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APPROXIMATE EIGENFUNCTIONS OF THE LAPLACIAN

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Introduction

Let M be a compact orientable $(n+1)$-dimensional riemannian manifold, and let Γ be a closed geodesic on M. We say Γ is stable if the Poincaré map associated with Γ (this is defined in § 1) splits into a direct sum of rotations through distinct angles $\theta_1, \cdots, \theta_n$, $0 < \theta_i < 2\pi$, $\theta_i \neq 2\pi - \theta_j$ for all i, j. Let Δ denote the Laplace-Beltrami operator on M. Guillemin and Weinstein [5] have recently proved the following.

Theorem I. *If there is a stable closed geodesic on M of length L, then, given any multi-index α, there are at least two eigenvalues λ_m of Δ, counted by multiplicity, satisfying $\sqrt{-\lambda_m} = k_m + 0(m^{-\frac{1}{2}})$, where $k_m = L^{-1}(2\pi m + (\alpha_1 + \frac{1}{2})\theta_1 + \cdots + (\alpha_n + \frac{1}{2})\theta_n + \pi p_0)$. Here $p_0 = 0$ or 1 and is determined by the behavior of the Jacobi fields along Γ.*

Since a rotation through an angle θ can turn into a rotation through $2\pi - \theta$ if one changes bases, there is a technical condition that determines which of these rotations one chooses in selecting $\theta_1, \cdots, \theta_n$ in Theorem I. We omit this here; see § 2.

Guillemin and Weinstein's proof of Theorem I is based on the construction of an isometry from $L^2(S^1)$ to $L^2(M)$ that approximately intertwines $d^2/d\theta^2$ and Δ. The isometry is a Fourier integral operator of a new type developed by Guillemin in [4].

Our objective here is to prove Theorem I and its analogue for nonorientable M by constructing approximate solutions u_m to the equations $(\Delta + k_m^2)u = 0$. The functions u_m are probably very close to the image of $\{e^{im\theta}\}_{m=1}^{\infty}$ under the isometry used by Guillemin and Weinstein. However, we construct them by beginning with the ansatz of geometrical optics and using a complex phase function ψ with $\operatorname{Im}\psi > 0$ off Γ and $\operatorname{Im}\psi = 0$ on Γ. The resulting u_m are very small outside a tube around Γ with radius $0(m^{-\frac{1}{2}})$. The construction is quite explicit, expressing the u_m in terms of the Jacobi fields along Γ.

Our approach is derived from the work of Babich and Lazutkin [2] who used a similar method to prove Theorem I in the case $n = 1$. We found the idea which enabled us to carry out the construction for general n in the paper [6] of Hörmander.

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In § 4 we sketch the extension of these results to the case of a closed piecewise geodesic arc Γ reflected off ∂M at its corners. In the case that Γ consists of two copies of a single arc invariant under reflection in ∂M, the functions u_m are known as "bouncing ball waves". Such waves were discussed by Keller and Rubinow in a paper [8] which introduced the idea that closed ray paths could be used to predict eigenvalues of the Laplacian. As in the case of a closed geodesic, when $n = 1$ the results in § 4 for bouncing ball waves are due to Lazutkin [9] and also Smith [10].

The significance of the approximate eigenfunctions u_m is not clear. In [1] Arnol'd has given an example (in the case of bouncing ball waves with $n = 1$) where they approximated no true eigenfunctions of Δ. In § 5 we point out that in all cases (even when M is not compact) the u_m for m large are the amplitudes of very long-lasting standing wave vibrations of M. This observation also appears in [1].

The author is indebted to Professor Alan Weinstein for many helpful discussions of this work.[1]

1. Description of method

We will look for approximate solutions to $(\Delta + k^2)u = 0$, which have the form

$$u = e^{ik\psi(x)}(a_0(x) + a_1(x)/k + \cdots + a_N(x)/k^N) . \tag{1.1}$$

When ψ is real-valued, (1.1) is the standard ansatz of geometrical optics. The only novelty here is that we allow $\mathrm{Im}\, \psi \geq 0$.

In local coordinates the principal symbol of Δ has the form

$$p(x, \xi) = \sum g^{ij}(x)\xi_i\xi_j ,$$

where the line element $ds^2 = \sum g_{ij}dx_idx_j$, and $g^{ij} = (g_{ij})^{-1}$. When u has the form (1.1), the coefficient c_s of $k^{2-s}e^{ik\psi}$ in $(\Delta + k^2)u$ is given by

$$c_0 = \underline{c}_0 a_0 = (1 - p(x, d\psi))a_0 ,$$

$$c_1 = i\left(\frac{\partial p}{\partial \xi}(x, d\psi)\cdot\frac{\partial a_0}{\partial x} + (\Delta\psi)a_0\right) + \underline{c}_0 a_1 ,$$

$$c_s = i\left(\frac{\partial p}{\partial \xi}(x, d\psi)\cdot\frac{\partial a_{s-1}}{\partial x} + (\Delta\psi)a_{s-1} - i\Delta a_{s-2}\right) + \underline{c}_0 a_s ,$$

where $a_{-2} = a_{-1} = a_{N+1} = a_{N+2} \equiv 0$. To solve $\underline{c}_0 \equiv 0$ one prescribes ψ and its normal derivative on a surface S transverse Γ so that $\underline{c}_0 = 0$ on S. Then one solves the characteristic equations $\dot{x} = \partial p/\partial \xi$, $\dot{\xi} = -\partial p/\partial x$ with $x(0) \in S$, $\xi(0) = d\psi(x(0))$. If one prescribes complex values for ψ on S, it is clear that

[1] **Added in proof.** For additional results and references see the author's article, Comm. Math. Phys. **51** (1976) 219-242.

the coordinates x must be complex, and the characteristic equations make no sense, since the metric g^{ij} is not necessarily analytic in x. Hence we will not attempt to solve $\underline{c}_0 \equiv 0$, but instead will try to make $\underline{c}_0 = 0$ to third order along Γ. To do this we will use an idea which we believe is due to Hörmander (see the remarks following Theorem 3.4.1 of [6]).

