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ORIENTATION OF DIFFERENTIABLE MANIFOLDS

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0. Introduction

We shall study compact, connected C^∞-manifolds M_n provided with a Riemannian metric. Homologically characterized, an "orientable" manifold M_n is a manifold whose n-th Betti number is 1, or equivalently a manifold whose n-th connectivity over the field $\mathbf{Q}$ of rational numbers is 1. If the manifold M_n is triangulated, another and equivalent characterization is that the simplicial cells of M_n can be *coherently oriented*, in the classical sense.

In [6] the authors concern themselves with a systematic development of singular homology on M_n without making use of any triangulation of M_n. Triangulations are avoided for two reasons. In a study of *ND* (abbreviating nondegenerate) functions on M_n it is found that a global triangulation is neither needed nor relevant. A deeper reason is that the methods of the critical point theory, if developed without any use of global triangulations of M_n, are extendable to compact, connected topological manifolds admitting a topologically *ND* function. See [4], [8] and [7]. For a definition of topologically *ND* functions see [1].

Objective. We shall give a *geometic* definition of the orientability of M_n. This definition has many consequences in the study of *ND* function on M_n. In particular one can show, without making use of any global triangulation of M_n, that M_n is geometrically orientable in our sense if and only if $\beta_n(M_n)=1$. It is believed, moreover, that the theory here developed for differentiable manifolds has an extension to topological manifolds admitting a topologically *ND* function. The theorems on "critical shells", introduced in § 7 when $n > 2$ and f is "biorderd" (§ 4), are believed to be fundamental both in the orientable and nonorintable case.

1. Inverting sequences of presentations

Definition 1.0. $\pm$*Compatibility*[1]. Two overlapping presentations Q and F in $\mathscr{D}M_n$ (see [6, § 13]) will be said to be $\overset{+}{\text{Com}}$ ($\overset{-}{\text{Com}}$) if the transition diff λ associated with Q and F (see [6, § 5]) has a positive (negative) Jacobian at each point of the euclidean domain of definition of λ. The intersection of the ranges

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[1] Compatible will be abbreviated by Com.

RQ and RF of Q and F may fail to be a connected set, and Q and F may accordingly fail to be $\overset{+}{\text{Com}}$ or $\overset{-}{\text{Com}}$. We shall, however, establish the following basic lemma.

Lemma 1.0. *If ε is a sufficiently small positive constant, any two overlapping presentations in $\mathscr{D}M_n$, whose ranges on $|M_n|$ are connected subsets of*[2] $|M_n|$ *which have diameters on $|M_n|$ at most ε, are either* $\overset{+}{\text{Com}}$ *or* $\overset{-}{\text{Com}}$.

Proof. A sufficiently small positive constant ε has the following property: corresponding to each point $p \in |M_n|$ there exists a presentation $F_p \in \mathscr{D}M_n$ whose range RF_p is an open topological n-ball which contains each point $q \in |M_n|$ at most a distance ε on $|M_n|$ from p. The proof of this statement can be given in many ways. It depends on the fact that $|M_n|$ is a compact manifold. For such an ε Lemma 1.0 is satisfied, as we shall now verify.

Let Q_1 and Q_2 be presentations in $\mathscr{D}M_n$ whose ranges are connected subsets of $|M_n|$, which have diameters on $|M_n|$ at most ε and which contain a common point p. Note that

$$RQ_1 \subset RF_p;\ RQ_2 \subset RF_p\ . \tag{1.0}$$

Since RQ_1 is a connected subset of RF_p, it follows that Q_1 and F_p are either $\overset{+}{\text{Com}}$ or $\overset{-}{\text{Com}}$. Similarly Q_2 and F_p are either $\overset{+}{\text{Com}}$ or $\overset{-}{\text{Com}}$. Since

$$RQ_1 \cap RQ_2 \subset RF_p\ , \tag{1.1}$$

it follows that Q_1 and Q_2 are $\overset{+}{\text{Com}}$ or $\overset{-}{\text{Com}}$ regardless of whether $RQ_1 \cap RQ_2$ has one or more components.

The ensemble $\mathscr{D}_\varepsilon M_n$. In accord with Lemma 1.0 we denote by $\mathscr{D}_\varepsilon M_n$ the subset of presentations in $\mathscr{D}M_n$ whose ranges are connected subsets of $|M_n|$ with diameters less than ε. We suppose ε conditioned by Lemma 1.0.

Definition 1.1. *Orientable manifolds.* The manifold M_n will be termed *orientable*[3] if it admits a covering Γ by a subset of presentations in $\mathscr{D}M_n$ such that any two overlapping presentations in Γ are $\overset{+}{\text{Com}}$. Such a covering Γ will be termed *orienting.*

If M_n is orientable, there always exists a finite orienting covering Γ of M_n, since $|M_n|$ is compact. If M_n admits an orienting covering Γ, it admits an orienting covering in $\mathscr{D}_\varepsilon M_n$, taken as the union of all presentations $P \in \mathscr{D}_\varepsilon M_n$ such that for some presentation $Q \in \Gamma$, $P \overset{+}{\text{Com}} Q$ and $RP \subset RQ$.

To find conditions sufficient that M_n be orientable various definitions are needed.

Definition 1.2. *Sequences $Q_1 * Q_\mu$.* A sequence

[2] $|M_n|$ is the carrier of M_n.
[3] Orientability in the sense of Def. 1.1 will be termed *geometric.*

(1.2) $$Q_1 : Q_2 : \cdots : Q_\mu \qquad (\mu > 2)$$

of presentations $Q_i \in \mathscr{D}_\varepsilon M_n$ such that each presentation except the last is $\overset{+}{\text{Com}}$ with its successor, will be denoted by $Q_1 * Q_\mu$. A sequence $Q_1 * Q_\mu$ is not uniquely defined by the presentations Q_1 and Q_μ.

We term a sequence $Q_1 * Q_\mu$ *admissible.*

As a prelude to our study of "inverting sequences" we shall inroduce two lemmas on what we shall call *conditioned transitivity* of the relation $\overset{+}{\text{Com}}$. If Q_1 and Q_2 are two presentations in $\mathscr{D}_\varepsilon M_n$ which are $\overset{+}{\text{Com}}$, or $\overset{-}{\text{Com}}$, we shall write $Q_1 \overset{+}{\text{Com}}\, Q_2$, or $Q_1 \overset{-}{\text{Com}}\, Q_2$, respectively.

Lemma 1.1. *If A,B,C are three presentations in $\mathscr{D}_\varepsilon M_n$ such that*

(1.3) $$A \overset{+}{\text{Com}}\, C\,, \qquad B \overset{+}{\text{Com}}\, C\,,$$

then $A \overset{+}{\text{Com}}\, B$ whenever

(1.4) $$RA \cap RB \cap RC \neq \emptyset\,.$$

The proof is trivial. We continue with an extension.

Lemma 1.2. *Let $F \in \mathscr{D}_\varepsilon M_n$ be given with a sequence*[4] *$Q_1 * Q_\mu$ of form* (1.2) *such that $Q_1 \overset{+}{\text{Com}}\, F$ and*

(1.5) $$RQ_{j-1} \cap RQ_j \cap RF \neq \emptyset$$

for j on the range $2, \cdots, \mu$. Then $Q_j \overset{+}{\text{Com}}\, F$ for each j.

Proceeding inductively we assume $F \overset{+}{\text{Com}}\, Q_{j-1}$ for some j on the range $2, \cdots, \mu$. Since

(1.6) $$Q_j \overset{+}{\text{Com}}\, Q_{j-1} \qquad \text{(by hypothesis)}$$

and (1.5) holds, Lemma 1.1 implies that $Q_j \overset{+}{\text{Com}}\, F$.

Lemma 1.2 follows.

Presentations F and $\hat{F}$. Let $(F\colon U, X)$ be a presentation in $\mathscr{D}_\varepsilon M_n$. The domain U of F is an open nonempty subset of a euclidean space E_n of coordinates $u_1, \cdots, u_n$. Let ρ be a reflection of U in the coordinate $(n-1)$-plane E_{n-1} of E_n on which $u_n = 0$. The presentation

(1.7) $$(F \circ \rho^{-1}\colon \rho(U), X) \in \mathscr{D}_\varepsilon M_n$$

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[4] All sequences $Q_1 * Q_\mu$ satisfy the conditions of Def. 1.2.

will be denoted by $\hat{F}$. It is clear that F and $\hat{F}$ are $\overset{+}{\text{Com}}$. We term $\hat{F}$ the *invert* of F.

Definition 1.3. *Inverting sequences.* A sequence $Q_1 * Q_\mu$ such that $Q_\mu = \hat{Q}_1$ or equivalently $Q_1 = \hat{Q}_\mu$ will be called an *inverting sequence.*

We shall prove the following lemma.

Lemma 1.3. *If an inverting sequence exists on M_n, then for each presentation $F \in \mathscr{D}_\varepsilon M_n$ there exists an inverting sequence $F * \hat{F}$.*

Proof. By hypothesis, for some presentation $Q \in \mathscr{D}_\varepsilon M_n$ there exists an inverting sequence $Q * \hat{Q}$. Because $|M_n|$ is arcwise connected, one at least of the two following cases arises:

Case I. A sequence $F * Q$ exists;

Case II. A sequence $F * \hat{Q}$ exists.

In Case I a sequence $\hat{Q} * \hat{F}$ also exists. Hence a sequence $F * \hat{F}$ exists of the form

$$(F * Q) : (Q * \hat{Q}) : (\hat{Q} * \hat{F}) \ . \tag{1.8}$$

Thus Lemma 1.3 is true in Case I.

In Case II a sequence $F * \hat{F}$ exists of the form

$$(F * \hat{Q}) : (\hat{Q} * Q) : (Q * \hat{F}) \ . \tag{1.9}$$

Thus Lemma 1.3 is true in both cases.

In preparation for a proof of Theorem 1.1 we shall give a definition and prove a lemma and a corollary.

Definition 1.4. An orienting covering of M_n in $\mathscr{D}_\varepsilon M_n$ is termed *maximal* in $\mathscr{D}_\varepsilon M_n$ if it is a subset of no other orienting covering of M_n in $\mathscr{D}_\varepsilon M_n$.

Lemma 1.4. *If there exists an orienting covering Δ of M_n in $\mathscr{D}_\varepsilon M_n$, there exists an orienting covering of M_n which is maximal in $\mathscr{D}_\varepsilon M_n$ and includes Δ.*

We shall show that the set of presentations

$$\Gamma = \{A \in \mathscr{D}_\varepsilon M_n \,|\, A \overset{+}{\text{Com}}\, Q \quad \text{for some} \quad Q \in \Delta\} \tag{1.10}$$

is a maximal orienting covering in $\mathscr{D}_\varepsilon M_n$. We first prove ($\alpha$):

(α) If F and G are two overlapping presentations in Γ, then $F \overset{+}{\text{Com}}\, G$.

Case 1. Suppose F and G are in Δ. Then $F \overset{+}{\text{Com}}\, G$ by definition of Δ.

Case 2. Suppose that just one of the presentations F and G, say G, is in Δ and the other F is in $\Gamma - \Delta$. Since Δ is a covering there exists a presentation $Q \in \Delta$ overlapping F. Because RF is arcwise connected, Δ contains a sequence $Q_1 * Q_\mu$ such that $Q_1 = Q$, $Q_\mu = G$ and

$$RQ_{j-1} \cap RQ_j \cap RF \neq \emptyset \qquad (j = 2, \cdots, \mu) \ . \tag{1.11}$$

From Lemma 1.2 it follows that $F \overset{+}{\text{Com}} Q_\mu$.

