$; $H_G^*(DN_p^-, SN_p^-; \mathbf{Z}_p)$ *is a free $H^*(BG; \mathbf{Z}_p)$-module on generators in* dim $\lambda_p, \lambda_p + 1$ *and*

$$i_p^*\colon H_G^*(DN^-, SN^-; \mathbf{Z}_p) \to H_G^*(DN_p^-, SN_p^-; \mathbf{Z}_p)$$

is an injection.

Proof. The lemma follows from the Thom isomorphism and §§1.6, 2.3, 2.5: the bundle N_0^- is self-completing mod p since the Euler class of $N_0 \to B\mathbf{Z}_m$ is $\prod_\alpha \alpha m/2\pi$ times a generator of $H^{2(\lambda-\lambda_p)}(B\mathbf{Z}_m) \approx \mathbf{Z}_m$ and since $\alpha = 2\pi j/m$ with $j/p \notin \mathbf{Z}$.

Lemma 3. *Let $p > 2$ be prime. Suppose $(\alpha, \beta]$ contains one critical value. If all geodesics on M are nondegenerate, and if all geodesics c of index k with $\alpha < f(c) \leqslant \beta$ have multiplicity divisible by p, then*

$$i_p^*\colon H_G^k(\Lambda^\beta, \Lambda^\alpha; \mathbf{Z}_p) \to H_G^k(\Lambda_p^\beta, \Lambda_p^\alpha; \mathbf{Z}_p)$$

is injective.

Proof. By Theorem B in §2.1, Λ^β is G-diffeomorphic to Λ^α with a handle bundle disjointly attached for each critical orbit in $\Lambda^\beta - \Lambda^\alpha$. By §2.5 this diffeomorphism restricts to a diffeomorphism of the same type on Λ_p^β. So we can assume that $\Lambda^\beta - \Lambda^\alpha$ intersects the critical set K of f in the G-orbit of a single closed geodesic of index λ and multiplicity m. In the unorientable case $H_G^*(DN^-, SN^-; \mathbf{Z}_p) = 0$ for $p \neq 2$ (§1.6). In the orientable case if $(p, m) = 1$ then by hypothesis $\lambda \neq k$. So

$$H_G^k(\Lambda^\beta, \Lambda^\alpha; \mathbf{Z}_p) = H^{k-\lambda}(B\mathbf{Z}_m; \mathbf{Z}_p) = 0.$$

If p divides m, i_p^* is injective by Lemma 2.

Proposition. *Suppose all geodesics on M are hyperbolic. Let p be prime; k a multiple of p. If all geodesics of index k in Λ^β have multiplicity divisible by p, then $i_p^*\colon H_G^k(\Lambda^\beta, \Lambda^\alpha; \mathbf{Z}_p) \to H_G^k(\Lambda_p^\beta, \Lambda_p^\alpha; \mathbf{Z}_p)$ is injective for all $\alpha \in \mathbf{R}$.*

Proof. From now on we assume $\mathbf{Z}_p$ coefficients. For this extension of Lemma 3 we shall need

Lemma 4. *Suppose all geodesics on M are hyperbolic and that $(\sigma, \tau]$ contains one critical value. Let w be a generator of $H^*(BG)$.*

(i) *If $x \in H_G^j(\Lambda_p^\tau, \Lambda_p^\sigma)$ is in the image of $i_p^* H_G^j(\Lambda^\tau, \Lambda^\sigma)$ then x is of the form $x = x_i w^i$ for $i \leqslant j(p-1)/2p$.*

(ii) *If $x \in H_G^j(\Lambda_p^\tau, \Lambda_p^\sigma)$ then xw^i is in the image of i_p^* for $i \geqslant j(p-1)/2$.*

Lemma 4 follows from Lemma 2 and the fact that $\lambda(c^p) = p\lambda(c)$.

The proposition is proved by induction, up on β and down on α. First it is true when $\beta = 0$ or, by Lemma 3, when $(\alpha, \beta]$ contains one critical value. Let γ be such that $(\alpha, \gamma]$ contains one critical value. We assume

(A) i_p^* is injective on $H_G^k(\Lambda^\tau, \Lambda^\sigma)$, $\tau < \beta$, $\sigma \in \mathbf{R}$.

(B) i_p^* is injective on $H_G^k(\Lambda^\beta, \Lambda^\gamma)$.

Note that by hypothesis all geodesics are hyperbolic and thus nondegenerate, so that the critical values of f are isolated.

Suppose $0 \neq x \in H_G^k(\Lambda^\beta, \Lambda^\alpha)$ and $i_p^* x = 0$. By (A) x comes from a class $x_\tau \in H_G^k(\Lambda^\beta, \Lambda^\tau)$ where $(\tau, \beta]$ contains one critical value. Let x_κ be the image of x_τ in $H_G^k(\Lambda^\beta, \Lambda^\kappa)$ and put $\bar{x}_\kappa = i_p^* x_\kappa$. By Lemma 4(i) and Lemma 3

$$0 \neq \bar{x}_\tau = \bar{y}_\tau w^{(p-1)k/2p} \in H_G^k\left(\Lambda_p^\beta, \Lambda_p^\tau\right).$$

Then also

$$0 \neq \bar{x}_\gamma = \bar{y}_\gamma w^{(p-1)k/2p} \in H_G^k\left(\Lambda_p^\beta, \Lambda_p^\gamma\right).$$