If ψ were real and one solved $\underline{c}_0 \equiv 0$, the submanifold $\Lambda \subset T^*(M)$ given by $\{(x, d_x\psi)\}$ would be invariant under the characteristic flow. Moreover, the conic submanifold $C = \{(x, cd_x\psi) : x \in \Gamma, c \in \boldsymbol{R}_+\}$ would be invariant under the characteristic flow, because Γ is a geodesic. Let σ denote the symplectic 2-form—in local coordinates

$$\sigma = \sum d\xi_i \wedge dx_i \ .$$

We define, for $p \in \gamma = \{(x, d_x\psi) : x \in \Gamma\}$,

$$J_p = \{v \in T_p(T^*(M)) : \sigma(v, t) = 0, t \in T_p(C)\} \ ,$$
$$L_p = \{v \in T_p(\Lambda) : \sigma(v, t) = 0, t \in T_p(C)\} \ ,$$

and let Φ denote the flow on $\{T_p(T^*(M)) : p \in \gamma\}$ induced by the characteristic flow on $T^*(M)$. Since Φ preserves σ and leaves $\{T_p(C) : p \in \gamma\}$ invariant, Φ leaves $\{J_p : p \in \gamma\}$ invariant. Moreover, if Λ is invariant under the characteristic flow, then Φ must leave $\{L_p : p \in \gamma\}$ invariant.

If we introduce coordinates $x_0, x_1, \cdots, x_n$ on M with $x' = (x_1, \cdots, x_n)$ vanishing on Γ and $\partial/\partial x_i$, $i = 1, \cdots, n$, perpendicular to $\partial/\partial x_0$ along Γ, then

$$L_p = \{(\delta x', \delta\xi') : \delta\xi' = d^2_{x'}\psi\delta x'\} \ .$$

Hence, knowing L_p, $p \in \gamma$, is equivalent to knowing $d^2_{x'}\psi$ on Γ.

Thus to solve $\underline{c}_0 = 0$ to third order along Γ we first choose ψ real-valued on Γ so that $\underline{c}_0 = 0$ on Γ (there are two possible choices of $d\psi$ here — leading to ψ_+ and ψ_-). Then we pick a complex n-dimensional isotropic subspace L_{p_0} of the complexification of J_{p_0}, for some $p_0 \in \gamma$, and consider its orbit under Φ (from this point on J_p is always complexified). Assuming this orbit is periodic, i.e., there is just one subspace in it sitting over each $p \in \gamma$, and assuming the projection of this subspace onto the complexification of $T_p(M)$ considered as a subspace of J_p is always nonsingular, we can determine $d^2_{x'}\psi = B(x_0)$ along Γ. Then we define $\psi = \psi|_{x'=0} + \frac{1}{2}x' \cdot B(x_0)x'$. The reason for this roundabout approach is that if one simply writes down the differential equation $B(x_0)$ must satisfy in order that $\underline{c}_0 = 0$ on Γ to third order, one gets a formidable nonlinear ordinary differential equation in matrices. In contrast the flow Φ arises from a linear ordinary differential equation—the differential equation satisfied by the Jacobi fields along Γ. Taking the approach of Hörmander described here, one finds it is easy to express $B(x_0)$ in terms of the (complex) Jacobi fields.[2]

The Poincaré map is the map of J_p to itself obtained by following to flow Φ once around Γ. The hypothesis that Γ is stable means that the Poincaré map has distinct eigenvalues $\lambda_1, \bar{\lambda}_1, \cdots, \lambda_n, \bar{\lambda}_n$ with $|\lambda_i| = 1$. As will be shown in § 2, this implies that for each of the two choices of $d\psi$ on Γ there is a unique choice of L_{p_0} such that the orbit of L_{p_0} is periodic and $\operatorname{Im} B(x_0)$ is positive definite for $x_0 \in \Gamma$. Thus we can construct two phase functions ψ_+ and ψ_-, with $\operatorname{Im} \psi_\pm = 0$ on Γ and $\operatorname{Im} \psi_\pm > 0$ off Γ, which are quadratic in x' and satisfy $\underline{c}_0 = 0$ to second order. Actually $\psi_- = -\bar{\psi}_+$ and we only construct ψ_+ in § 2.

Let $\| \; \|$ denote the L^2-norm over a fixed k-independent neighborhood of Γ. Since we are solving $\underline{c}_0 = 0$ only to third order on Γ, we will not attempt to solve $(\Delta + k^2)u = 0$ more accurately than

$$\|(\Delta + k^2)u\| = O(\|k^2 \underline{c}_0 a_0 e^{-k \operatorname{Im} \psi}\|) = O(k^{-\frac{1}{4}n - \frac{1}{2}(l-1)}) , \tag{1.2}$$

where a_0 vanishes to order l on Γ. Noting that multiplying a function of the form $f(x_0, x')e^{-k|x'|^2}$ by a linear function in x' essentially multiplies its norm by $k^{-\frac{1}{2}}$, we see the contributions to $(\Delta + k^2)u$ from terms in a_0 which vanish to order $l + 1$ on Γ will be negligible. Applying this reasoning to $a_1, \cdots, a_N$ one arrives at the ansatz: a_s is a homogeneous polynomial in x' of degree $l - 2s$ with coefficients depending on x_0, and a_N is degree 0 or 1 in x'. Then, assuming this ansatz, one solves $c_s = 0$ to order $l - 2s + 3$ on Γ, $s = 1, \cdots, N + 1$, which implies (1.2).

