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REGULARITY THEOREMS FOR PARTIAL DIFFERENTIAL OPERATORS

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1. In this paper we introduce the notion of a regular space and a regular linear or non-linear map. This is done in such a way as to abstract the notion of a space of smooth sections of a vector bundle and a linear or non-linear partial differential operator with smooth coefficients. The abstraction depends upon the notion of being able to take covariant derivatives of the sections as well as of the operators. This creates a category of spaces and maps, which is closed under composition and also inversion. These regular spaces, while being Fréchet spaces in one sense, have enough extra structures so that we may retain a number of the important theorems of the corresponding theory for Banach spaces. We prove for example that the set of regular linear maps with a regular inverse is open. We also prove an inverse function theorem: if a regular non-linear map has a derivative at a point which has a regular inverse, then the non-linear map has a regular inverse in a neighborhood of the point. It is hoped that the reader will find this a useful framework for passing from results for Banach spaces to results for smooth sections.

2. A regular space is a Fréchet space whose topology is defined in the following way by a norm and a finite collection of linear operators. Let E be a real (or complex) vector space with norm $\| \;\|$, and let $\nabla_1, \cdots, \nabla_N$ be a finite collection of linear operators which map E into itself and have closed graphs in the norm topology. If $I = (i_1, \cdots, i_k)$ is a multi-index of length $|I| = k$ with $1 \leq i_1, \cdots, i_k \leq N$, we define the higher-order operator $\nabla_I \colon E \to E$ as the composition

$$\nabla_I = \nabla_{i_1} \circ \nabla_{i_2} \circ \cdots \circ \nabla_{i_k} .$$

For each integer r we define the higher-order norm

$$\|f\|_r = \sum_{|I| \leq r} \frac{1}{|I|!} \|\nabla_I f\| .$$

Note that $\|f\|_0 = \|f\|$. Let $\mathscr{T}_r$ be the topology on E induced by the norm $\| \;\|_r$. If the topology $\mathscr{T}_\infty = \bigcup_{r=0}^{\infty} \mathscr{T}_r$ is complete (and hence Fréchet) we say that E, or more precisely the triple

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$$\{E; \nabla_1, \cdots, \nabla_N; \| \, \|\} ,$$

is a *regular space*. As an example, if M is a compact manifold and A a vector bundle over M, then let $\Gamma_2^r(A)$ denote the regular space whose vector space E is the space of smooth sections of A with a norm measuring the L_2 norms of r derivatives, and with the operators $\nabla_1, \cdots, \nabla_N$ being covariant differentiation with respect to vector fields $z_1, \cdots, z_N$ having the property that they span the tangent space of M at each point. Note that $L_2^r(A)$ is the completion of $\Gamma_2^r(A)$ with respect to the norm of $\Gamma_2^r(A)$. In general it should be clear that at many points it would be possible to use norms other than the L_2 norms on our function spaces.

Let E and F be two regular spaces with the "same" operators $\nabla_1, \cdots, \nabla_N$. If $L: E \to F$ is linear, we define $\nabla_i L: E \to F$ by

$$(\nabla_i L)(x) = \nabla_i\{L(x)\} - L(\nabla_i x) ,$$

and define $\nabla_I L$ for all multi-indices I by a repeated application of this definition. We say that the map $L: E \to F$ is *regular* if the maps $\nabla_I L$ are norm-bounded for all I. We shall show in the next section that if L is a linear partial differential operator of degree r then (for any $k \geq 0$) L is a regular linear map of $\Gamma_2^{r+k}(A)$ into $\Gamma_2^k(B)$ in this sense. For this purpose we first prove some basic properties of regular linear maps.

If L is norm-bounded, then $\|L\|$ denotes the smallest number with $\|Lx\| \leq \|L\| \, \|x\|$. Consequently, the vector space $RL(E, F)$ of regular linear maps of E into F admits the operators $\nabla_1, \cdots, \nabla_N$ and a norm. We shall show that $RL(E, F)$ is itself a regular space.

Lemma 1.

$$\nabla_I\{L(x)\} = \sum_{J,K} \varepsilon_I^{JK} (\nabla_J L)(\nabla_K x) ,$$

$$\nabla_I\{L \circ M\} = \sum_{J,K} \varepsilon_I^{JK} (\nabla_J L) \circ (\nabla_K M) ,$$

where ε_I^{JK} is the number of different ways in which the multi-indices J and K can be combined to form I without disturbing their internal orders. (*Thus* $\varepsilon_I^{JK} = 0$ *unless* $|I| = |J| + |K|$.)

Proof. If $|I| = 1$, then $\nabla_i\{L(x)\} = (\nabla_i L)(x) + L(\nabla_i x)$ follows directly from the definition, while

$$\begin{aligned} \{\nabla_i(L \circ M)\}(x) &= \nabla_i\{L \circ M(x)\} - (L \circ M)(\nabla_i x) \\ &= (\nabla_i L)\{M(x)\} + L\{\nabla_i(M(x))\} - (L \circ M)(\nabla_i x) \\ &= \{(\nabla_i L) \circ M\}(x) + \{L \circ (\nabla_i M)\}(x) , \end{aligned}$$

so $\nabla_i(L \circ M) = (\nabla_i L) \circ M + L \circ (\nabla_i M)$. The cases $|I| > 1$ now follow by an induction which is as easy to believe as it is clumsy to write.

Lemma 2.

$$\|L(x)\|_r \leq \sum_{j+k=r} \|L\|_j \, \|x\|_k \ ,$$
$$\|L \circ M\|_r \leq \sum_{j+k=r} \|L\|_j \, \|M\|_k \ .$$

Proof. These estimates follow from the previous formulas (and explain the presence of the factor $1/|I|!$ in the definition of $\| \ \|_r$). Thus

$$\begin{aligned}
\|L(x)\|_r &= \sum_{|I|\leq r} \frac{1}{|I|!} \|\nabla_I\{L(x)\}\| \\
&= \sum_{|I|\leq r} \frac{1}{|I|!} \sum_{J,K} \varepsilon_I^{JK} \|\nabla_J L\| \, \|\nabla_K x\| \\
&\leq \sum_{j+k=r} \sum_{|J|\leq j} \sum_{|K|\leq k} \sum_{I} \frac{1}{|I|!} \varepsilon_I^{JK} \|\nabla_J L\| \, \|\nabla_K x\|
\end{aligned}$$

$$\left(\text{since } \sum_I \varepsilon_I^{JK} = \frac{|I|!}{|J|! \, |K|!}\right)$$

$$\begin{aligned}
&\leq \sum_{j+k=r} \left(\sum_{|J|\leq j} \frac{1}{|J|!} \|\nabla_J L\|\right) \left(\sum_{|K|\leq k} \frac{1}{|K|!} \|\nabla_K x\|\right) \\
&= \sum_{j+k=r} \|L\|_j \, \|x\|_k \ ,
\end{aligned}$$

and the other formula is proved in the same way. Note that the composition of two regular linear maps is a regular linear map by the second formula. Moreover, these two formulas show the very useful fact that the evalution maps

$$RL(E, F) \times E \to F \ , \qquad RL(E, F) \times RL(F, G) \to RL(E, G)$$

are continuous in the Fréchet topologies, a property which in general is lacking for Fréchet spaces.

Also we see that L has a unique continuous extension $\bar{L}_r \colon \bar{E}^r \to \bar{F}^r$.