Here $\bar{x}_\gamma \neq 0$ by (B). Now $i_p^* x = \bar{x}_\alpha = 0$ implies $\bar{x}_\gamma \to 0$ in $H_G^k(\Lambda_p^\beta, \Lambda_p^\alpha)$; thus by Lemma 1(ii) $\bar{y}_\gamma \to 0$ in $H_G^{k/p}(\Lambda_p^\beta, \Lambda_p^\alpha)$. So there is a $\bar{v} \in H^{k/p-1}(\Lambda_p^\gamma, \Lambda_p^\alpha)$ with $\partial \bar{v} = \bar{y}_\gamma$. By Lemma 4(ii) there exists $u \in H_G^{k-1}(\Lambda^\gamma, \Lambda^\alpha)$ with

$$i_p^* u = \bar{v} w^{(p-1)k/2p}.$$

Consider $\partial u \in H_G^*(\Lambda^\beta, \Lambda^\gamma)$. By commutativity $i_p^*(\partial u) = \bar{x}_\gamma$. But then by (B) $\partial u = x_\gamma$ so that $0 = x_\alpha = x$, a contradiction.   q.e.d.

The theorem follows immediately by Lemma 1(ii).

6.2 Simply connected rational CROSS's. The goal of this section is the

Theorem. *Let M be a simply connected Riemannian manifold with the rational homotopy type of a compact rank 1 symmetric space. Suppose all geodesics on M are hyperbolic. Let $n(l)$ be the number of prime closed geodesics*

on M of length $\leq l$. *Then*

$$\liminf n(l)\frac{\log(l)}{l} > 0.$$

The theorem will follow from the

Lemma. *Let* M_s *be a simply connected CROSS with standard metric. There exists an integer b so that, for each prime* $p > \chi(M)$, *there is a nonzero class* $x_p \in H_G^k(\Lambda, \Lambda^0; \mathbf{Z}_p)$ *for some positive multiple* $k \leq bp$ *of p so that*:

(i) x_p *is annihilated by some nonzero class in* $H^*(BG)$.

(ii) $x_p \nrightarrow 0$ *in* $H_G^k(\Lambda^{2\pi^2 p^2}, \Lambda^0; \mathbf{Z}_p)$.

Assume the lemma. Using obstruction theory one can find a differentiable map $\alpha: M_s \to M$ inducing rational homotopy equivalence. This follows from a theorem of Sullivan (Inst. Hautes Études Sci. Publ. Math. no. 47) since the spaces are formal. One can also give a "direct" proof. Then for some $\kappa \in \mathbf{R}$, $\alpha(\Lambda^a M_s) \subset \Lambda^{\kappa a} M$ for all a. For large primes p we will have an isomorphism $H^*(M; \mathbf{Z}_p) \to H^*(M_s; \mathbf{Z}_p)$ and an isomorphism of $H^*(BG)$-modules

$$H_G^*\left(\Lambda M, \Lambda^0 M; \mathbf{Z}_p\right) \xrightarrow{\sim} H_G^*\left(\Lambda M_s, \Lambda^0 M_s; \mathbf{Z}_p\right).$$

For each such prime the lemma and the theorem of §6.1 give a geodesic of length $\leq 2\pi p\sqrt{\kappa}$ and index k whose multiplicity is prime to p. By Bott's formula the multiplicity of such a geodesic is $\leq b$ and the theorem is proved.

It remains to prove the lemma. By Lemma 1(i) of §6.1, the Poincaré polynomial of §4.2 gives $P_G(\Lambda, \Lambda^0; \mathbf{Z}_p, t)$ accurate to order $t^{\lambda(A_p)}$ if p does not divide the Euler characteristic of M. Since $\lambda(A_m) = a + b(m - 1)$ with $a - b \not\equiv 0$ and $a \not\equiv 0 \bmod p$ for large p, p will divide $\lambda(A_m)$ for some $2 \leq m \leq p - 1$. Let $k = \lambda(A_m)$. Then $p \leq k \leq bp$, and $H^k(\Lambda, \Lambda^0; \mathbf{Z}_p) \to H^k(\Lambda^{2\pi^2 p^2}, \Lambda^0; \mathbf{Z}_p)$ is injective.

When $M_s \neq S^3$, $\mathbf{C}P^n$, it is easy to see from the table that Rank $H_G^k(\Lambda, \Lambda^0; \mathbf{Z}_p) > $ Rank $H_G^{k+2}(\Lambda, \Lambda^0; \mathbf{Z}_p)$.

When $M_s = S^3$ or $\mathbf{C}P^n$, Rank $H_G^j(\Lambda, \Lambda^0; \mathbf{Z}_p) = $ Rank $H_G^{j+2}(\Lambda, \Lambda^0; \mathbf{Z}_p)$ for $\lambda(A_2) \leq j < \lambda(A_p) - 1$. Since the energy function is perfect for $H^*(\Lambda; \mathbf{Z}_p)$, the map

$$i^*: H_G^\lambda\left(\Lambda, \Lambda^0; \mathbf{Z}_p\right) \to H^\lambda\left(\Lambda; \mathbf{Z}_p\right)$$

is nonzero for $\lambda = \lambda(A_{m+1})$; the Thom class of the negative bundle over $(A_{m+1})_G$ pulls back to the Thom class for A_{m+1}. But since Λ is the fiber of $\Lambda_G \to BG$, $i^* \equiv 0$ on classes of the form $x \cdot w$. It follows that the kernel of

$$\cup w^{(\lambda(A_{m+1}) - \lambda(A_m))/2}: H_G^k\left(\Lambda, \Lambda^0; \mathbf{Z}_p\right) \to H_G^{\lambda(A_{m+1})}\left(\Lambda, \Lambda^0; \mathbf{Z}_p\right)$$

is nonzero and the lemma is proved.

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