We solve the equations $c_k = 0$, $k = 1, \cdots, N + 1$ to the required accuracy on Γ in § 3. The resulting coefficients $a_0, \cdots, a_N$ must be chosen so that they are multipled by the same factor $e^{-i\beta}$, β real, as one goes once around Γ. However, once a_0 has been chosen so that this holds, it will hold for $a_1, \cdots, a_N$. Since ψ_+ increases by the length L of Γ as one goes once around Γ, the functions

$$u_m = e^{ik_m\psi_+}(a_0 + a_1/k + \cdots + a_N/k^N) ,$$

where $k_m = (2\pi m + \beta)/L$, $m \in Z_+$ will be well-defined near Γ.

2. Construction of the phase function

In this section and the next we will use the following coordinate system (x_0, x') near Γ. At a point p_0 on Γ choose an orthonormal frame $v_1(p_0), \cdots, v_n(p_0)$ orthogonal to Γ and, using parallel transport along Γ, extend $v_1, \cdots, v_n$ to parallel vector fields along Γ. If the length of Γ is L, we assign coordinates

[2] N. Grossman has pointed out that the "formidable nonlinear ordinary differential equation" is a matrix Riccati equation and the process described here is related to the method used to reduce such equations to linear equations.

$x_0, x_1, \cdots, x_n$ where $0 \leq x_0 < L$ to the image of $x_1 v_1(p) + \cdots + x_n v_n(p)$ under the exponential map at p where p is the point on Γ at distance x_0 from p_0. Hence the orthonormal frames $\partial/\partial x_1, \cdots, \partial/\partial x_n$ on Γ at $x_0 = 0$ and $x_0 = L$ are related by the orthogonal holonomy matrix O.

In these coordinates the metric g^{ij} satisfies $g^{ij} = \delta_{ij} + O(|x'|)$, $g^{0j} = O(|x'|^2)$ for $j \neq 0$ and $g^{00} = 1 + O(|x'|^2)$. Moreover, if we define $\psi = x_0$ on Γ, the differential equation governing the flow Φ on $\{J_p : p \in \gamma\}$ is given by

$$\delta \dot{x}' = \delta \xi' , \qquad \delta \dot{\xi}' = -\frac{1}{2} \frac{\partial^2 g^{00}}{\partial x' \partial x'}(x_0, 0)\delta x' , \tag{2.1}$$

where $\cdot$ over a letter denotes differentiation with respect to x_0. To verify (2.1) we note that in these coordinates J_p is the (complex) span of

$$\frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p, \frac{\partial}{\partial \xi_1}\Big|, \cdots, \frac{\partial}{\partial \xi_n}\Big|_p .$$

Then we consider a one-parameter family of solutions of the characteristic equations $(x(t,h), \xi(t,h))$ satisfying

$$(x(t,0), \xi(t,0)) = (2t, 0, \cdots, 0, 1, 0, \cdots, 0) .$$

Then letting $x_h(t) = (\partial x/\partial h)(t,0)$, $\xi_h(t) = (\partial \xi/\partial h)(t,0)$ and differentiating the characteristic equations we have

$$\dot{x}_h = \frac{\partial^2 p}{\partial \xi^2}(x(t,0), \xi(t,0))\xi_h + \frac{\partial^2 p}{\partial x \partial \xi}(x(t,0), \xi(t,0))x_h ,$$

$$\dot{\xi}_h = -\frac{\partial^2 p}{\partial x \partial \xi}(x(t,0), \xi(t,0))\xi_h - \frac{\partial^2 p}{\partial x^2}(x(t,0), \xi(t,0))x_h .$$

Using the properties of g^{ij} given earlier these equations reduce to

$$\dot{x}_h = 2\xi_h , \qquad \dot{\xi}_h = -\frac{\partial^2 g^{00}}{\partial x^2}(2t, 0)x_n .$$

Now, changing the curve parameter to x_0 and restricting to the (x', ξ') components, one has (2.1).

The hypothesis that the eigenvalues of the Poincaré map lie on the unit circle and are distinct implies that we can find complex solutions $f_i = (\varphi_i, \dot{\varphi}_i)$, $i = 1, \cdots, n$, of (2.1) satisfying

$$(\varphi_i(L), \dot{\varphi}_i(L)) = \lambda_i(O\varphi_i(0), O\dot{\varphi}_i(0)) , \tag{2.2}$$

where $\lambda_1, \bar{\lambda}_1, \cdots, \lambda_n, \bar{\lambda}_n$ are the eigenvalues of the Poincaré map.

Let $L(x_0)$ denote the complex span of $f_1(x_0), \cdots, f_n(x_0)$. $L(x_0)$ is periodic in the sense described in § 1. To carry out the construction of a phase function from $L(x_0)$, we need to show $L(x_0)$ is isotropic and $\{\varphi_1(x_0), \cdots, \varphi_n(x_0)\}$ is linearly independent $0 \leq x_0 \leq L$.

We have

$$\varphi_i \cdot \dot{\varphi}_j - \dot{\varphi}_i \cdot \varphi_j = c_{ij}\ , \qquad \varphi_i \cdot \dot{\bar{\varphi}}_j - \dot{\varphi}_i \cdot \bar{\varphi}_j = d_{ij}\ . \tag{2.3}$$

where "$\cdot$" denotes the vector dot product, "—" over a letter denotes conjugation, and c_{ij} and d_{ij} are constants. (2.3) is just the statement that Φ is a real symplectic map, but one can verify it directly using (2.1). Now (2.2) implies $c_{ij} = \lambda_i \lambda_j c_{ij}$ and $d_{ij} = \lambda_i \bar{\lambda}_j d_{ij}$, and we conclude $c_{ij} = 0$, $\forall i, j$ and $d_{ij} = 0, i \neq j$. If $d_{ii} = 0$ for any i, we have $(\varphi_i, \dot{\varphi}_i) = 0$, a contradiction. Thus normalizing φ_i and interchanging λ_i and $\bar{\lambda}_i$ where necessary we can achieve $d_{ii} = -2\sqrt{-1}$. Note that now $L(x_0)$ is uniquely determined.