We now prove that $RL(E, F)$ is itself a regular space. For let $\{L_n\}$ be any sequence in $RL(E, F)$ which is Cauchy in each norm $\| \ \|_r$. By Lemma 2, for each $x \in E$

$$\begin{aligned}
\|L_n(x) - L_m(x)\|_r &= \|(L_n - L_m)x\|_r \\
&\leq \sum_{j+k=r} \|L_n - L_m\|_j \, \|x\|_k \ .
\end{aligned}$$

Hence $\{L_n(x)\}$ is a Cauchy sequence in F for each $x \in E$. Let $L_n(x) \to L(x)$. This defines a linear map L of E into F. Moreover since $\nabla_I\{L_n(x)\} = \sum_{K,J} \varepsilon_I^{JK}(\nabla_J L_n)(\nabla_K x)$ it follows by induction on $|I|$ that $(\nabla_I L_n)(x) \to (\nabla_I L)(x)$ for each x. But since $\{\nabla_I L_n\}$ is Cauchy in the norm $\| \ \|_0$ it follows that $\nabla_I L_n \to \nabla_I L$

in the norm $\| \|_0$ for each $|I|$. Hence $L_n \to L$ in each norm $\| \|_r$. Thus $L_n \to L$ in the Fréchet topology of $RL(E, F)$, so $RL(E, F)$ is complete and hence a regular space.

We shall also define regular multi-linear maps. If $L\colon E_1 \times \cdots \times E_n \to F$ is multi-linear we define $\nabla_i L\colon E_1 \times \cdots \times E_n \to F$ by

$$(\nabla_i L)(x_i, \cdots, x_n) = \nabla_i\{L(x_1, \cdots, x_n)\} - L(\nabla_i x_1, x_2, \cdots, x_n) - \cdots \\ - L(x_1, \cdots, x_{n-1}, \nabla_i x_n) \ .$$

Then the space of regular multi-linear maps of $E_1 \times \cdots \times E_n$ into F is again a regular space, denoted $RL(E_1 \times \cdots \times E_n, F)$, or $RL^n(E, F)$ if $E_1 = \cdots = E_n = E$. The evaluation maps $\varepsilon\colon RL(E, F) \times E \to F$ and $\tilde{\varepsilon}\colon RL(E, F) \times RL(F, G) \to RL(E, G)$ are regular bilinear maps; in fact, $\nabla_i \varepsilon = 0$ and $\nabla_i \tilde{\varepsilon} = 0$, as follows directly from the definition.

3. Let A and B be vector bundles over M, and λ be a functor mapping the category V of finite-dimensional vector spaces and linear maps into itself. We say that λ is *smooth* if for any vector spaces E, E', F, F' in V the map

$$\lambda\colon L(E', E) \times L(E, F') \to L(\lambda(E, F), \lambda(E', F'))$$

is always smooth, where for convenience we assume λ to be contravariant in the first variable and covariant in the second. Then we can define a new vector bundle $\lambda(A, B)$ in the following way. If $p \in M$, then the fibre $\lambda(A, B)_p$ is just $\lambda(A_p, B_p)$, while if $\Phi\colon U \times E \to A$ and $\Psi\colon U \times F \to B$ are charts on A and B over an open set $U \subseteq M$, then $\lambda(\Phi^{-1}, \Psi)\colon U \times \lambda(E, F) \to \lambda(A, B)$ is a chart on $\lambda(A, B)$, where of course

$$\lambda(\Phi^{-1}, \Psi)_p = \lambda(\Phi_p^{-1}, \Psi_p) \ .$$

In order to do covariant differentiation we need to have connections on our bundles. If $\pi\colon A \to M$ is the projection map for the bundle A, and if $a \in A$, then $VTA_a = \text{Ker}\, T\pi_a\colon TA_a \to TM_{\pi a}$ is the subspace of vertical tangent vectors at a. A connection on the bundle A selects at each point $a \in A$ a complementary subspace HTA_a of "horizontal" tangent vectors. For each tangent vector $z \in TM_{\pi a}$ let $C(z, a)$ be the unique vector in HTA_a with $T\pi\{C(z, a)\} = z$. Clearly C is linear in z for each fixed a. If C is also linear in a for each fixed z, then the connection C is called an affine connection. If $\Phi\colon U \times E \to A$ is a chart on A, then $T\Phi\colon TU \times TE \to TA$ is a chart on TA and we can write $a = \Phi(u, e)$, and for $z \in TM_u$

$$C(z, a) = T\Phi(z, (e, \gamma(z)e)) \ ,$$

where $TE \approx E \times E$ and $\gamma\colon TU \to L(E, E)$. We call γ the local representative of the connection in the chart Φ. In case the reader is lost in all the linear algebra, let $x^1, \cdots, x^n$ be coordinates on U, so that $\partial/\partial x^1, \cdots, \partial/\partial x^n$ form a basis for TU at each point, and let $e_1, \cdots, e_n$ be a basis for E. Then we can write

$$\gamma\left(\frac{\partial}{\partial x^k}\right)(e_j) = \sum \Gamma^i_{jk} e_i ,$$

where the Γ^i_{jk} are smooth functions on U and are called the Christoffel symbols of the affine connection. It is an easy matter to show that for any connection and any point $p \in U$ we can choose the chart $\Phi: U \times E \to A$ so that $\gamma_p = 0$. Moreover, for any symmetric connection on the tangent bundle TM of M, we can choose around any point $p \in M$ a chart $\varphi: W \subseteq R^n \to U \subseteq M$ such that if $\Phi: U \times R^n \to TM$ is the natural chart $\Phi(\varphi w, v) = T\varphi(w, v)$ then $\gamma_p = 0$ in the natural chart Φ; this is just the familiar theorem on the existence of geodesic coordinates. A symmetric connection on TM is one with $C(z, w) = C(w, z)$. We say that the chart Φ is flat at p if $\gamma_p = 0$. This means just that $HTA_a = T\Phi(TU_{\pi a} \times \{0\})$ and $VTA_a = T\Phi(\{0\} \times E)$.

Now it is not hard to prove that if A and B are bundles with connections and if λ is a smooth functor of V into itself, then there exists a unique connection on $\lambda(A, B)$ with the property that if the charts Φ and Ψ are flat at p then so is the chart $\lambda(\Phi^{-1}, \Psi)$. This can be expressed more generally by the following formula; if $\gamma: TU \to L(E, E)$ and $\delta: TU \to L(F, F)$ are the representatives of the connections on A and B in the charts $\Phi: U \times E \to A$ and $\Psi: U \times F \to B$, then the representative of the connection on $\lambda(A, B)$ in the chart $\lambda(\Phi^{-1}, \Psi)$ is given by

$$\varphi: TU \to L(\lambda(E, F), \lambda(E, F)) ,$$
$$\varphi(z) = D\lambda(id, id)(-\gamma(z), \delta(z)) ,$$

where $D\lambda(id, id)$ is the derivative of

$$\lambda: L(E, E) \times L(F, F) \to L(\lambda(E, F), \lambda(E, F))$$

evaluated at the identity maps of E and F.

Suppose now that we take λ to be L. Let A and B be bundles over M with connections, and let ∇_z represent covariant differentiation with respect to the vector field z. If l is a section of $L(A, B)$ and x is a section of A, then $l(x)$ is a section of B defined by $l(x)_p = l_p(x_p)$.