Case 3. Suppose that F and G are in $\Gamma - \Delta$. Since F and G overlap and Δ is a covering, Δ contains a presentation Q such that $RF \cap RG \cap RQ \neq \emptyset$. By Case 2, $F \overset{+}{\text{Com}} Q$ and $G \overset{+}{\text{Com}} Q$. Hence by Lemma 1.1, $F \overset{+}{\text{Com}} G$.

Since $\Delta \subset \Gamma$, Γ is a covering of M_n and it follows from (α) that Γ is an orienting covering. If $A \in \mathscr{D}_\varepsilon M_n$, then exactly one of the pair $\{A, \hat{A}\}$ is in Γ by virtue of (α). Therefore, if $\hat{\Gamma}$ denotes the set of inverts of elements of Γ, then $(\Gamma, \hat{\Gamma})$ is a partition of $\mathscr{D}_\varepsilon M_n$. It follows that Γ is maximal.

Thus Lemma 1.4 is true.

We state a corollary.

Corollary 1.1. *If M_n is orientable, there exists a unique partition $(\Gamma, \hat{\Gamma})$ of $\mathscr{D}_\varepsilon M_n$ in which Γ and $\hat{\Gamma}$ are maximal orienting coverings of M_n in $\mathscr{D}_\varepsilon M_n$.*

Theorem 1.1. *A necessary and sufficient condition that M_n be orientable is that there exist no inverting sequence in $\mathscr{D}_\varepsilon M_n$.*

(α) *The condition is necessary.* In fact, if M_n is orientable $\mathscr{D}_\varepsilon M_n$ admits a partition $(\Gamma, \hat{\Gamma})$ as in Corollary 1.1. If $Q_1 * Q_\mu$ is a given sequence, then $Q_1 \in \Gamma$ or $Q_1 \in \hat{\Gamma}$. If $Q_1 \in \Gamma$, then each presentation in the sequence $Q_1 * Q_\mu$, including Q_μ, must be in Γ. The sequence cannot then be an inverting sequence. If $Q_1 \in \hat{\Gamma}$ the proof is similar.

The condition of Theorem 1.1 is accordingly necessary.

(β) *The condition of Theorem* 1.1 *is sufficient.* We seek a subset of $\mathscr{D}_\varepsilon M_n$ which is an "orienting covering" Γ of M_n (Def. 1.1). We shall define Γ as a special subset of $\mathscr{D}_\varepsilon M_n$ whose presentations cover M_n, and prove that any two overlapping presentations in Γ are $\overset{+}{\text{Com}}$.

The definition of Γ. Let A be a finite or countably infinite range of indices α. Let

$$\Pi = (Q_\alpha)_{\alpha \in A}\ , \qquad \hat{\Pi} = (\hat{Q}_\alpha)_{\alpha \in A}$$

be subsets of $\mathscr{D}_\varepsilon M_n$ such that the presentations of Π (and hence of $\hat{\Pi}$) cover M_n. Let a presentation $H_0 \in \Pi$ be prescribed and fixed. For each $\alpha \in A$ one, at least, of the presentations Q_α and $\hat{Q}_\alpha$, say H_α, is such that a sequence $H_0 * H_\alpha$ exists. With H_α so defined we set

$$\Gamma = (H_\alpha)_{\alpha \in A}\ . \tag{1.12}$$

We understand that 0 is an index in A so that H_0 is in Γ. It is clear that the presentations of Γ cover M_n.

It remains to prove the following:

I. *If H_α and H_β are two overlapping presentations in Γ, and no inverting sequences exist in $\mathscr{D}_\varepsilon M_n$, then H_α and H_β are* $\overset{+}{\text{Com}}$.

Suppose on the contrary that H_α and H_β are $\overset{-}{\text{Com}}$. Then one has the relation

$$H_\beta \overset{+}{\text{Com}}\, \hat{H}_\alpha \,. \tag{1.13}$$

By definition of Γ there exist sequences $H_0 * H_\alpha$ and $H_0 * H_\beta$. There accordingly exist a sequence $H_\alpha * H_0$ and, by virtue of (1.13), a sequence of the form

$$H_0 * H_\beta : \hat{H}_\alpha \,. \tag{1.14}$$

Hence there exists an inverting sequence of the form

$$H_\alpha * H_0 : H_0 * \hat{H}_\alpha \tag{1.15}$$

contrary to hypothesis in I.

Hence I is true and the conditions of Theorem 1.1 are sufficient. This completes the proof of Theorem 1.1.

We shall make use of the following lemma, leaving its verification to the reader.

Lemma 1.5. *Two overlapping presentations in $\mathscr{D}_\varepsilon M_n$ which are restrictions respectively of presentations F and Q in $\mathscr{D}_\varepsilon M_n$ are* $\overset{+}{\text{Com}}$ *if and only if* $F \overset{+}{\text{Com}}\, Q$.

Definition 1.5. *Inversion invariant coverings of M_n. A* subset Λ of presentations in $\mathscr{D}_\varepsilon M_n$, which covers M_n and is such that $\hat{Q}$ is in Λ whenever Q is in Λ, will be called an *inversion invariant covering* of M_n.

The following lemma is immediate.

Lemma 1.6. *If M_n admits an orienting covering, and Λ is an inversion invariant covering of M_n in $\mathscr{D}_\varepsilon M_n$, then the following is true.*

If $(\Gamma, \hat{\Gamma})$ is the partition of $\mathscr{D}_\varepsilon M_n$ given in Corollary 1.1, *then Λ admits a unique partition*

$$(\Gamma \cap \Lambda, \hat{\Gamma} \cap \Lambda)$$

into two subsets each of which is a maximal orienting covering of M_n in Λ.

Theorem 1.1 can be generalized as follows.

Theorem 1.2. *If Λ is an inversion invariant covering of M_n, then a necessary and sufficient condition that M_n be orientable is that there exist no inverting sequence $F * \hat{F}$ in Λ.*

The condition is necessary since an inverting sequence in Λ would be an inverting sequence in $\mathscr{D}_\varepsilon M_n$, contrary to Theorem 1.1. A proof that the condition is sufficient is given by the proof that the condition of Theorem 1.1 is sufficient, on replacing $\mathscr{D}_\varepsilon M_n$, wherever it occurs in the latter proof, by Λ.

Definition 1.6. *Extended geometric orientability.* Let M'_n be a differentiable manifold such that $|M'_n|$ is an open arcwise connected subspace of $|M_n|$ and M'_n has a differentiable structure induced on $|M'_n|$ by M_n. Definitions 1.1 to 1.5 can be extended to M'_n, by replacing M_n by M'_n. The lemmas and theorems of § 1 remain valid if M_n is replaced by M'_n.

2. f-level manifolds on M_n

Definition 2.1. *Admissible ND functions f.* We *admit ND* functions f of class C^∞ on M_n such that f has different values a at different critical points and just one critical point of index 0 and one of index n. See [2]. The critical point of f at the f-level a is denoted by p_a.

Given a value c of f, a subset of $|M_n|$ of the form

$$f^c = \{x \in |M_n| \,|\, f(x) = c\} \tag{2.1}$$

is called an *f-level set* on M_n.

If c is an ordinary value of f, then f^c is a compact topological $(n-1)$-manifold which is the union of a finite number of disjoint compact, connected, topological $(n-1)$-manifolds. It is well-known that when c is ordinary, f^c is the carrier of a C^∞-manifold $\mathbf{f}^c$ which is C^∞-embedded in M_n by the inclusion mapping of f^c into $|M_n|$. See (ii) of the proof of [6, Theorem 20.1].

When c is a critical value a of f of index[5] $k \neq 0$ or n, the topological manifold $\dot{f}^a = f^a - p_a$ is also the carrier of a C^∞-manifold $\dot{\mathbf{f}}^a$ which is C^∞-embedded in M_n by the inclusion mapping of $\dot{f}^a$ into $|M_n|$.

Notation. Suppose that c is an ordinary value of f. Let M_{n-1} be a component of $\mathbf{f}^c$. Let a metric on M_{n-1} be induced by the Riemannian metric on M_n, and $\mathscr{D}_e M_{n-1}$ be a subset of the presentations in $\mathscr{D}M_{n-1}$ with e conditioned relative to M_{n-1} as ε was conditioned by Lemma 1.0 relative to M_n. We suppose $e < \varepsilon$.

A trajectory on M_n which is orthogonal to the non-singular level manifolds of f will be called an *ortho-f*-arc on M_n. Each such arc γ will be parameterized by the values of f on γ.

***ff*-Presentations.** The proof in § 20 of [6] that the non-singular level manifolds $\mathbf{f}^c$ are C^∞-embedded in M_n by the inclusion mapping of f^c into $|M_n|$ makes use of specialized presentations $\mathscr{F} \in \mathscr{D}M_n$, termed *ff-presentations*, whose nature we shall briefly recall.

The euclidean domain of an $\mathscr{F}$ is taken as a product $J \times V$ of a bounded open interval J containing the value c of f and an open subset V of the coordinate $(n-1)$-plane E_{n-1} of E_n on which the coordinate $u_n = 0$. A point in J will be represented by its coordinate t, a point in V by rectangular coordinates $v_1, \cdots, v_{n-1}$, a point in the domain $J \times V$ of $\mathscr{F}$ by a set of coordinates

$$(t, v_1, \cdots, v_{n-1}) = (t, v) .$$

An ff-presentation $\mathscr{F}$ is a presentation in $\mathscr{D}M_n$ which can be given the form

$$(t, v) \to \mathscr{F}(t, v) \colon J \times V \to |M_n| \tag{2.2}$$

subject to the condition that each partial mapping $t \to \mathscr{F}(t, v) \colon J \to |M_n|$ be

[5] A critical value a of f is said to have the *index* of the corresponding critical point p_a.

an ortho-f-arc. When c is an ordinary value of f, J is restricted to an open interval of ordinary values of f containing c. In this case, partial mappings of the form $v \to \mathscr{F}(c, v): V \to f^c$ cover f^c and define a differential structure of class C^∞ on f^c. With this structure f^c becomes a differentiable manifold $\mathbf{f}^c$, C^∞-embedded in M_n by the inclusion mapping of f^c into $|M_n|$. We suppose $\dot{\mathbf{f}}^a$ similarly defined when a is a critical value.

"Bases" of ff-presentations. If c is ordinary, let M_{n-1} be a component of $\mathbf{f}^c$. If c is a critical value a of f let M_{n-1} be a component $\dot{\mathbf{f}}^a$. If a presentation

$$(Q: V, X) \in \mathscr{D}M_{n-1} \tag{2.3}$$

is given, an ff-presentation

$$(H: J \times V, Y) \in \mathscr{D}M_n \tag{2.4}$$

based on Q is defined as follows. If c is an ordinary value of f, then J shall be an interval of ordinary values of f as above. If $c = a$ is critical, J shall be an interval of values of f of which a alone is critical. For $(t, v) \in J \times V, H(t, v)$ shall be the point at the f-level t on the ortho-f-arc which meets the point $Q(v)$ of M_{n-1} when $t = f(Q(v))$.

We can now prove the following theorem.

Theorem 2.1. (i). *Suppose that $n > 2$ and that c is an ordinary value of f. If a component M_{n-1} of $\mathbf{f}^c$ is non-orientable, M_n is non-orientable.*

(ii) *If M_n is orientable, then for each ordinary value c of f each component of $\mathbf{f}^c$ is orientable.*

Proof of (i). Since M_{n-1} is non-orientable, there exists (Theorem 1.1) an inverting sequence $F * \hat{F}$ of presentations

$$Q_1 : Q_2 : \cdots : Q_\mu \qquad (F = Q_1;\ \hat{F} = Q_\mu) \tag{2.5}$$

in $\mathscr{D}_e M_{n-1}$. For i on the range $1, \cdots, \mu$ suppose that Q_i has the form

$$(Q_i: V_i, X_i)\ . \tag{2.6}$$

We are taking $e < \varepsilon$ so that if J is a sufficiently small open interval containing c, then an ff-presentation

$$(H_i: J \times V_i, Y_i) \in \mathscr{D}_\varepsilon M_n \tag{2.7}$$

based on Q_i exists. The sequence

$$H_1: H_2 : \cdots : H_\mu \qquad (\text{in } \mathscr{D}_\varepsilon M_n) \tag{2.8}$$

is readily seen to be admissible on M_n. The hypothesis that $Q_\mu = \hat{Q}_1$ implies that $H_\mu = \hat{H}_1$ so that (2.8) is an inverting sequence $H_1 * \hat{H}_1$ in $\mathscr{D}_\varepsilon M_n$.

