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ON THE QUANTUM EXPECTED VALUES OF INTEGRABLE METRIC FORMS

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1. Introduction

Let (M^n, g) be a compact, real-analytic, Riemannian manifold, P_0 a first order, self-adjoint, real-analytic, elliptic pseudodifferential operator with principal symbol,

$$H(x,\xi) = \sqrt{g^{ij}(x)\xi_i\xi_j}$$

generating geodesic flow. We will assume that P_0 is quantum integrable; that is, there exist $n-1$ first order, jointly elliptic, real-analytic, classical pseudodifferential operators $P_1, ..., P_{n-1}$ such that, for all $i,j = 0,1,...,n-1$,

$$[P_i, P_j] = 0. \tag{1}$$

Given the Hamilton vector field,

$$\Xi_H = \sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j}\frac{\partial}{\partial \xi_j},$$

we denote the associated geodesic flow by $\exp t\Xi_H : C^\infty(S^*M) \to C^\infty(S^*M)$. Suppose γ is a simple, periodic orbit of $\exp t\Xi_H$ (i.e., a

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closed geodesic). Under the assumption that the associated linearized Poincaré map, P_γ, has eigenvalues of the form $e^{\pm i\theta_j}, j = 1, ..., n-1$ with θ_j rationally independent, one can explicitly construct [15], [11] a sequence of L^2-normalized functions (the so-called "quasimodes"), $\phi_k \in C^\infty(M)$, with

$$-\Delta\phi_k = \lambda_k\phi_k + \mathcal{O}(k^{-\infty}).$$

Here, the ϕ_k have very sharp localization properties along the configuration space projection, $\pi(\gamma)$, of the geodesic γ as $\lambda_k \to \infty$. When γ is unstable, it is well-known that there is no such quasimode construction available. Nevertheless, the question of whether or not there exist actual sequences of eigenfunctions with mass asymptotically accumulating along γ seems to depend on the nature of the geodesic flow: In the ergodic example of arithmetic surfaces, Rudnick and Sarnak [16] have shown that periodic orbits do not support mass in the quasiclassical limit. The general ergodic case is still open. On the other hand, it is known that in the integrable case, such orbits can and do support mass. However, there are few rigorous results (see [6], [20]) along these lines and the analysis in each example has been somewhat ad hoc, usually depending on separation of appropriate variables and a detailed analysis of the corresponding special functions. This approach is unsatisfactory since one is often faced with the very difficult problem of studying the spectral asymptotics of coupled systems of multiparameter O.D.E. with automorphic coefficients.

The purpose of this paper is to present a more systematic analysis in the integrable case using microlocal techniques and in particular, quantum Birkhoff normal form (QBNF) (see [10], [24], [26], [27] and Section 3). Our main result (see Theorem 1) can be summarized as follows: Suppose that the level set

$$\Sigma_E = \{z \in T^*M; p_0(z) - E_0 = ... = p_n(z) - E_n = 0\}$$

contains a finite number of nondegenerate, unstable, periodic geodesics $\gamma_1, ..., \gamma_k$ and that Σ_E is smooth outside a union of tubular neighbourhoods of the γ_j's. Roughly speaking, Theorem 1 says that, under a joint non-resonance condition (H1) (see below), the bicharacteristics $\gamma_j; j = 1, ..., k$ always support eigenfunction mass in the semiclassical limit. In order to state Theorem 1 more precisely, we will now describe the contents of the paper in more detail.

Section 2 consists of some salient facts on the symplectic geometry of periodic orbits (see [1], [9], [10], [26], [27]). Here, we review some basic symplectic linear algebra as well symplectic normal form for quadratic Hamiltonians.

In Section 3, we show that under hypotheses (H1) and (H2) below, there exists a convergent joint quantum Birkhoff normal form (Theorem 3) for the quantum integrals $P_0, ..., P_{n-1}$ (see also [6], [24]). To state these hypotheses, we introduce some notation here: Let (s, σ, y, η) denote the Birkhoff normal coordinates near γ (see Section 2), where, in particular, $\sigma = y = \eta = 0$ along γ. Let I^h, I^{ch} denote the respective real and complex hyperbolic classical actions associated with the Poincaré mapping P_γ (see Section 2). We will assume that:

$$(H1) \quad \det \begin{pmatrix} \nabla_\sigma p_0 & \nabla_{I^h} p_0 & \nabla_{I^{ch}_{Re}} p_0 & \nabla_{I^{ch}_{Im}} p_0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \nabla_\sigma p_{n-1} & \nabla_{I^h} p_{n-1} & \nabla_{I^{ch}_{Re}} p_{n-1} & \nabla_{I^{ch}_{Im}} p_{n-1} \end{pmatrix} \neq 0$$

when $\sigma = y = \eta = 0$, and also,

(H2) The geodesics $\gamma_j;\ j = 1, ..., k$ are forward limit sets for the bicharacteristics of the Hamilton vector field, Ξ_H, on the variety, Σ_E. That is, for any $(x, \xi) \in \Sigma_E$, there is a γ_j with $1 \leq j \leq k$ such that $\exp t\Xi_H(x, \xi) \to \gamma_j$ as $t \to \infty$.

The proof of Theorem 3 will hinge on establishing a convergent classical Birkhoff normal form near each of the γ_j's (Theorem 2). This will follow from a result of Ito [13] (see also Vey [23] and Eliasson [7]) on the convergence of classical Birkhoff normal form near a critical point, together with a result of Francoise-Guillemin [9] (see also Guillemin [10]) relating the symplectic data associated with the Poincaré cross section to the contact geometry of the mapping cylinder.

In Section 4, we work out the example of the quantized Euler top in detail and show that Theorem 1 applies in this case. We should point out that our results apply in many examples, including Liouville tori, Clebsch-Gordon spinning tops and geodesic flow on quadrics among others. The analogue of Theorem 1 also applies in inhomogeneous examples such as Neumann oscillators, Lagrange and Kowalevsky tops among others. We hope to return to this elsewhere.

Section 5 is concerned with time asymptotics of the classical geodesic flow. This will play a crucial role in the microlocalization problem in the next section.

In Section 6, we carry out the necessary microlocalization near the γj's to enable us to use the quantum Birkhoff normal form construction described in Section 3. The microlocalization is accomplished by first establishing an a priori mass estimate near the level variety Σ_E (Lemma 3), and then applying the semiclassical Egorov Theorem. It is in this last step that the time asymptotics of Section 5 enters in a pivotal way.

One is then faced with the problem of explicitly estimating the semiclassical expected values of various model distributions which arise in the Birkhoff construction. This, we do in Section 7, where we treat the real hyperbolic case (see also [6]), and Section 8, where the estimates for the complex hyperbolic case are given.

Finally, in Section 9, we prove Theorem 1 below:

Theorem 1. *Let $P_0, ..., P_{n-1}$ be a real-analytic quantum integrable system on a compact, real-analytic Riemannian manifold, M, with P_0 given above. Let Σ_E be a fixed level set*

$$\{(x,\xi) \in T^*M; p_0(x,\xi) - E_0 = ... = p_{n-1}(x,\xi) - E_{n-1} = 0\},$$

and let $\gamma_1, ..., \gamma_k \subset \Sigma_E$ be k non-degenerate, unstable, periodic geodesics for the metric form $p_0(x,\xi) = \sqrt{g^{ij}(x)\xi_i\xi_j}$. Assume moreover, that hypotheses $(H1)$ and $(H2)$ are satisfied and that, for convenience, the periods are normalized to be 2π. Then, given $\hbar^{-1} \in Spec\ (P_0)$, ψ_j, an L^2-normalized joint eigenfunction satisfying

$$\hbar P_k \psi_j = E_k \psi_j + \mathcal{O}(\hbar)\psi_j$$

and any $q \in C_0^\infty(T^*M)$, there exist non-negative real numbers $\alpha_1, ..., \alpha_k$ with

$$\sum_{j=1}^{k} \alpha_j = 1$$

such that,

$$(Op_\hbar(q)\psi_j, \psi_j) = (2\pi)^{-1} \sum_{j=1}^{k} \alpha_j \int_0^{2\pi} q(\gamma_j(t))dt + \mathcal{O}(|\log \hbar|^{-1/2}).$$

Remarks. **1)** It follows from Theorem 1 that in many integrable examples including Euler tops and geodesic flow on quadrics, one can find unstable periodic bicharacteristics that support eigenfunction mass (see Section 5). However, if the singularities of the level variety are more complicated than those permitted in hypotheses (H1) and (H2), it is unclear whether this accumulation phenomenon persists. This is a very interesting question which we hope to address elsewhere.

2) Although we have stated Theorem 1 for periodic orbits, the result holds equally well when the limit sets are points.

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2. Some symplectic geometry

It will be useful to review some symplectic geometry here which will be used in the implementation of Birkhoff normal form. The treatment here will be rather brief; we refer the reader to Guillemin [10] and Zelditch [26], [27] for further details. In the first part, we will follow quite closely the exposition in [27] Section 1.

Let (M^n, g) be a compact, Riemannian manifold and γ a closed geodesic of g. In the case of such a metric form, there is a rather explicit recipe for putting $H = \sqrt{g^{ij}\xi_i\xi_j}$ into Birkhoff normal form in a tubular neighbourhood of γ. To describe this procedure, following Zelditch [27, Section 1.1] we denote the space of real orthogonal Jacobi fields along γ by $\mathcal{J}_\gamma^\perp$. Then, $Y \in \mathcal{J}_\gamma^\perp$ if and only if,

$$g(\frac{\partial}{\partial s}, Y) = 0 \text{ and } \frac{D^2}{ds^2}Y + R\left(\frac{\partial}{\partial s}, Y\right)\frac{\partial}{\partial s} = 0. \tag{2}$$

There is a natural symplectic structure on $\mathcal{J}_\gamma^\perp$ given by

$$\omega(X,Y) = g\left(X, \frac{D}{ds}Y\right) - g\left(\frac{D}{ds}X, Y\right). \tag{3}$$

The linearized Poincaré map P_γ is just the symplectic mapping on $(\mathcal{J}_\gamma^\perp, \omega)$ defined by $P_\gamma Y(t) := Y(t + L_\gamma)$, where L_γ denotes the length of γ. By complexification, we get an induced complex linear map, $P_\gamma^{\mathbb{C}} \in Sp(\mathcal{J}^\perp\gamma \times \mathbb{C}, \omega_{\mathbb{C}})$. Since it is symplectic (see [1]), its eigenvalues occur either as complex conjugates on the unit circle (i.e., the *elliptic* subspace), real pairs $\lambda, \lambda^{-1}; \lambda \in \mathbb{R}$ (i.e., the *real hyperbolic* subspace), or complex quadruples of the form $\rho, \rho^{-1}, \overline{\rho}, \overline{\rho}^{-1}; |\rho| \neq 1$ (i.e., the *loxodromic* subspace). We will henceforth make the usual non-degeneracy assumption on γ; that is,

$$\rho_1^{m_1} \cdots \rho_n^{m_n} = 1 \Rightarrow m_i = 0 \ (\forall i, m_i \in \mathbb{N}). \tag{4}$$

Since (4) implies that the eigenvalues are in particular, simple, there exists a decomposition:

$$\mathcal{J}_\gamma^\perp = \mathcal{J}_\gamma^e + \mathcal{J}_\gamma^h + \mathcal{J}_\gamma^{ch}, \tag{5}$$

where, $\mathcal{J}_\gamma^e, \mathcal{J}_\gamma^h, \mathcal{J}_\gamma^{ch}$ denote the elliptic, real hyperbolic and complex hyperbolic subspaces. Here, $\mathcal{J}_\gamma^e$ is characterized by the condition:

$$P_\gamma|_{\mathcal{J}_\gamma^e} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$

for some $\alpha \in \mathbb{R} - 0$. The real hyperbolic subspace $\mathcal{J}_\gamma^h$ is defined by the condition that

$$P_\gamma|_{\mathcal{J}_\gamma^h} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix},$$

for some $\rho \in \mathbb{R}$. Finally, the loxodromic subspace $\mathcal{J}_\gamma^{ch}$ is a four-dimensional real symplectic subspace

$$\mathcal{J}_\gamma^{ch} = \mathcal{J}_\gamma^{ch}(\rho) + \mathcal{J}_\gamma^{ch}(\rho^{-1}),$$

where $\rho = e^{-\mu + i\nu} \in \mathbb{C} - \mathbb{R}$ with $\mu, \nu \in \mathbb{R}$, and

$$P_\gamma|_{\mathcal{J}_\gamma^{ch}(\rho)} = e^{-\mu} \begin{pmatrix} \cos\nu & \sin\nu \\ -\sin\nu & \cos\nu \end{pmatrix}.$$

Following Zelditch [26], [27], we say that the geodesic γ has type (p, q, c) if it has p pairs of stable eigenvalues, $\{e^{i\alpha}, e^{-i\alpha}\}$, q pairs of real inverse eigenvalues, $\{e^\lambda, e^{-\lambda}\}$ and c quadruples of totally complex eigenvalues, $\{e^{\pm\mu\pm i\nu}\}$. Before stating the variant of the classical Birkhoff normal form about γ, we recall certain salient facts about the symplectic geometry of quadratic Hamiltonians (see [1] and [27] Section 1.1 c).

Let $(\mathbb{R}^{2n}, \omega)$ be the symplectic vector space with symplectic coordinates $z = (x_1, x_2, ..., x_n, \xi_1, ..., \xi_n)$ and symplectic form $\omega = \sum_{j=1}^n dx_j \wedge d\xi_j$. A quadratic Hamiltonian is by definition, of the form:

$$H(x, \xi) = \langle Az, z \rangle = \omega(JAz, z), \tag{6}$$

where, $J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$. Since JA is a $2n \times 2n$ symplectic matrix, its spectrum decomposes into purely imaginary pairs $(i\alpha, -i\alpha)$ (the elliptic set), real pairs $(\lambda, -\lambda)$ and complex quadruples $(\pm\mu \pm i\nu)$. By a theorem of Williamson, one can characterize the normal form of the functions $H(x, \xi)$ up to symplectic equivalence (see [1]). The general case is rather complicated to state since it involves the Jordan normal form of JA, but under the nondegeneracy assumption (4), this result says that there exist symplectic coordinates $(y_1, ..., y_n, \eta_1, ..., \eta_n)$ in terms of which,

$$\begin{aligned} H(y, \eta) =& \sum_{j=1}^{p} I_j^e(y_j, \eta_j) + \sum_{j=p+1}^{p+q+1} I_j^h(y_j, \eta_j) \\ &+ \sum_{j=p+q+2}^{p+q+2c+2} I_j^{ch}(y_j, y_{j+1}, \eta_j, \eta_{j+1}). \end{aligned} \tag{7}$$

The classical action operators I_j^e, I_j^h, I_j^{ch} are given by:

$$I_j^e(y_j, \eta_j) = \frac{1}{2}\alpha_j(y_j^2 + \eta_j^2), \tag{8}$$

$$I_j^h(y_j, \eta_j) = \lambda y_j \eta_j, \tag{9}$$

$$\begin{aligned} I_j^{ch}(y_j, y_{j+1}, \eta_j, \eta_{j+1}) =& \mu(y_j \eta_j + y_{j+1}\eta_{j+1}) \\ &+ \nu(y_j \eta_{j+1} - y_{j+1}\eta_j). \end{aligned} \tag{10}$$

The corresponding $\hbar$-Weyl quantizations, which we will refer to as the "model operators" are then just

$$\hat{I}_j^e = \frac{1}{2}\alpha_j(\hbar^2 D_{y_j}^2 + y_j^2), \tag{11}$$

$$\hat{I}_j^h = \frac{1}{2}\lambda(\hbar D_{y_j} y_j + \hbar y_j D_{y_j}), \tag{12}$$

$$\hat{I}_j^{ch} = \frac{1}{2}\mu(\hbar r_j D_{r_j} + \hbar D_{r_j} r_j) + \nu \hbar D_{\theta_j}. \tag{13}$$

Here, we have used (r_j, θ_j) to denote polar variables in the (y_j, y_{j+1}) plane. It will also be convenient to introduce the following notation:

$$\hat{I}^{ch}_{jRe} = \frac{1}{2}\mu\hbar(r_j D_{r_j} + D_{r_j} r_j) \text{ and } \hat{I}^{ch}_{jIm} = \nu\hbar D_{\theta_j}. \tag{14}$$

Finally, (see also [27]) note that, if one extends the classical action functions I^e, I^h, I^{ch} to $T^*\mathbb{C}^n$ in the natural way, each of them may be written in the form

$$I(z,\zeta) = \Re s z \zeta \tag{15}$$

for $s \in \mathbb{C}$ and (z,ζ) symplectically dual complex linear coordinates. In the elliptic case, $z = y_1 + i\eta_1, \zeta = y_1 - i\eta_1, s = \alpha$, in the hyperbolic case $z = y_1, \zeta = \eta_1, s = \lambda$ and, in the loxodromic case, $s = \mu + i\nu, z = y_1 + iy_2, \zeta = \eta_1 - i\eta_2$.