Lemma 3. *There exists a connection on $L(A, B)$ such that*

$$\nabla_z\{l(x)\} = (\nabla_z l)(x) + l(\nabla_z x) .$$

Proof. If the chart Φ is flat at p and if x also denotes the representative of the section x in the chart Φ, then the covariant derivative is given at p by

$$\nabla_z x = Dx(z) .$$

Choose charts Φ and Ψ on A and B which are flat at p, and let $L(A, B)$ have the unique connection such that $\lambda(\Phi^{-1}, \Psi)$ is flat at p. Then by the product rule at p

$$\begin{aligned} \nabla_z\{l(x)\} &= D[l(x)](z) = Dl(z)(x) + l[D(x)(z)] \\ &= (\nabla_z l)(x) + l(\nabla_z x) . \end{aligned}$$

Now let $\Gamma_2^k(l)$ denote the induced map of $\Gamma_2^k(A) \to \Gamma_2^k(B)$ defined by

$$\Gamma_2^k(l)(x) = l(x) \quad \text{for} \quad x \in \Gamma_2^k(A) .$$

Remembering that ∇_i is covariant differentiation with respect to z_i we see that

$$\begin{aligned} \Gamma_2^k(\nabla_i l)(x) &= (\nabla_i l)(x) = \nabla_i\{l(x)\} - l(\nabla_i x) \\ &= \nabla_i\{\Gamma_2^k(l)(x)\} - \Gamma_2^k(l)(\nabla_i x) \\ &= [\nabla_i \Gamma_2^k(l)](x) . \end{aligned}$$

Hence $\Gamma_2^k(\nabla_i l) = \nabla_i \Gamma_2^k(l)$. In fact, this is what motivated our original definition for the covariant derivative of a linear map. If l is a smooth section, then so is $\nabla_I l$ for every I. Consequently $\nabla_I \Gamma_2^k(l) = \Gamma_2^k(\nabla_I l)$ is always a norm-bounded linear map, and the map $\Gamma_2^k(l)$ is regular.

Next we claim that the r-jet extension map $j^r : \Gamma_2^{r+k}(A) \to \Gamma_2^r(j^r A)$ is regular To see this we must recall the definition of the jet bundle. If we choose connections on A and TM, then

$$j^r A \approx P^r(TM, A) ,$$

where $P^r(E, F)$ is the functor of polynomial maps of E into F of degree $\leq r$, and the r-jet extension is given by

$$j^r x(z) = x + \nabla x(z) + \cdots + \frac{1}{r!} \nabla^r x(z, \cdots, z) .$$

In particular, if we choose flat coordinates on A and geodesic coordinates on M, then the local representative of $j^r x$ is just the r^{th} order Taylor polynomial for the local representative of x. We can write (in local coordinates)

$$j^r(x)(z) = \frac{1}{r!} D^r x(z, \cdots, z) + \cdots ,$$

where the dots denote derivatives of x of degree $< r$. Consequently

$$\begin{aligned}(\nabla_i j^r)(x)(z) &= [\nabla_i\{j^r(x)\} - j^r(\nabla_i x)](z)\\ &= [D\{j^r(x)\}(z_i) - j^r(Dx(z_i))](z)\\ &= \frac{1}{r!}D^{r+1}x(z, \cdots, z, z_i) - \frac{1}{r!}D^{r+1}(x)(z_i, z, \cdots, z) + \cdots ,\end{aligned}$$

where the dots denote terms involving derivatives of f of degree $\leq r$. Now ordinary derivatives are symmetric, so the two terms we have written cancel. Moreover any derivative of f of degree $\leq r$ can be recovered from $j^r f$. Hence there exists a smooth section l_i of $L(j^r A, j^r A)$ with

$$\nabla_i j^r = l_i(j^r) .$$

Hence

$$\begin{aligned}\nabla_{ki} j^r &= \nabla_k[l_i(j^r)] = (\nabla_k l_i)(j^r) + l_i(\nabla_k j^r)\\ &= (\nabla_k l_i + l_i \circ l_k)(j^r) = l_{ki}(j^r) ,\end{aligned}$$

and in general we can show by induction on $|I|$ that there always exists a smooth section l_I of $L(j^r A, j^r A)$ with $\nabla_I j^r = l_I(j^r)$. Consequently $j^r\colon \Gamma_2^{r+k}(A) \to \Gamma_2^k(A)$ is regular. Now the composition of two regular linear maps is regular. Therefore any linear partial differential operator $L = \Gamma_2^k(l) \circ j^r$ is a regular linear map of $\Gamma_2^{r+k}(A) \to \Gamma_2^k(B)$, if l is a smooth section of $L(j^r A, B)$.

4. Now we shall prove some theorems about the inverses of regular linear maps. Let E and F be regular spaces with the "same" operators and let $L\colon E \to F$ be a regular linear map.

Theorem 1. *If L is invertible, and L^{-1} is norm-bounded, then L^{-1} is regular.*

Proof. We will show that $\nabla_I L^{-1}$ is norm-bounded for all I by induction on $|I|$. This is assumed to hold if $I = \emptyset$. If $|I| \geq 1$,

$$0 = \nabla_I(L \circ L^{-1}) = L \circ \nabla_I L^{-1} + \sum_{|K| < |I|} \varepsilon_I^{JK}(\nabla_J L) \circ (\nabla_K L^{-1})$$

by Lemma 1. Thus

$$\nabla_I L^{-1} = -L^{-1} \circ \sum_{|K| < |I|} \varepsilon_I^{JK}(\nabla_J L) \circ (\nabla_K L^{-1}) .$$

Now $\nabla_K L^{-1}$ is norm-bounded for $|K| < |I|$ by the induction hypothesis. Since the composition of two norm-bounded maps is norm-bounded, the Theorem is true.

Theorem 2. *If A is a regular linear map of E into itself and $\|A\|_0 < 1$, then $I - A$ has a regular inverse.*

Proof. Consider the power series

$$I + A + A^2 + \cdots + A^n + \cdots .$$

By a multiple appplication of Lemma 2, if $n \geq r$, then

$$\begin{aligned} \|A^n\|_r &\leq \sum_{j_1+\cdots+j_n=r} \|A\|_{j_1} \cdots \|A\|_{j_n} \\ &\leq n^r \|A\|_r^r \|A\|_0^{n-r} , \end{aligned}$$

since at most r of the $j_1, \cdots, j_n$ can be non-zero and there are n^r terms altogether. Consequently

$$\limsup_{n\to\infty} \sqrt[n]{\|A^n\|_r} = \|A\|_0 < 1 .$$

Therefore the above infinite series converges by the root test in each norm $\| \|_r$. Since $RL(E, E)$ is complete, the series converges to a regular linear map in $RL(E, E)$, which we denote by B. Since the series converges absolutely in each norm $\| \|_r$, we can rearrange terms, so $B(I - A) = (I - A)B = I$. Hence $I - A$ has a regular inverse.

Corollary 1. *If L and A are regular, L has a regular inverse, and $\|A\|_0 < \|L^{-1}\|_0^{-1}$, then $L - A$ has a regular inverse, and hence the set of maps in $RL(E, F)$ with a regular inverse is norm-open.*

Proof. $(L - A)^{-1} = L^{-1}(I - AL^{-1})^{-1}$.

Corollary 2. *If $A \in L(\bar{E}^r, \bar{E}^r)$ and $\|A\|_0 < \|L^{-1}\|_0^{-1}$, then $\bar{L}_r - A$ has an inverse in $L(\bar{E}^r, \bar{E}^r)$.*

Proof. This follows from the convergence of the series in $\| \|_r$. We shall use this fact later.

We say that the regular map L has a *left* (*or right*) *quasi-inverse* M, if M is regular and for each r there is a constant C_r with

$$\|(ML - I)x\|_{r+1} \leq C_r \|x\|_r \text{ (or } \|(LM - I)x\|_{r+1} \leq C_r \|x\|_r) .$$

Theorem 3. *If L is regular and has a left* (*or right*) *quasi-inverse M, and A is regular with $\|A\|_0 < \|M\|_0^{-1}$, then $L - A$ has a left* (*or right*) *quasi-inverse given by*

$$N = M + MAM + MAMAM + \cdots .$$

Proof. For the n^{th} term, if $n \geq r + 1$, then

$$\|MA \cdots AM\|_r \leq (2n + 1)^r \|A\|_r^r \|M\|_r^r \|A\|_0^{n-r-1} \|M\|_0^{n-r}$$

as before, so the series converges absolutely in each norm $\| \|_r$ by the root test. Rearranging terms we see that

$$\begin{aligned} N(L - A) - I &= (M + MAM + \cdots)(L - A) - I \\ &= (I + MA + MAMA + \cdots)(ML - I) . \end{aligned}$$

The rest of the proof is obvious.