Thus Theorem 2.1 (i) is true. The proof of Theorem 2.1 (ii) is similar.

3. f-Preferred Riemannian structures on M_n

In [6, § 22] we have shown that a Riemannian structure on M_n can always be modified near the respective critical points of f in such a manner as to leave M_n and f unchanged and yield a new Riemannian structure on M_n of the type which we have termed *f-preferred*. We shall recall the theorem which characterizes such structures.

To that end let D_σ be an open origin-centered n-ball of radius σ in the euclidean space E_n of coordinates $u_1, \cdots, u_n$. Let[6]

$$(3.1) \qquad (F\colon D_\sigma, X) \in \mathscr{D}M_n \qquad (F(\mathbf{O}) = p_a)$$

be a presentation of a neighborhood X of a critical point p_a of f. We term the presentation F "isometric" if the euclidean length in E_n of any rectifiable arc $\gamma \subset D_\sigma$ equals the Riemannian length on X of $F(\gamma)$. The following theorem is a special consequence of [6, Theorem 22.2].

Theorem 3.1. *If $\sigma > 0$ is sufficiently small, there exists a Riemannian metric*[7] *on M_n such that corresponding to each critical point p_a of f of index k there exists an isometric presentation $F = I^a$ of form* (3.1) *of a neighborhood X of p_a on M_n such that*

$$(3.2) \quad f(I^a(u)) - a = -u_1^2 - \cdots - u_k^2 + u_{k+1}^2 + \cdots + u_n^2 \qquad (u \in D_\sigma) .$$

Ortho-f-arcs near p_a. Set

$$(3.3) \qquad \varphi_k(u) = -u_1^2 - \cdots - u_k^2 + u_{k+1}^2 + \cdots + u_n^2 ,$$

and

$$(3.4) \qquad X - p_a = \dot{X} , \qquad D_\sigma - \mathbf{O} = \dot{D}_\sigma .$$

Since I^a is isometric, the ortho-f-arcs on X are images under I^a of the ortho-φ_k-arcs on $\dot{D}_\sigma$. Orthogonality in $\dot{D}_\sigma$ is euclidean.

The case $k = n - 1$. In this case let E_{n-1} denote the coordinate $(n-1)$-plane in E_n on which $u_n = 0$. With $0 < \rho < \sigma$ set

$$(3.5) \qquad \Delta_\rho = \{u \in E_{n-1} \mid \|u\| < \rho\} .$$

Let $\dot{\Delta}_\rho$ be Δ_ρ with the origin deleted. Ortho-φ_{n-1}-arcs are radial in $\dot{\Delta}_\rho$ and have images under I^a which radiate[8] from p_a on M_n on the $(n-1)$-manifold $I^a(\Delta_\rho)$. The intervals $(0, \rho)$, $(-\rho, 0)$ on the u_n-axis represent radial ortho-φ_{n-1}-arcs. All other ortho-φ_{n-1}-arcs in $\dot{D}_\rho$ are arcs of rectangular hyperbolas with either the positive u_n-axis or negative u_n-axis as an asymptote. See [6, § 22].

[6] We denote the origin by $\mathbf{O}$.
[7] Obtained by a modification of a given Riemannian metric.
[8] Strictly tend to p_a as a limiting end point.

The cone Λ_{n-1}. In § 7 we shall refer to the $(n-1)$-cone

$$\Lambda_{n-1} = \{u \in E_n \mid u_n^2 = u_1^2 + \cdots + u_{n-1}^2\} \tag{3.6}$$

and to the subsets Λ_{n-1}^+ and Λ_{n-1}^- of Λ_{n-1} on which $u_n \geq 0$ and $u_n \leq 0$ respectively. The subsets Λ_{n-1}^+ and Λ_{n-1}^- intersect only in the origin.

Definition 3.1. *A neighborhood*[9] *U_e^k of* **O** *in E_n.* Corresponding to the function $u \to \varphi_k(u)$ defined by (3.3) when $0 < k < n$, we shall define for future use an arbitrarily small neighborhood

$$U_e^k \subset D_\sigma \qquad (0 < e < \sigma) \tag{3.7}$$

in E_n of the origin **O** in E_n. Corresponding to a positive constant $e < \sigma$ we introduce the *truncated* $(n-1)$*-cone*

$$\Omega_e = \{u \in E_n \mid \varphi_k(u) = 0, \|u\| < e\} . \tag{3.8}$$

The neighborhood U_e^k shall be the union of the origin and all ortho-φ_k-arcs which meet Ω_e or have the origin as limiting end point and on which φ_k is in absolute value less than e. A final condition is that e be so small that (3.7) holds.

The set U_e^k is connected and by virtue of condition (3.7) I^a is defined on U_e^k and

$$I^a \mid U_e^k \in \mathscr{D}M_n . \tag{3.9}$$

Subsets of U_e^k, with $\varphi_k > 0$ or $\varphi_k < 0$. Set

$$U_{\underset{e}{-}}^k = \{u \in U_e^k \mid \varphi_k(u) < 0\} , \tag{3.10}$$

$$U_{\underset{e}{+}}^k = \{u \in U_e^k \mid \varphi_k(u) > 0\} . \tag{3.11}$$

The following lemma is easily verified.

Lemma 3.1. *$U_{\underset{e}{-}}^k$ is connected when $1 < k < n$, and $U_{\underset{e}{+}}^k$ is connected when $0 < k < n-1$. Neither $U_{\underset{e}{-}}^1$ nor $U_{\underset{e}{+}}^{n-1}$ is connected.*

$U_{\underset{e}{-}}^1$ is the union of two components

$$'U_{\underset{e}{-}}^1 , \quad ''U_{\underset{e}{-}}^1 \tag{3.12}$$

on which $u_1 > 0$ and $u_1 < 0$, respectively, while $U_{\underset{e}{+}}^{n-1}$ is the union of two components

[9] The image $I^a(U_e^k)$ is a subset of $f_{a+e} = \{x \in M_n \mid f(x) \leq a + e\}$ which is sometimes called a "*handle*" of f_{a+e} associated with the critical point p_a. Such "handles" were first introduced by Morse in 1925 although not called by this name. See the first reference to Morse in [6].

$$(3.13) \qquad {}'U^{n-1}_{\underset{e}{+}}, \quad {}''U^{n-1}_{\underset{e}{+}}$$

on which $u_n > 0$ and $u_n < 0$, respectively.

4. *ND* functions f of biordered type

A *ND* function f shall be admissible in the sense of Def. 2.1.

Definition 4.1. *f of biordered type.* An admissible *ND* function f on M_n will be said to be of *biordered* type if each critical value of index 1 is less than each critical value of index $k > 1$ and, dually, each critical value of index $n - 1$ is greater than each critical value of index $k < n - 1$.

We shall prove the following theorem.

Theorem 4.1. *There exists on M_n an admissible ND function f of biordered type.*

This theorem is here proved with the aid of the theory of "bowls" of f as developed in papers [3] and [5].

Theorem 4.1 is a special case of the theorem that there exists on M_n an admissible *ND* function the numerical order of whose critical values is in accord with the indices of these critical values. This result was *first* formulated by Smale [9]. It was discovered independently by Morse and is readily verified with the aid of Morse [3] and [5]. The proof here given of Theorem 4.1 illustrates one mode of proof.

Proof of Theorem 4.1. The reader is asked to refer to [6, § 22] for the definition of "bowls ascending or descending" from a critical point with index k. The following lemma implies Theorem 4.1.

Lemma 4.1. *For $n > 2$ the Riemannian form on M_n can be infinitesimally modified*[10] *near a finite number of ordinary points of f in such a manner that the following is true.*

(i) *The* 1*-bowls ascending from the critical points of f of index $n - 1$* (if any exist) *have as upper limiting end points the critical point p_M at which f assumes its maximum M.*

(ii) *The* 1*-bowls descending from the critical points of index* 1 (if any exist) *have as lower limiting end point the critical point p_m at which f assumes its minimum m.*

Satisfaction of (i). To satisfy (i) the modification of the Riemannian form need be made in the neighborhood of at most a finite set of points chosen as follows.

Let $B_-(z, k)$ be a k-bowl descending from a critical point z of index k such that $0 < k < n$, and c be an ordinary value of f such that $c < f(z)$ and the interval $(c, f(z))$ contains no critical values of f. If a 1-bowl γ ascending from

[10] The modified Riemannian form is to be admissible over *all* of M_n.

a critical point p of index $n - 1$ meets $B_-(z, k)$, it meets $B_-(z, k)$ in a point $w \in f^c$. Note that

$$\dim (B_-(z, k) \cap f^c) = k - 1 < n - 1 . \tag{4.1}$$

If the Riemannian form is suitably altered on f^c sufficiently near w, the modified 1-bowl γ will not meet $B_-(z, k) \cap f^c$. Hence the modified 1-bowl γ will "by-pass" z.

After modifications of this type in number less than the number of critical points of f above p, the modified 1-bowl γ will ascend to p_M as a limiting upper end point. If each subsequent modification is made so small as not to alter the "by-passing" of critical points by an original or modified ascending 1-bowl, all 1-bowls will ascend to p_M as a limiting end point.

Satisfaction of (ii). One can similarly satisfy the condition (ii) of Lemma 4.1 by an additional modification of the metric, so made that (i) remains satisfied.

Thus Lemma 4.1 is true.

Completion of proof of Theorem 4.1. The theorem is trivial if $n = 2$. We suppose then that $n > 2$.

By virtue of Lemma 4.1 we can suppose that conditions (i) and (ii) of Lemma 4.1 are satisfied. Theorems 4.1 and 4.2 of Morse [3] or [5] then imply the following. The function f can be further modified in open disjoint neighborhoods N_γ of the respective 1-bowls γ in such a manner that the critical values of f of index $n - 1$ differ arbitrarily little from M, while those of index 1 differ arbitrarily little from m, while other critical values of f remain unaltered. Theorem 4.1 follows.

5. Unipartite functions f

Definition 5.1. *Unipartite functions f.* An admissible *ND* function f (Def. 2.1) will be termed *unipartite* if each level set f^c of f is connected.

We shall show that when $n > 2$ a *ND* f of "biordered" type (Def. 4.1) is "unipartite". To this end we first extend the comparison of the singular homology groups of f_a and of $\dot{f}_a$ made in [6, § 29] for each critical value a of f. Here

$$f_a = \{x \in M_n \mid f(x) \leq a\} , \tag{5.1}$$

and $\dot{f}_a = f_a - p_a$.

As in [6, § 29] the homology groups are taken over a field $\mathscr{K}$. Among fields the field **Q** of rational numbers is for us the most important.

ND **functions** g. If c and e are arbitrary values of f with $c < e$, set

$$f_{(c,e)} = \{x \in M_n \mid c < f(x) < e\} , \tag{5.2}$$

$$f_{[c,e]} = \{x \in M_n \mid c \leq f(x) \leq e\} . \tag{5.3}$$

If c and e are ordinary values of f, then $f_{[c,e]}$ is a bounded topological manifold with the non-singular $(n-1)$-manifolds f^c and f^e as boundaries. To give a differential structure to f_c of to $f_{[c,e]}$ is to give a differential structure to some open neighborhood of f_c or $f_{[c,e]}$ relative to M_n. A differential structure will be induced on $f_{(c,e)}$ and on open neighborhoods of f_c and $f_{[c,e]}$ by M_n. For simplicity of notation we shall hereafter denote the differentiable manifolds carried by

$$f_c, f_{(c,e)}, f_{[c,e]} \tag{5.4}$$

by the *same* symbols.