3. Birkhoff normal form

In the course of the proof of Theorem 1, we will have to establish the existence of a sequence of joint eigenfunctions ψ_j which are microlocally of a specific form when expressed in terms of the model eigenfunctions (see Proposition 5). This will be done by establishing a *convergent*, joint, semiclassical quantum Birkhoff normal form (QBNF) for the first-order operators $P_0, ..., P_{n-1}$ under the hypotheses (H1) and (H2). We should point out that a similar normal form has recently been obtained by San Vu Ngoc [24], [25]. However, our normal form holds in a neighbourhood of a closed geodesic, and since we use a classical result of Ito[13], the integrals in involution $p_1, .., p_{n-1}$ can have degenerate behaviour along γ, provided (H2) is satisfied and all integrals are taken to be real-analytic. In this section, it will be convenient to work with the operators $H_0 = \hbar P_0 - E_0, ..., H_{n-1} = \hbar P_{n-1} - E_{n-1}$ rather than $P_0, ..., P_{n-1}$. When there the context is clear, we shall also denote the respective semiclassical principal symbols by $H_0, ..., H_{n-1}$. The starting point here is the existence of a convergent classical Birkhoff normal form (CBNF) which is valid in a sufficiently small tubular neighbourhood $\Omega \times \gamma$ of the geodesic, γ. This result will follow from a theorem of Francoise and Guillemin [9](see also Guillemin [10]) together with a result of Ito [13] on the convergence of canonical Birkhoff transformation around a fixed point in the real-analytic, integrable case. With regards to the last result, we should also point out that related results have been proved in the analytic case by Russman [15], Vey [23] and in the C^∞ setting by Eliasson [7].

Theorem 2. *Let $H_0, H_1, ..., H_{n-1}$ be real-analytic integrals in involution and γ be a closed, non-degenerate unstable geodesic for $p_0 = g^{ij}\xi_i\xi_j$. Then, for $j = 0, 1, ..., n-1$, there exists a neighbourhood of the origin $U \in \mathbb{R}^n$, $f_j \in C^\omega(U)$, and a real-analytic symplectic diffeomorphism $\kappa : \Omega \times \gamma \to \Omega_0 \times \mathbb{S}^1$ such that:*

$$\kappa^* H_j = f_j(\sigma, I^h, I^{ch}).$$

Here, to simplify notation, we have written $I^h = (I_1^h, ..., I_q^h)$ and $I^{ch} = (I_{q+1}^{ch}, ..., I_{q+c+1}^{ch})$. The induced symplectic, modified Fermi coordinates (see [27]) on $\Omega_0 \times \mathbb{S}^1$ will be denoted by (s, σ, y, η).

Proof. We fix an unstable geodesic γ_j and to simplify the writing somewhat, we will drop the subscript j in the following. Let

$$M = \{z \in T^*M; H_0(z) = 0\}$$

and take as our contact form, α, the restriction to M of the canonical one-form $\sum_j \xi_j dx_j$ on $T^*M - 0$. Let $W \subset M$ be an open submanifold that is transversal to the flow $\exp t\Xi_{H_0}$ at $p_0 \in \gamma$. Then, there exists an open submanifold $W_0 \subset W$ such that:

$$f : (W_0, p_0) \longrightarrow (W, p_0). \tag{16}$$

Here, f is the Poincaré map corresponding to the flow $\exp t\Xi_{H_1}$. By a well-known result of Poincaré ([9], [10]),

$$f^*\alpha - \alpha = d\phi, \tag{17}$$

where ϕ denotes the "first return time" function. In particular, f is symplectic with respect to the symplectic form $\omega = d\alpha$, with an unstable fixed point at p_0. Let $\iota : W \longrightarrow M$ denote the inclusion map. Then, since $\{H_i, H_j\} = 0$, it follows that:

$$H_k(f(z)) = H_k(z) \tag{18}$$

for all $z \in W$ and $k = 1, ..., n-1$. Since p_0 is a non-resonant fixed point of f, by a theorem of Ito [13], there exists a *convergent*, real-analytic, canonical mapping $\phi : (W_0, \Omega) \to (W_0, \Omega)$ under which, the Poincaré mapping, f, is put into classical Birkhoff normal form. A precise statement of this theorem is most easily given by complexifying the Hamiltonian, H_0, as well as W and then imposing a reality condition [13]. Denote the complex, canonical coordinates on $W^{\mathbb{C}}$ by (z, ζ) and the holomorphic continuation of $f(y, \eta)$ by $f(z, \zeta)$. Then, the above result of Ito says that there exist a function, $H(\omega)$, holomorphic in the variables $\omega_0 = z_0\zeta_0, ..., \omega_{n-1} = z_{n-1}\zeta_{n-1}$, and a complex, canonical mapping $\phi : (W_0^{\mathbb{C}}, \Omega^{\mathbb{C}}) \to (W_0^{\mathbb{C}}, \Omega^{\mathbb{C}})$, such that

$$\phi f \phi^{-1}(z, \zeta) = (z \exp(\partial_\omega H), \zeta \exp(-\partial_\omega H)). \tag{19}$$

The identity in (19) implies that there is a corresponding convergent *real* Birkhoff normal form for $f(y, \eta)$. However, to state this one must decompose W_0 into hyperbolic and loxodromic blocks (see Section 2): If (y_1, η_1) denotes symplectic dual coordinates on a hyperbolic block, (19) implies that there exists a real-analytic, canonical map ϕ acting on this block, with the property that:

$$\phi_h f \phi_h^{-1}(y_1, \eta_1) = \begin{pmatrix} \exp(\partial_{I^h} H) & 0 \\ 0 & \exp(-\partial_{I^h} H) \end{pmatrix} \begin{pmatrix} y_1 \\ \eta_1 \end{pmatrix},$$

where, $H = H(I^h, I^{ch}_{Re}, I^{ch}_{Im})$. Finally, when the symplectic 4-plane $(y_1, y_2, \eta_1, \eta_2)$ is loxodromic, there exists ϕ_{ch} such that:

$$\begin{aligned} &\phi_{ch} f \phi_{ch}^{-1}(y_1, y_2, \eta_1, \eta_2) \\ &= \begin{pmatrix} e^{-\mu}\cos\nu & e^{-\mu}\sin\nu & 0 & 0 \\ -e^{-\mu}\sin\nu & e^{-\mu}\cos\nu & 0 & 0 \\ 0 & 0 & e^{\mu}\cos\nu & e^{\mu}\sin\nu \\ 0 & 0 & -e^{\mu}\sin\nu & e^{\mu}\cos\nu \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \eta_1 \\ \eta_2 \end{pmatrix}. \end{aligned}$$

Here, we have written, $\mu = \partial_{I^{ch}_{Re}} H$ and $\nu = -\partial_{I^{ch}_{Im}} H$. Now, define

$$\tau(I^h, I^{ch}_{Re}, I^{ch}_{Im}) = \tau(0) + H(I^h, I^{ch}_{Re}, I^{ch}_{Im}). \tag{20}$$

The function τ plays an important role in determining the contact manifold (M, α) from the symplectic data (W, Ω, f). Namely, recall that by a theorem of Guillemin-Francoise [9], [10], there exists a contact isomorphism mapping (M, α) onto $(\mathbb{R}^{2n} \times \mathbb{S}^1, \alpha_0)$, where,

$$\alpha_0 = \tau(I)ds + \eta dy. \tag{21}$$

Here, we have denoted the angle variable on $\mathbb{S}^1$ by $2\pi s$ and, by a slight abuse of notation, $I = (I^h, I^{ch})$. Note that, following the usual convention, we will also denote the action variables near regular Lagrangian tori by $I = (I_1, ..., I_n)$ and so, the two cases should not be confused. Following Guillemin ([10, Section 2]), one can show that there exists an extension (by homogeneity) of this contact isomorphism to a locally-defined canonical mapping

$$\psi^{-1} : \Omega \times \gamma \longrightarrow \Omega_0 \times \mathbb{S}^1$$

such that,

$$\psi^* \alpha = \eta dy + \tau ds \tag{22}$$

and

$$\psi^* H_0 = f_0(I^h, I^{ch}_{Re}, I^{ch}_{Im}, \sigma) = \sigma + \lambda I^h + \mu I^{ch}_{Re} + \nu I^{ch}_{Im} + \mathcal{O}(|I|^2). \tag{23}$$

This completes the first part of the proposition.

To show that $H_k; k = 1, ..., n-1$ must simultaneously also be in normal form, we simply use the analyticity of the H_k's together with fact that $\{H_0, H_k\} = 0$. Since $T^*\mathbb{R}^n \times T^*\mathbb{S}^1 - 0$ splits into complementary symplectic subspaces corresponding to the action functions I^h, I^{ch}, it suffices to assume that (y, η) corresponds to a single summand. To begin, we assume that this summand is real-hyperbolic. Then, since H_k is assumed to be analytic, make a power series development for H_k up to total order 2 in (y, η, σ) and denote the resulting polynomial by H_k^2. Therefore, $\kappa^* H_k^2$ equals

$$\begin{aligned} &f_0(s)\sigma + f_1(s)y + f_2(s)\eta + f_3(s)y\eta + f_4(s)y^2 \\ &\quad + f_5(s)\eta^2 + f_6(s)y\sigma + f_7(s)\eta\sigma + f_8(s)\sigma^2. \end{aligned} \tag{24}$$

As a consequence of the error term $\mathcal{O}(y^2\eta^2)$ in (23), it follows that:

$$\{\sigma + \lambda y\eta, H_k^2(s, \sigma, y, \eta)\} = 0. \tag{25}$$

By matching the different coefficients of the various monomials in y, η, σ, we get:

$$\begin{array}{lll} \partial_s f_0(s) = 0 & \partial_s f_1(s) + \lambda f_1(s) = 0 & \partial_s f_2 - \lambda f_2(s) = 0 \\ \partial_s f_3 = 0 & \partial_s f_4(s) + 2\lambda f_4(s) = 0 & \partial_s f_5 - 2\lambda f_5(s) = 0 \\ \multicolumn{3}{c}{\partial_s f_6(s) + \lambda f_6(s) = 0 \qquad \partial_s f_7(s) - \lambda f_7(s) = 0,} \end{array}$$

and $\partial_s f_8(s) = 0$. However, all the f_j's are required to be 2π-periodic functions in the s variable and so, this forces $f_0 = const., f_1 = 0, f_2 = 0, f_3 = const., f_4 = f_5 = f_6 = f_7 = 0, f_8 = const.$. Therefore, H_k^2 is resonant.

We now repeat the above argument and apply induction: To simplify the writing, we denote the maximum total order of a polynomial in the variables (σ, y, η) by *ord*. Since $\{H_0, H_k - H_k^2\} = 0$, it follows that:

$$\{\sigma + \lambda y\eta, H_k - H_k^2\} + \{e(y, \eta, \sigma), H_k - H_k^2\} = 0.$$

Now,

$$ord\{e(y, \eta, \sigma), H_k - H_k^2\} \geq 5 \text{ and } ord\{\sigma + \lambda y\eta, H_k^5 + H_k^6 + ...\} \geq 5.$$

Therefore, since

$$ord\{\sigma + \lambda y\eta, H_k^3 + H_k^4\} \leq 4,$$

it follows that:

$$\{\sigma + \lambda y\eta, H_k^3 + H_k^4\} = 0.$$

Finally, repeat the above argument with H_k^2 replaced by $H_k^3 + H_k^4$ and apply induction.

Suppose now that the summand is complex hyperbolic and denote the symplectic coordinates corresponding to the summand in question by $(y_1, y_2, \eta_1, \eta_2)$. By introducing the complex variables $z = y_1 + iy_2$ and $\eta = \eta_1 - i\eta_2$, we can write

$$I_{Re}^{ch} = \frac{1}{2}(z\eta + \overline{z\eta}) \text{ and } I_{Im}^{ch} = \frac{i}{2}(\overline{z\eta} - z\eta).$$

Repeating the argument for the real-hyperbolic case, we write

$$\kappa^* H_k^2 = \sum_{\alpha+\beta+\gamma+\delta \leq 2} f_{\alpha\beta\gamma\delta}(s, \sigma) z^\alpha \overline{z}^\beta \eta^\gamma \overline{\eta}^\delta.$$

One readily shows as in (25) that $f_{\alpha\beta\gamma\delta} = 0$ provided either $\alpha \neq \gamma$ or $\beta \neq \delta$ and moreover, $f_{\alpha\beta\alpha\beta} = f_{\alpha\beta\alpha\beta}(\sigma)$. The general case follows by applying the above argument successively to each summand. q.e.d.