5. Let V be a real finite-dimensional vector space, and E and F be complex ones. The Fourier transform defines an isomorphism of the regular space $\Gamma_2^r(V, E)$ of maps of V into E, all of whose derivatives lie in L_2, and with operators $\partial/\partial x^j$, onto another regular space $\Gamma_2^{(r)}(V^*, E)$ of functions which are in L_2 with respect to any polynomial weight function, and with operators multiplication by ix^j. In the first case the norm is $\sum_{|I|\leq r}\int_V |\partial_I f(x)|^2 dx$ where $\partial_I = \frac{\partial}{\partial x^{i_1}\cdots\partial x^{i_k}}$, and in the second case the norm is

$$\sum_{|I|\leq r}\int_{V^*} |g(x)|^2 x^{2I} dx \ .$$

If Q is a partial differential operator from E to F of degree r with constant coefficients, then $Q \in L(P^r(V, E), F)$ and there is a unique $\hat{Q} \in P^r(V^*, L(E, F))$ with

$$\widehat{Q(f)}(v) = \hat{Q}(v)\hat{f}(v) \qquad \text{for all } v \in V^* \ .$$

We say that Q is *elliptic* if $\hat{Q}_r(v)$ is invertible for all $v \neq 0$, where $\hat{Q}_r$ is the r^{th} order part of $\hat{Q}$.

If Q is elliptic, then $\hat{Q}\colon \Gamma_2^{(r)}(V^*, E) \to \Gamma_2^{(0)}(V^*, F)$ clearly has a two-sided quasi-inverse given by $\phi(v)\hat{Q}(v)^{-1}$ where ϕ is a smooth function which is $\equiv 1$ for large v and vanishes for those v for which $\hat{Q}(v)$ is not invertible. This is a quasi-inverse, since it is a true inverse except for v in a bounded set, where all polynomial weight functions are comparable. By applying the inverse Fourier transform we see that $Q\colon \Gamma_2^r(V, E) \to \Gamma_2^0(V, F)$ has a two-sided quasi-inverse if Q is elliptic with constant coefficients. By Theorem 3 it follows that any linear partial differential operator whose coefficients are sufficiently close to constant elliptic ones in the supremum sense has a two-sided quasi-inverse as a map of $\Gamma_2^r(V, E) \to \Gamma_2^0(V, F)$.

Suppose M is a compact manifold and that Q is a section of the bundle $L(P^r(TM, A), B)$, so that Q defines a linear partial differential operator of degree r from A to B (remember $j^r A \approx P^r(TM, A)$). Then there is a natural transformation of functors which takes Q into a section $\hat{Q}$ of the bundle $P^r(TM^*, L(A, B))$. The highest order homogeneous part $\hat{Q}_r \in L^r(TM^*, L(A, B))$ is called the symbol of Q, and Q is elliptic if and only if $\hat{Q}_r(v)$ is invertible for all non-zero cotangent vectors $v \in TM^*$. If $p \in M$ and we choose charts in a neighborhood of p, then we can write $Q = Q_p - B_p$ where Q_p has constant coefficients equal to those of Q at p and B_p vanishes at p. If φ_p is chosen to be $\equiv 1$ on a neighborhood W_p^1 of p but to vanish outside a neighborhood W_p^2 of p which is sufficiently small, then regarding $Q_p - \varphi_p B_p$ as an operator on a whole vector space by setting it equal to Q_p outside the domain of the coordinate chart we can make $Q_p - \varphi_p B_p$ have a two-sided quasi-inverse N_p by the previous remark, provided we make W_p^2 sufficiently small. Choose a finite number of points p such that the $\{W_p^1\}$ cover M, and let $\{\lambda_p\}$ be a partition of unity with respect to this cover. Finally, let μ_p be chosen to be $\equiv 1$ on the support of λ_p and to have compact support in W_p^1.

Let $N = \sum_p \mu_p N_p \lambda_p$. Then

$$NQF - f = \sum_p \mu_p N_p \lambda_p Qf - f$$
$$= \sum_p \mu_p N_p \lambda_p (Q_p - \varphi_p B_p) f - f$$

(since $\lambda_p \varphi_p = \lambda_p$)

$$= \sum_p \mu_p N_p (Q_p - \varphi_p B_p) \lambda_p f + \sum_p \mu_p N_p [\lambda_p, Q_p - \varphi_p B_p] f - f$$
$$= \sum_p \mu_p [N_p (Q_p - \varphi_p B_p) - I] \lambda_p f + \sum_p \mu_p N_p [\lambda_p, Q_p - \varphi_p B_p] f$$

(since $\mu_p \lambda_p = \lambda_p$ and $\sum \lambda_p \equiv 1$).

But N_p is a two-sided quasi-inverse for $Q_p - \varphi_p B_p$, and $[\lambda_p, Q_p - \varphi_p B_p]$ is a differential operator of degree at most $r - 1$. Hence for each k we can find a constant C_k with

$$\|(NQ - I)f\|_{r+k+1} \leq C_k \|f\|_{r+k} \, .$$

Likewise,

$$QNg - g = \sum_p (Q_p - \varphi_p B_p) \mu_p N_p \lambda_p g - g$$

(since the operator $(1 - \varphi_p) B_p \mu_p$ vanishes everywhere)

$$= \sum_p [Q_p - \varphi_p B_p, \mu_p] N_p \lambda_p g + \sum_p \mu_p (Q_p - \varphi_p B_p) N_p \lambda_p g - g$$
$$= \sum_p [Q_p - \varphi_p B_p, \mu_p] N_p \lambda_p g + \sum_p \mu_p [(Q_p - \varphi_p B_p) N_p - I] \lambda_p g \, ,$$

and hence also

$$\|(QN - I)g\|_{k+1} \leq C_k \|g\|_k \, .$$

Let $\bar{Q}_k$ and $\bar{N}_k$ denote the unique continuous extensions of Q and N to $L_2^{r+k}(A)$ and $L_2^k(B)$ respectively. It now follows from the Rellich selection theorem that $\bar{N}_k \bar{Q}_k - I: L_2^{r+k}(A) \to L_2^{r+k}(A)$ and $\bar{Q}_k \bar{N}_k - I: L_2^k(B) \to L_2^k(B)$ are compact for every k. Hence $\bar{Q}_k: L_2^{r+k}(A) \to L_2^k(B)$ is Fredholm for every k, as in $\bar{N}_k: L_2^k(B) \to L_2^{r+k}(A)$. Moreover $\deg N_k Q_k = 0$ so $\deg N_k = -\deg Q_k$. Since $\dim \ker Q_k$ cannot increase with k while $\dim \operatorname{coker} Q_k$ cannot decrease with k, we have that $\deg Q_k$ is a non-increasing function of k. But the same applies to $\deg N_k$, so $\deg Q_k$ is constant. This means that $\dim \ker Q_k$ and $\dim \operatorname{coker} Q_k$ must be constant. Consequently $Q: \Gamma_2^{r+k}(A) \to \Gamma_2^k(B)$ is Fredholm. This proves the well-known result that every linear elliptic operator is Fredholm on the smooth functions. It also shows that if $f \in L_2^r(A)$ and Qf is smooth, then f is smooth.

6. We add a few remarks on duality. Let E be a regular space with operators $\nabla_1, \cdots, \nabla_N$, and let these operators act trivially on R, the real numbers. Then $E^* = RL(E, R)$ is a regular space. If $L\colon E \to F$ is regular, it is easy to show that the adjoint L^* defines a regular linear map of F^* into E^* with $\nabla_I L^* = (\nabla_I L)^*$.