In Theorems 5.1 and 5.2 below, g shall be one of the three restrictions

$$g = f \mid f_{(c,e)}\,;\; g = f \mid f_{[c,e]}\,;\; g = f \mid f_c \tag{5.5}$$

of f, with c and e ordinary values of f.

In the comparisons in [6, § 29], one can replace f by g subject to proper interpretations of the theorems involved. The principal theorems of [6, § 29] are proved by induction with respect to an integer denoted by **m**. The induction is *completed* in [6] so that in stating our extensions we can properly omit the inductive integer **m**. A first theorem extends [6, Corollary 29.1].

Theorem 5.1. *If p_α is a critical point of g of index k with critical value α, and if one sets $\dot{g}_\alpha = g_\alpha - p_\alpha$, then over an arbitrary field $\mathscr{K}$ the connectivity*

$$R_q(g_\alpha, \dot{g}_\alpha) = \delta_k^q \qquad (q = 0, 1, \cdots)\,, \tag{5.6}$$

where $\dot{g}_\alpha$ serves as a modulus.

The proof of this theorem is identical in form with the proof of [6, Theorem 29.1].

Definition 5.2. *q-caps.* A relative q-cycle on $g_\alpha \bmod \dot{g}_\alpha$ which is nonbounding on $g_\alpha \bmod \dot{g}_\alpha$ will be called a *q-cap* of p_α *relative* to g. Cf. [6, Def. 29.1].

By virtue of Theorem 5.1 there are no q-caps of p_α relative to g other than k-caps, nnd any such k-cap of p_α is a "homology prebase"[11] for the homology group $H_k(g_\alpha, \dot{g}_\alpha)$ over $\mathscr{K}$.

Definition 5.3. *k-caps of linking type.* A k-cap y^k of a critical point p_α of g of index k is said to be of *linking type* relative to g and p_α, if ∂y^k is bounding (over $\mathscr{K}$) on $\dot{g}_\alpha$. Otherwise, y^k is said to be of *nonlinking type*. Cf. [6, Def. 29.1].

Definition 5.4. *Linking k-cycles.* If p_α has the index k, a k-cap of p_α which is an absolute k-cycle λ^k will be called a *linking* k-cycle of p_α (relative to g).

Notation. Given the critical point p_α of g let

[11] If a homology group H_q over $\mathscr{K}$ has a base B_q, a set of q-cycles, one and only one from each "homology class" in B_q is called a "*prebase*" of H_q. See [6, Def. 24.7].

$$(5.7) \qquad \mathbf{b}_q(g_\alpha), \mathbf{b}_q(\dot{g}_\alpha) \qquad (q = 0, 1, \cdots)$$

denote *homology prebases*, possibly empty, of the singular homology groups $H_q(g_\alpha)$ and $H_q(\dot{g}_\alpha)$, respectively, over the given field $\mathscr{K}$.

Note. A critical point p of g is a critical point of f. A k-cap of p, relative to g, is a k-cap of p relative to f. A critical point p of g which is of linking type, relative to g, is of linking type, relative to f. However, the converses of these three statements obtained by interchanging g and f are not in general true, as examples show.

The fundamental theorem follows. We continue with the field $\mathscr{K}$.

Theorem 5.2. (i) *If a critical point p_α of g has the index k, then a homology prebase $\mathbf{b}_q(\dot{g}_\alpha)$ is a homology prebase $\mathbf{b}_q(g_\alpha)$ unless*

Case I. *$q = k$ and p_α is of linking type, or*

Case II. *$q = k - 1$ and p_α is of non-linking type.*

(ii) *In Case* I *a homology prebase $\mathbf{b}_k(g_\alpha)$ is given by any*[12] *set of absolute k-cycles of the form*

$$(5.8) \qquad \mathbf{b}_k(\dot{g}_\alpha) \cup \lambda^k ,$$

where λ^k is a linking k-cycle of p_α (relative to g).

(iii) *In Case* II *a homology prebase $\mathbf{b}_{k-1}(g_\alpha)$ is given by any*[13] *set of absolute $(k-1)$-cycles, $k > 0$, of the form*

$$(5.9) \qquad \mathbf{b}_{k-1}(\dot{g}_\alpha) - w^{k-1}$$

in which w^{k-1} is the algebraic boundary of a k-cap of p_α and $\mathbf{b}(\dot{g}_\alpha)$ contains w^{k-1}.

Theorem 5.2, as formulated above, differs from [6, Theorem 29.3] only in that g here replaces f of [6]. When stated in terms of g or f, all terms must be understood relative to g or f respectively.

A review by the reader of the proof of [6, Theorem 29.3] will make clear the proof of Theorem 5.2.

We shall use Theorem 5.2 to prove the following lemma, true for $n \geq 2$.

Lemma 5.1. *Let a be a critical value of f with index k such that $0 < k < n$ and such that f^a is connected. Let $[\mu, \nu]$ be an interval of values of f such that $\mu < a < \nu$ and $[\mu, \nu]$ contains no critical values of f other than a.*

(i) *If f^μ is not connected, then $k = 1$,*

(ii) *If f^ν is not connected, then $k = n - 1$.*

Proof of (i). Returning to (5.5) set $g = f | f_{[\mu,\nu]}$. The critical point z of f at the f-level a is a critical point of g at the g-level a. By hypothesis the set f^μ, and hence the set g^μ, are not connected.

[12] Such a set exists in Case I.

[13] Such a set exists in Case II.

As in [6, § 23], a proper use of ortho-g-arcs leads to a deformation[14] retracting $\dot{g}_a$ onto g^μ. Hence $\dot{g}_a$ is not connected.

The set f^a and hence g^a are connected by hypothesis. Each point of g_a is either on g^a or can be deformed on an ortho-g-arc so as to ascend to a point of g^a, or to z as a limiting end point. The last statements follow from Theorem 3.1. Hence g_a is connected and $\dot{g}_a$ is not connected.

We infer that a prebase of the homology group $H_0(g_a)$ contains just one 0-cycle while a prebase of the homology group $H_0(\dot{g}_a)$ contains more than one 0-cycle. It follows from Theorem 5.2 that z must be a critical point of g of non-linking type with index 1. This completes the proof of (i).

Proof of (ii). Statement (ii) follows readily if one applies Lemma 5.1 (i) to $-f$ in place of f.

Lemma 5.1 will lead to a proof of the following theorem.

Theorem 5.3. *If $n > 2$, and the ND f on M_n is of biordered type* (Def. 4.1), *then f is unipartite* (Def. 5.1), *that is each level set of f is connected.*

Proof. Let m and M be respectively the minimum and maximum values of f on M_n, and a be the largest of the values $\alpha > m$ of f such that f^c is connected for $m < c \leq \alpha$. Then a must be a critical value of f. We shall prove Theorem 5.3 by showing that

$$a = M \; . \tag{5.10}$$

Proof of (5.10). If $a < M$, the index k of a is such that $0 < k < n$. It then follows from Lemma 5.1 (ii) that $k = n - 1$.

If there are *no* critical values of f of index $n - 1$, it is impossible that index $a = n - 1$ and (5.10) must be true.

If there are critical values of f of index $n - 1$, then the hypothesis that f is of "*biordered*" type implies that the critical values of index $n - 1$ form a sequence

$$a_1 > a_2 > \cdots > a_r \tag{5.11}$$

of values which exceed each critical value of index less than $n - 1$. It is impossible that $a = a_1$ since for $M > c > a_1$, f^c is connected, contrary to the definition of a.

It is impossible that $a = a_2$. Were $a = a_2$ then for $a_1 > c > a_2$, f^c would be nonconnected by virtue of the definition of a, and hence a_1 would be of index 1 by Lemma 5.1. If, however, index a_1 is both 1 and $n - 1$, $n = 2$, contrary to hypothesis. Continuing, one proves inductively that a is no one of the numbers (5.11) of index 1. From this contradiction we infer the truth of Theorem 5.3.

Theorem 5.3 follows.

[14] We term such a deformation a *g-deformation.*

We shall make repeated use of the following lemma and its variations.

Lemma 5.2. *If a and b, with $a < b$, are successive critical values of f on M_n, then*

$$H_q(f_a, \mathbf{Q}) \approx H_q(\dot{f}_b, \mathbf{Q}) \qquad (q = 0, 1, \cdots) . \tag{5.12}$$

As in [6, § 23] there exists an f-deformation d retracting $\dot{f}_b$ onto f_a. It follows from [6, Theorem 28.4], with the moduli A and A' empty sets, that the deformation d induces the isomorphisms (5.12).

6. Homological orientability

The manifold M_n will be termed *homologically orientable* if and only if the connectivity

$$R_n(|M_n|, \mathbf{Q}) = 1 . \tag{6.1}$$

In this section we shall show that if the manifold M_n is geometrically orientable in the sense of § 1, then M_n is homologically orientable. The proof is given for the case $n > 2$. It can be shown in many ways without use of any global triangulation of M_n that geometrical orientability of M_n implies (6.1) when $n = 2$.

The proof is inductive in character and will make use of an inductive hypothesis formulated as follows:

Inductive hypothesis. We shall assume that if $n > 2$, then for each integer r such that $1 < r < n$, a connected, compact, differentiable, orientable[15] manifold M_r is such that

$$R_r(|M_r|, \mathbf{Q}) = 1 . \tag{6.2}$$

From this point on we shall assume that f is of biordered type on M_n (Def. 4.1) and so is a *ND unipartite* function on M_n (Def. 5.1).

If c is an ordinary value of f and M_n is orientable, then the level manifold f^c has but one component, since f is unipartite, and f^c is orientable in accord with Theorem 2.1 (ii). By the inductive hypothesis (6.2),

$$R_{n-1}(f^c, \mathbf{Q}) = 1 . \tag{6.3}$$

There accordingly exists a rational $(n-1)$-cycle y_c^{n-1} carried by f^c which is a "prebase" for the singular homology group $H_{n-1}(f^c, \mathbf{Q})$.

Theorem 6.1. *Suppose that $n > 2$ and that M_n is orientable. If c and e are ordinary values of f such that $c < e$, and if y_c^{n-1} and y_e^{n-1} are respectively suitably chosen* $(n-1)$st *RHP*[16] *of f^c and f^e, then*

[15] Orientable shall mean orientable in the sense of § 1.

[16] A "rational homology prebase" of $H_r(\chi, \mathbf{Q})$ will be termed an "rth *RHP* of χ."

$$y_c^{n-1} \sim y_e^{n-1} \qquad (\text{over } \mathbf{Q} \text{ on } f_{[c,e]}) . \tag{6.4}$$

We shall begin the proof of Theorem 6.1 by showing that Theorem 6.1 is true in a special case.

Lemma 6.1. *In the special case in which the interval* $[c, e]$ *contains just one critical value* a *of* f, *Theorem* 6.1 *is true.*

We shall prove a proposition bearing on Lemma 6.1.

Proposition 6.1. *Under the hypotheses of Lemma* 6.1 *an* $(n-1)$*-cycle which is an* $(n-1)$*st RHP of* f^c *or of* f^e *is an* $(n-1)$*st RHP of* $f_{[c,e]}$.

Notation. Let k be the index of the critical value a in the interval (c, e). We shall verify the following.