Given hypothesis (H1), it follows by the Inverse Function Theorem that locally near $\sigma = y = \eta = 0$,

$$(26) \quad \begin{aligned} \sigma =& g_0(H_0, ..., H_{n-1}), I_1^h = g_1(H_0, ..., H_{n-1}), ..., I_{Im,n-1}^{ch} \\ =& g_{n-1}(H_0, ..., H_{n-1}) \end{aligned}$$

where, for $j = 0, 1, ..., n-1$, g_j are locally-defined, real-analytic functions with $g_j(0, ..., 0) = 0$. Let $\Omega_1 \times \gamma_1, ..., \Omega_k \times \gamma_k$ denote tubular neighbourhoods of the geodesics, $\gamma_1, ..., \gamma_k$ respectively, Moreover, we assume that they are sufficiently small, so the identity (26) holds and the CBNF in Theorem 2 is valid in these neighbourhoods. Let $\chi_1, ..., \chi_k$ denote cutoff functions supported in $\Omega_1 \times \gamma_1, ..., \Omega_k \times \gamma_k$ respectively. In the following, for notational simplicity, we assume that the Maslov indices $m(\gamma_l); l = 1, ...k$ are all zero. Otherwise, the exponentials e^{ins} in the Fourier series expansions below are to be replaced by $e^{i(n+\frac{m(\gamma_l)}{4})s}$. We are now in a position to prove the quantum analogue of Theorem 2:

Theorem 3. *Suppose hypothesis (H1) is satisfied. Then, for $j = 0, ..., n-1$ and $l = 1, ..., k$, there exist a microlocally unitary $\hbar$-Fourier integral operator, $F_l : C_0^\infty(\Omega_l \times \gamma_l) \longrightarrow C_0^\infty(\Omega_0 \times \mathbb{S}^1)$, and a locally analytic symbol $g_j^l(x_1, ..., x_n; \hbar) \sim \sum_k g_{jk}^l(x_1, ..., x_n)\hbar^k$ such that,*

$$\|\chi_l(F_l g_j^l(H_0, ..., H_{n-1}; \hbar)F_l^{-1} - Q_j)\| = \mathcal{O}(\hbar^\infty).$$

Here $Q_1 = -i\hbar\partial_s$ and

$$Q_j = \begin{cases} \hat{I}^h(y_j, \hbar D_{y_j}) & 2 \leq j \leq q+1, \\ \hat{I}^{ch}(y_j, \hbar D_{y_j}) & q+2 \leq j \leq q+2c+2, \end{cases}$$

where, $\hat{I}^h, \hat{I}^{ch}$ are the microlocal action operators given in (11)-(13) and $q + 2c + 2 = n - 1$.

Proof. For simplicity, we denote microlocal equivalence on Ω by $=_\Omega$ and will use $\hbar$-Weyl quantizations since we need only work microlocally near a fixed γ. To simplify the writing we will drop the index l. The ansatz is a variant of that given in [6] (see also [24]) with some modifications. To simplify the writing somewhat, we first assume that there exist two commuting operators $Q_1 = -i\hbar\partial_s$ and $Q_2 = -i\frac{\hbar}{2}(y\partial_y - \partial_y y)$ corresponding to a single real hyperbolic summand. Then, by the CBNF result above together with the semiclassical Egorov theorem, there exists a microlocally unitary $\hbar$-Fourier integral operator, F_0, with the property that:

$$F_0 g_1(H_0, H_1)F_0^{-1} =_\Omega -i\hbar\partial_s + \hbar R_1, \tag{27}$$

$$F_0 g_2(H_0, H_1)F_0^{-1} = -i\frac{\hbar}{2}(y\partial_y - \partial_y y) + \hbar R_2, \tag{28}$$

where, $R_j \in Op_\hbar(S^{0,-\infty})$ for $j = 1,2$. The proof proceeds by an inductive argument: Define

$$F_{k+1} := F_k(Id + \hbar^k V_k)$$

with $V_k \in Op_\hbar(S^{0,-\infty})$ for $k > 0$. Here, we say that $a(s,\sigma,y,\eta;\hbar) \in C^\infty(\mathbb{R}^n \times \mathbb{S}^1)$ is in $S^{m,k}(\mathbb{R}^n \times \mathbb{S}^1)$ if

$$|\partial_x^\alpha \partial_\xi^\beta a(s,\sigma,y,\eta;\hbar)| \leq C_{\alpha\beta}\hbar^m(1+|y|+|\eta|+|\sigma|)^{k-|\alpha|-|\beta|},$$

and A denotes the corresponding $\hbar$-Weyl quantization. Suppose that $g_{k,j}$ and F_k have been constructed so that

$$F_k g_{k,j}(H_0,H_1;\hbar)F_k^{-1} =_\Omega Q_j + \hbar^k R_{k,j}, \tag{29}$$

$$F_k - F_{k-1} = \mathcal{O}(\hbar^k), \tag{30}$$

$$g_{k,j} - g_{k-1,j} = \mathcal{O}(\hbar^{k-1}). \tag{31}$$

Then, the $(k+1) - st$ step involves constructing a pseudodifferential operator, $V_k \in Op_\hbar(S^{0,-\infty})$, and symbols, $g_{k+1,j}; j = 1,2$, with the property that:

$$F_{k+1}^{-1} g_{k+1,j}(H_0,H_1;\hbar)F_{k+1} =_\Omega -i\hbar\partial_s + \hbar^{k+1}R_{k+1,j}.$$

Let $r_{k,j}$ be the semiclassical principal symbol of $R_{k,j}$. By Theorem 2, there exist real analytic functions $e_{k,j}(\sigma,y,\eta)$ such that:

$$\begin{aligned} r_{k,j} - e_{k,j} = &\sum_{n\neq 0}\left(\sum_{\alpha,\beta,\gamma} c^j_{\alpha\beta\gamma}(n) y^\alpha \eta^\beta \sigma^\gamma\right) e^{ins} \\ &+ \sum_{\gamma,\alpha\neq\beta} c^j_{\alpha\beta\gamma}(0) y^\alpha \eta^\beta \sigma^\gamma. \end{aligned} \tag{32}$$

Assume now that the k-th step of the induction has been verified. Define

$$g_{k+1,j}(H_0,H_1;\hbar) = g_{k,j}(H_0,H_1;\hbar) - \hbar^k e_{k.j}(H_1,H_2). \tag{33}$$

Then, by the symbolic calculus,

$$F_{k+1}^{-1} g_{k+1,j}(H_0,H_1;\hbar)F_{k+1} = Q_j + \hbar^k S_{k,j} + \hbar^{k+1}R_{k+1,j}, \tag{34}$$

where, $s_{k,j} = r_{k,j} - e_{k,j} - \{q_j, v_k\}$. Therefore, we must solve the system of homological equations:

$$-r_{k,1} + e_{k,1} = \partial_s v_k, \tag{35}$$

$$-r_{k,2} + e_{k,2} = (y\partial_y - \eta\partial_\eta)v_k. \tag{36}$$

To see that this system can be locally solved for v_k, simply note that since $[H_0, H_1] = 0$, it follows from the symbolic calculus that:

$$(y\partial_y - \eta\partial_\eta)(r_{k,1} - e_{k,1}) = \partial_s(r_{k,2} - e_{k,2}). \tag{37}$$

Written out explicitly, this consistency condition is:

$$\begin{aligned}
&\sum_{n\neq 0}(in)\left(\sum_{\alpha\beta\gamma} c^2_{\alpha\beta\gamma} y^\alpha \eta^\beta \sigma^\gamma\right) e^{ins} \\
&\quad = \sum_{\gamma,\alpha\neq\beta} c^1_{\alpha\beta\gamma}(0)(\alpha-\beta) y^\alpha \eta^\beta \sigma^\gamma \\
&\quad\quad + \sum_{k\neq 0}\left(\sum_{\alpha\beta\gamma} c^1_{\alpha\beta\gamma}(\alpha-\beta) y^\alpha \eta^\beta \sigma^\gamma\right) e^{iks}.
\end{aligned} \tag{38}$$

Comparing coefficients of the Fourier series in (38) yields

$$c^1_{\alpha\beta\gamma}(0) = 0 \text{ and } (\alpha-\beta)c^1_{\alpha\beta\gamma}(n) = inc^2_{\alpha\beta\gamma}(n) \tag{39}$$

for all $n \neq 0$. Since $r_{k,1} - e_{k,1}$ has no zeroth Fourier coefficient and by (39), $c^2_{\alpha\alpha\gamma}(n) = 0$ for all $n \neq 0$, it follows that $r_{k,2} - e_{k,2}$ contains no resonant terms of the form $y^\alpha \eta^\alpha$. Thus the systems in (35) and (36) can indeed be solved by Fourier series, and the inductive step has been proved.

Suppose now that we are in the loxodromic case. Consequently, we now assume that there are three commuting operators H_0, H_1, H_2 corresponding to a single complex hyperbolic summand with model operators:

$$Q_1 = -i\hbar\partial_s,\ Q_2 = \hat{I}^{ch}_{Re},\ Q_3 = \hat{I}^{ch}_{Im}.$$

Note that, if we introduce the complex variables $z = y_1 + iy_2$ and $\eta = \eta_1 - i\eta_2$, then

$$\begin{aligned}
I^{ch}_{Re} &= \frac{1}{2}(z\eta + \overline{z\eta}), \\
I^{ch}_{Im} &= \frac{i}{2}(\overline{z\eta} - z\eta),
\end{aligned} \tag{40}$$

and the corresponding Hamilton vector fields are $\Xi^{ch}_{Re} = \Re(z\partial_z - \eta\partial_\eta)$ and $\Xi^{ch}_{Im} = -i\Im(z\partial_z - \eta\partial_\eta)$. Now, just as in the real hyperbolic case above, we make Fourier series decompositions in s in all symbols and replace the coordinates y_1, y_2, η_1, η_2 with $z, \overline{z}, \eta, \overline{\eta}$. The consistency relations analogous to (37) are:

$$\partial_s(r_{2,k} - e_{2,k}) = -i(z\partial_z - \eta\partial_\eta - \overline{z}\partial_{\overline{z}} + \overline{\eta}\partial_{\overline{\eta}})(r_{1,k} - e_{1,k}), \tag{41}$$

$$\partial_s(r_{3,k} - e_{3,k}) = (z\partial_z - \eta\partial_\eta + \overline{z}\partial_{\overline{z}} - \overline{\eta}\partial_{\overline{\eta}})(r_{1,k} - e_{1,k}), \tag{42}$$

$$\begin{aligned} &- i(z\partial_z - \eta\partial_\eta - \overline{z}\partial_{\overline{z}} + \overline{\eta}\partial_{\overline{\eta}})(r_{3,k} - e_{3,k}) \\ &\qquad = (z\partial_z - \eta\partial_\eta + \overline{z}\partial_{\overline{z}} - \overline{\eta}\partial_{\overline{\eta}})(r_{2,k} - e_{2,k}). \end{aligned} \tag{43}$$

The relevant system of equations for v_k is:

$$\partial_s v_k = -r_{k,1} + e_{k,1}, \tag{44}$$

$$(z\partial_z - \eta\partial_\eta + \overline{z}\partial_{\overline{z}} - \overline{\eta}\partial_{\overline{\eta}})v_k = -r_{k,2} + e_{2,k}, \tag{45}$$

$$-i(z\partial_z - \eta\partial_\eta - \overline{z}\partial_{\overline{z}} + \overline{\eta}\partial_{\overline{\eta}})v_k = -r_{k,3} + e_{k,3}. \tag{46}$$

Note that the Hamilton vector fields Ξ^{ch}_{Re} and Ξ^{ch}_{Im} preserve monomials of the form

$$c_{\alpha\beta\gamma} z^\alpha \overline{z}^\beta \eta^\gamma \overline{\eta}^\delta.$$

In fact, a direct computation gives:

$$\Xi^{ch}_{Re}(z^\alpha \overline{z}^\beta \eta^\gamma \overline{\eta}^\delta) = (\alpha - \gamma + \beta - \delta)(z^\alpha \overline{z}^\beta \eta^\gamma \overline{\eta}^\delta), \tag{47}$$

$$\Xi^{ch}_{Im}(z^\alpha \overline{z}^\beta \eta^\gamma \overline{\eta}^\delta) = -i(\alpha - \gamma - \beta + \delta)(z^\alpha \overline{z}^\beta \eta^\gamma \overline{\eta}^\delta). \tag{48}$$

The first two consistency equations (41) and (42) ensure that the nonzero Fourier coefficients in the expansion of $-r_{k,2} + e_{k,2}$ and $-r_{k,3} + e_{k,3}$ do not contain resonant terms of the form $z^\alpha \overline{z}^\beta \eta^\gamma \overline{\eta}^\delta$, where $\alpha - \gamma + \beta - \delta = 0$ and $\alpha - \gamma - \beta + \delta = 0$ respectively. Thus, the system of equations (44)-(46) can be solved in the case of non-zero Fourier coefficients. Note that equations (41) and (42) also imply that the zeroth Fourier coefficient of $-r_{k,1} + e_{k,1}$ can only contain terms of the form $z^\alpha \overline{z}^\beta \eta^\gamma \overline{\eta}^\delta$ where both equations $\alpha - \gamma + \beta - \delta = 0$ and $\alpha - \gamma - \beta + \delta = 0$ are satisfied. Therefore, the zeroth Fourier coefficient of $-r_{k,1} + e_{k,1}$ is a sum of terms of the form

$$c_{\alpha\beta\kappa} z^\alpha \eta^\alpha \overline{z}^\beta \overline{\eta}^\beta \sigma^\kappa. \tag{49}$$

However, by (40), $z\eta = I^{ch}_{Re} + iI^{ch}_{Im}$ and $\overline{z\eta} = I^{ch}_{Re} - iI^{ch}_{Im}$ and so, by Theorem 2, we can choose $e_{1,k}(H_0, H_1, H_2)$ to ensure that no terms of the form (49) appear in the zeroth Fourier coefficient of $-r_{k,1}+e_{k,1}$. The arguments for the other terms $-r_{k,2} + e_{k,2}$ and $-r_{3,k} + e_{k,3}$ are similar and finally, the general case follows by repeating the above arguments for all the real and complex hyperbolic summands. Clearly, the above argument can also be carried out for elliptic summands, but this will not concern us here. q.e.d.

4. An example

In [21], we showed that the geodesic corresponding to rotation about the middle-length inertial axis supports eigenfunction mass by explicit separation of variables and an analysis of the resulting special functions. The purpose of this section is to show that [21] emerges simply as a special case of Theorem 1.