Suppose that the norm on E comes from a regular bilinear inner product

$$B\colon E \times E \to R \ .$$

We say that E is a *regular Hilbert space* if $E = E^*$, i.e., if every $l \in E^*$ is given by $l(x) = B(x, y)$ for some $y \in E$.

Theorem. *$\Gamma_2^r(A)$ is a regular Hilbert space with the inner product*

$$B_r(f, g) = \sum_{|I| \leq r} \int_M \langle \nabla_I f, \nabla_I g \rangle dV \ ,$$

where $\langle \ , \ \rangle$ is a Riemannian metric on the bundle A, and dV is a smooth measure on M.

Proof. First we show that B_r is regular. The map $\Gamma_2^r(A) \times \Gamma_2^r(A) \to \Gamma_2^0(I)$ given by $(f, g) \to \sum_{|I| \leq r} \langle \nabla_I f, \nabla_I g \rangle$ is regular by the same reasoning we applied to linear partial differential operators. We must show that any map of the form $f \to \int_M f \cdot \varphi dV$ where φ is a smooth function is a regular map of $\Gamma_2^r(I)$ into R. We can always write locally

$$\int f \cdot \varphi dV = \int f(x) \varphi(x) dx \ ,$$

$$\nabla_z f = z^i \frac{\partial f}{\partial x^i} - \phi f \ ,$$

where ϕ is a function depending on the connection. If

$$L(f) = \int_M f \cdot \varphi dV \ ,$$

then

$$\begin{aligned}(\nabla_i L)(f) &= \nabla_i \{L(f)\} - L(\nabla_i f) = -L(\nabla_i f) \\ &= -\int \left(z^i \frac{\partial f}{\partial x^i} - \phi f \right) \varphi(x) dx \\ &= \int f(x) \left\{ \frac{\partial}{\partial x^i} [z^i \varphi(x)] + \varphi(x) \phi(x) \right\} dx\end{aligned}$$

$$= \int f(x)\eta(x)dx = \int_M f\cdot\eta dv \ ,$$

for some globally defined function η. Hence covariant differentiation of a map of the type of L always gives another map of the same type. Since such a map is norm-bounded, it is regular.

Finally, suppose $L\colon \Gamma_2^r(A) \to R$ is regular. Then L is norm-bounded, so $L(f) = B_r(f, g)$ for some $g \in L_2^r(A)$. Moreover, $B_r(f, g) = B_0(f, Qg)$ by Green's theorem, where Q will be an elliptic (self-adjoint) operator of degree $2r$. Since L is regular,

$$L(\nabla_I f) = (-1)^{|I|}(\nabla_{I*}L)(f) \ ,$$

where I^* is I backwards; this formula follows easily by induction on $L(\nabla_i f) = -(\nabla_i L)(f)$, and is true because the operators on R are trivial. Therefore $|L(\nabla_I f)| \leq C_I \|f\|_{L^r_2}$ for some constant C_I and all I. Hence

$$\left|\int_M \langle\nabla_I f, Qg\rangle dV\right| \leq C_I \|f\|_{L_2^r} \ .$$

It is a standard argument in distribution theory that if $h \in L_2^0(A)$, and

$$\left|\int_M \langle\nabla_I f, h\rangle\right| \leq C_I \|f\|_{L_2^0}$$

for all I, then h is smooth. Hence Qg is smooth, and g will be smooth since Q is elliptic. Thus $L(f) = B_r(f, g)$ for $g \in \Gamma_2^r(A)$.

Regular Hilbert spaces have the following beautiful property.

Theorem. *Let E and F be regular Hilbert spaces, and $L\colon E \to F$ be regular. If $\bar{L}_0\colon \bar{E}^0 \to \bar{F}^0$ is invertible, then so is L, and L^{-1} is regular.*

Proof. Form the adjoint $L^*\colon F^* \to E^*$. Since E and F are regular Hilbert spaces, $E = E^*$ and $F = F^*$. Hence L^*L maps E into itself and is regular. Moreover,

$$(L^*Lx, x) = (Lx, Lx) \geq \varepsilon(x, x)$$

for some $\varepsilon > 0$, since $\bar{L}_0$ is invertible. Therefore

$$\begin{aligned}&((I - kL^*L)x, (I - kL^*L)x)\\ &\quad = (x, x) - 2k(L^*Lx, x) + k^2(L^*Lx, L^*Lx)\\ &\quad \leq (1 - 2k\varepsilon + k^2 \|L\|^2) \|x\|^2 \ .\end{aligned}$$

Choose $k < 2\varepsilon/\|L\|^2$. Then $\|I - kL^*L\| < 1$. It follows from Theorem 2 that kL^*L has a regular inverse. Hence L^*L and likewise LL^* have regular inverses

A and B. Since $AL^*L = I$ and $LL^*B = I$, L is invertible and has a regular inverse $L^{-1} = AL^* = L^*B$.

Corollary. *If the linear partial differential operator Q is invertible as a map of Banach spaces $Q\colon L_2^{r+k}(A) \to L_2^k(B)$, then it is invertible as a map of Fréchet spaces $Q\colon C^\infty(A) \to C^\infty(B)$.*

7. We now turn our attention to non-linear partial differential operators. Let U be an open set in the vector bundle A, and $\alpha\colon U \subseteq A \to B$ be a smooth map which takes fibres into fibres, i.e., $\pi_B \circ \alpha = \pi_A$. If we choose connections on A and B, then for any point $a \in U$ we have direct sum decompositions $TA_a \approx HTA_a \oplus VTA_a$ and $TB_{\alpha(a)} \approx HTB_{\alpha(a)} \oplus VTB_{\alpha(a)}$. Consequently we can write $T\alpha_a$ as a matrix

$$T\alpha_a = \begin{pmatrix} I & 0 \\ \nabla\alpha_a & D\alpha_a \end{pmatrix}.$$

If $\pi_A a = p \in M$, then $HTA_a \approx TM_p$, $VTA_a \approx A_p$, $HTB_{\alpha(a)} \approx TM_p$, and $VTB_{\alpha(a)} \approx B_p$. Hence $T\alpha_a$ is determined by two linear maps $\nabla\alpha_a\colon TM_p \to B_p$ and $D\alpha_a\colon A_p \to B_p$. This defines new maps $\nabla\alpha\colon U \subseteq A \to L(TM, B)$ and $D\alpha\colon U \subseteq A \to L(A, B)$ which also take fibres into fibres. We can think of $\nabla\alpha$ and $D\alpha$ as being the horizontal and vertical derivatives of α.

The map α induces a map $L_2^k(\alpha)$ of an open set W in $L_2^k(A)$ into $L_2^k(B)$ for k greater than the dimension of M by the Sobolev inequalities. This map has a Fréchet derivative; in fact it is easy to see that $DL_2^k(\alpha) = L_2^k(D\alpha)$. If we regard $\Gamma_2^k(A)$ as a normed vector space, we can also write $D\Gamma_2^k(\alpha) = \Gamma_2^k(D\alpha)$ with the derivative being taken in the norm topologies. (Recall that $L_2^k(A)$ is the completion of $\Gamma_2^k(A)$, which as a space contains only smooth sections.)