Under the hypotheses of Lemma 6.1, an $(n-1)$st *RHP* of $f_{[c,e]}$ is given

(i) by y_c^{n-1} when $0 < k < n-1$,
(ii) by y_e^{n-1} when $1 < k < n$,
(iii) by y_c^{n-1} when $k = n-1$,
(iv) by y_e^{n-1} when $k = 1$.

Proof of (i). As in (5.5) we set $g = f \mid f_{[c,e]}$. Since there is an f-deformation retracting $\dot{g}_a$ onto g^c, we infer that y_c^{n-1} is an $(n-1)$st *RHP* of $\dot{g}_a$. Since $0 < k < n-1$ in the case at hand, Theorem 5.2 (i) implies that y_c^{n-1} is an $(n-1)$st *RHP* of g_a. Since there exists an f-deformation retracting $f_{[c,e]}$ onto g_a, y_c^{n-1} is also an $(n-1)$st *RHP* of $f_{[c,e]}$.

Proof of (ii). On replacing f by $-f$, (i) implies (ii).

Proof of (iii). By virtue of (ii), for some rational number r, possibly 0,

$$y_c^{n-1} \sim r y_e^{n-1} \qquad (\text{over } \mathbf{Q} \text{ on } f_{[c,e]}) . \tag{6.5}$$

Hence (iii) is valid if $r \neq 0$.

Proof that $r \neq 0$. On setting $g = f \mid f_{[c,e]}$ as in the proof of (i), we see that y_c^{n-1} is an $(n-1)$st *RHP* of $\dot{g}_a$. The critical point p_a has the index $n-1$. Whether p_a is of linking or non-linking type relative to g, Theorem 5.2 implies that y_c^{n-1} is an element in an $(n-1)$-st *RHP* of g_a and hence of $f_{[c,e]}$. Hence $r \neq 0$ in (6.5), and (iii) is true.

Proof of (iv)· On replacing f by $-f$, (iii) implies (iv).

Lemma 6.1 follows from (i), (ii), (iii), (iv).

Note. The proof of (iii) shows that p_a is of non-linking type relative to g since it shows that y_c^{n-1} is an $(n-1)$-st *RHP* of both $\dot{g}_a$ and g_a when $k = n-1$.

Completion of proof of Theorem 6.1. Let

$$\alpha_1 < \alpha_2 < \cdots < \alpha_t \tag{6.6}$$

be ordinary values of f on $[c, e]$, such that $c = \alpha_1$ and $e = \alpha_t$ and such that between successive values in (6.6) there is just one critical value of f. In case there are no critical values of f between c and e, Theorem 6.1 is trivial.

For j on the range $1, 2, \cdots, t$ let $y^{n-1}_{\alpha_j}$ be an $(n-1)$st *RHP* of f^{α_j}. It follows from Lemma 6.1 that if r_j is a suitably chosen non-null rational number, then

$$r_1 y^{n-1}_{\alpha_1} \sim r_2 y^{n-1}_{\alpha_2} \sim \cdots \sim r_t y^{n-1}_{\alpha_t} \qquad (\text{over } \mathbf{Q} \text{ on } f_{[c,e]}) \;. \tag{6.7}$$

Hence Theorem 6.1 is true.

Before coming to the principal theorem of this section we shall show how to associate a special n-cap with p_M.

A simply-carried n-cap. If β is an ordinary value of f such that $M - \beta$ is sufficiently small and positive, then the special isometric presentation of a neighborhood of p_M given by Theorem 3.1 shows that $f_{[\beta, M]}$ is a topological hemisphere H^n bounded by f^β as a topological $(n-1)$-sphere. It follows[17] that there is an n-cap z^n of p_M, which is defined by a homeomorphic map of a vertex-ordered euclidean n-simplex onto H^n. Such a cap has been termed "simply-carried". See [6, Def. 30.2]. It follows then from [6, Lemma 30.3] that ∂z^n is an $(n-1)$-cycle y^{n-1}_β which is an $(n-1)$st *RHP* of f^β.

Theorem 6.2. *If M_n is orientable, then*

$$R_n(|M_n|, \mathbf{Q}) = 1 \;. \tag{6.8}$$

We suppose $n > 2$. According to the definition of a critical point of linking type, p_M is of linking type if and only if the $(n-1)$-cycle y^{n-1}_β, introduced in the paragraph preceding the theorem, satisfies the homology

$$y^{n-1}_\beta \sim 0 \qquad (\text{over } \mathbf{Q} \text{ on } \mathring{f}_M) \;. \tag{6.9}$$

Because M_n is orientable, (6.9) holds in accord with (6.4) of Theorem 6.1, since in (6.4) $y^{n-1}_c \sim 0$ on f_c, if $c - m$ is sufficiently small.

The critical point p_M is accordingly of linking type. It follows from [6, Theorem 29.3 (ii)] that (6.8) holds.

The proof of Theorem 6.1 shows that it has the following useful extension.

Theorem 6.3. *Suppose $n > 2$. Let c and e be ordinary values such that $c < e$. Suppose further that some open neighborhood of $f_{[c,e]}$ is orientable. If then y^{n-1}_c and y^{n-1}_e are suitably chosen $(n-1)$st RHP's of f^c and f^e respectively, then*

$$y^{n-1}_c \sim y^{n-1}_e \qquad (\textit{over } \mathbf{Q} \textit{ on } f_{[c,e]}) \cdot \tag{6.10}$$

We shall use this extension of Theorem 6.1 in proving that $R_n(|M_n|, \mathbf{Q}) = 0$ when M_n is nonorientable.

[17] From [6, "Carrier Theorem" (36.2)].

7. Orientability and critical shells

Let a be a critical value of f with index k such that $0 < k < n$, and μ, ν be ordinary values of f such that a is the only critical value of f in (μ, ν). Set

$$f^a_{(\mu,\nu)} = \{x \in |M_n| \,|\, \mu < f(x) < \nu\} \,. \tag{7.1}$$

It will be convenient to suppose that

$$\nu - a = a - \mu = e > 0 \,. \tag{7.2}$$

We term $f^a_{(\mu,\nu)}$ a *critical shell, based on* f^a, *with index* k, provided e is so small that the following conditions are satisfied.

Conditions on e *of* (7.2). Note that

$$f^a_{(\mu,\nu)} = f_{(\mu,a)} \cup f^a \cup f_{(a,\nu)} \,. \tag{7.3}$$

We impose two conditions on e each of which is satisfied if e is sufficiently small. A *first* condition is that the closures of ortho-f-arcs on $f_{(a,\nu)}$ and $f_{(\mu,a)}$ have diameters less than $\varepsilon/2$, where ε is conditioned by Lemma 1.0. A *second* condition on e is a reimposition of the condition

$$U^k_e \subset D_\sigma \qquad \text{(of 3.7))} \,. \tag{7.4}$$

This condition implies that $e < \sigma$.

The choice of μ, ν. It will simplify our theorems if the parameters μ and ν associated with a critical value a in the definition of a critical shell $f^a_{(\mu,\nu)}$ are chosen once and for all when a is given.

We shall define a special covering of critical shells $f^a_{(\mu,\nu)}$ by special presentations in $\mathscr{D}_\varepsilon M_n$.

A covering $\Gamma^a_{(\mu,\nu)}$ *of* $f^a_{(\mu,\nu)}$. A first presentation has the form

$$\mathbf{F}^k = I^a \,|\, U^k_e \qquad (k = \text{index } a) \,, \tag{7.5}$$

where I^a is the isometry of Theorem 3.1, and e is given by (7.2). The range of $\mathbf{F}^k$ is an open neighborhood in $f^a_{(\mu,\nu)}$ of p_a and of the union of p_a and the ortho-f-arcs on $f^a_{(\mu,\nu)}$ which have p_a as a limiting end point.

Corresponding to an arbitrary point $p \in \dot{f}^a$ let

$$(Q_p \colon V_p, X_p) \in \mathscr{D}\dot{f}^a \qquad \text{(cf. (2.3))} \tag{7.6}$$

be a presentation of an open connected subset X_p of $\dot{f}^a$, which contains p. We can suppose that X_p is so small a neighborhood of $p \in \dot{f}^a$ that the ff-presentation

$$(H_p \colon (\mu, \nu) \times V_p, Y_p) \qquad \text{(cf. (2.4))} \tag{7.7}$$

based on Q_p satisfies the condition

$$H_p \in \mathscr{D}_\varepsilon M_n \ . \tag{7.8}$$

A covering $\Gamma^a_{(\mu,\nu)}$ of the critical shell $f^a_{(\mu,\nu)}$ is thus afforded by the presentations

$$\mathbf{F}^k, H_p \qquad (\text{index } a = k) \tag{7.9}$$

and their "inverts", as p ranges over $\dot{f}^a$.

$\hat{H}$ interpreted. The domain V_p of Q_p in (7.6) is an open connected subset of E_{n-1} with coordinates $v_1, \cdots, v_{n-1}$. Let ρ denote a reflection of V_p in the $(n-2)$-plane of E_{n-1} on which $v_{n-1} = 0$. One can obtain $\hat{H}_p$ by first replacing Q_p by

$$Q_p = (Q_p \circ \rho^{-1} \colon \rho(V_p), X_p) \in \mathscr{D}\dot{f}^a \ . \tag{7.10}$$

$\hat{H}_p$ is then a presentation

$$(\hat{H}_p \colon (\mu, \nu) \times \rho(V_p), Y_p) \in \mathscr{D}_\varepsilon M_n \tag{7.11}$$

"based" on $\hat{Q}_p$ with a euclidean domain $(\mu, \nu) \times \rho(V_p)$ which is a reflection of the euclidean domain $(\mu, \nu) \times V_p$.

The role of $\mathbf{F}^k$. Let K_e be the subset of U^k_e which is the union of $\mathbf{O}$ and the ortho-φ_k-arcs which tend in U^k_e to $\mathbf{O}$. The union of the ranges of the presentations H_p admitted in (7.9) cover

$$f^a_{(\mu,\nu)} - I^a(K_e) \ , \tag{7.12}$$

while $I^a(K_e)$ is covered by $\mathbf{F}^k$.

Given a critical shell $f^a_{(\mu,\nu)}$ it is essential that we define "inversion invariant coverings" of $f_{(a,\nu)}$ and $f_{(\mu,a)}$ by presentations which are simply related to the presentations in the set $\Gamma^a_{(\mu,\nu)}$ covering $f^a_{(\mu,\nu)}$. These presentations should be in

$$\mathscr{D}_\varepsilon f_{(a,\nu)} \ , \quad \mathscr{D}_\varepsilon f_{(\mu,a)} \tag{7.13}$$

respectively and so, in particular, have *connected* ranges.

To cover $f_{(a,\nu)}$ and $f_{(\mu,a)}$ we shall make use of presentations which are restrictions of the presentations H_p and $\hat{H}_p$ of $\Gamma^a_{(\mu,\nu)}$ and of the special presentation $\mathbf{F}^k$ in $\Gamma^a_{(\mu,\nu)}$ and of $\hat{\mathbf{F}}^k$.

Restrictions of H_p. For each point $p \in \dot{f}^a$ there exist unique presentations

$$H^+_p \in \mathscr{D}_\varepsilon f_{(a,\nu)} \ , \quad H^-_p \in \mathscr{D}_\varepsilon f_{(\mu,a)} \ , \tag{7.14}$$

which are the restrictions of H_p with ranges

$$RH^+_p = RH_p \cap f_{(a,\nu)} \ , \quad RH^-_p = RH_p \cap f_{(\mu,a)} \tag{7.15}$$

respectively. The set of presentations H_p^+, as p ranges over $\mathring{f}^a$, cover all points of $f_{(a,\nu)}$ except points of $f_{(a,\nu)}$ on ortho-f-arcs tending to p_a as a limit point. Each of the presentations H_p^+ has a connected range. To cover the set of residual points of $f_{(a,\nu)}$ we make use of restrictions of $\mathbf{F}^k$.