That $c_{ij} \equiv 0$ is simply the statement that $L(x_0)$, $0 \leq x_0 \leq L$, is isotropic. Suppose $\sum_{i=l}^n a_i \varphi_i(x_0) = 0$. Then by (2.3)

$$(\textstyle\sum a_i \dot{\varphi}_i) \cdot \bar{\varphi}_j = 2\sqrt{-1} a_j\ ,$$

$$0 = \sum_{j=1}^n \left(\sum_{i=1}^n a_i \dot{\varphi}_i\right) \cdot \bar{a}_j \bar{\varphi}_j = 2\sqrt{-1}\left(\sum_{j=1}^n |a_j|^2\right)\ ,$$

and we have $a_j = 0$, $j = 1, \cdots, n$, and $\{\varphi_i(x_0)\}_{i=1}^n$ is linearly independent.

The results of the preceding paragraph show that the equations

$$B\varphi_i = \dot{\varphi}_i\ , \qquad i = 1, \cdots, n\ ,$$

define a symmetric $n \times n$ matrix function. The argument of § 1 shows the phase function

$$\psi = x_0 + \tfrac{1}{2} x' \cdot B(x_0) x'$$

satisfies the eichonal equation $\underline{c}_0 = 0$ to third order on Γ. This can also be checked by direct computation.

For our construction it is also necessary that Im B be positive definite. However, we have, writing $B = C + iA$,

$$\begin{aligned} \bar{\varphi}_j \cdot A\varphi_i &= \frac{1}{2\sqrt{-1}}(\bar{\varphi}_j \cdot B\varphi_i - \bar{B}\bar{\varphi}_j \cdot \varphi_i) \\ &= \frac{1}{2\sqrt{-1}}(\bar{\varphi}_j \cdot \dot{\varphi}_i - \dot{\bar{\varphi}}_j \cdot \varphi_i) = \delta_{ij}\ . \end{aligned} \tag{2.4}$$

3. Amplitudes and the eigenvalue condition

Following the ansatz that a_s is a homogeneous polynomial in x' of degree $l - 2s$, and ignoring terms vanishing on Γ to order $l + 1$, the equation $c_1 = 0$ becomes

$$0 = 2\frac{\partial a_0}{\partial x_0} + 2Bx' \cdot \frac{\partial a_0}{\partial x'} + (\text{trace } B)a_0 \ . \tag{3.1}$$

In ordinary geometrical optics the equation $c_1 = 0$ (given $\underline{c}_0 = 0$) means that a_0, considered as a $\frac{1}{2}$-density, is invariant under the flow defined by

$$\dot{x} = \frac{\partial p}{\partial \xi}(x, d\psi) \ . \tag{3.2}$$

If we continue the approach taken in § 1, we should interpret this invariance infinitesimally along Γ to solve (3.1). To do this, we note that the $\varphi_i, i = 1, \cdots, n$, are the x' components of derivatives of the flow defined by (3.2) in directions normal to Γ and the derivative of the flow with respect to x_0 is just $(1, 0, \cdots, 0)$ on Γ. Hence, if we define Z to be the $n \times n$ matrix with columns $\varphi_1, \cdots, \varphi_n$ on Γ, $\det Z$ is a constant multiple of the Jacobian of the flow defined by (3.2).

To check this argument one notes that $B = (dZ/dx_0)Z^{-1}$ and uses the identity

$$\text{trace}\, \frac{dZ}{dx_0}Z^{-1} = \frac{d}{dx_0} \log \det Z \ . \tag{3.3}$$

Defining $e_0 = (\det Z)^{\frac{1}{2}}a_0$, it then follows that

$$0 = 2\frac{\partial e_0}{\partial x_0} + 2Bx' \cdot \frac{\partial e_0}{\partial x'} \tag{3.4}$$

as is implied by the invariance of a_0, considered as a $\frac{1}{2}$-density, under a flow whose Jacobian is a constant multiple of $\det Z$.

(3.4) means that the function e_0 is invariant under the flow defined by (3.2), modulo terms vanishing to order $l + 1$ on Γ. Since e_0 is a homogeneous polynomial in x' and invariance under a flow is preserved under sums and products, it will suffice to solve (3.3) in the case where e_0 is homogeneous linear function of x' (with coefficients depending on x_0).

Let Λ_t denote the flow defined by (3.2). The differential version of $e_0 \circ \Lambda_t = e_0$ is

$$\Lambda_t^*(de_0) = de_0 \ . \tag{3.5}$$

We will only use (3.5) on Γ. Since de_0 on Γ annihilates vectors tangent to Γ, (3.5) simply means

$$\frac{\partial e_0}{\partial x'} \cdot \varphi_i = c_i \quad \text{on} \quad \Gamma \ , \qquad i = 1, \cdots, n \ ,$$

where c_i is a constant. Hence referring to (2.4) one sees $\partial e_0/\partial x'$ is in the span of $\{A\bar{\varphi}_1, \cdots, A\varphi_n\}$.

Turning now to the case where e_0 is homogeneous of degree l, the preceding paragraph implies that e_0 has the form (in multi-index notation)

$$e_0 = \sum_{|\alpha|=l} a_\alpha (x'\cdot A\bar{\varphi}_1)^{\alpha_1} \cdots (x'\cdot A\bar{\varphi}_n)^{\alpha_n} .$$

Each "monomial" $e_{0\alpha} = (x'\cdot A\bar{\varphi}_1)^{\alpha_1} \cdots (x'\cdot A\bar{\varphi}_n)^{\alpha_n}$ in this sum satisfies $e_{0\alpha}(L, Ox') = \bar{\lambda}_1^{\alpha_1} \cdots \bar{\lambda}_n^{\alpha_n} e_{0,\alpha}(0, x') = e^{-i\beta} e_{0\alpha}(0, x')$ with real β, and we will see each $e_{0\alpha}$ leads to a distinct approximate solution to $(\Delta + k^2)u = 0$. Thus, without loss of generality, we assume e_0 is one of these monomials. Therefore we have

$$a_0 = (\det Z)^{-\frac{1}{2}} (x'\cdot A\bar{\varphi}_1)^{\alpha_1} \cdots (x'\cdot A\bar{\varphi}_n)^{\alpha_n} , \qquad |\alpha| = l .$$

One may also check directly that that a_0 satisfies (3.1).