Let x be a section of A whose range lies in U. Then

$$Tx_p = \begin{pmatrix} I \\ \nabla x_p \end{pmatrix}$$

in terms of the horizontal and vertical decomposition of $TA_{x(p)}$. Here ∇x is a section of $L(TM, A)$; if z is a vector field, then $\nabla_z x = \nabla x(z)$. Since

$$T(\alpha \circ x)_p = T\alpha_{x(p)} \circ Tx_p ,$$

we have

$$\begin{pmatrix} I \\ \nabla(\alpha \circ x)_p \end{pmatrix} = \begin{pmatrix} I & 0 \\ \nabla\alpha_{x(p)} & D\alpha_{x(p)} \end{pmatrix} \begin{pmatrix} I \\ \nabla x_p \end{pmatrix},$$

so $\nabla(\alpha \circ x)_p = \nabla\alpha_{x(p)} + D\alpha_{x(p)} \nabla x_p$. Hence, if z is a vector field, then

$$\nabla_z(\alpha \circ x) = \nabla_z\alpha \circ x + (D\alpha \circ x) \circ \nabla_z x .$$

Consequently, since $\alpha \circ x = \Gamma_2^k(\alpha)(x)$, we have

$$\nabla_z\{\Gamma_2^k(\alpha)(x)\} = \Gamma_2^k(\nabla_z\alpha)(x) + \Gamma_2^k(D\alpha)(x)(\nabla_z(x)) \ .$$

Now $\Gamma_2^k(D\alpha) = D\Gamma_2^k(\alpha)$. We would also like to write $\Gamma_2^k(\nabla_z\alpha) = \nabla_z\Gamma_2^k(\alpha)$. This suggests the following definition.

Let E and F be regular spaces with the "same" operators $\nabla_1, \cdots, \nabla_N$. Let U be a norm-open set in E, and let $P\colon U \subseteq E \to F$. We say that P is *norm-smooth*, if all the derivatives $D^nP\colon U \subseteq E \to L^n(E, F)$ exist in the norm-topologies and are norm-continuous. The map P extends in this case uniquely to a map $\bar{P}^r\colon \bar{U}^r \subseteq \bar{E}^r \to \bar{F}^r$ on the closure of U in $\| \ \|_r$, and this map $\bar{P}_r$ will be differentiable and $D^n(\overline{P^r}) = (\overline{D^nP})^r$. We define $\nabla_iP\colon U \subset E \to F$ by the formula

$$(\nabla_iP)(x) = \nabla_i\{P(x)\} - DP(x)(\nabla_ix) \ .$$

If ∇_iP is norm-smooth, we can define $\nabla_j\nabla_iP = \nabla_{ji}P$, and so on. If ∇_IP is defined and norm-smooth for every multi-index I, we say that P is *regular*.

We now prove a series of lemmas leading up to the assertions that $D^kP(x)$ is a regular k-multilinear map, and that $D^kP\colon U \subset E \to RL^k(E, F)$ is regular with $D_I(D^kP) = D^k(D_IP)$.

Lemma 4.

$$\nabla_i\{D^kP(x)\} = D^k(\nabla_iP)(x) + D^{k+1}P(x)(\nabla_ix) \ .$$

Proof. If $k = 0$, the lemma follows from the definition of ∇_iP. We proceed by induction on k.

$$\begin{aligned} &D^{k+1}P(x)(v_1, \cdots, v_k, y) \\ &\quad = \lim_{t\to 0} [D^kP(x + ty)(v_1, \cdots, v_k) - D^kP(x)(v_1, \cdots, v_k)]/t \ , \end{aligned}$$

where the convergence is in the norm topology. Since ∇_i has a closed graph in the norm topology, we must have

$$\begin{aligned} &\nabla_i\{D^{k+1}P(x)(v_1, \cdots, v_k, y)\} \\ &\quad = \lim_{t\to 0} \nabla_i[D^kP(x + ty)(v_1, \cdots, v_k) - D^kP(x)(v_1, \cdots, v_k)]/t \ , \end{aligned}$$

which by the induction hypothesis equals

$$\begin{aligned} &\lim_{t\to 0} [\nabla_i\{D^kP(x + ty)(v_1, \cdots, v_k)\} - \nabla_i\{D^kP(x)(v_1, \cdots, v_k)\}]/t \\ &\quad = \lim_{t\to 0} [\nabla_i\{D^kP(x + ty)\}(v_1, \cdots, v_k) \\ &\qquad\qquad + D^kP(x + ty)(\nabla_iv_1, v_2, \cdots, v_k) + \cdots \\ &\qquad\qquad + D^kP(x + ty)(v_1, \cdots, \nabla_iv_k) - \nabla_i\{D^kP(x)\}(v_1, \cdots, v_k) \\ &\qquad\qquad - D^kP(x)(\nabla_iv_1, v_2, \cdots, v_k) - \cdots \\ &\qquad\qquad - D^kP(x)(v_1, \cdots, \nabla_iv_k)]/t \end{aligned}$$

$$\begin{aligned}
&= \lim_{t\to 0} [D^k(\nabla_i P)(x + ty)(v_1, \cdots, v_k) \\
&\qquad + D^{k+1}P(x + ty)(v_1, \cdots, v_k, \nabla_i x + t\nabla_i y) \\
&\qquad - D^k(\nabla_i P)(x)(v_1, \cdots, v_k) - D^{k+1}P(x)(v_1, \cdots, v_k, \nabla_i x)]/t \\
&\quad + D^{k+1}P(x)(\nabla_i v_1, v_2, \cdots, v_k, y) + \cdots + D^{k+1}P(x)(v_1, \cdots, \nabla_i v_k, y) \\
&= \lim_{t\to 0} [\{D^k(\nabla_i P)(x + ty) - D^k(\nabla_i P)(x)\}(v_1, \cdots, v_k)]/t \\
&\quad + \lim_{t\to 0} [\{D^{k+1}P(x + ty) - D^{k+1}P(x)\}(v_1, \cdots, v_k, \nabla_i x)]/t \\
&\quad + D^{k+1}P(x)(\nabla_i v_1, v_2, \cdots, v_k, y) + \cdots + D^{k+1}P(x)(v_1, \cdots, \nabla_i v_k, y) \\
&\quad + \lim_{t\to 0} D^{k+1}P(x + ty)(v_1, \cdots, v_k, \nabla_i y) \\
&= D^{k+1}(\nabla_i P)(x)(v_1, \cdots, v_k, y) + D^{k+2}P(x)(v_1, \cdots, v_k, \nabla_i x, y) \\
&\quad + D^{k+1}P(x)(\nabla_i v_1, v_2, \cdots, v_k, y) + \cdots + D^{k+1}P(x)(v_1, \cdots, v_k, \nabla_i y).
\end{aligned}$$

Hence recalling the formula for applying ∇_i to multi-linear maps we have

$$\nabla_i\{D^{k+1}P(x)\} = D^{k+1}(\nabla_i P)(x) + D^{k+2}P(x)(\nabla_i x) ,$$

which is the formula for $k + 1$. This completes the induction.

Lemma 5.

$$\nabla_I\{D^kP(x)\} = \sum \varepsilon_I^{JK_1\cdots K_n} D^{k+n}(\nabla_J P)(x)(\nabla_{K_1}x, \cdots, \nabla_{K_n}x) ,$$

where $\varepsilon_I^{JK_1\cdots K_n}$ is the number of ways of combing the multi-indices $J, K_1, \cdots, K_n$ to form I without changing their internal orders, and such that the last number in K_p preceeds the last number in K_{p+1}. Here we must always have $|K_p| \geq 1$, but J can be empty.

Proof. We merely use the above formula and induction on $|I|$. Indeed, if $|I| = 1$ this is Lemma 4, while

$$\begin{aligned}
\nabla_{iI}\{D^kP(x)\} &= \nabla_i\{\nabla_I(D^kP(x)\} \\
&= \nabla_i\{\sum \varepsilon_I^{JK_1\cdots K_n} D^{n+k}(\nabla_J P)(x)(\nabla_{K_1}x, \cdots, \nabla_{K_n}x)\} \\
&= \sum \varepsilon_I^{JK_1\cdots K_n}\{D^{n+k}(\nabla_{iJ} P)(x)(\nabla_{K_1}x, \cdots, \nabla_{K_n}x) \\
&\qquad + D^{n+k+1}(\nabla_J P)(x)(\nabla_i x, \nabla_{K_1}x, \cdots, \nabla_{K_n}x) \\
&\qquad + D^{n+k}(\nabla_J P)(x)(\nabla_{iK_1}x, \nabla_{K_2}x, \cdots, \nabla_{K_n}x) \\
&\qquad + \cdots + D^{n+k}(\nabla_J P)(x)(\nabla_{K_1}x, \cdots, \nabla_{iK_n}x)\} .
\end{aligned}$$

Consequently $\nabla_{iI}\{D^kP(x)\}$ is composed of the right sort of terms, and we merely need to count the multiplicities. As we apply the numbers in I, each time we can do one of three things:

(a) add i to the front of J,
(b) add i to the front of some K_p,
(c) form a new index with i.