Restrictions of $\mathbf{F}^k$. Recall that index $a = k$. Referring to the subsets

$$U^k_{\underset{e}{-}}\ , \quad U^k_{\underset{e}{+}} \tag{7.16}$$

of U^k_e defined in (3.10) and (3.11), and to the isometry I^a of Theorem 3.1, we introduce restrictions of $\mathbf{F}^k$ of the form

$$F^k_- = I^a \mid U^k_{\underset{e}{-}}\ , \quad F^k_+ = I^a \mid U^k_{\underset{e}{+}} \qquad (0 < k < n)\ , \tag{7.17}$$

and state the following lemma.

Lemma 7.1. *By virtue of Lemma* 3.1

$$F^k_- \in \mathscr{D}_\varepsilon f_{(\mu,a)} \qquad (1 < k < n)\ , \tag{7.18$'$}$$

$$F^k_+ \in \mathscr{D}_\varepsilon f_{(a,\nu)} \qquad (0 < k < n-1)\ . \tag{7.18$''$}$$

A first condition that (7.18) hold is that the sets $\mathrm{Cl}\, RF^k_-$ and $\mathrm{Cl}\, RF^k_+$ have diameters on M_n less than ε. This condition is satisfied as a consequence of the relation $\mathbf{F}^k \in \mathscr{D}_\varepsilon M_n$. That RF^k_- and RF^k_+ are connected subject to the conditions on k in (7.18) follows from Lemma 3.1.

Note. The relation in (7.18)$'$ is not valid if $k=1$, nor the relation in (7.18)$''$ if $k = n - 1$, because RF^1_- and RF^{n-1}_+ are not connected.

The case $k = 1$. In this case we refer to (3.12) and set

$$'F^1_- = I^a \mid 'U^1_{\underset{e}{-}}\ , \qquad ''F^1_- = I^a \mid ''U^1_{\underset{e}{-}}\ . \tag{7.19}$$

The ranges of these two restrictions of F^1_- are connected sets. These ranges are disjoint and have RF^1_- as their union. Cf. (3.12).

The case $k = n - 1$. In this case we refer to (3.13) and set

$$'F^{n-1}_+ = I^a \mid 'U^{n-1}_{\underset{e}{+}}\ , \qquad ''F^{n-1}_+ = I^a \mid ''U^{n-1}_{\underset{e}{+}}\ . \tag{7.20}$$

The ranges of these two restrictions of F^{n-1}_+ are connected sets. These ranges are disjoint and have RF^{n-1}_+ as their union. Cf. (3.13).

We summarize as follows:

Theorem 7.1. *"Inversion invariant coverings"* $\Gamma_{(\mu,a)}$ *of* $f_{(\mu,a)}$ *and* $\Gamma_{(a,\nu)}$ *of* $f_{(a,\nu)}$ *are given respectively by the second and third columns in the following table, provided these presentations are supplemented by their inverts and the point p ranges over* $\mathring{f}^a$.

Table Γ

$k =$ Index a	$\Gamma_{(\mu,a)}$	$\Gamma_{(a,\nu)}$
$k = 1$	$H_p^- : 'F_-^1, ''F_-^1$	$H_p^+ : F_+^1,$
$1 < k < n-1$	$H_p^- : F_-^k$	$H_p^+ : F_+^k,$
$k = n-1$	$H_p^- : F_-^{n-1}$	$H_p^+ : 'F_+^{n-1}, ''F_+^{n-1}$

Note. Each entry in Table Γ should properly bear the critical value a as an index.

Given a critical shell $f_{(\mu,\nu)}^a$ we shall term $f_{(a,\nu)}$ the *upper* shell of $f_{(\mu,\nu)}^a$, and $f_{(\mu,a)}$ the *lower* shell of $f_{(\mu,\nu)}^a$. We term these subshells *auxiliary* shells of $f_{(\mu,\nu)}^a$.

Definition 7.1. *Inverting critical shells.* A critical shell $f_{(\mu,\nu)}^a$ will be termed *orientation inverting* if one of its two auxiliary shells is orientable and the other nonorientable.

Theorem 7.2. *If $n > 2$, the following is true.*

(i) *A necessary and sufficient condition that a critical shells $f_{(\mu,\nu)}^a$ be orientable is that both its upper and lower auxiliary shells be orientable.*

(ii) *If index $a = k$, and the shell $f_{(\mu,\nu)}^a$ is inverting, then $k = 1$ or $n - 1$.*

(iii) *If $k = 1$, and the shell $f_{(\mu,\nu)}^a$ is inverting, then its upper shell is nonorientable.*

(iv) *If $k = n - 1$, and the shell $f_{(\mu,\nu)}^a$ is inverting, then its lower shell is nonorientable.*

We begin the proof of Theorem 7.2 by establishing the following:

The condition of (i) *is necessary.* Since

$$\mathscr{D}_\varepsilon f_{(\mu,a)} \subset \mathscr{D}_\varepsilon f_{(\mu,\nu)}^a ,$$

an inverting sequence in $\mathscr{D}_\varepsilon f_{(\mu,a)}$ is an inverting sequence in $\mathscr{D}_\varepsilon f_{(\mu,\nu)}^a$. It follows from Theorem 1.1 that if $f_{(\mu,\nu)}^a$ is orientable, then $f_{(\mu,a)}$ is orientable.

One proves similarly that if $f_{(\mu,\nu)}^a$ is orientable, then $f_{(a,\nu)}$ is orientable.

The condition of (i) *is sufficient.* We shall establish this and the remainder of Theorem 7.2 by proving the following lemma. We are supposing that $n > 2$.

Lemma 7.2 (α). *$f_{(\mu,\nu)}^a$ is orientable, if its lower shell $f_{(\mu,a)}$ is orientable, and index $a = k$ is on the range $2, \cdots, n-1$.*

(β). *$f_{(\mu,\nu)}^a$ is orientable, if its upper shell $f_{(a,\nu)}$ is orientable, and index $a = k$ is on the range $1, \cdots, n-2$.*

Proof of (α). When $1 < k < n$, a covering $\Gamma_{(\mu,\nu)}^a$ of $f_{(\mu,\nu)}^a$ is given by the presentations

$$\text{(7.21)} \qquad \mathbf{F}^k, H_p \qquad (p \text{ ranging over } \dot{f}^a)$$

and their inverts, while a covering $\Gamma_{(\mu,a)}$ of $f_{(\mu,a)}$ is given by the presentations

$$(7.22) \qquad F_-^k \ , \ H_p^- \qquad (p \text{ ranging over } \dot{f}^a)$$

and their inverts. (See Table Γ.) The presentatations in $\Gamma^a_{(\mu,\nu)}$ and $\Gamma_{(\mu,a)}$ admit a biunique *matching* ξ: $\Gamma_{(\mu,a)} \leftrightarrow \Gamma^a_{(\mu,\nu)}$ under which

$$(7.23) \qquad \mathbf{F}^k \leftrightarrow F_-^k \ , \qquad H_p \leftrightarrow H_p^-$$

and

$$(7.24) \qquad \hat{\mathbf{F}}^k \leftrightarrow \hat{F}_-^k \ , \qquad \hat{H}_p \leftrightarrow \hat{H}_p^- \ .$$

Under ξ each presentation in $\Gamma^a_{(\mu,\nu)}$ is matched with a restriction in $\Gamma_{(\mu,a)}$.

By hypothesis of (α) there exists a subset Λ of presentations in $\Gamma_{(\mu,a)}$ which form an orienting covering of $f_{(\mu,a)}$. The subset $\xi\Lambda$ of presentations in $\Gamma^a_{(\mu,\nu)}$ covers $f^a_{(\mu,\nu)}$ and, by virtue of Lemma 1.5, is an orienting covering of $f^a_{(\mu,\nu)}$.

Proof of (β). The proof of (β) is similar to that of (α). The index k is on the range $1, \cdots, n-2$. One makes use of $F_+^k, \hat{F}_+^k, H_p^+$ and $\hat{H}_p^+$.

Statements (ii), (iii) and (iv) follow from Lemma 7.2. Hence the proof of Theorem 7.2 is complete.

The following special lemma is needed in § 8.

Lemma 7.3. *Let a be a critical value with* $0 <$ index $a < n$, *and a' be a critical value such that (a', a) is an interval of ordinary values of f. If the shell $f^a_{(\mu,\nu)}$ is such that $f_{(\mu,a)}$ is orientable, then $f_{(a',a)}$ is orientable.*

Proof. We introduce a C^∞-diff

$$(7.25) \qquad T\colon f_{(\mu,a)} \to f_{(a',a)}$$

under which each maximal ortho-f-arc γ on $f_{(\mu,a)}$ is mapped onto the maximal ortho-f-arc γ' on $f_{(a',a)}$ extending γ, so that a point $p \in \gamma$ goes into the point $p' \in \gamma'$ such that

$$(7.26) \qquad \frac{f(p') - a}{f(p) - a} = \frac{a' - a}{\mu - a} \ .$$

If ε' is a sufficiently small positive constant, and $f_{(\mu,a)}$ is covered (as is possible) by an "orienting" set of presentations $Q \in \mathscr{D}_{\varepsilon'} f_{(\mu,a)}$, then the presentations $(T \mid RQ) \circ Q$ will be an orienting set in $\mathscr{D}_\varepsilon f_{(a',a)}$ covering $f_{(a',a)}$.

8. The existence of inverting critical shells

We begin with new notation.

Submanifolds $M^-_{n,\alpha}$ and $M^+_{n,\beta}$. Let α and β be values of f such that $\alpha > m$ and $\beta < M$ respectively. We shall denote by $M^-_{n,\alpha}$ and $M^+_{n,\beta}$ open submanifolds of M_n with differential structures induced by that of M_n and carriers

$$\begin{aligned} |M_{n,\alpha}^-| &= \{x \in |M_n| \,|\, f(x) < \alpha\} \ , \\ |M_{n,\beta}^+| &= \{x \in |M_n| \,|\, f(x) > \beta\} \ . \end{aligned} \tag{8.0}$$

We suppose $n > 2$ *in* § 8.

Lemma 8.1. *Let* a *be a critical value of* f *with* $0 <$ index $a < n$. *If* $M_{n,a}^+$ *and the critical shell* $f_{(\mu,\nu)}^a$ *are orientable, then* $M_{n,\mu}^+$ *is orientable.*

Proof. Let $e < \varepsilon$ be a positive constant such that the minimum distance from f^ν to f^a on M_n exceeds $3e$. Set

$$\Gamma = \mathscr{D}_e f_{(a,\nu)} \ . \tag{8.1}$$

Let Γ' be an inversion (see Def. 1.5) invariant covering of $f_{(\mu,\nu)}^a$ obtained by adding to Γ presentations in $\mathscr{D}_e f_{(\mu,\nu)}^a$ the range of each of which meets $f_{(\mu,a)}$. Let Γ'' be an inversion invariant covering of $M_{n,\mu}^+$ obtained by adding to Γ' presentations in $\mathscr{D}_e M_{n,a}^+$ the range of each of which meets $H_{n,\nu}^+$.

Suppose Lemma 8.1 *false.* By an extension of Theorem 1.2 to the sub-manifold $M_{n,\mu}^+$ of M_n there then exists an inverting $F * \hat{F}$ in Γ''. By Lemma 1.3 we can suppose that F is in Γ. If $F * \hat{F}$ *is in* Γ', we have a contradiction to the hypothesis that $f_{(\mu,\nu)}^a$ is orientable. If the sequence $F * \hat{F}$ *is not in* Γ', we shall show that it can be admissibly modified without changing F or $\hat{F}$, so that the modified sequence is in Γ'.