In solving $c_s = 0$ on Γ to order $l - 2s + 3$ for $s = 2, \cdots, N + 1$ we will abandon the geometrical approach used to solve $\underline{c}_0 = 0$ and $c_1 = 0$. This will be done primarily because we do not know how to solve them geometrically, but also because, given our solutions to $\underline{c}_0 = 0$ and $c_1 = 0$, the rest of the equations are easy to solve.

We note first that (2.4) implies $(d/dx_0)(A\bar{\varphi}_j) + BA\bar{\varphi}_j = 0$. Then we have

$$\text{(3.6)} \qquad \frac{d}{dx_0}(\bar{\varphi}_i\cdot A\bar{\varphi}_j) = \bar{B}\bar{\varphi}_i\cdot A\bar{\varphi}_j - \bar{\varphi}_i\cdot BA\bar{\varphi}_j = -2\sqrt{-1}A\bar{\varphi}_i\cdot A\bar{\varphi}_j .$$

Let C denote the differential operator

$$Cu = \frac{\partial}{\partial x'}\left(A^{-1}\frac{\partial u}{\partial x'}\right) .$$

Ignoring terms vanishing on Γ to order $l - 2s + 3$ the equation $c_s = 0$ becomes

$$\text{(3.7)} \quad 2\frac{\partial a_{s-1}}{\partial x_0} + 2Bx'\cdot\frac{\partial a_{s-1}}{\partial x'} + (\operatorname{trace} B)a_{s-1} = \sqrt{-1}\left(\frac{\partial^2 a_{s-2}}{\partial x_1^2} + \cdots + \frac{\partial^2 a_{s-2}}{\partial x_n^2}\right) .$$

Then using (3.6) and the definition of C one sees (3.7) will be satisfied if $a_s = -\frac{1}{4}Ca_{s-1}/s$, $s = 1, \cdots, N$.

To sum up we have shown that if

$$u = e^{ik(x_0+\frac{1}{2}x'\cdot Bx')}\left(a_0 + \frac{-1}{4k}Ca_0 + \cdots + \frac{1}{N!}\left(\frac{-1}{4k}\right)^N C^N a_0\right) ,$$

where $a_0 = (\det Z)^{-\frac{1}{2}}(x'\cdot A\bar{\varphi}_1)^{\alpha_1}\cdots(x'\cdot A\bar{\varphi}_n)^{\alpha_n}$, then $\|(\Delta + k^2)u\| = O(k^{-\frac{1}{4}n-\frac{1}{2}(l-1)})$. However, u is not well-defined in a neighborhood of Γ unless $u(0, x') = u(L, Ox')$. As noted in § 1, this leads to the eigenvalue condition on k.

To determine the relation between $u(0, x')$ and $u(L, Ox')$ we note that (see 3.3)

$$\frac{d}{dx_0}(\operatorname{Arg}\det Z) = \operatorname{trace} A \ ,$$

and (2.4) implies $\operatorname{trace} A > 0$. Hence we may choose $p \in Z$, $p \geq 0$ and θ_0, $0 \leq \theta_0 < 2\pi$, so that

$$\int_0^L d(\operatorname{Arg}\det Z) = 2\pi p + \theta_0 \ .$$

Let $\lambda_i = e^{\sqrt{-1}\theta_i}$, $0 < \theta_i < 2\pi$. We have

$$Z(L) = OZ(0)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

and hence $\det Z(L) = \lambda_1 \cdots \lambda_n \det Z(0) \det O$. Let $(-1)^\nu = \det O$, $\nu = 0$ or 1. $\nu = 0$ if a neighborhood of Γ is orientable, $\nu = 1$ if it is not. We then have

$$\theta_0 \equiv (\theta_1 + \cdots + \theta_n + \pi\nu) \ , \qquad \bmod 2\pi \ ,$$

and

$$u(L, Ox') = e^{-\sqrt{-1}(\pi p + \theta_0/2)}\lambda_1^{-\alpha_1} \cdots \lambda_n^{-\alpha_n} e^{\sqrt{-1}kL} u(0, x') \ .$$

Thus, setting $\theta_0 = \theta_1 + \cdots + \theta_n + \pi\nu + 2\pi p'$, we see u is a well-defined smooth function near Γ provided

$$(3.8) \qquad \begin{aligned} k = k_{r,\alpha} = \frac{1}{L}\Big(2\pi r &+ \Big(\alpha_1 + \frac{1}{2}\Big)\theta_1 \\ &+ \cdots + \Big(\alpha_n + \frac{1}{2}\Big)\theta_n + \pi(p + p') + \frac{\pi}{2}\nu\Big) \ . \end{aligned}$$

Finally, using the change of variables $(y_0, y') = (x_0, k^{\frac{1}{2}}x')$ one checks easily that $\|u_{r,\alpha}\| \geq c_\alpha k_{r,\alpha}^{-\frac{1}{2}l - \frac{1}{4}n}$ for $r \gg 0$. If we cut u smoothly off to zero outside a k-independent neighborhood of Γ, the construction is complete.

The integer p_0 in the statement of Theorem I is simply $p + p'$, mod 2. In the case when $n = 1$ and $\nu = 0$, $p_0 \equiv [\frac{1}{2}\mu]$, mod 2 where μ is the Morse index of Γ considered as a geodesic arc with fixed end points $(0, 0)$ and $(L, 0)$. For more information about the role of Morse indices in this sort of problem the reader should see [3].