Recall, the Euler top is governed by a left-invariant Hamiltonian on $T^*SO(3)$ associated with a rigid body with distinct moments of inertia $0 < \alpha_1 < \alpha_2 < \alpha_3$ (see [1]). Let E_1, E_2, E_3 denote the standard basis of the Lie algebra $so(3)$ corresponding to the vectors e_1, e_2, e_3 in $\mathbb{R}^3$. The associated left-invariant vector fields, L_1, L_2, L_3 are defined by

$$L_i(f)(x) = \frac{d}{dt}\{f(x \exp tE_i)\}|_{t=0}.$$

One immediately verifies the commutation relations,

$$[L_1, L_2] = L_3, [L_2, L_3] = L_1, [L_3, L_1] = L_2. \tag{50}$$

Let $e = (0, 0, 1) \in \mathbb{R}^3$ and for $x \in SO(3)$, define

$$q_i(x) = (xe_i, e). \tag{51}$$

The quantum Euler top is governed by the partial differential operators:

$$P_0 = \sum_{j=1}^{3} \alpha_j L_j^2, \quad P_1 = \textstyle\sum_{j=1}^{3} L_j^2, \quad P_2 = \displaystyle\sum_{j=1}^{3} q_j L_j. \tag{52}$$

Note that the pairwise commutators $[P_i, P_j]$ all vanish for $i, j = 0, 1, 2$ and the quantum Hamiltonian is the left-invariant Laplacian, P_0, on

$SO(3)$. We denote the principal symbols of the operators L_1, L_2, L_3 by l_1, l_2, l_3 respectively. In this case, the classical Euler equations are:

$$\frac{dl_1}{dt} = (\alpha_3 - \alpha_2) l_2 l_3, \ \frac{dl_2}{dt} = (\alpha_1 - \alpha_3) l_1 l_3, \ \frac{dl_3}{dt} = (\alpha_2 - \alpha_1) l_1 l_2, \tag{53}$$

$$\begin{aligned} \frac{dq_1}{dt} &= \alpha_3 l_3 q_2 - \alpha_2 l_2 q_3, \ \frac{dq_2}{dt} = \alpha_1 l_1 q_2 - \alpha_3 l_3 q_1, \\ \frac{dq_3}{dt} &= \alpha_2 l_2 q_1 - \alpha_1 l_1 q_2. \end{aligned} \tag{54}$$

Consider the following parametrized submanifold of $T^*SO(3)$:

$$\begin{aligned} \Gamma(t) = \{(l(t), q(t)); q_1(t) = \cos \alpha_2 t, q_3(t) = \sin \alpha_2 t, \\ q_2(t) = l_1(t) = l_3(t) = 0, l_2(t) = 1\}. \end{aligned} \tag{55}$$

Since we will eventually need to compute Poincaré maps, we write down the first variation of the Euler system in (53) and (54) along $\Gamma(t)$. In particular,

$$\frac{d\delta l_1}{dt} = (\alpha_3 - \alpha_2)\delta l_3, \ \frac{d\delta l_2}{dt} = 0, \ \frac{d\delta l_3}{dt} = (\alpha_2 - \alpha_1)\delta l_1, \tag{56}$$

$$\frac{d\delta q_2}{dt} = \alpha_1 \delta l_1 \sin \alpha_2 t - \alpha_3 \delta l_3 \cos \alpha_2 t. \tag{57}$$

Therefore, from (56) and (57) it follows that the map $P_\gamma := d \exp T\Xi_{p_0}$ restricted to the span of $l_1(t), l_2(t), l_3(t), q_2(t), \delta l_1(t), \delta l_2(t), \delta l_3(t), \delta q_2(t)$ where $T = 2\pi/\alpha_2$, is of the form:

$$P_\gamma = \begin{pmatrix} e^{TA} & 0 \\ B & Id \end{pmatrix}.$$

Hence, P_γ has 1 as a double eigenvalue and the other eigenvalues λ are determined by solving the characteristic equation

$$\det(e^{TA}) - \operatorname{trace}(e^{TA})\lambda + \lambda^2 = 0. \tag{58}$$

However, since

$$TA = \begin{pmatrix} 0 & 2\pi\alpha_2^{-1}(\alpha_3 - \alpha_2) \\ 2\pi\alpha_2^{-1}(\alpha_2 - \alpha_1) & 0 \end{pmatrix},$$

it follows that $\det(e^{TA}) = 1$ and that

$$\operatorname{trace}(e^{TA}) = 2 \sum_{j=0}^{\infty} \frac{x^j}{(2j)!}, \tag{59}$$

where, $x = 4\pi^2\alpha_2^{-2}(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)$. Therefore,

$$\lambda = \cosh\left(\sqrt{2\pi\alpha_2^{-1}(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}\right) \pm \sinh\left(\sqrt{2\pi\alpha_2^{-1}(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}\right). \tag{60}$$

Thus, the eigenvalues of P_γ are of the form $1, 1, \lambda, \lambda^{-1}$ where λ is given by (60). It turns out that the degenerate subspace corresponding to $1, 1$ can be dispensed with by performing reduction: Recall, the set

$$\mathcal{O}_0 = \{(q,l); q_1^2+q_2^2+q_3^2 = 1, q_1l_1+q_2l_2+q_3l_3 = 0\} \cong T^*\mathbb{S}^2 \tag{61}$$

can be naturally identified with the null orbit of the left action of $SO(2)$ on $SO(3)$, where we consider $SO(2) \subset SO(3)$ as an upper 2×2 submatrix. It is clear that the submanifold, $\Gamma(t)$ descends to a periodic geodesic, $\gamma(t)$, in the reduced system, where the reduced Hamiltonian $H = p_1$ and the integral in involution p_2 are given by the expressions:

$$p_1 = \alpha_3(x_1\xi_2 - x_2\xi_1)^2 + \alpha_2(x_1\xi_3 - x_3\xi_1)^2 + \alpha_1(x_3\xi_2 - x_2\xi_3)^2, \tag{62}$$

$$p_2 = (x_1\xi_2 - x_2\xi_1)^2 + (x_1\xi_3 - x_3\xi_1)^2 + (x_3\xi_2 - x_2\xi_3)^2. \tag{63}$$

Here, we identify T^*S^2 with the set of points

$$\{(x,\xi) \in \mathbb{R}^6;\ |x| = 1, x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = 0\}.$$

It is not hard to see that [20], [21] the corresponding quantum Hamiltonian, P_1, is a second order, elliptic partial differential operator that is the radial part of a left-invariant Laplacian on $SO(3)$. The quantum commutant is just the standard constant curvature Laplacian on $\mathbb{S}^2$; that is, $P_2 = -\Delta$. Without loss of generality, we suppose $\alpha_2 = 1$ and consider the level set,

$$\Sigma_E = \{(x,\xi) \in T^*S^2; p_1(x,\xi) - 1 = p_2(x,\xi) - 1 = 0\}. \tag{64}$$

Lemma 1. *The curve,*

$$\gamma(t) = \{x_2(t) = \xi_2(t) = 0, x_1(t) = -\xi_3(t) = \cos t,\ x_3(t) = \xi_1(t) = -\sin t; 0 \le t \le 2\pi\}$$

is a joint geodesic of p_1, p_2 which is real-hyperbolic for p_1 and satisfies the hypotheses of Theorem 1.

Proof. The result follows from the computations in (58)-(60), taking into account that the $1,1$ subspace disappears upon reduction. In particular, the eigenvalues of the Poincaré map of p_1 are λ, λ^{-1}, where λ is given in (60). As far as p_2 is concerned, it is just the unit constant curvature metric form on $\mathbb{S}^2$ and so, its Poincaré map has eigenvalues $1,1$. The end result is that in terms of modified Fermi coordinates (y, η, s, σ),

$$p_1 - 1 = \sigma + \lambda y\eta + ..., \tag{65}$$

$$p_2 - 1 = \sigma + y\eta + ..., \tag{66}$$

where, the dots indicate terms which are of total order at least three in (y, η, σ). Therefore, when $y = \eta = \sigma = 0$,

$$\det \begin{pmatrix} \nabla_\sigma p_1 & \nabla_{y\eta} p_1 \\ \nabla_\sigma p_2 & \nabla_{y\eta} p_2 \end{pmatrix} = 1 - \lambda \neq 0.$$

Thus, hypothesis (H1) is verified. Finally, it is not difficult to show that (H2) is also satisfied by constructing the action-angle variables explicitly (see [21]). The lemma follows. q.e.d.

By a similar argument, it can be shown that the middle-length axial ellipse on a triaxial ellipsoid also satisfies the hypotheses of Theorem 1. Also, one can generalize the argument above to higher dimensions; i.e., to $SO(n)$ for arbitrary n and get many examples of unstable orbits that satisfy the hypotheses of Theorem 1. In fact, even inhomogeneous examples such as the quantum Lagrange top fall under this rubric. Note that, when the angular momentum of the Lagrange top is below a certain threshold, the periodic geodesic in $T^*SO(3)$ corresponding to simple nutation turns out to be complex hyperbolic (see [7]). One can show that Theorem 1 also applies in this case. We hope to return to some of these examples elsewhere.

5. Asymptotic properties of the classical flow

Let M^n be a compact, Riemannian manifold and

$$p_0(x, \xi) = \sqrt{g^{ij}(x)\xi_i\xi_j}$$

be the Hamiltonian function for geodesic flow. Suppose there exist C^∞ functions $p_1, ..., p_{n-1}$ with the property that:

$$\{p_k, p_l\} = 0 \tag{67}$$

for all $k, l = 0, 1, ..., n-1$. Assume moreover, that the differentials $dp_0, ..., dp_{n-1}$ are linearly independent almost everywhere with respect to Liouville measure on T^*M. In equation (67), the Poisson bracket is taken with respect to the canonical symplectic form on T^*M. By a theorem of Arnol'd, for regular energy levels $E_0, ..., E_{n-1}$, there exist canonical coordinates $(I_1, ..., I_n, \theta_1, ..., \theta_n)$ (action-angle coordinates) near each connected component of $\Sigma_E = \{(x,\xi) \in T^*M; p_0(x,\xi) - E_1 = ... = p_{n-1}(x,\xi) - E_n = 0\}$. Moreover, each component is an n-dimensional torus and locally $H = H(I_1, ..., I_n)$. Thus, in terms of (I, θ), the equations for the geodesic flow on Σ_E are given by:

$$\frac{dI_j}{dt} = 0, \tag{68}$$

$$\frac{d\theta_j}{dt} = \frac{\partial H}{\partial I_j}. \tag{69}$$

In particular, the conditionally periodic flow on such a torus in linearized in the coordinates (I, θ). However, generically there exist exceptional level varieties, Σ_E, which are singular. Moreover, there are no action-angle variables near Σ and thus, the classical flow $\exp t\Xi_H$ is much more complicated. These singular varieties are precisely the objects of interest here.

Let $\gamma_1 \cup ... \cup \gamma_k$ be a collection of k, non-degenerate, unstable, periodic geodesics all contained in the level variety

$$\Sigma_E = \{(x,\xi) \in T^*M; p_0(x,\xi) - E_0 = ... = p_{n-1}(x,\xi) - E_{n-1} = 0\}.$$

Note that, by homogeneity considerations, there is no loss of generality in taking $E_0 = 1$. Recall, $\Omega_1 \times \gamma_k, ..., \Omega_k \times \gamma_k$ are arbitrarily small, but fixed tubular neighbourhoods of the geodesics $\gamma_1, ..., \gamma_k$ respectively. Our objective here is to discuss a simple criterion for analyzing the long-time flow on Σ_E and to give a sufficient condition to determine when hypothesis (H2) is satisfied:

Lemma 2. *Let $U_1, .., U_m$ be the connected components of the complement, $\Sigma_E - \cup_j \Omega_j \times \gamma_j$, where for each $j; 1 \leq j \leq k$, γ_j is an unstable, nondegenerate periodic geodesic. Suppose that there exist symplectic Darboux coordinates $(x_1^{(k)}, ..., x_n^{(k)}, \xi_1^{(k)}, ..., \xi_n^{(k)})$ in a neighbourhood of each component, U_k, and that $H|_{U_k} = H(\xi_1^{(k)}, ..., \xi_n^{(k)})$ in this neighbourhood with $\nabla_{\xi_1} H \neq 0$. Then, for each $z \in \Sigma_E$ there exists some γ_j such that:*

$$\exp t\Xi_H(z) \to \gamma_j \ \ as \ \ t \to \infty.$$

Thus, in particular, (H2) is satisfied.

Proof. Without loss of generality, we assume that there is a single connected component, U_1, and will drop the superscript in the Darboux coordinates. By the Smale theorem, there exist transversal, open manifolds $W_s(\gamma_1), W_u(\gamma_1)$ defined near γ_1 such that $W_s(\gamma_1) \cap W_u(\gamma_1) = \gamma_j$. Moreover, W_s and W_u are both invariant under the flow with $\exp t\Xi_H(z) \to \gamma_1$ for $z \in W_s$ (resp. W_u) when $t \to \infty$ (resp. $-\infty$). It is not hard to show that W_s and W_u are both Lagrangian manifolds where for each $m \in \gamma_1$, $T_m(W_s)$ (resp. $T_m(W_u)$) is the sum of eigenspaces of P_{γ_1} corresponding to eigenvalues of modulus < 1 (resp. > 1). $W_s(\gamma_1)$ and $W_u(\gamma_1)$ are commonly referred to as the respective stable and unstable manifolds of γ_1, where,

$$\Omega_1 \times \gamma_1 \cap \Sigma_E = \Omega_1 \times \gamma_1 \cap (W_s \cup W_u). \tag{70}$$

Suppose now that the lemma is false; that is, there exists $z \in \Sigma_E$ such that $\exp t\Xi_H(z)$ does not converge to the geodesic γ_1. We now show that this leads to a contradiction by constructing a very simple Liapunov function on the complement U_1: Indeed, consider the function

$$g(x, \xi) = x_1. \tag{71}$$

By assumption, $g(x,\xi)$ is a well-defined, smooth function on U_1, with the property that:

$$\{g, H(\xi)\} = \frac{\partial H}{\partial \xi_1} \neq 0 \tag{72}$$

on U_1. To simplify the notation a little bit, we will denote the flow $\exp t\Xi_H$ simply by ϕ_t. Fix $z \in U_1$ and consider the integral,

$$I_t(z) := \int_0^t \frac{d}{ds} \phi_s^* g(z)\, ds. \tag{73}$$

Clearly,

$$I_t(z) = \int_0^t \{g, H\}(\exp s\Xi_H(z))\, ds. \tag{74}$$

Since we are assuming the flow does not converge to a geodesic, it must never reach a stable manifold, $W_s(\gamma_j)$. As a consequence, the flow either stays in the complement U_1 or intersects the unstable manifold, W_u.

Suppose that the latter scenario holds and that $\phi_T(z) \in W_u(\gamma_j)$ for some γ_j. Then, there exist constants $c_1, c_2 > 0$ such that:

$$\frac{d}{dt} d(\gamma_j, \phi_t(z))|_{t=T} \geq c_1 e^{c_2 |T|},$$

where $d(,)$ denotes a distance function for some Riemannian metric on T^*M. Thus, for $s > T$, $d(\gamma_j, \phi_s(z)) > d(\gamma_j, \phi_T(z))$ and so, the flow reenters the complement U_1. As a consequence, the function g continues to be well-defined and in particular, (72) is satisfied. Therefore, it then follows that for all $t > 0$

$$(75) \qquad I_t(z) = \frac{\partial H}{\partial \xi_1} \cdot \int_0^t ds = \frac{\partial H}{\partial \xi_1} \cdot t.$$

On the other hand, (73) together with the compactness of Σ_E implies that:

$$(76) \qquad I_t(z) = g(\exp t\Xi_H(z)) - g(z) = \mathcal{O}(1)$$

uniformly in $z \in U_1$ as $t \to \infty$. Clearly, (76) contradicts (75) and the lemma follows. q.e.d.