Modification of $F * \hat{F}$. A presentation in $F * \hat{F}$ which is not in Γ' meets $M_{n,\nu}^+$ and is a nonterminal presentation in a subsequence of consecutive presentations

$$Q_1 : Q_2 : \cdots : Q_r \tag{8.2}$$

of $F * \hat{F}$ such that Q_1 and Q_r are in Γ, while the remaining Q_i's have ranges which meet $M_{n,\nu}^+$. Each presentation in (8.2) is in $\mathscr{D}_e M_{n,a}^+$ as a consequence of our choice of e. Since $M_{n,a}^+$ is orientable by hypothesis, and since $f_{(a,\nu)}$ is arc-wise connected and

$$f_{(a,\nu)} \subset M_{n,a}^+ \ ,$$

each subsequence (8.2) of $F * \hat{F}$ can be replaced by an admissible sequence

$$Q_1 : P_1 : \cdots : P_s : Q_r$$

in which the presentations P_j are in Γ.

The sequence $F * \hat{F}$ so modified will be in Γ'. This is impossible since $f_{(\mu,\nu)}^a$ is orientable by hypothesis. Thus Lemma 8.1 is true.

We continue with the following lemma, referring to the value μ in $f_{(\mu,\nu)}^a$.

Lemma 8.2. *Let* a *be a critical value with* $0 <$ index $a < n$, *and* $a' < a$ *be the critical value next below* a. *If* $M_{n,\mu}^+$ *is orientable, then* $M_{n,a'}^+$ *is orientable.*

Proof. If $M^+_{n,\mu}$ is orientable, its submanifold $f^a_{(\mu,\nu)}$ is orientable. According to Lemma 7.3, $f_{(a',a)}$ is then orientable. By an argument similar to that used in proving Lemma 8.1 one then proves that $M^+_{n,a'}$ is orientable. The roles of the values μ, a, ν in the proof of Lemma 8.1 are played by the respective values a', μ, a in the proof of Lemma 8.2.

Lemmas 8.1 and 8.2 combine to give the following.

Lemma 8.3. *Let a be a critical value with $0 <$ index $a < n$, and a' be the critical value of f next below a. If $M^+_{n,a}$ and $f^a_{(\mu,\nu)}$ are orientable, then $M^+_{n,a'}$ is orientable.*

By an induction with respect to the critical values of f of index $< n$ enumerated in the order of decreasing values, one proves the following theorem. Theorem 7.2 is essential for the proof.

Theorem 8.1. *If $f^\alpha_{(\mu,\nu)}$ is orientable for each critical value α of index $n - 1$ which exceeds a critical value a, then $M^+_{n,a}$ is orientable.*

We turn to nonorientable manifolds M_n.

Theorem 8.2. (i) *If M_n is nonorientable, there is a greatest value ω among critical values a with nonorientable critcal shells $f^a_{(\mu,\nu)}$.*

(ii) *The value ω is the least of the critical values a such that $M^+_{n,a}$ is orientable.*

(iii) *For this value ω, $f_{(\omega,\nu)}$ is orientable and $f_{(\mu,\omega)}$ nonorientable.*

(iv) *Index $\omega = n - 1$.*

Proof. Statement (i) is a consequence of Theorem 8.1. It follows from Lemma 8.3 that $M^+_{n,\omega}$ is orientable. Statement (ii) follows from (i) and Lemma 8.3. Since $f_{(\omega,\nu)}$ is orientable and $f^\omega_{(\mu,\nu)}$ orientation inverting, $f_{(\mu,\omega)}$ is nonorientable, so that (iii) is true. That index $\omega = n - 1$ follows from Theorem 7.2.

Theorems dual to Theorems 8.1 and 8.2 are obtained by reasoning with $-f$ as with f.

Dual of Theorem 8.1. *If $f^\alpha_{(\mu,\nu)}$ is orientable for each critical value α of index 1 which is less than a critical value a, then $M^-_{n,a}$ is orientable.*

Dual of Theorem 8.2. (i) *If M_n is nonorientable there is a least value ω' among critical values with nonorientable critical shells.*

(ii) *The value ω' is the greatest of the critical values a such that $M^-_{n,a}$ is orientable.*

(iii) *For this value ω', $f_{(\mu,\omega')}$ is orientable and $f_{(\omega',\nu)}$ nonorientable.*

(iv) *Index $\omega' = 1$.*

The theorems of this section together with Theorem 7.2 imply the following theorem. Recall that $n > 2$ and that f is biordered.

Theorem 8.3. *If M_n is nonorientable, then there exist just one inverting critical shell $f^\omega_{(\mu,\nu)}$ such that index $\omega = n - 1$ and just one inverting critical shell $f^{\omega'}_{(\mu,\nu)}$ such that index $\omega' = 1$. Each critical shell $f^a_{(\mu,\nu)}$ for which $\omega' < a < \omega$ has a nonorientable upper and lower shell. Of the three open submanifolds*

$$f_{(m,\omega')},\ f_{(\omega',\omega)},\ f_{(\omega,M)}$$

of M_n the first and third are geometrically orientable and the second is geometrically nonorientable.

9. The equivalence of geometric and homological orientability

In this section we shall prove that if $n > 2$ and M_n is nonorientable, then $R_n(|M_n|, \mathbf{Q}) = 0$. It can be shown by elementary methods without any use of triangulation that when a C^∞-manifold M_2 is compact, connected and nonorientable, then $R_2(|M_2|, \mathbf{Q}) = 0$. Proceeding inductively we shall make the following hypothesis.

Inductive hypothesis. We shall assume that if $n > 2$ then for each integer r such that $1 < r < n$ an admissible differentiable nonorientable manifold M_r is such that

$$R_r(|M_r|, \mathbf{Q}) = 0 \ . \tag{9.1}$$

In Lemmas 9.1, 9.2, 9.3 *we suppose that M_n is nonorientable and that* $n > 2$. The following lemma is essential.

Lemma 9.1. *Let $f^\omega_{(\mu,\nu)}$ be the orientation inverting critical shell of M_n with index $\omega = n - 1$. If one sets*

$$g = f \,|\, f^\omega_{(\mu,\nu)} \ , \tag{9.2}$$

then p_ω is a critical point of g of linking type over $\mathbf{Q}$, relative to g, and hence relative to f.

We shall prove Lemma 9.1 by verifying the following statements:

I. If $\omega < e < \nu$, then f^e is orientable.
II. If $\mu < c < \omega$, then f^c is nonorientable.
III. $R_{n-1}(f^c, \mathbf{Q}) = 0$.
IV. $R_{n-1}(\dot{g}_\omega, \mathbf{Q}) = 0$, $\quad (\dot{g}_\omega = g_\omega - p_\omega)$.
V. $R_{n-1}(g_\omega, \mathbf{Q}) = 1$.

Proof of I. According to Theorem 7.2, $f_{(\omega,\nu)}$ is orientable. Were f^e a nonorientable manifold M_{n-1}, it would follow, as in the proof of Theorem 2.1 (i), that $f_{(\omega,\nu)}$ would be nonorientable contrary to fact. Hence I is true.

Proof of II. According to Theorem 7.2, $f_{(\mu,\omega)}$ is nonorientable. Suppose II false, that is, suppose that f^c is orientable. By Def. 1.1 of geometric orientability there exists an orienting covering Γ of f^c in $\mathscr{D}_{\varepsilon'} f^c$, where $\varepsilon' < \varepsilon$ can be taken as an arbitrarily small positive constant. If ε' is sufficiently small, a set of ff-presentations based on presentations in $\mathscr{D}_{\varepsilon'} f^c$ (cf. (2.4)) will give an orienting covering Γ' of $f_{(\mu,\omega)}$ in $\mathscr{D}_\varepsilon f_{(\mu,\omega)}$. (See conditions on e of (7.2).) From this contradiction to the nonorientability of $f_{(\mu,\omega)}$ we infer that f^c is nonorientable.

Proof of III. By our inductive hypothesis, III is true since f^c is nonorientable.

Proof of IV. There exists an f-deformation on M_n retracting $\dot{g}_\omega$ onto f^c. See [6, § 23]. Since $R_{n-1}(f^c, \mathbf{Q}) = 0$, IV is true.

Proof of V. We shall again apply Theorem 5.2, replacing g therein by

$$\gamma = -f \,|\, f^\omega_{(\mu,\nu)} \ . \tag{9.3}$$

The critical value ω of f yields a critical value $\alpha = -\omega$ of γ. The critical point $z = p_\omega$ of f is a critical point of γ of index 1. We set $\dot{\gamma}_\alpha = \gamma_\alpha - z$.

Since f^e is orientable by I, Theorem 6.2 implies that $R_{n-1}(f^e, \mathbf{Q}) = 1$. However, there exists an f-deformation retracting $\dot{\gamma}_\alpha$ onto f^e, implying that

$$R_{n-1}(\dot{\gamma}_\alpha, \mathbf{Q}) = 1 \ . \tag{9.4}$$

Since $n - 1 > 1$, it follows from Theorem 5.2 that

$$R_{n-1}(\gamma_\alpha, \mathbf{Q}) = 1 \ . \tag{9.5}$$

There are an f-deformation retracting $f^\omega_{(\alpha,\nu)}$ onto γ_α and an f-deformation retracting $f^\omega_{(\omega,\nu)}$ onto g_ω so that V is true.

The application of Theorem 5.2 to g shows that when IV and V are true, p_ω is of linking type relative to g. This completes the proof of Lemma 9.1.

Lemma 9.1 is supplemented by Lemma 9.2.

Lemma 9.2. *Under the hypotheses of Lemma* 9.1 *let* λ^{n-1} *be a rational linking cycle of* p_ω *relative to* g. *Then the following is true*:

(i) *There exists a rational cycle* y^{n-1}_ω *on* g^ω *such that*

$$\lambda^{n-1} \sim y^{n-1}_\omega \qquad (\text{over } \mathbf{Q} \text{ on } g_\omega) \ . \tag{9.6}$$

(ii) *The cycle* y^{n-1}_ω *is an* $(n-1)$*st RHP of* g^ω, g_ω *and* $f^\omega_{(\mu,\nu)}$.

Proof of (i). Because g_ω admits an f-deformation D retracting g_ω onto g^ω, there exists an $(n-1)$-cycle y^{n-1}_ω on g^ω such that (9.6) holds.

Proof of (ii). By V of the proof of Lemma 9.1, $R_{n-1}(g_\omega, \mathbf{Q}) = 1$. It follows from Theorem 5.2 (ii) that λ^{n-1} is an $(n-1)$st *RHP* of g_ω. Since (9.6) holds, y^{n-1}_ω is similarly an $(n-1)$st *RHP* of g_ω and hence of g^ω (since D exists). Since there exists an f-deformation retracting $f^\omega_{(\mu,\nu)}$ onto g^ω, y^{n-1}_ω is also an $(n-1)$st *RHP* of $f^\omega_{(\mu,\nu)}$.

The following lemma is essential in proving, without triangulation of M_n, that geometrical and homological orientability of M_n are equivalent.

Lemma 9.3. *Continuing Lemmas* 9.1 *and* 9.2 *let* e *be any ordinary value of* f *such that* $\omega < e < M$. *Then the following is true*:

(i) *If* y^{n-1}_ω *is an* $(n-1)$*st RHP of* f^ω, *then*

$$y^{n-1}_\omega \not\sim 0 \qquad (\text{over } \mathbf{Q} \text{ on } \dot{f}_M) \ . \tag{9.7}$$

(ii) *If* y^{n-1}_e *is a suitably chosen* $(n-1)$st *RHP of* f^e, *then*

$$(9.8) \qquad y_e^{n-1} \sim y_\omega^{n-1} \qquad (\text{over } \mathbf{Q} \text{ on } f_{[\omega,e]}) .$$

Proof of (i). The cycle λ^{n-1} of Lemma 9.2 is a linking cycle of p_ω relative to g and hence relative to f. It follows from [6, Theorem 29.3 (ii)] that λ^{n-1} is in an $(n-1)$st *RHP* of f_ω. Hence y_ω^{n-1} of (9.6) is in an $(n-1)$st *RHP* of f_ω. Since each of the critical values α of f between ω and M (if any exist) is of index $n-1$, successive application of Lemma 5.2 and [6, Theorem 29.3] to the successive sets $\dot{f}_\alpha$ shows that y_ω^{n-1} is in an $(n-1)$st *RHP* of each set $\dot{f}_\alpha$ and f_α. A final application of Lemma 5.2 to $\dot{f}_M$ shows that y_ω^{n-1} is in an $(n-1)$st *RHP* of $\dot{f}_M$. Hence (9.7) holds.