4. Reflected waves

In this section we assume Γ is a closed ray path which is reflected off ∂M at a finite number of points. More precisely, $\Gamma = \bigcup_{i=1} \Gamma_i$ where Γ_i is the geodesic arc which is the projection of the curve $(x^i(t), \xi^i(t))$, $t \in [t_i, t_{i+1}]$ in the characteristic flow with $p(x^i(t), \xi^i(t)) = 1$. We assume

(i) $x^i(t) \in M$, for $t \in (t_i, t_{i+1})$, $x^i(t_i)$ and $x^i(t_{i+1})$ are on ∂M, and $x^i(t_i) = x^{i-1}(t_i)$, $i = 2, \cdots, m$,

(ii) $x^m(t_{m+1}) = x^1(t_1)$.

In view of (i) and (ii) we set $x^i(t_i) = x^i$, $\xi^i(t_i) = \xi^i_-$, $\xi^i(t_{i+1}) = \xi^i_+$ and adopt the convention that indices are reduced mod m where necessary. We assume further

(iii) $\xi^i_- - \xi^{i-1}_+$ is normal to ∂M at x^i, i.e., it annihilates $T_{x^i}(M)$,

(iv) $\xi^i_- - \xi^{i-1}_+ \neq 0$.

(iii) is the reflection condition, and (iv) implies Γ never touches Γ tangentially.

Assuming a stability condition analogous to that used previously, we will sketch the construction of a sequence of approximate eigenfunctions concentrated near Γ, satisfying

$$\|(\Delta + k_m^2)u_m\| = O(k^{\frac{1}{2}-\frac{1}{4}n}) , \qquad \|u_m\| > ck_m^{-n/4} ,$$

and $u_m = 0$ on ∂M. This sequence will correspond to the "fundamental" sequence in the previous construction, i.e., the sequence with $\alpha = 0$. For simplicity we assume M is orientable.

To construct an analogue of the Poincaré map for Γ we choose functions ρ_i defined near x^i such that

(i) $\rho_i = 0$ on ∂M,

(ii) $p(x, d\rho_i) = 1$.

Then in ordinary geometrical optics a ray hitting ∂M near x^i with data (x, ξ_+) will be reflected to a ray with date (x, ξ_-), where $x \in \partial M$ and

$$\xi_- = \xi_+ - \left(\xi_+ \cdot \frac{\partial p}{\partial \xi}(x, d\rho_i)\right) d\rho_i .$$

Hence, considering the induced map on the tangent space to $T^*(M)$, we define "reflection" maps

$$R_i \colon \Lambda^i_+ = \{(\delta x, \delta\xi) \in T_{(x^i, \xi^{i-1}_+)}(T^*(M)) \colon d\rho_i \cdot \delta x = 0\}$$
$$\to \Lambda^i_- = \{(\delta x, \delta\xi) \in T_{(x^i, \xi^i_-)}(T^*(M)) \colon d\rho_i \cdot \delta x = 0\} ,$$

$$R_i \colon (\delta x, \delta\xi) \to (\delta x, \delta\xi) - \left(0, \left(\delta\xi \cdot \frac{\partial p}{\partial \xi}(x, d\rho_i) d\rho_i \Big|_{x=x^i}\right)\right)$$
$$- \left(0, d_x\left(\xi^{i-1}_+ \cdot \frac{\partial p}{\partial \xi}(x, d\rho_i) d\rho_i\right)\Big|_{x=x^i} \cdot \delta x\right) .$$

In defining R_i we have made use of the natural identification of Λ^i_+ and Λ^i_-. A short computation shows R_i is a real symplectic map of Λ^i_+ to Λ^i_-.

Along each arc $(x^i(t), \xi^i(t))$, $t \in [t_i, t_{i+1}]$, we can introduce the flow Φ_i and define P_i to be the real symplectic map of

$$J^i_- = \{v \in T_{(x^i,\xi^i_-)}(T^*(M)) : \sigma(v,t) = 0, \forall t \in T_{(x^i,\xi^i_-)}(C_i)\}$$

to

$$J^i_+ = \{v \in T_{(x^{i+1},\xi^i_+)}(T^*(M)) : \sigma(v,t) = 0, \forall t \in T_{(x^{i+1},\xi^i_+)}(C)\}$$

obtained from Φ_i as in § 1. Here $C_i = \{(x^i(t), c\xi^i(t)) : c \in \boldsymbol{R}_+, t \in [t_i, t_{i+1}]\}$.

To construct a Poincaré map we must redefine R_i so that it maps J^{i-1}_+ onto J^i_-. To do this we identify $v \in J^{i-1}_+$ and $v' \in \Lambda^i_+$ if $v - v' \in T_{(x^i,\xi^{i-1}_+)}(C_{i-1})$, and identify $w \in J^i_-$ and $w' \in \Lambda^i_-$ if $w - w' \in T_{(x^i,\xi^i_-)}(C_i)$. Since R_i maps $T_{(x^i,\xi^{i-1}_+)}(C_{i-1}) \cap \Lambda^i_+$ onto $T_{(x^i,\xi^i_-)}(C_i) \cap \Lambda^i_-$, with the preceding identifications R_i becomes a well-defined real symplectic map of J^{i-1}_+ onto J^i_-. Hence we can finally define the Poincaré map $P: J^1_- \to J^1_-$ by

$$P = R_1 P_m \cdots P_2 R_2 P_1 \ .$$

P is a real symplectic map. If P has distinct eigenvalues $\lambda_1, \bar\lambda_1, \cdots, \lambda_n, \bar\lambda_n$, we can carry out the construction of §§ 1 and 2, getting phase functions ψ_i, $i = 1, \cdots, m$, where $\operatorname{Im} \psi_i > 0$ off Γ_i and ψ_i satisfies the eichonal equation $\underline{c}_0 = 0$ to second order on Γ_i. Adding constants to the ψ_i, one can arrange $\psi_i(x^i) = \psi_{i-1}(x^i)$, $i = 2, \cdots, m$. Then $\psi_m(x^1) = \psi_1(x^1) + L$, where L is the length of Γ. The reflections R_i were defined so that $d\psi_i = d\psi_{i-1} - \left(d\psi_{i-1} \cdot \frac{\partial p}{\partial \xi}(x, d\rho_i)\right) d\rho_i$ on ∂M to second order at x^i. Since $d\rho_i$ vanishes on the tangent space to ∂M, it follows that $\psi_i - \psi_{i-1}$ restricted to ∂M vanishes to third order at x^i for $i = 2, \cdots, m$, and $\psi_m - \psi_1 - L$ restricted to ∂M vanishes to third order at x^1.