6. Microlocalization near γ_j

Let ψ_j be an L^2-normalized joint eigenfunction of $\hbar P_0, ..., \hbar P_{n-1}$ corresponding to an eigenvalues $\lambda_j(\hbar) = E_j + o(1); j = 0, ..., n-1$ and $\chi(t_1, ..., t_n) \in C_0^\infty(\mathbb{R}^n)$ a cutoff function which is identically 1 near $(0, ..., 0)$. As before, we will assume that $\gamma_1, ..., \gamma_k \subset \Sigma_E$ are unstable, nondegenerate, periodic bicharacteristics of $H = p_0$ and that (H1) and (H2) are satisfied. Our first objective here is to microlocalize the analysis of expected values of the ψ_j to arbitrarily small tubular neighbourhoods of the γ_k's. To begin, we introduce the appropriate classes of $\hbar$-pseudodifferential operators: Let $\Omega \subset \mathbb{R}^n$ be an open set. We say $a(x, \xi; \hbar) \in S^{m,k}_{cl}(\Omega \times \mathbb{R}^n)$ if $a \sim \hbar^{-m} \sum_j a_j \hbar^j$, where, for any $\alpha, \beta \in \mathbb{N}^n$,

$$(77) \qquad |\partial_x^\alpha \partial_\xi^\beta a_j(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{k-j-|\beta|}.$$

As is customary, we will henceforth write $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$. The corresponding Kohn-Nirenberg $\hbar$-pseudodifferential operators are defined locally by:

$$(78) \qquad Op_\hbar u(x) := (2\pi\hbar)^{-n} \int e^{i(x-y)\xi/\hbar} a(x, \xi; \hbar) u(y) dy d\xi.$$

Such operators form a calculus, provided we work modulo operators in $Op_{\hbar}(S^{-\infty,-\infty})$. In particular,

$$Op_{\hbar}(S_{cl}^{m,k}) \circ Op_{\hbar}(S_{cl}^{m',k'}) \subset Op_{\hbar}(S_{cl}^{m+m',k+k'}),$$

and there exists the usual composition formula for symbols: If

$$Op_{\hbar}(c) = Op_{\hbar}(a) \circ Op_{\hbar}(b),$$

then:

$$c(x,\xi;\hbar) \sim \sum_{\alpha} \frac{(i\hbar)^{|\alpha|}}{\alpha!} \partial_x^{\alpha} a(x,\xi;\hbar) \partial_{\xi}^{\alpha} b(x,\xi;\hbar). \tag{79}$$

Without loss of generality, we will assume that the operators $\hbar P_0, ..., \hbar P_{n-1}$ are all in $Op_{\hbar}(S_{cl}^{0,1})$ and since we are interested in spectral asymptotics, we will choose the artificial semiclassical parameter, $\hbar$, so that $\hbar^{-1} \in \text{Spec } P_0$. Consider the cutoff function,

$$\chi_E(x,\xi) := \chi(p_0 - E_0, ..., p_{n-1} - E_{n-1}) \in S^{0,-\infty}. \tag{80}$$

First, to microlocalize the analysis to a neighbourhood Ω_E of Σ_E, we consider the action of the operator $Op_{\hbar}(\chi_E)$ on the eigenfunctions ψ_j. When there is no risk of confusion, we will denote both the cutoff function and the associated operator by χ.

Lemma 3. *Let ψ_j be an L^2-normalized, joint eigenfunction satisfying*

$$(H_0 - \lambda_0)\psi_j = ... = (H_{n-1} - \lambda_{n-1})\psi_j = 0,$$

where $\lambda_j(\hbar) = E_j + o(1)$. Then,

$$\|(1-\chi_E)\psi_j\| = \mathcal{O}(\hbar^{\infty}).$$

Proof. Since

$$(\hbar P_0 - \lambda_0)\psi_j = ... = (\hbar P_{n-1} - \lambda_{n-1})\psi_j = 0,$$

we can write,

$$P(1-\chi_E)\psi_j = [P, 1-\chi_E]\psi_j. \tag{81}$$

where, for simplicity of notation, we have written

$$P := \sum_{j=0}^{n-1} (\hbar P_j - \lambda_j)^* \cdot (\hbar P_j - \lambda_j). \tag{82}$$

Note that the operator P is $\hbar$-elliptic on supp $(1-\chi_E)$. Thus, if χ'_E is a cutoff function which is identically 1 on supp χ_E (we will henceforth use the notation $\chi_E \prec \chi'_E$), by a parametrix construction in the calculus, one can construct a symbol, $r(x,\xi;\hbar) \in S^{0,-1}_{cl}$ depending locally smoothly on $(\lambda_1, \dots, \lambda_n)$, such that:

$$(83) \qquad (1-\chi'_E)PR(1-\chi_E)\psi_j = (1-\chi'_E)\psi_j + \mathcal{O}(\hbar^\infty).$$

On the other hand, by the symbolic calculus,

$$(84) \qquad \sigma([P,1-\chi_E]) \sim \sum_\alpha \frac{(i\hbar)^{|\alpha|}}{\alpha!}(\partial_x^\alpha \sigma(P)\partial_\xi^\alpha(1-\chi_E) - \partial_x^\alpha(1-\chi_E)\partial_\xi^\alpha \sigma(P)).$$

Since for any $|\alpha| \geq 1$, $\partial^\alpha_{x,\xi}(1-\chi_E) = 0$ on supp $(1-\chi'_E)$, we have

$$(85) \qquad \|\,(1-\chi'_E)\,[P,1-\chi_E]\,\| = \mathcal{O}(\hbar^\infty).$$

Finally, note that the initial choice of the cutoff function χ_E was arbitrary. Therefore, by working with a cutoff $\chi''_E \prec \chi_E$ instead of χ_E, the result follows. q.e.d.

Given a classical observable $q \in C_0^\infty(T^*M)$ as above, our aim is to study the asmyptotics of the expected values $(Q\psi_j, \psi_j)$ of the associated quantum observable, $Q := Op_\hbar^F(q)$, where ^{F} denotes a semiclassical, anti-Wick (Friedrichs) quantization. The result of Theorem 1 is independent of the particular anti-Wick quantization but, for the sake of concreteness, we fix such a quantization once and for all as follows: Given $q \in C_0^\infty(T^*M)$, we define:

$$(86) \qquad Op_\hbar^F(q)u(x;\hbar) = (2\pi\hbar)^{-3n/2}\int e^{i(\phi(z,x,\xi)-\overline{\phi(z,y,\xi)})/\hbar} \cdot q(x,\xi)\chi(x-z)\chi(x-y)u(y)dydzd\xi,$$

where,

$$(87) \qquad \phi(x,y,\xi) = \exp_x^{-1}(y)\cdot\xi + \frac{i}{2}d^2(x,y).$$

Here, $\exp_x : T_xM \to M$ is the geodesic exponential map for the metric form $g^{ij}\xi_i\xi_j$, $d(\cdot,\cdot)$ is the Riemann distance function and χ is a cutoff function supported in a sufficiently small neighbourhood of 0. It is readily shown that, given $q = q(x,\xi) \in C_0^\infty(T^*M)$,

$$Op_\hbar^F(q) = Op_\hbar(q) + Op_\hbar(r_2), \tag{88}$$

where $r_2(x,y,\xi;\hbar) \in C_0^\infty$ and is $\mathcal{O}(\hbar)$. The identity (88) is proved as follows: Let $\chi_j; j = 1, ..., N$ be a partition of unity on M subordinate to a covering by coordinate charts: By performing an iterated integral, we get that $Op_\hbar^F(q)(\chi_j u)(x)$ equals:

$$=(2\pi\hbar)^{-3n/2} \int e^{i(x-y)\xi/\hbar} \Big(\int e^{i[(y-x)\xi + \phi(z,x,\xi) - \overline{\phi(z,y,\xi)}]/\hbar} \cdot q(x,\xi)\chi(x-z)\chi(x-y)dz\Big)\chi_j(y)dyd\xi \tag{89}$$

$$=(2\pi\hbar)^{-n} \int e^{i(x-y)\xi/\hbar}\tilde{q}(x,y,\xi)\chi_j(y)dyd\xi. + (2\pi\hbar)^{-n} \cdot \int e^{i(x-y)\xi/\hbar} r_1(x,y,\xi;\hbar)\chi_j(y)dyd\xi. \tag{90}$$

Here, we have used the fact that, by definition (87), the phase in (89) is non-degenerate in z and, as a consequence, $r_1 \in C_0^\infty$ and $r_1(x,y,\xi;\hbar) = \mathcal{O}(\hbar)$. Finally, by making a Taylor expansion about $x = y$ in (90) and an integration by parts:

$$Op_\hbar^F(q) = Op_\hbar(q) + Op_\hbar(r_2), \tag{91}$$

where, $r_2(x,y,\xi;\hbar) \in C_0^\infty$ and $r_2 = \mathcal{O}(\hbar)$. As a consequence of the above argument combined with Lemma 3, it follows that:

$$\|(1 - Op_\hbar^F(\chi_E))\psi_j\| = \mathcal{O}(\hbar^\infty). \tag{92}$$

Henceforth, when there is no risk of confusion, we drop the superscript ^{F}, with the understanding that unless otherwise stated, anti-Wick quantization is implied. Summing up, we so far have shown that:

$$(Q\psi_j, \psi_j) = (\chi_E Q\psi_j, \psi_j) + \mathcal{O}(\hbar^\infty). \tag{93}$$

To refine the microlocalization in (93), we now introduce a few other cutoff functions about the geodesics γ_k: Let $d(,)$ denote a Riemannian distance function on T^*M. Consider the 1-parameter family of tubular neighbourhoods of the form:

$$\Omega_k(r) = \{(x,\xi) \in T^*M; d((x,\xi),\gamma_k) < r\epsilon\},$$

where, $r \in \mathbb{R}$ and $\epsilon > 0$ is a fixed constant. Let $\chi_1^k \in C_0^\infty(\Omega_k(2))$ be identically 1 in $\Omega_k(1)$ and $\chi_{12} \in C_0^\infty(\Omega_k(13/8)-\Omega_k(11/8))$ be identically 1 in $\Omega_k(7/4) - \Omega_k(5/4)$. Write

$$(Q\psi_j, \psi_j) = \sum_{l=1}^{k} (\chi_1^l Q\psi_j, \psi_j) + \sum_{l=1}^{k} ((1-\chi_1^l)Q\psi_j, \psi_j). \tag{94}$$

Now, if $U(t) = \exp(itP_0)$, since ψ_j is an eigenfunction of P_0, by the unitarity of $U(t)$ together with Lemma 3 it follows that:

$$((1-\chi_1^l)Q\psi_j, \psi_j) = ((1-\chi_1^l)\chi_E Q\psi_j, \psi_j) + \mathcal{O}(\hbar^\infty) \tag{95}$$

$$= (U(t)(1-\chi_1^l)\chi_E Q U(-t)\psi_j, \psi_j) + \mathcal{O}(\hbar^\infty). \tag{96}$$

Then, by the semiclassical Egorov Theorem [14],

$$\begin{aligned}(U(t)&(1-\chi_1^l)\chi_E Q U(-t)\psi_j, \psi_j) \\ &= (Op_\hbar(\exp t\Xi_{p_0}^*(1-\chi_1^l)\chi_E q)\psi_j, \psi_j) + \mathcal{O}(\hbar).\end{aligned} \tag{97}$$

Note that in Lemma 3, we can choose supp χ arbitrarily small. Thus, it follows from hypothesis (H2) that for $\hbar$ sufficiently small, there exists $T \in \mathbb{R}$, such that for all $l = 1, .., k$,

$$\bigcup_{l=1}^{k} \mathrm{supp}(\chi_{12}^l) \supset \mathrm{supp}(\exp T\Xi_{p_0}^*(1-\chi_1^l)\chi_E q). \tag{98}$$

We can in fact assume that $\chi_{12}^l = 1$ on $\mathrm{supp}(\exp T\Xi_{p_0}^*(1-\chi_1^l)\chi_E q)$. As a consequence, we have proved

Proposition 1. *Let $\psi_j \in C^\infty(M)$ be an L^2-normalized joint eigenfunction satisfying $\hbar P_k \psi_j = \lambda_k(\hbar)\psi_j$ for $k = 0, ..., n-1$, where $\lambda_k(\hbar) = E_k + o(1)$. Then, for $\hbar$ sufficiently small,*

$$(Q\psi_j, \psi_j) = \sum_{l=1}^{k} (\chi_1^l \chi_E Q\psi_j, \psi_j) + E(\hbar) + \mathcal{O}(\hbar),$$

where, $|E(\hbar)| \leq \|q\|_\infty \left(\sum_{l=1}^{k} \|\chi_{12}^l \psi_j\|\right)$.

To estimate the terms $(\chi_1^l \chi_E Q\psi_j, \psi_j)$ and $E(\hbar)$ appearing in Proposition 1, we will need the microlocal Birkhoff normal form proved in

Theorem 3. In particular, recall that there exists a microlocally unitary $F_l : C_0^\infty(\Omega_l \times \gamma_l) \to C_0^\infty(\Omega_0 \times \mathbb{S}^1)$ such that

$$\|\chi_1^l (F_l g_0(H_0, ..., H_{n-1}; \hbar) F_l^{-1} - \frac{\hbar}{i}\partial_s)\| = \mathcal{O}(\hbar^\infty) \tag{99}$$

and

$$\|\chi_1^l (F_l g_j(H_0, ..., H_{n-1}; \hbar) F_l^{-1} - Q_j)\| = \mathcal{O}(\hbar^\infty) \tag{100}$$

for $j = 1, ..., n-1$. Moreover, for each γ_l, there exist canonical generalized Fermi coordinates $(s, \sigma, y, \eta) \in T^*(\mathbb{S}^1) \times T^*(\mathbb{R}^{n-1})$ in terms of which,

$$\gamma = \{(s, 0, 0, 0); s \in \mathbb{S}^1\}. \tag{101}$$

Clearly, we would like to replace the ψ_j by $u_j := F_l \psi_j$ and then estimate the latter by using the explicit form of the model problem given by the microlocal Birkhoff normal form above. To estimate the first term on the RHS of Proposition 1, note that, since F_l is microlocally unitary on Ω_2^l,

$$\begin{aligned} (\chi_1^l \chi_E Q\psi_j, \psi_j) &= (\chi_1^l Q\psi_j, \psi_j) + \mathcal{O}(\hbar^\infty) \quad (102) \\ &= (\chi_1^l Q F_l u_j, \chi_2 F_l u_j) + \mathcal{O}(\hbar^\infty) \\ &= (F_l^* \chi_2^* F_l F_l^* \chi_1 Q F_l u_j, u_j) + \mathcal{O}(\hbar^\infty). \quad (103) \end{aligned}$$

Therefore,

$$(\chi_1^l \chi_E Q\psi_j, \psi_j) = \sum_{l=1}^{k} (\chi_1^{l\prime} Q_l' u_j, u_j) + \mathcal{O}(\hbar^\infty),$$

where $Q_l' := F_l^* Q F_l$. Moreover, since χ_{12}^l is supported on the annular region $\Omega_{13/8}^l - \Omega_{11/2}^l$, it also follows that

$$|E(\hbar)| \leq \|q\|_\infty (\sum_{l=1}^{k} \|\chi_{12}^{l\prime} u_j\|),$$

where $\chi_{12}^{l\prime} := F_l^* \chi_{12}^l F_l$. Hence, as a consequence of Proposition 1, we have:

$$(Q\psi_j, \psi_j) = \sum_{l=1}^{k} (\chi_1^{l\prime} Q_l' u_j, u_j) + E(\hbar) + \mathcal{O}(\hbar), \tag{104}$$

where, $|E(\hbar)| \leq \|q\|_\infty (\sum_{l=1}^{k} \|\chi_{12}^{l\prime} u_j\|)$.