Proof of (ii). Given $f^\omega_{(\mu,\nu)}$ let $c < e$ be an ordinary value of f such that $\omega < c < \nu$. An $(n-1)$st *RHP* of f^c is an $(n-1)$st *RHP* of $f_{(\omega,\nu)}$, $f_{[\omega,\nu)}$ and $f^\omega_{(\mu,\nu)}$. Hence, by (ii) of Lemma 9.2, if y_c^{n-1} is properly chosen, then

$$(9.9) \qquad y_\omega^{n-1} \sim y_c^{n-1} \qquad (\text{over } \mathbf{Q} \text{ on}^{18}\ f_{[\omega,c]}) .$$

According to Theorem 8.2, $M^+_{n,\omega}$ is geometrically orientable. It follows from Theorem 6.3 that if y_e^{n-1} is a suitably chosen $(n-1)$st *RHP* of f^e, then

$$(9.10) \qquad y_c^{n-1} \sim y_e^{n-1} \qquad (\text{over } \mathbf{Q} \text{ on } f_{[c,e]}) .$$

The homology (9.8) follows from (9.9) and (9.10). This completes the proof of Lemma 9.3.

Theorem 6.2 and the following theorem complete the proof that geometric and homological orientability are equivalent.

Theorem 9.1. *If M_n is nonorientable, then*

$$(9.11) \qquad R_n(|M_n|, \mathbf{Q}) = 0 .$$

In § 6 we have seen that if β is such that $M - \beta$ is positive and sufficiently small, then f^β is the carrier of the algebraic boundary y_β^{n-1} of a "simply carried" n-cap associated with p_M. So defined y_β^{n-1} is an $(n-1)$st *RHP* of f^β. According to [6, Theorem 29.3], (9.11) holds if and only if

$$(9.12) \qquad y_\beta^{n-1} \not\sim 0 \qquad (\text{on } \dot{f}_M) .$$

The above value of β is admissible as e in Lemma 9.3, so that it follows from (9.7) and (9.8) that (9.11) is true.

Since the n-th Betti number of M_n equals its n-th connectivity over $\mathbf{Q}$, Theorems 9.1 and 6.2 have the following corollary.

Corollary 9.1. *The manifold M_n is geometrically orientable if and only if its n-th Betti number is* 1.

[18] The homology (9.9) holds on $f_{(\mu,\nu)}$ by virtue of Lemma 9.2, and hence on $f_{[\omega,c]}$, since there is an f-deformation retracting $f_{(\mu,\nu)}$ onto $f_{[\omega,c]}$.

10. The Table Γ of § 7

When M_n is nonorientable, ω and ω' are respectively the inverting critical values of f of indices $n-1$ and 1. It will be useful to obtain further information concerning the pair of monoverlapping presentations

$$'F^{n-1}_{+}, \quad ''F^{n-1}_{+} \qquad (\text{in } \Gamma_{(\omega,\nu)}) \tag{10.1}$$

and the pair

$$'F^{1}_{-}, \quad ''F^{1}_{-} \qquad (\text{in } \Gamma_{(\mu,\omega')}) \tag{10.2}$$

as introduced in Table Γ. The relations

$$\mathbf{F}^{n-1} \overset{+}{\text{Com}}\, 'F^{n-1}_{+}; \ \mathbf{F}^{n-1} \overset{+}{\text{Com}}\, ''F^{n-1}_{+} \qquad (\text{in } \mathscr{D}_{\varepsilon} f^{\omega}_{(\mu,\nu)}) \tag{10.3}$$

are a consequence of the fact that the presentations (10.1), as defined in (7.20), are restrictions of the presentations $\mathbf{F}^{n-1}$ in $\Gamma^{\omega}_{(\mu,\nu)}$, as defined in (7.5). The relations

$$\mathbf{F}^{1} \overset{+}{\text{Com}}\, 'F^{1}_{-}, \qquad \mathbf{F}^{1} \overset{+}{\text{Com}}\, ''F^{1}_{-} \qquad (\text{in } \mathscr{D}_{\varepsilon} f^{\omega'}_{(\mu,\nu)}) \tag{10.4}$$

are valid for a similar reason. We shall make use of the coverings of $f^{\omega}_{(\mu,\nu)}$ and $f^{\omega'}_{(\mu,\nu)}$, as given by the presentations (7.9) and their inverts, when $a = \omega$ and ω' respectively.

Recall that the submanifolds $M^{+}_{n,\omega}$ and $M^{-}_{n,\omega'}$ of M_n are orientable. We shall prove the following theorem. We suppose $n > 2$.

Theorem 10.1. *When M_n is non-orientable and ω is the inverting critical value of index $n-1$, there is no orienting covering of $M^{+}_{n,\omega}$ which contains both of the presentations* (10.1).

To prove Theorem 10.1 it is sufficient to prove

(i) *There exists an admissible sequence in $\Gamma^{\omega}_{(\omega,\nu)}$ of the form*

$$'F^{n-1}_{+} * ''\hat{F}^{n-1}_{+} . \tag{10.5}$$

Proof of (i). Since the critical shell $f^{\omega}_{(\mu,\nu)}$ is nonorientable, there exists in the covering $\Gamma^{\omega}_{(\mu,\nu)}$ of $f^{\omega}_{(\mu,\nu)}$ an inverting sequence

$$\mathbf{F}^{n-1} * \hat{\mathbf{F}}^{n-1} \tag{10.6}$$

of the form

$$Q_0 : Q_1 : \cdots : Q_r : Q_{r+1} \qquad (Q_0 = \mathbf{F}^{n-1}, Q_{r+1} = \hat{\mathbf{F}}^{n-1}) \tag{10.7}$$

in which the non-terminal presentations are of the type of H_p or $\hat{H}_p$ in $\Gamma^{\omega}_{(\mu,\nu)}$. We can suppose that the ranges of Q_1 and Q_r are such that neither Q_1 nor Q_r overlaps both of the presentations (10.1).

Corresponding to the sequence (10.7) in $\Gamma^{\omega}_{(\mu,\nu)}$ an admissible sequence

$$Q_0^* : Q_1^* : \cdots : Q_r^* : Q_{r+1}^* \qquad (\text{in } \Gamma_{(\omega,\nu)}) \tag{10.8}$$

is uniquely determined in the covering $\Gamma_{(\omega,\nu)}$ of $f_{(\omega,\nu)}$ by the sequence (10.7) and by the condition that Q_j^* be a restriction in $\Gamma_{(\omega,\nu)}$ of Q_j in $\Gamma^{\omega}_{(\mu,\nu)}$ for j on the range $0, 1, \cdots, r+1$. More specifically, Q_k^* will be the unique restriction in $\Gamma_{(\omega,\nu)}$ of Q_k in $\Gamma^{\omega}_{(\mu,\nu)}$ for k on the range $1, \cdots, r$. Then Q_0^* must be that one of the two restrictions (10.1) in $\Gamma_{(\omega,\nu)}$ of $\mathbf{F}^{n-1}$ in $\Gamma^{\omega}_{(\mu,\nu)}$ which overlaps Q_1^*, while Q_{r+1}^* must be that one of the two restrictions of $\hat{\mathbf{F}}^{n-1}$

$$'\hat{F}_+^{n-1}, \quad ''\hat{F}_+^{n-1} \qquad (\text{in } \Gamma_{(\omega,\nu)}) \tag{10.9}$$

which overlaps Q_r^*. The sequence (10.8) is admissible by virtue of Lemma 1.5.

With (10.8) thereby uniquely determined we shall conclude the proof of (i) and Therem 10.1 by verifying the following.

(ii) *The sequence* (10.8) *in* $\Gamma_{(\omega,\nu)}$ *is either a sequence of the form* (10.5) *or a sequence of the form*

$$''F_+^{n-1} * '\hat{F}_+^{n-1} \qquad (\text{in } \Gamma_{(\omega,\nu)}) \ . \tag{10.10}$$

Note that the existence of a sequence of the form (10.10) implies the existence of a sequence in $\Gamma_{(\omega,\nu)}$ of the form (10.5).

Proof of (ii). As we have seen, Q_0^* in $\Gamma_{(\omega,\nu)}$ is $'F_+^{n-1}$ or $''F_+^{n-1}$ while Q_{r+1}^* is either $'\hat{F}_+^{n-1}$ or $''\hat{F}_+^{n-1}$. If $Q_0^* = 'F_+^{n-1}$ then $Q_{r+1}^* = ''\hat{F}_+^{n-1}$; otherwise, the sequence (10.8) would be an inverting sequence

$$'F_+^{n-1} * '\hat{F}_+^{n-1} \qquad (\text{in } \Gamma_{(\omega,\nu)}) \ ,$$

contrary to the fact that $f_{(\omega,\nu)}$ is orientable. Similarly, if $Q_0^* = ''F_+^{n-1}$, then $Q_{r+1}^* = '\hat{F}_+^{n-1}$. Thus (ii) is true, and Theorem 10.1 follows.

The following is the dual of Theorem 10.1.

Theorem 10.2. *When M_n is nonorientable and ω' is the inverting critical value of index* 1, *there is no orienting covering of $M^-_{n,\omega'}$ which contains both of the presentations* (10.2).

We supplement Theorems 10.1 and 10.2 by the following two theorems.

Theorem 10.3. *When M_n is nonorientable and a is a critical value of f of index $n-1$ between ω and M, there is an orienting covering of $M^+_{n,\omega}$ which contains both of the presentations*

$$'F_+^{n-1}, \quad ''F_+^{n-1} \qquad (\text{in } \Gamma_{(a,\nu)}) \ . \tag{10.11}$$

Theorem 10.4. *When M_n is nonorientable and a is a critical value of f of index* 1 *between m and ω', there is an orienting covering of $M^-_{n,\omega'}$ which contains both of the presentations*

$$(10.12) \qquad {}'F^1_- , \quad {}''F^1_- \qquad (\text{in } \Gamma_{(\mu,a)}) .$$

Proof of Theorem 10.3. It is sufficient to show that the following statement is true.

(i). *Under the hypotheses of Theorem* 10.3 *there is no admissible sequence in* $\Gamma_{(a,\nu)}$ *of the form*

$$(10.13) \qquad {}'F^{n-1}_+ * {}''\hat{F}^{n-1}_+$$

in which the non-terminal presentations are of the type H^+_p *or* $\hat{H}^+_p$ *in* $\Gamma_{(a,\nu)}$.

As in the proof of Theorem 10.1, preceeding however from a sequence in $\Gamma_{(a,\nu)}$ to a sequence in $\Gamma^a_{(\mu,\nu)}$, one shows that the existence of a sequence in $\Gamma_{(a,\nu)}$ of form (10.13) implies the existence of a sequence $\Gamma^a_{(\mu,\nu)}$ of the form

$$\mathbf{F}^{n-1} * \hat{\mathbf{F}}^{n-1} ,$$

contrary to the fact that the critical shell $f^a_{(\mu,\nu)}$ is orientable when a is a critical value between ω and M.

The proof of the dual Theorem 10.4 is similar.

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