On each curve Γ_i the construction of ψ_i also yields a matrix Z_i analogous to Z in § 3, and we build u as

$$\begin{aligned} u = {} & e^{ik\psi_1}(\det Z_1)^{-\frac12} - e^{ik\psi_2}(\det Z_2)^{-\frac12} \\ & + \cdots + (-1)^{m+1} e^{ik\psi_m}(\det Z_m)^{-\frac12} \ . \end{aligned} \tag{4.1}$$

The Z_i satisfy $\det Z_i = \det Z_{i-1}$ at x^i, $i = 2, \cdots, m$. The eigenvalue condition is imposed by requiring $u(x^1) = 0$. This leads to $k = k_r$ exactly as in (3.8) with $\alpha = 0$ except that ν is defined by $\nu = 0$ if m is even, and $\nu = 2$ if m is odd.

Since $u(x^i) = 0$ and $\psi_i - \psi_{i-1}$ vanishes on ∂M to third order at x^i, we can modify u near x^i so that $u = 0$ on ∂M and still maintain $\|(\Delta + k_r^2)u\| = O(k_r^{(-n+2)/4})$. As in § 3, cutting u off to zerooutside a k-independent neighborhood of Γ completes the construction.

In the case of "bouncing ball" waves, i.e., when Γ consists of two copies of the same geodesic arc traced in opposite directions, there is an interesting simplification in (4.1). In this case the spaces J^+_{i-1} and J^-_i may be identified and we call the resulting space J_i, $i = 1, 2$. The reflection map R_i becomes a map of J_i onto J_i. We introduce coordinates (x_0, x') near Γ like those used in § 2 with $x^1 \sim (0, 0)$ and $x^2 \sim (\frac{1}{2}L, 0)$, and let $(\varphi_i(x_0), \dot{\varphi}_i(x_0))$, $i = 1, \cdots, n$, be the solution of (2.1) with data at $x_0 = 0$ equal to an eigenvector of P. As in § 2 this eigenvector is chosen so that $\varphi_i \cdot \dot{\bar{\varphi}}_i - \dot{\varphi}_i \cdot \bar{\varphi}_i = -2\sqrt{-1}$. Let $(\eta_i(x_0), \dot{\eta}_i(x_0))$, $i = 1, \cdots, n$, be the solution of (2.1) such that $R_1((\eta_i(0), -\dot{\eta}_i(0))) = ((\varphi_i(0), \dot{\varphi}(0))$.

Let T be the map $(\delta x, \delta\xi) \to (\delta x, -\delta\xi)$. The maps TR_i, $i = 1, 2$, are involutions. By the definition of P

$$R_2((\varphi_i(\tfrac{1}{2}L), \dot{\varphi}_i(\tfrac{1}{2}L)) = \lambda_i(\eta_i(\tfrac{1}{2}L), -\dot{\eta}_i(\tfrac{1}{2}L))$$

and, since TR_2 is an involution and $|\lambda_i| = 1$,

$$\bar{\lambda}_i(\varphi_i(\tfrac{1}{2}L), \dot{\varphi}_i(\tfrac{1}{2}L)) = R_2(\eta_i(\tfrac{1}{2}L), \dot{\eta}_i(\tfrac{1}{2}L)) .$$

Imitating the procedure in § 2,

$$\varphi_i \cdot \dot{\eta}_j - \dot{\varphi}_i \cdot \eta_j = e_{ij} , \qquad \bar{\varphi}_i \cdot \dot{\eta}_j - \dot{\bar{\varphi}}_i \cdot \eta_j = f_{ij} .$$

As before e_{ij} and f_{ij} are constants, and since R_2 is a real symplectic map, $e_{ij} = \lambda_i \bar{\lambda}_j e_{ij}$ and $f_{ij} = \bar{\lambda}_i \bar{\lambda}_j f_{ij}$. Hence $e_{ij} = 0$ for $i \neq j$ and $f_{ij} \equiv 0$. Since we also have (see § 2)

$$\begin{aligned} \varphi_i \cdot \dot{\bar{\varphi}}_j - \dot{\varphi}_i \cdot \bar{\varphi}_j &= 0 , \qquad i \neq j , \\ \varphi_i \cdot \dot{\varphi}_j - \dot{\varphi}_i \cdot \varphi_j &\equiv 0 , \end{aligned}$$

and $\{(\varphi_i, \dot{\varphi}_i)\}_{i=1}^n \cup \{\bar{\varphi}_i, \dot{\bar{\varphi}}_i\}_{i=1}^n$ forms a basis for C^{2n}, it follows

$$(\eta_i, \dot{\eta}_i) = c_i(\bar{\varphi}_i, \dot{\bar{\varphi}}_i)$$

for constants c_i determined by $\varphi_i(0) = \eta_i(0)$. The matrix B derived from the η_i is defined by $B\eta_i = -\dot{\eta}_i$.