To simplify the writing somewhat, we will drop the primes indicating microlocally defined operators in the model variables (s, σ, y, η): For example, the pseudodifferential operators $\chi_{1'}^{l}$ will be denoted simply by $\chi_1^{l'}$. Before going on to estimate the RHS of (104) we will make one further standard simplification involving the so-called averaging technique. The point of this final simplification is that in the course of proving Theorem 1, we will need to make a Taylor expansion of the symbol $q_l(s, \tau, y, \eta)$ about γ_l. The averaging argument will enable us to assume without loss of generality that q is *constant* along γ_l. To describe the ansatz, consider the $\hbar$-pseudodifferential operator:

$$\mathcal{H}_l := F_l^* \hbar P_0 F_l \tag{105}$$

acting on $C_0^\infty(\mathbb{S}^1 \times \mathbb{R})$. Denote the Weyl symbol of $\mathcal{H}_l$ by $h_l(s, \sigma, y, \eta)$. Since $\sigma \sim 0$ and $\eta^2 + y^2 < \epsilon$ on Ω_l, it follows that, for $\epsilon > 0$ sufficiently small, $h_l(s, \tau, y, \eta)$ is $\hbar$-elliptic on Ω_1^l. Therefore, microlocally on supp χ_1^l, the unitary operator $W_l(t) := \exp(it\mathcal{H}_l/\hbar)$ is an $\hbar$-Fourier integral operator with Schwartz kernel of the form:

$$\begin{aligned} &W_l(y', y, s', s; t) \\ &\quad = (2\pi\hbar)^{-2} \int e^{i(\phi(y', s', \eta, \tau) - y\eta - s\tau)/\hbar} \, a(y', y, s', s, \eta, \tau; \hbar) d\eta d\tau, \end{aligned} \tag{106}$$

where ϕ is the generating function of the time t bicharacteristic flowout of h_l. Note that, in particular, the geodesic $\gamma = \{(t, 0, 0, 0)\}$ is a bicharacteristic of period 2π. Define

$$q_l^{av} = \frac{1}{2\pi} \int_0^{2\pi} \exp t\Xi_{h_l}^* q_l \, dt. \tag{107}$$

Then,

$$(\chi_1^l Q_l u_j, u_j) = \frac{1}{2\pi} \int_0^{2\pi} (W_l(-t) \chi_1^l Q_l W_l(t) u_j, u_j) dt + \mathcal{O}(\hbar^\infty). \tag{108}$$

By an application of the Fubini theorem to interchange orders of integration combined with the semiclassical Egorov theorem, it follows that:

$$(\chi_1^l Q_l u_j, u_j) = (Op_\hbar(\chi_1^l q_l^{av}) u_j, u_j) + \mathcal{O}(\hbar). \tag{109}$$

Clearly, q_l^{av} is constant along γ_l and is in fact equal to $(2\pi)^{-1} \int_{\gamma_l} q_l$. Therefore, by (109) we can assume that q_l is constant on γ_l, modulo an error term that is $\mathcal{O}(\hbar)$ in L^2.

7. The real hyperbolic case

Let $\Omega' = [-1,+1]^2$ and $|t| < 1$. Consider the operator

$$P = \frac{\hbar}{2}(D_y y + y D_y) = -i\hbar(y\partial_y + 1/2). \tag{110}$$

Let $v \in \mathcal{D}'(\Omega')$ solve the eigenfunction equation:

$$Pv = tv. \tag{111}$$

A distributional basis of generalized solutions of (111) [6] is given by:

$$v_\pm(y;\hbar) = |\log \hbar|^{-1/2} H(\pm y)|y|^{-1/2+it/\hbar}. \tag{112}$$

Here, $H(y)$ denotes the Heaviside function, and the reason for introducing the normalizing constant $|\log \hbar|^{-1/2}$ will become apparent later on. Recall, a family of distributions $\phi_\hbar(x)$ is said to be *admissible* if there exist real numbers C_1, C_2, C_3 such that:

$$\forall u \in C_0^\infty,\ |\int u(x)\phi_\hbar(x)dx| \leq \hbar^{-C_1} \sup_{k \leq C_3} |\partial_x^k(\langle x\rangle^{C_2} u)|.$$

We shall need the following characterization of the admissible microlocal solutions of equation (111) due to Colin de Verdière and Parisse [6]:

Proposition 2 ([6]). *Let $|t| \leq 1$ and $u \in \mathcal{D}'(\Omega')$ be an admissible solution of the equation (111). Then, there exist $\alpha_\pm(\hbar) \in \mathbb{C}$ such that:*

$$\|\chi_1'(\alpha_+(\hbar)v_+(y;\hbar) + \alpha_-(\hbar)v_-(y;\hbar) - u)\| = \mathcal{O}(\hbar^\infty).$$

Here, $\chi_1'(x,y,\xi) \in C_0^\infty$ is a cutoff function which is identically equal to 1 on $[-1/2,1/2]^3$ which, for convenience, we take to be of the form:

$$\chi_1'(x,y,\xi) = \chi(x)\chi(y)\chi(\xi),$$

where, $\chi \in C_0^\infty(\mathbb{R})$ is identically 1 on $[-1/2,1/2]$. For our present purposes, we will really only need to understand the case $E = \mathcal{O}(\hbar)$ here. Nevertheless, with a view towards future applications, it will be useful to obtain certain order of magnitude estimates for the microlocal masses of the $v_\pm$ which are uniform in E and in $\hbar$, provided $\hbar$ is sufficiently small. Lemma 4 can be found in a slightly different form in [6]. However, for the sake of completeness and to collect the various estimates we shall need later on, we will sketch the arguments.

Lemma 4. *Let $|t| < 1$. Then, for $\hbar$ sufficiently small, there exist constants $C_1, C_2 > 0$ independent of $\hbar$ and t such that:*

$$(i)\ C_1 \leq (\chi_1' v_\pm, v_\pm) \leq C_2,$$

$$(ii)\ |(\chi_1' v_\pm, v_\mp)| \leq 10^{-3} C_1.$$

Proof. It will be convenient to split the analysis into two separate cases: Let $\epsilon > 0$ be a fixed constant the size of which will be determined in due course. First, we assume that $|t| \leq \frac{1}{\epsilon}\hbar$. Then,

$$\begin{aligned}(\chi_1' v_\pm, v_\pm) =& (2\pi \log \hbar)^{-1} \int_0^{1/\hbar} \frac{d\eta}{\eta} \left| \int_0^\infty e^{-i(y - t/\hbar \log y)} \chi(y/\eta) y^{-1/2} dy \right|^2 \\ =& (2\pi \log \hbar)^{-1} \int_1^{1/\hbar} \frac{d\eta}{\eta} \left| \int_0^\infty e^{-i(y - t/\hbar \log y)} \chi(y/\eta) y^{-1/2} dy \right|^2 \\ &+ \mathcal{O} |\log \hbar|^{-1}. \end{aligned} \tag{113}$$

where, the last line in (113) follows by noting that when $0 \leq \eta \leq 1$,

$$\int_0^\infty e^{-iy} \chi(y/\eta) y^{-1/2 + it/\hbar} dy = \mathcal{O}(\eta^{1/2}). \tag{114}$$

Thus, by the standard asymptotic expansion for the indefinite Gamma functions [3] and a change of integration contour, it follows that:

$$\begin{aligned}(\chi_1' v_\pm, v_\pm) =& \frac{1}{2\pi} |\Gamma(1/2 + it/\hbar)|^2 e^{t/\hbar\pi} + \mathcal{O}(|\log \hbar|^{-1}) \\ =& (1 + e^{-2t\pi/\hbar})^{-1} + \mathcal{O}(|\log \hbar|^{-1}),\end{aligned} \tag{115}$$

where, in the last line we have used the identity

$$|\Gamma(1/2 + iy)|^2 = \pi (\cosh \pi y)^{-1}.$$

By a similar argument, one can show that:

$$|(\chi_1' v_\pm, v_\mp)| = \mathcal{O}(|\log \hbar|^{-1}). \tag{116}$$

This concludes the argument in the case where $|t| \leq \frac{1}{\epsilon}\hbar$.

Assume now that $|t| \geq \frac{1}{\epsilon}\hbar$. In such a case, it is generally too ambitious to look for asymptotic expansions in $\hbar$ which are uniformly valid in t. Instead, we treat $\frac{1}{\epsilon}$ as the large asymptotic parameter and apply stationary phase expansions with error terms. More precisely, by making the change of variables $y = \frac{t}{\hbar} z$ in (113), we get

$$
\begin{aligned}
(\chi_1' v_\pm, v_\pm) = &(2\pi \log \hbar)^{-1} \frac{t}{\hbar} \\
&\cdot \int_1^{1/\hbar} \frac{d\eta}{\eta} \left| \int_0^\infty e^{-it/\hbar(z - \log z)} \chi(tz/\hbar\eta) z^{-1/2} dz \right|^2 \\
&+ \mathcal{O}(|\log \hbar|^{-1}).
\end{aligned} \tag{117}
$$

We will estimate (117) by subdividing the domain of integration and applying the lemma of stationary phase. Let $\chi_1 \in C_0^\infty([0,2])$ be identically 1 on $[1/2, 3/2]$ and choose χ_2 so that $\chi_1 + \chi_2 = 1$ with χ_2 identically 1 on $[0, 1/4]$. For $|t| \geq \epsilon^{-1}\hbar$ and $\epsilon < 1/2$, it follows that:

$$
\begin{aligned}
&\int_0^\infty e^{it/\hbar(z - \log z)} \chi(tz/\hbar\eta) z^{-1/2} dz \\
&\quad = \int_0^2 e^{it/\hbar(z - \log z)} \chi_1(z) z^{-1/2} dz \\
&\quad\quad + \int_2^{\hbar\eta/t} e^{it/\hbar(z - \log z)} \chi_2(z) z^{-1/2} dz \\
&\quad\quad + \mathcal{O}\left(\left(\frac{\eta t}{\hbar} \right)^{-1/2} \right) \\
&\quad = I_1 + I_2.
\end{aligned} \tag{118}
$$

To estimate the the first integral I_1, note that the phase function

$$
\phi(z) := z - \log z \tag{119}
$$

has a nondegenerate critical point at $z = 1$. Thus, by the Lemma of Stationary Phase [12]:

$$
\begin{aligned}
&\left| \int_0^\infty e^{it/\hbar(z - \log z)} \chi_1(z) z^{-1/2} dz - (2\pi i t/\hbar)^{-1/2} e^{it/\hbar} \right| \\
&\qquad\qquad \leq C(t/\hbar)^{-3/2} \| \chi_1(z) z^{-1/2} \|_{C^2}.
\end{aligned} \tag{120}
$$

To bound the second integral, I_2, we perform an integration by parts as in (118):

$$
\begin{aligned}
|I_2| =&(t/\hbar)^{-1} \left| \int_2^{\hbar\eta/t} \partial_z(e^{it/\hbar(z-\log z)})\chi_2(z) z^{1/2}(z-1)^{-1} dz \right| \\
\leq& C\left(\frac{\eta t}{\hbar}\right)^{-\frac{1}{2}}.
\end{aligned} \tag{121}
$$

The last step involves an estimation of the off-diagonal terms $(\chi_1' v_+, v_-)$. Repeating the argument in (118)-(121) and using the fact that the principal term in the stationary phase expansion (120) vanishes, it follows that:

$$
|(\chi_1' v_\pm, v_\mp)| \leq C\left(\frac{t}{\hbar}\right)^{-1}. \tag{122}
$$

The lemma now follows for appropriate $\epsilon > 0$ and $\hbar < \hbar_0(\epsilon)$ sufficiently small. q.e.d.

We will henceforth assume that $t = \mathcal{O}(\hbar)$. Note that, given $q(x, y, \xi) \in C_0^\infty$, one can always find a constant $C > 0$ and a cutoff function $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$, such that $q(x, y, \xi) \leq C\chi_1(x, y, \xi)$, where, just as before, $\chi_1(x, y, \xi) = \chi(x)\chi(y)\chi(\xi)$. Therefore, as a consequence of the non-negativity of anti-Wick quantization, it follows by Lemma 4 that:

$$
(Op_\hbar(q)v_\pm, v_\pm) = \mathcal{O}(1). \tag{123}
$$

Our next order of business will be to investigate in more detail the action of certain specific pseudodifferential operators on the model distributions $v_\pm$. This will play a crucial role in the proof of Theorem 1 (see Section 9). Suppose $r(x, \xi) \in C_0^\infty(\mathbb{R}^2)$ and $r(0, 0) = 0$. We would like to estimate the expected values $(Op_\hbar(r\chi_1)v_\pm, v_\pm)$ as $\hbar \to 0$. By Taylor expansion,

$$
r(x, \xi) = \nabla_x r(0, 0)x + \nabla_\xi r(0, 0)\xi + R(x, \xi).
$$

Here, $|R(x, \xi)| = |\sum_{i+j=2} \nabla^2_{x_i, \xi_j} r(z_1, z_2)|$ for some (z_1, z_2) with $|z_1| \leq |x|$ and $|z_2| \leq |\xi|$. Thus, there exist constants $c_1, c_2 > 0$ such that $|R(x, \xi)| \leq c_1 x^2 + c_2 \xi^2$. Since we are working with anti-Wick quantizations, it follows that:

$$
(Op_\hbar(R\chi_1)v_\pm, v_\pm) \leq c_1(Op_\hbar(x^2\chi_1)v_\pm, v_\pm) + c_2(Op_\hbar(\xi^2\chi_1)v_\pm, v_\pm).
$$

Thus, to estimate $(Op_\hbar(r\chi_1)v_\pm, v_\pm)$ it suffices to estimate expected values of the form $(Op_\hbar(x^k\chi_1)v_\pm, v_\pm)$ and $(Op_\hbar(\xi^k\chi_1)v_\pm, v_\pm)$ where $k \in \mathbb{Z}$ with $k \geq 1$. To begin, suppose $k \geq 1$ and

$$r(x,y,\xi) = x^k\chi(x)\chi(y)\chi(\xi). \tag{124}$$

Consider,

$$I_\pm = \int_{\mathbb{R}} Op_\hbar(r)v_\pm(x) \cdot \overline{v_\pm(x)}dx. \tag{125}$$

Note that, by the argument in (86)-(91), together with (123), we can work with a standard $\hbar$-pseudodifferential operator with symbol $x^k\chi(x)\chi(y)\chi(\xi)$ modulo an error that is $\mathcal{O}(\hbar)$. By the Cauchy-Schwartz inequality and the Fubini theorem,

$$\begin{aligned} |I_\pm| \leq & C\hbar^{-1/2}\left(\int_0^1 d\xi \left|\int_0^\infty e^{ix\xi/\hbar}x^k\chi(x)v_\pm(x)dx\right|^2\right)^{1/2} \\ & + \mathcal{O}(|\log\hbar|^{-1/2}) \\ = & C|\log\hbar|^{-1/2}\Bigg(\int_1^{1/\hbar} \eta^{-1-2k}d\eta \\ & \qquad \cdot \left|\int_0^\infty e^{iy}y^k\chi(y/\eta)y^{-1/2+it/\hbar}dy\right|^2\Bigg)^{1/2} \\ & + \mathcal{O}(|\log\hbar|^{-1/2}). \end{aligned} \tag{126}$$

Due to the presence of y^k in the integral (126), the lower boundary term in the last integral in (126) vanishes when $y = 0$. Therefore, by integrating by parts once in y, we get:

$$|I_\pm|^2 \leq \frac{C}{\log\hbar}\int_1^{1/\hbar}\frac{d\eta}{\eta^2} = \mathcal{O}(|\log\hbar|^{-1}). \tag{127}$$

Suppose now that $r(x,y,\xi) = \xi^k\chi(x)\chi(y)\chi(\xi)$. Then, by the Fubini theorem,

$$\begin{aligned} I_\pm = & (2\pi\hbar)^{-1}\int e^{i(x-y)\xi/\hbar}\,\xi^k\chi(\xi)\chi(x)\overline{v_+(x)}\chi(y)v_+(y)dyd\xi dx \\ = & (2\pi\hbar)^{-1}\int \xi^k\chi(\xi)d\xi\,\left|\int e^{ix\xi/\hbar}v_+(x)\chi(x)dx\right|^2. \end{aligned} \tag{128}$$

By change of variables $w = \hbar^{-1}x\xi$, we get:

$$
\begin{aligned}
(129)\qquad |I_+| \leq & C|\log\hbar|^{-1}\int_0^1 \xi^{k-1}d\xi \left|\int_0^\infty e^{iw}w^{-1/2+it/\hbar}\chi(\hbar w/\xi)dw\right|^2 \\
= & \mathcal{O}(|\log\hbar^{-1}|).
\end{aligned}
$$

Consequently, we have proved:

Lemma 5. *Assume* $|t| \leq \epsilon^{-1}\hbar$ *and suppose*

$$\tilde{q} = q(x,\xi)\chi(x)\chi(y)\chi(\xi) \in C_0^\infty$$

with $q(x,\xi) = \mathcal{O}(|x,\xi|)$ *near* $(0,0)$. *Then,*

$$(Op_\hbar(\tilde{q})v_\pm, v_\pm) = \mathcal{O}(|\log\hbar|^{-1/2}).$$

Recall, we also have to estimate terms of the form $\|\chi_{12}u\|$, where χ_{12} is a cutoff supported in an annular region around the periodic geodesic, γ (see Proposition 1). In terms of the model problem at hand, after making a Fourier series in the periodic variable along the geodesic, γ, this corresponds to measuring microlocal mass in an annular region around the critical point $(0,0)$.