Parts (a) and (b) require that we can combine $J, K_1, \cdots, K_n$ to form I without changing their internal order (since i is always added to the front of J or K_p from the front of I), while part (c) requires that the first number in K_p should preceed the first number in K_{p+1}. This completes the proof.

As a corollary we have the following very important formula.

Corollary 2.

$$\nabla_I\{P(x)\} = DP(x)(\nabla_I x) + \cdots ,$$

where the dots denote terms of the form

$$D^n(\nabla_J P)(x)(\nabla_{K_1}x, \cdots, \nabla_{K_n}x)$$

with $|J| + |K_1| + \cdots + |K_n| = |I|$ *and* $|K_p| < |I|$ *for all* K_p, *i.e., the dots involve only derivatives of* x *of degree* $< |I|$.

It is clear from Lemma 5 that $\nabla_I\{D^kP(x)\}$ is a norm-bounded multi-linear map for all I. Hence $D^kP(x)$ is a regular multi-linear map and $D^kP\colon U \subseteq E \to RL^k(E, F)$. It therefore makes sense to ask if D^kP is regular. Since D^kP is smooth as a map into the larger space $L^k(E, F)$, it is smooth as a map into $RL^k(E, F) \subseteq L^k(E, F)$ in the norm topology. From the formula of Lemma 4,

$$\{\nabla_i(D^kP)\}(x) = \nabla_i\{D^kP(x)\} - D^{k+1}P(x)(\nabla_i x) = D^k(\nabla_i P)(x) .$$

Thus $\nabla_i(D^kP) = D^k(\nabla_i P)$. It follows that $\nabla_I(D^kP) = D^k(\nabla_I P)$ for all I. Thus D^kP is regular for all R.

8. Let $\bar{E}^r$ denote the completion of E in the norm $\| \|_r$. If U is a $\| \|_0$-open set in E, we define the completion $\bar{U}^r$ of U in the norm $\| \|_r$ to be the unique open set in $\bar{E}^r$ with $\bar{U}^r \cap E = U$. If $P\colon U \subseteq E \to F$ is regular, then by the formula of Lemma 5 the maps D^kP all have continuous extensions

$$\overline{D^kP^r}\colon \bar{U}^r \subseteq \bar{E}^r \to L^k(\bar{E}^r, \bar{F}^r) .$$

Lemma 6. $\bar{P}^r$ *is smooth and* $D^k\bar{P}^r = \overline{D^kP^r}$.

Proof. Assume we have shown that $D^k\bar{P}^r$ exists and equals $\overline{D^kP^r}$ for some $k \geq 0$. For $x \in U$ and $y \in E$ sufficiently small we will have

$$D^kP(x + y) - D^kP(x) = \int_0^1 D^{k+1}P(x + ty)(y)dt .$$

By continuity, for all $x \in \bar{U}^r$ and $y \in \bar{E}^r$ sufficiently small we will have

$$\overline{D^kP^r}(x + y) - \overline{D^kP^r}(x) = \int_0^1 \overline{D^{k+1}P^r}(x + ty)(y)dt ,$$

and hence

$$\overline{D^kP^r}(x+y) - \overline{D^kP^r}(x) - \overline{D^{k+1}P^r}(x)(y)$$
$$= \int_0^1 [\overline{D^{k+1}P^r}(x+ty) - \overline{D^{k+1}P^r}(x)](y)dt \; .$$

Now if $\|y\|_r$ is sufficiently small, we can make

$$\|\overline{D^{k+1}P^r}(x+ty) - \overline{D^{k+1}P^r}(x)\|_r < \varepsilon$$

for any $\varepsilon > 0$. Then we will have

$$\|\overline{D^kP^r}(x+y) - \overline{D^kP^r}(x) - \overline{D^{k+1}P^r}(x)(y)\|_r < \varepsilon \|y\|_r \; .$$

This proves that $\overline{D^kP^r}$ is differentiable and $D\overline{D^kP^r} = \overline{D^{k+1}P^r}$. Hence by induction on k, $\bar{P}^r$ is smooth and $D^k\bar{P}^r = \overline{D^kP^r}$.

Theorem 4. *If P and Q are regular, so is $P\circ Q$. If $\nabla_I P$ and $\nabla_I Q$ exist and are norm-smooth for $|I| \leq r$, then the same holds for $P\circ Q$. Also $\nabla_i(P\circ Q) = \nabla_i P\circ Q + (DP\circ Q)(\nabla_i Q)$.*

Proof. If P and Q are norm-smooth, then so is $P\circ Q$ and

$$\begin{aligned}\{\nabla_i(P\circ Q)\}(x) &= \nabla_i\{(P\circ Q)(x)\} - D(P\circ Q)(x)(\nabla_i X) \\ &= (\nabla_i P)(Qx) + DP(Qx)[\nabla_i(Qx)] - DP(Qx)[DQ(x)(\nabla_i x)] \\ &= (\nabla_i P)(Qx) + DP(Qx)[(\nabla_i Q)(x)] \; .\end{aligned}$$

So $\nabla_i(P\circ Q) = \nabla_i P\circ Q + (DP\circ Q)(\nabla_i Q)$. Hence if $\nabla_i P$ and $\nabla_i Q$ are both norm-smooth, so will be $\nabla_i(P\circ Q)$. Moreover by applying the formula repeatedly, we see that if $\nabla_I P$ and $\nabla_I Q$ exist for $|I| \leq r$, so will $\nabla_I(P\circ Q)$ which can be expressed as a formula involving terms of the form

$$(D^n\nabla_J P\circ Q)(\nabla_{K_1}Q, \cdots, \nabla_{K_n}Q)$$

with $|J| + |K_1| + \cdots + |K_n| = |I|$.

9. Now we prove an inverse function theorem for regular maps.

Theorem 5. *Suppose that $P\colon U \subseteq E \to F$ is regular, and that for some $x \in U$, $DP(x)$ has a regular inverse. Then P gives a bijection of a norm-open neighborhood V of x onto a norm-open neighborhood W of $P(x)$, and $P^{-1}\colon W \subseteq F \to V \subseteq E$ is regular.*

Proof. First we observe that $D\bar{P}^0(x)$ will be invertible, so by the inverse function theorem for Banach spaces, $\bar{P}^0$ gives a diffeomorphism of an open set $\bar{V}^0$ containing x onto an open set $\bar{W}^0$. Let $V = \bar{V}^0 \cap E$ and $W = \bar{W}^0 \cap F$. With no loss we can assume that $\bar{W}^0$ is the ball of radius ρ around $P(x)$, and hence is convex. Then $\bar{W}^r = \bar{W}^0 \cap \bar{E}^r$ will be convex, and hence connected, for every r.

Consider the completions $\bar{P}^r\colon \bar{V}^r \subseteq \bar{E}^r \to \bar{W}^r \subseteq \bar{F}^r$.

Lemma 7. *$\bar{P}^r(\bar{V}^r)$ is open in $\bar{W}^r$ for all r, if ρ is sufficiently small (in-*

dependent of r).