Continuing with the construction of the approximate eigenfunctions as in § 2, we eventually get

$$u = e^{ik\psi_1}(\det Z_1)^{-\frac{1}{2}} - e^{ik\psi_2}(\det Z_2)^{-\frac{1}{2}} .$$

However, the observations of the preceding paragraph imply $\psi_2 = -\bar{\psi}_1$, and for suitable $\beta \in R$, $e^{i\beta} \det Z_2 = e^{-i\beta} \det \bar{Z}_1$. Thus replacing u by $e^{-\frac{1}{2}i\beta}u$ and modifying Z_1 appropriately we have

$$u = w - \bar{w} , \tag{4.2}$$

where $w = e^{ik\psi_1}(\det Z_1)^{-\frac{1}{2}}$. This is consistent with the form of u in the case $n = 1$ which was found in [10].

The chief implication of (4.2) is that the argument that will be used in § 5 to show there are *two* eigenvalues of Δ in an interval about $-k_{r,\alpha}^2$ of radius $O(k_{r,\alpha}^{\frac{1}{2}})$ *fails* for bouncing ball waves.

5. Implications

For the domain of Δ we take the subspace of $C_0^\infty(M \cup \partial M)$ consisting of functions vanishing on ∂M. With this domain Δ is symmetric, densely defined and nonpositive. Thus, by Friedrichs' theorem, (cf. [7, pp. 325–326]), it has a nonpositive self-adjoint extension Δ_M. The norm of $(\Delta_M - \lambda I)^{-1}$ equals the distance from λ to the spectrum of Δ_M (cf. [7, p. 272]).

If $M \cup \partial M$ is compact, then Δ_M is the graph closure of Δ and has pure point spectrum. Since the $u_{r,\alpha}$ constructed in §§ 3 and 4 satisfy

$$\|(\Delta + k_{r,\alpha}^2)u_{r,\alpha}\| = O(k_{r,\alpha}^{-n/4-|\alpha|/2+1/2}) \ ,$$
$$\|u_{r,\alpha}\| \geq c_\alpha k_{r,\alpha}^{-n/4-|\alpha|/2} \ ,$$

it follows the distance from $k_{r,\alpha}^2$ to an eigenvalue of Δ_M is $O(k_{r,\alpha}^{1/2})$.

In the case where Γ is a closed geodesic, as in §§ 1–3, we claim that there are at least two eigenvalues of Δ_M, counted by multiplicity, whose distance to $k_{r,\alpha}^2$ is $O(k_{r,\alpha}^{1/2})$. To see this consider

$$(u_{r,\alpha}, \bar{u}_{r,\alpha}) = \int_M u_{r,\alpha}^2 dx = \int dx' \int_0^L e^{i2k_{r,\alpha}x_0}(e^{-i2k_{r,\alpha}x_0}u_{r,\alpha}^2)dx_0 \ .$$

Integration by parts once in x_0 shows $(u_{r,\alpha}, \bar{u}_{r,\alpha}) = O(k_{r,\alpha}^{-n/2-|\alpha|-1})$. Thus we have

$$\left\|(\Delta + k_{r,\alpha}^2)\frac{u_r}{\|u_r\|}\right\| = O(k_{r,\alpha}^{1/2}) \ ,$$
$$\left\|(\Delta + k_{r,\alpha}^2)\frac{\bar{u}_r}{\|\bar{u}_r\|}\right\| = O(k_{r,\alpha}^{1/2}) \ ,$$
$$\left(\frac{u_{r,\alpha}}{\|u_{r,\alpha}\|}, \frac{\bar{u}_{r,\alpha}}{\|\bar{u}_{r,\alpha}\|}\right) = O(k_{r,\alpha}^{-1}) \ .$$

Now an elementary argument, which we leave to the reader, gives the desired result.

Suppose Δ_M has only two eigenvalues in an interval about $-k_{r,\alpha}^2$ of length d_r, where $\lim_{r\to\infty} (k_{r,\alpha}^{1/2})/d_r = 0$. Let P_r denote the orthogonal projection on the subspace spanned by $u_{r,\alpha}$, and let $\tilde{P}_r$ denote the spectral projection for Δ_M on the interval $[-d_r - k_{r,\alpha}^2, -k_{r,\alpha}^2 + d_r]$. Then one can show $\lim_{r\to\infty} \|P_r - \tilde{P}_r\| = 0$.

However, Arnold has given an example where none of the $u_{r,\alpha}$'s are close to true eigenfunctions (see [1]). We offer the following argument that, in a practical sense, it is impossible to distinguish the $u_{r,\alpha}$'s from true eigenfunctions when r is sufficiently large. For functions $u(x, t)$ on $M \times \boldsymbol{R}$ we introduce the energy

$$E(u) = \int_M \left(\sum g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} + \left|\frac{\partial u}{\partial t}\right|^2\right) d\omega ,$$

where $d\omega$ is the volume form on M. Then, if u is a solution to $\partial^2 u/\partial t^2 - \Delta u = 0$, $E(u)$ is independent of t.

Let $u(x, t)$ be the solution to the mixed problem: $\partial^2 u/\partial t^2 - \Delta u = 0$, $u(x, 0) = u_{r,\alpha}(x)$, $(\partial u/\partial t)(x, 0) = ik_{r,\alpha} u_{r,\alpha}(x)$ and $u(x, t) = 0$ on ∂M. A simple estimate from Duhamel's formula (cf. [7, pp. 486–487]) shows $E(u(x, t) - e^{ik_{r,\alpha}t} u_{r,\alpha}) = O(k_{r,\alpha}^{-(n/2+|\alpha|-1)} |t|)$. One checks easily $E(u(x, t)) = E(u(x, 0)) \geq c_\alpha k_{r,\alpha}^{-(n/2+|\alpha|-3)}$. Hence, given T and ε, there is an r_0 such that for $r \geq r_0$, the standing wave $v(x, t) = c_r e^{ik_{r,\alpha}t} u_{r,\alpha}(x)$ differs from a true solution to $\partial^2 u/\partial t^2 - \Delta u = 0$ by ε in energy norm for $|t| < T$. The constant c_r is chosen so that $E(v(x, 0)) = 1$. Note that this argument applies equally well when $M \cup \partial M$ is not compact.

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