Lemma 6. *Let* $|E| \leq 1/\epsilon\hbar$ *and* $\chi_{12} \in C_0^\infty(\Omega)$ *be supported in the annulus* $\{(x,\xi); c_0^2 \leq x^2+\xi^2 \leq 1; 0 < c_0 < 1\}$. *Then,*

$$\|\chi_{12}v_\pm\| = \mathcal{O}(|\log\hbar|^{-1/2}).$$

Proof. Without loss of generality, we suppose that χ_{12} is supported in the region where $c_0 \leq x \leq 1$ and $|\xi| \leq 1$. Then,

$$
\begin{aligned}
(130)\qquad \|\chi_{12}v_+\|^2 = & \frac{1}{\log\hbar}\int_1^{1/\hbar}\frac{d\eta}{\eta}\left|\int_{c_0\eta}^{\eta} e^{iy}\chi_{12}(y/\eta,\hbar\eta)y^{-1/2+it/\hbar}dy\right|^2 \\
& + \mathcal{O}(|\log\hbar|^{-1}).
\end{aligned}
$$

Notice that in the inner integral we can make an integration by parts:

$$
\begin{aligned}
(131)\qquad & \int_{c_0\eta}^{\eta} e^{iy}\chi_{12}(y/\eta,\hbar\eta)y^{-1/2+it/\hbar}dy \\
& = i\int_{c_0\eta}^{\eta} e^{iy}\partial_y(\chi_{12}(y/\eta,\hbar\eta)y^{-1/2+it/\hbar})dy \\
& \quad + \mathcal{O}(\eta^{-1/2}) \\
& = \mathcal{O}(\eta^{-1/2}).
\end{aligned}
$$

The other cases are handled in the same way. Note that when estimating masses in regions where $c_0 \leq |\xi| \leq 1$ and $|x| \leq 1$, we interchange the roles of the x and ξ variables. The result follows. q.e.d.

8. The complex hyperbolic case

Here, the model operator is

$$P(\hbar) := \frac{\hbar}{2i}(x_1\partial_{x_1} + x_2\partial_{x_2}) + \frac{\hbar}{i}(x_1\partial_{x_2} - x_2\partial_{x_1}), \tag{132}$$

where $(x_1, x_2) \in \mathbb{R}^2$. To simplify the writing somewhat, we have put the constants (see Section 2) $\mu = \nu = 1$ in (132). This has no bearing on the estimates to follow. To begin, it is useful to exploit the obvious spherical symmetry of the model operator from the outset. In terms of polar coordinates (r, θ),

$$P(\hbar) = \frac{\hbar}{2i}(r\partial_r + \partial_r r) + \frac{\hbar}{i}\partial_\theta, \tag{133}$$

where (see Section 2),

$$\hat{I}^{ch}_{Re} = \frac{\hbar}{2i}(r\partial_r + \partial_r r) \text{ and } \hat{I}^{ch}_{Im} = \frac{\hbar}{i}\partial_\theta.$$

The model distributions of interest are:

$$v_k(r,\theta) = |\log \hbar|^{-1/2}\, r^{it/\hbar - 1} e^{ik\theta}, \tag{134}$$

where $k \in \mathbb{Z}$ and $t \in \mathbb{R}$. Again, the reason for introducing the logarithmic normalizing constant in (134) will become clear later on. To estimate the microlocal masses as well as the action of relevant pseudodifferential operators, we introduce radial cutoffs of the form

$$\chi_1(r, \xi_r) = \chi(r)\chi(\xi_r). \tag{135}$$

In the following, (ξ_r, ξ_θ) denote polar variables in the (ξ_1, ξ_2)-space and are not to be confused with the symplectically dual coordinates to (r, θ). Note that there is a fundamental difference between the distributions v_k above and the real hyperbolic eigenfunctions $v_\pm$; namely, the former are in $L^1(\mathbb{R})$, since Lebesgue measure on $\mathbb{R}^2$ is $rdrd\theta$. Just as in the previous section, we start by computing $(\chi_1 v_k, v_k)$. Letting (r,θ) denote the incoming variables and (r',θ') the outgoing variables, we write:

$$
\begin{aligned}
(\chi_1 v_k, v_k) =& \frac{(2\pi\hbar)^{-2}}{\log\hbar} \int e^{i\phi(r,r',\theta,\theta',\xi)/\hbar} \chi(r')\chi(r)\chi(\xi_r) \\
&\cdot \left(\frac{r}{r'}\right)^{it/\hbar} e^{ik(\theta-\theta')} \xi_r \, dr d\theta d\xi_r d\xi_\theta dr' d\theta'
\end{aligned}
\tag{136}
$$

where,

(137)

$$\phi(r',r,\theta',\theta,\xi) = (r'\xi_1\cos\theta' + r'\xi_2\sin\theta') - (r\xi_1\cos\theta + r\xi_2\sin\theta).$$

By performing an iterated integral in (136), it follows that:

$$
\begin{aligned}
(\chi_1 v_k, v_k) =& \frac{(2\pi\hbar)^{-2}}{\log\hbar} \int_0^{2\pi}\int_0^\infty \chi(\xi_r) I_k(\xi)\overline{I_k(\xi)}\,\xi_r\, d\xi_r d\xi_\theta \\
&+ \mathcal{O}(\hbar^\infty)
\end{aligned}
\tag{138}
$$

where,

$$
\begin{aligned}
I_k &= \int_0^{2\pi}\int_0^\infty e^{i/\hbar[r\xi_1\cos\theta + r\xi_2\sin\theta]} \chi(r) r^{it/\hbar} e^{ik\theta} dr d\theta \\
&= \int_0^\infty \chi(r) r^{it/\hbar} dr \left(\int_0^{2\pi} e^{ir\xi_r[\cos(\theta-\alpha)]/\hbar} e^{ik\theta} d\theta \right)
\end{aligned}
\tag{139}
$$

and $\alpha = \arccos(\xi_1/|\xi|)$. This last integral in parentheses is easily seen to be a k-th order Bessel function (see [3]). Recapping, we have shown that:

$$(\chi_1 v_k, v_k) = \frac{(2\pi\hbar)^{-2}}{\log\hbar} \int_0^{2\pi}\int_0^\infty \xi_r\,\chi(\xi_r) I_k(\xi)\overline{I_k(\xi)} d\xi_r d\xi_\theta \tag{140}$$

where,

$$I_k = 2\pi i^k e^{ik\arccos(\xi_1/|\xi|)} \int_0^\infty \mathcal{J}_k(-\xi_r r/\hbar) r^{it/\hbar} \chi(r)\, dr. \tag{141}$$

Here, $\mathcal{J}_k$ denotes the k-th order Bessel function of the first kind [3]. We will assume here that $k \in \mathbb{Z}$. Moreover, since,

$$\mathcal{J}_{-k}(x) = (-1)^k \mathcal{J}_k(x),$$

we also take the integers $k \geq 0$. Before going on to actually estimate the integrals in (140) and (141), in complete analogy with the analysis in Section 7, we make the final change of variables

$$\eta_r = \frac{\xi_r}{\hbar} \quad \rho = \frac{r\xi_r}{\hbar} \tag{142}$$

in (140) and (141). In terms of these new variables,

$$\begin{aligned}(\chi_1 v_k, v_k) \\ = \frac{1}{\log \hbar} \int_0^\infty \chi(\eta_r \hbar) \frac{d\eta_r}{\eta_r} \left| \int_0^\infty \mathcal{J}_k(-\rho) \chi(\rho/\eta_r) \rho^{it/\hbar}\, d\rho \right|^2 .\end{aligned} \tag{143}$$

Unfortunately, when estimating the integral in (143) it is difficult to apply the direct argument as in the real hyperbolic model. The reason for this is that the functions $\mathcal{J}_k$ and $\mathcal{J}_k'$ are both asymptotic to $e^{i\rho}\rho^{-1/2}$ as $\rho \to \infty$. Thus, a direct integration by parts does not improve the integrability properties of (143) as $|\mathcal{J}_k|$ and $|\mathcal{J}_k'|$ are comparable at radial infinity. To circumvent this problem, we use the following well-known recurrence formulae for Bessel functions [3]: Recall, for $k = 1, 2, 3, ..$ the Bessel functions $\mathcal{J}_k$ and $\mathcal{J}_{k-1}$ are related to each other by the integral formulae:

$$\begin{aligned}\int_0^\eta x^k \mathcal{J}_{k-1}(x) dx &= \eta^k \mathcal{J}_k(\eta), \\ \int_0^\eta x^{-k} \mathcal{J}_{k+1}(x) dx &= -\, \eta^{-k} \mathcal{J}_k(\eta) + \frac{2^{-k}}{\Gamma(k+1)}.\end{aligned} \tag{144}$$

One can think of these equations as raising and lowering identities in the representation theory of the Galilean group on $L^2(r dr d\theta)$. We claim that in complete analogy with Section 7 it suffices to assume $\eta_r \geq 1$, modulo an error that is $\mathcal{O}(|\log \hbar|^{-1})$. This is easily seen by noting that

$$|\mathcal{J}_k(\rho)| \leq 1 \tag{145}$$

for all $k \in \mathbb{Z}$ and $\rho \in \mathbb{R}$. Therefore, if $\eta_r \leq 1$,

$$\left| \int_0^\infty \mathcal{J}_k(-\rho) \chi(\rho/\eta_r) \rho^{it/\hbar} d\rho \right| \leq |\eta_r|. \tag{146}$$

Thus, the contribution to $(\chi_1 v_k, v_k)$ coming from the interval $\eta_r \leq 1$ is $\mathcal{O}(\log \hbar^{-1})$. Henceforth, we will assume that $\eta_r \geq 1$. It follows from the stationary phase estimate (see (154) below), together with an integration by parts that:

(147)
$$\int_0^\infty \mathcal{J}_k(\rho)\chi(\rho/\eta_r)\rho^{it/\hbar}d\rho = \int_0^{\eta_r} \mathcal{J}_k(\rho)\,\rho^{it/\hbar}d\rho + \mathcal{O}(|\eta_r|^{-1/2}).$$

Thus, it follows that [3]:

$$\int_0^\infty \mathcal{J}_k(\rho)\chi(\rho/\eta_r)\rho^{it/\hbar}d\rho = \frac{2^{\frac{it}{\hbar}}\Gamma(\frac{k+1+it/\hbar}{2})}{\Gamma(\frac{k+1-it/\hbar}{2})} + \mathcal{O}(\eta_r^{-1/2}).$$

Since $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ for any $z \in \mathbb{C}$,

(148)
$$\left|\int_0^\infty \mathcal{J}_k(\rho)\chi(\rho/\eta_r)\rho^{it/\hbar}d\rho\right|^2 = 1 + \mathcal{O}(\eta_r^{-1/2}).$$

So, for $t = \mathcal{O}(\hbar)$ and $k = 0, 1, ..., N_0$, there exists a uniform constant $C > 0$ such that:

(149)
$$\frac{1}{C} \leq (\chi_1 v_k, v_k) \leq C.$$

The next step is to estimate the action of pseudodifferential operators on v_k again under the assumption that the symbol vanishes to first order at $(0,0)$. To wit, suppose $q(x,y,\xi) \in C_0^\infty(\Omega)$ with $q(x,y,\xi) = \mathcal{O}(|r, r', \xi_r|)$. By a Taylor expansion argument (see Section 7), it suffices to consider the two cases: $q(x,y,\xi) = r^m\chi(r)\chi(r')\chi(\xi_r)$, $q(x,y,\xi) = \xi_r^m\chi(r)\chi(r')\chi(\xi_r)$ with $m \in \mathbb{Z}$ and $m \geq 1$. To begin, suppose

(150)
$$q(x,y,\xi) = r^m\chi(r)\chi(r')\chi(\xi_r).$$

It follows by the Cauchy-Schwartz inequality that:

(151)
$$(\chi_1 Op_\hbar(q)v_k, v_k) \leq C|\log\hbar|^{-1/2}\left(\int_1^{1/\hbar}\eta_r^{-1-2m}d\eta_r\left|\int_0^\infty \rho^m\rho^{it/\hbar}\chi(\rho/\eta_r)\mathcal{J}_k(\rho)\,d\rho\right|^2\right)^{1/2} + \mathcal{O}(|\log\hbar|^{-1/2}).$$

Suppose for the moment that $k = 0$. Then, by the recurrence formulae in (144),

$$I_0 = \int_0^\infty \rho^{it/\hbar} \rho^m \chi(\rho/\eta_r) \mathcal{J}_0(\rho) d\rho = \int_0^\infty \rho^{it/\hbar - 1} \rho^m \chi(\rho/\eta_r) \frac{d}{d\rho} \left(\rho \mathcal{J}_1(\rho)\right) d\rho. \tag{152}$$

Now, recall [3]

$$\mathcal{J}_k(\rho) = \frac{1}{\Gamma(k+1)} \left(\frac{\rho}{2}\right)^k + \mathcal{O}(\rho^{k+1}) \tag{153}$$

near $\rho = 0$, and by stationary phase,

$$\mathcal{J}_k(\rho) = \left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}} \left[\cos(\rho - k\pi/2 - \pi/4) + \mathcal{O}(\rho^{-1})\right] \tag{154}$$

as $\rho \to \infty$. Therefore, we can make an integration by parts in (152) and get that:

$$I_0 = -\int_0^\infty \frac{d}{d\rho} \left(\rho^{it/\hbar - 1 + m} \chi_1(\rho/\eta_r)\right) \rho \mathcal{J}_1(\rho)\, d\rho = I_{01} + I_{02}, \tag{155}$$

where,

$$I_{01} = \int_0^\infty \rho^{it/\hbar - 1 + m} \chi(\rho/\eta_r) \mathcal{J}_1(\rho)\, d\rho, \tag{156}$$

$$I_{02} = \frac{1}{\eta_r} \int_0^\infty \rho^{it/\hbar + m} \chi'(\rho/\eta_r) \mathcal{J}_1(\rho)\, d\rho. \tag{157}$$

Since, by (154), $\mathcal{J}_1(\rho) = \mathcal{O}(\rho^{-1/2})$, it follows that,

$$I_{01} \leq C \int_0^{\eta_r} \rho^{-1+m-1/2} d\rho = \mathcal{O}(\eta_r^{m-1/2}). \tag{158}$$

By similar reasoning, the other term I_{02} is also seen to be $\mathcal{O}(\eta^{m-1/2})$. The end result is that:

$$(Op_\hbar(q) v_0, v_0) = \mathcal{O}(|\log \hbar|^{-1/2}), \tag{159}$$

provided $q(x,y.\xi)=r^m\chi(r)\chi(r')\chi(\xi_r)$. The result then follows for general finite $k\in\mathbb{Z}^+$ by the same sort of argument. That is, we write:

$$\begin{aligned}I_k &= \int_0^\infty \rho^{it/\hbar}\rho^m\chi(\rho/\eta_r)\mathcal{J}_{k-1}(\rho)d\rho \\ &= \int_0^\infty \rho^{it/\hbar-k+m}\chi(\rho/\eta_r)\frac{d}{d\rho}(\rho^k\mathcal{J}_k(\rho))d\rho,\end{aligned}$$

and then estimate the latter integral as in (155)-(158). This completes the case where $q=\mathcal{O}(r^m)$.