Proof. Choose ρ so small that for all $y \in \bar{V}^0$,

$$\|D\bar{P}^0(y) - D\bar{P}^0(x)\|_0 < 1/\|D\bar{P}^0(x)^{-1}\|_0 .$$

By Corollary 2, if $y \in \bar{V}^r$, then the map $D\bar{P}^r(y)$ will be invertible. Hence by the inverse function theorem for Banach spaces, $\bar{P}^r(\bar{V}^r)$ will contain a neighborhood of $\bar{P}^r(y)$ for every $y \in \bar{V}^r$. Hence $\bar{P}^r(\bar{V}^r)$ is open in $\bar{W}^r$.

Lemma 8. *$\bar{P}^r(\bar{V}^r)$ is also relatively closed in $\bar{W}^r$, and hence $\bar{P}^r(\bar{V}^r) = \bar{W}^r$.*

Proof. Suppose $z_n \in \bar{V}^r$ and $\bar{P}^r(z_n) \to w \in \bar{W}^r$. Choose $y_n \in V$ with $\|y_n - z_n\|_r < 1/n$. Then $P(y_n) \to w \in \bar{W}^r$ in $\| \|_r$. We proceed by induction on r. Suppose $\bar{P}^{r-1}(\bar{V}^{r-1}) = \bar{W}^{r-1}$. Since $\bar{P}^0$ is one-to-one, $\bar{P}^{r-1}$ must be as well. Therefore $\bar{P}^{r-1}$ will have a smooth inverse (recall that $D\bar{P}^{r-1}(y)$ is always invertible) so the sequence y_n will converge to some point $y \in \bar{V}^{r-1}$ with $\bar{D}^{r-1}(y) = w$. Moreover by Corollary 2, if $|I| = r$, then

$$\nabla_I\{P(y_n)\} = DP(y_n)(\nabla_I y_n) + \cdots ,$$

where the dots denote terms of the form

$$D^k(\nabla_J P)(y_n)(\nabla_{K_1} y_n, \cdots, \nabla_{K_k} y_n)$$

with $|K_p| < r$. Since $P(y_n)$ converges in $\| \|_r$, $\nabla_I\{P(y_n)\}$ will converge in $\| \|_0$; as will each sequence $\nabla_{K_p} y_n$ with $|K_p| < r$, and hence each expression denoted by the dots. Therefore the sequence $DP(y_n)(\nabla_I y_n) = u_n$ converges in $\| \|_0$. Moreover each $DP(y_n)$ is invertible and $DP(y_n)^{-1}$ converges to $DP(y)^{-1}$ in $\| \|_0$. Therefore the sequence $\nabla_I y_n = DP(y_n)^{-1} u_n$ converges in $\| \|_0$ for every I of length r. But this implies that y_n converges to y in $\| \|_r$. Thus $w = \bar{P}^r(y) \in \bar{P}^r(\bar{V}^r)$, which proves that $\bar{P}^r(\bar{V}^r)$ is relatively closed in $\bar{W}^r$, and hence $\bar{P}^r(\bar{V}^r) = \bar{W}^r$ since $\bar{W}^r$ is connected.

It follows immediately that $P(V) = W$. Also P is one-to-one, since $\bar{P}^0$ is. We must show that $P^{-1}\colon W \subseteq F \to V \subseteq E$ is also regular.

Lemma 9. *P^{-1} is norm-smooth.*

Proof. We know from the inverse function theorem that $\bar{P}^0$ is invertible and $(\bar{P}^0)^{-1}$ is norm-smooth. Moreover, $D(\bar{P}^0)^{-1}(x) = D\bar{P}^0((\bar{P}^0)^{-1}x)^{-1}$, so if $x \in F$, then $DP^{-1}(x)$ exists and equals $DP(P^{-1}x)^{-1}$. Since $DP(P^{-1}x)$ has a regular inverse, $DP^{-1}(x)$ maps F into E. Hence $P^{-1}\colon W \subseteq F \to V \subseteq E$ is C^1 in the norm topology. Since

$$DP^{-1}(x) = DP(P^{-1}x)^{-1} ,$$

is follows that if P^{-1} is of class C^k, then so is DP^{-1}; so P^{-1} is of class C^{k+1}. Hence P^{-1} is norm-smooth.

Lemma 10. *$\nabla_i P^{-1}$ is norm smooth and*

$$\nabla_i P^{-1} = -(DP \circ P^{-1})^{-1}(\nabla_i P \circ P^{-1}) .$$

Proof. We showed before that

$$\nabla_i(P \circ Q) = \nabla_i P \circ Q + (DP \circ Q)(\nabla_i Q) ,$$

if P and Q are both norm-smooth. Hence letting $Q = P^{-1}$,

$$0 = \nabla_i(P \circ P^{-1}) = \nabla_i P \circ P^{-1} + (DP \circ P^{-1})(\nabla_i P^{-1}) ,$$

so $\nabla_i P^{-1} = -(DP \circ P^{-1})^{-1}(\nabla_i P \circ P^{-1})$. Since $\nabla_i P$, DP and P^{-1} are all norm-smooth, so is $\nabla_i P$.

We now conclude that P is regular. For if $\nabla_I P^{-1}$ exists and is norm-smooth for all I with $|I| \leq r$, then it will also exist and be norm-smooth for all I with $|I| \leq r + 1$ by the above formula, once we have shown the following.

Lemma 11. *Let $U \subseteq RL(E, F)$ be the set of elements in $RL(E, F)$ with a regular inverse, and let Q: $U \subseteq RL(E, F) \to RL(F, E)$ be given by $Q(L) = L^{-1}$. Then Q is regular.*

Proof. We know that $Q(L)$ is smooth and $DQ(L)(M) = -Q(L) \circ M \circ Q(L)$. Therefore

$$\begin{aligned}(\nabla_i Q)(L) &= \nabla_i\{Q(L)\} - DQ(L)(\nabla_i L) \\ &= -Q(L) \circ \nabla_i L \circ Q(L) + Q(L) \circ \nabla_i L \circ Q(L) = 0 .\end{aligned}$$

Hence $\nabla_i Q = 0$.

10. Finally we point out how this result can be applied to non-linear partial differential operators. If α is a smooth map of an open set U in the bundle $j^r A$ into the bundle B, then the induced map $\Gamma_2^k(\alpha)$: $\Gamma_2^k(U) \subseteq \Gamma_2^k(j^r A) \to \Gamma_2^k(B)$ is regular, since for every multi-index I we can find a smooth map $\nabla_I \alpha$: $U \subseteq j^r A \to B$ which also takes fibres such that

$$\nabla_I \Gamma_2^k(\alpha) = \Gamma_2^k(\nabla_I \alpha) ;$$

this follows from the remarks at the beginning of the last section. Moreover the r-jet extension j^r: $\Gamma_2^{r+k}(A) \to \Gamma_2^k(j^r A)$ is regular, and the composition of two regular maps is regular. Hence any non-linear partial differential operator $Q = \Gamma_2^k(\alpha) \circ j^r$ is a regular map of an open set in $\Gamma_2^{r+k}(A)$ into $\Gamma_2^k(B)$. Moreover for any section x with Im $j^r x \subseteq U$ the derivative of Q at x

$$DQ(x) = \Gamma_2^k(D\alpha(x)) \circ j^r$$

will be a linear partial differential operator and hence a regular linear map of $\Gamma_2^{r+k}(A)$ into $\Gamma_2^k(B)$. Suppose that its completion

$$\overline{DQ(x)}_0 \colon L_2^{r+k}(A) \to L_2^k(B)$$

is invertible. It follws from the results of § 6 that

$$DQ(x) \colon \Gamma_2^{r+k}(A) \to \Gamma_2^k(B)$$

is invertible and has a regular inverse. Therefore by Theorem 5 the non-linear partial differential operator Q gives a bijection of a neighborhood of x onto a neighborhood of $Q(x)$ and the inverse will be regular. Consequently the condition for inverting the operator Q on the smooth functions is just the same as that for inverting it on an appropriate L_2 completion.

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