Suppose now that:

$$q(x,y,\xi)=\xi_r^m\chi(r)\chi(r')\chi(\xi_r). \tag{160}$$

Then,

$$\begin{aligned}(Op_\hbar(q)v_k,v_k) =&\frac{2\pi\hbar^{-2}}{\log\hbar}\int_0^\infty \xi_r^{m+1}\chi(\xi_r)d\xi_r\left|\int_0^\infty \mathcal{J}_k(-\xi_r r/\hbar)r^{it/\hbar}\chi(r)dr\right|^2 \\ =&\frac{1}{\log\hbar}\int_0^\infty \xi_r^{m-1}\chi(\xi_r)d\xi_r\left|\int_0^\infty \mathcal{J}_k(-\rho)\rho^{it/\hbar}\chi(\hbar\rho/\xi_r)d\rho\right|^2 \\ =&\mathcal{O}(|\log\hbar|^{-1}).\end{aligned} \tag{161}$$

The final step is to evaluate expressions of the form $(\chi_{12}v_k,v_k)$. This argument here proceeds as in the previous section. Note that in the $d\eta_r$ integral in (143) one integrates over the range $c_0/\hbar\le\eta_r\le 1/\hbar$ for some c_0 with $1>c_0>0$. By the asymptotic formula (153) and an integration by parts, it follows that

$$(\chi_{12}v_k,v_k)=\mathcal{O}(|\log\hbar|^{-1/2}). \tag{162}$$

Proposition 3. *Let $k=1,...,N_0$ and $\chi_1,\chi_{12}\in C_0^\infty(\mathbb{R}^2)$ be cutoff functions as above. Then,*

$$(\chi_{12}v_k,v_k)=\mathcal{O}(|\log\hbar|^{-1/2})(\chi_1 v_k,v_k).$$

Moreover, if $\tilde{q}=q(x,\xi)\chi(x)\chi(y)\chi(\xi)$, where $q(x,\xi)=\mathcal{O}(|x,\xi|)$ near $(0,0)$, then,

$$(Op_\hbar(\tilde{q})v_k,v_k)=\mathcal{O}(|\log\hbar|^{-1/2})(\chi_1 v_k,v_k).$$

In complete analogy with the previous section, the next result gives a characterization of microlocal solutions of $Pu=Eu$ in terms of the model distributions, v_k (see [24, Proposition 6.4]).

Proposition 4. *Suppose* $\phi_\hbar \in \mathcal{D}'(\Omega)$ *is an admissible family of solutions to the equations:*

$$\hat{I}_{ch}^{Re}\phi_\hbar =_\Omega E_1\phi_\hbar,$$
$$\hat{I}_{ch}^{Im}\phi_\hbar =_\Omega E_2\phi_\hbar.$$

Then, there exist integers $k = k(\hbar)$ *and constants* $c_k(\hbar) \in \mathbb{C}$ *such that:*

$$\phi_\hbar =_\Omega c_k r^{-1+iE_1/\hbar} e^{ik\theta}.$$

Moreover, $k\hbar = E_2 + \mathcal{O}(\hbar^\infty)$.

Proof. As in Section 7 (see also [6]) the main idea is to build a *local* distributional solution $\tilde{u}$ to the eigenfunction equation that microlocally agrees with u near the critical point, $(0,0)$. The final step involves invoking the variation of parameters formula for first-order system of linear PDE. For further details, we refer the reader to [24, Proposition 6.4.]. q.e.d.

9. The main theorem

We now turn to the proof of Theorem 1. In the course of proving the theorem, we will have to pick model eigenfunctions $u(s,y;\hbar)$ of a specific form in the modified Fermi coordinates (s,y) since Lemma 5 and Proposition 3 only provide asymptotic mass estimates for a range of model eigenfunctions. To formulate this more clearly, we define the following sets: In the real-hyperbolic case, define:

$$\Omega^h(\hbar) = \{(t,n); |n\hbar - 1| \leq C_1\hbar, |t| \leq C_2\hbar\} \tag{163}$$

whereas, in the complex hyperbolic case we define:

$$\Omega^{ch}(\hbar) = \{(t,k,n); |n\hbar - 1| \leq C_1\hbar, |k| \leq C_2, |t| \leq C_3\hbar\}. \tag{164}$$

Here, $C_1, C_2, C_3 > 0$ are constants independent of $\hbar$.

Proposition 5. *Under the hypotheses (H1), there exist joint eigenfunctions* ψ_j *of* $H_0, ..., H_{n-1}$ *such that for* $u = F\psi_j$,

$$u(y,s;\hbar) =_\Omega \prod_{j=2}^{q+1} \Big[c_+ v_+(y_j;t_j) + c_- v_-(y_j;t_j)\Big] \cdot \prod_{j=q+2}^{q+2c+2} c\, r_j^{-1+it_j/\hbar} e^{ik_j\theta_j} e^{ins}.$$

Here,

$$(t_j, n) \in \Omega^h(\hbar), ((t_j, k_j, n) \in \Omega^{ch}(\hbar)$$

and

$$c_\pm = c_\pm(t_j, n; \hbar), c = c(t_j, k_j, n; \hbar).$$

Proof. By separation of variables, it suffices to consider individual summands: To begin, suppose the coordinates (y, η) correspond to a single real-hyperbolic summand. Then, assuming (H1), it follows from Theorem 3 that there exist real-analytic symbols $g_k(x_1, x_2; \hbar) \sim_j \sum_{j=0}^{\infty} g_{kj}(x_1, x_2)\hbar^j; k = 0, 1$ with $g_{0,0}(0,0) = g_{1,0}(0,0) = 0$, such that:

$$-i\hbar\partial_s - 1 =_\Omega Fg_1(H_0, H_1; \hbar)F^{-1}, \tag{165}$$

$$-i\hbar(y\partial_y + \partial_y y) =_\Omega Fg_2(H_0, H_1; \hbar)F^{-1}. \tag{166}$$

Suppose ψ_j are a family of joint eigenfunctions of H_0, H_1 satisfying:

$$H_k\psi_j = \mathcal{O}(\hbar)\psi_j. \tag{167}$$

By the results of [5], such functions indeed exist under the hypotheses (H1) and (H2) above. Therefore, (165)-(166) imply that

$$\|\chi_1(\hbar D_s u - u + \mathcal{O}(\hbar)u)\| = \mathcal{O}(\hbar^\infty), \tag{168}$$

$$\|\chi_1(\hbar(yD_y + D_y y) + \mathcal{O}(\hbar))u\| = \mathcal{O}(\hbar^\infty). \tag{169}$$

Consequently, Proposition 2 combined with equations (168) and (169) imply that for a real hyperbolic summand, there exists integers $n = n(\hbar)$ and constants $c_\pm(\hbar)$ such that:

$$\|\chi_1(u - (c_+v_+(y;t) + c_-v_-(y,t))e^{ins})\| = \mathcal{O}(\hbar^\infty),$$

where $(t, n) \in \Omega(\hbar)$. The case of a complex hyperbolic summand is treated similarly: In this situation, we have:

$$i\hbar\partial_s - 1 =_\Omega Fg_1(H_0, H_1, H_2; \hbar)F^{-1}, \tag{170}$$

$$i\hbar\partial_\theta =_\Omega Fg_2(H_0, H_1, H_2; \hbar)F^{-1}, \tag{171}$$

$$i\hbar(r\partial_r + \partial_r r) =_\Omega Fg_3(H_0, H_1, H_2; \hbar)F^{-1}. \tag{172}$$

Then, by the microlocal characterization result in Proposition 4, it follows that there exist $(t, k, n) \in \Omega^{ch}(\hbar)$ and $c \in \mathbb{C}$, such that,

$$\|\chi_1(u - c\, r^{-1+it/\hbar}e^{ik\theta}e^{ins})\| = \mathcal{O}(\hbar^\infty).$$

q.e.d.

Recall, the result we want to prove is:

Theorem 1. *Suppose the L^2-normalized, joint eigenfunctions ψ_j satisfy*

$$H_k\psi_j = \lambda_k(\hbar)\psi_j,$$

where $\lambda_k(\hbar) = \mathcal{O}(\hbar)$. Assume moreover that both hypotheses (H1) and (H2) are satisfied. Then, there exist non-negative real numbers $\alpha_1, ..., \alpha_k$ with $\sum_{j=1}^k \alpha_j = 1$, such that, for any $q \in C_0^\infty(T^*M)$,

$$(Op_\hbar(q)\psi_j, \psi_j) = (2\pi)^{-1}\sum_{j=1}^k \alpha_j \int_0^{2\pi} q(\gamma_j(t))dt + \mathcal{O}(|\log\hbar|^{-1/2}).$$

Proof. By the microlocalization result in Proposition 1, we know that:

$$(Op_\hbar(q)\psi_j, \psi_j) = \sum_{l=1}^k (\chi_1^l Op_\hbar(q)\psi_j, \psi_j) + E(\hbar) + \mathcal{O}(\hbar) \tag{173}$$

where,

$$|E(\hbar)| \leq \|q\|_\infty \left(\sum_{l=1}^k \|\chi_{12}^l\psi_j\|\right). \tag{174}$$

Here, the cutoff functions χ_1^l and χ_{12}^l are defined as in Section 1. Roughly speaking, χ_1^l is supported in a tubular neighbourhood, $\Omega_l \times \gamma_l$ of γ_l and χ_{12}^l is supported in an annular region about γ_l. The idea is then to use Lemma 5, in the real-hyperbolic case and Proposition 3 in the complex-hyperbolic case to estimate the two terms on the RHS of (173). To simplify the writing a bit, we will write $u = F\psi_j$ and denote the microlocally conjugated operators $FOp_\hbar(a)F^*$, simply by $Op_\hbar(a)$. First, we must settle the question of L^2-normalization. By the microlocalization result in (173)-(174) (with $q = 1$), it follows that

$$\|\psi_j\|^2 = \sum_{l=1}^k (\chi_1^l u, u) + E(\hbar) + \mathcal{O}(\hbar). \tag{175}$$

Note that the condition $H_k\psi_j = \mathcal{O}(\hbar)\psi_j$ ensures that $(t_j, n) \in \Omega^h(\hbar)$ in the real-hyperbolic case and $(t_j, k_j, n) \in \Omega^{ch}(\hbar)$ in the loxodromic case. Therefore, the respective asymptotic, microlocal estimates in Lemma 5 and Proposition 3 hold: In particular, for some $C = C(\hbar) > 0$:

(176)
$$\sum_{l=1}^{k}(\chi_1^l u, u) = C\prod_{j=2}^{q+1}|c_\pm(n_j)|^2 \cdot \prod_{j=q+2}^{q+2c+2}|c(k_j, n_j|^2 + \mathcal{O}(|\log\hbar|^{-1/2}),$$

(177)
$$\sum_{l=1}^{k}(\chi_{12}^l u, u) \leq C|\log\hbar|^{-1/2}\prod_{j=2}^{q+1}|c_\pm(n_j)|^2 \cdot \prod_{j=q+2}^{q+2c+2}|c(k_j, n_j|^2.$$

Since $\|\psi_j\| = 1$, (176) and (177) imply that

$$\prod_{j=2}^{q+1}|c_\pm(n_j)|^2 \prod_{j=q+2}^{q+2c+2}|c(k_j, nj)|^2 = 1/C + \mathcal{O}(|\log\hbar|^{-1}).$$

Therefore, $E(\hbar) = \mathcal{O}(|\log\hbar|^{-1/2})$ and so (175) becomes,

$$(Op_\hbar(q)\psi_j, \psi_j) = \sum_{l=1}^{k}(\chi_1^l Op_\hbar(q)u, u) + \mathcal{O}(|\log\hbar|^{-1/2}). \tag{178}$$

By the averaging argument in Section 6, it in turn follows that:

$$(Op_\hbar(q)\psi_j, \psi_j) = \sum_{l=1}^{k}(\chi_1^l Op_\hbar(q^{av})u, u) + \mathcal{O}(|\log\hbar|^{-1/2}) \tag{179}$$

where,

$$q^{av}(\sigma, y, \eta) = \frac{1}{2\pi}\int_0^{2\pi} q(s, \sigma, y, \eta)ds. \tag{180}$$

Then, by Taylor expansion in (y, η, σ) about $y = \eta = \sigma = 0$, we obtain:

$$(\chi_1^l Op_\hbar(q^{av})u, u) = \frac{1}{2\pi}\int_0^{2\pi} q(\gamma_l(t))dt + \sum_{j=1}^{3}(\chi_1^l Op_\hbar(q_j^{av})u, u) \tag{181}$$

where, $q_1^{av} = \mathcal{O}(y)$ and $q_2^{av} = \mathcal{O}(\eta)$ and $q_3^{av} = \mathcal{O}(\sigma)$. But then, again by Lemma 5 and Proposition 3,

$$(Op_\hbar(\chi_1^l q_j^{av})u, u) = \mathcal{O}(|\log\hbar|^{-1/2}) \tag{182}$$

for $j = 1, 2$. So, we are left with estimating the last term $(\chi_1^l Op_\hbar(q_3^{av})u, u)$: Let $\zeta_1(s), \zeta_2(s) \in C_0^\infty(\mathbb{S}^1)$ be a partition of unity on $\mathbb{S}^1$, so that $\zeta_1 + \zeta_2 = 1$. Recall, the canonical coordinates arising from the Birkhoff transformation are $(y, \eta; s, \tau)$ where $\sigma := \tau - 1$. Thus, to understand the last term in (181), it suffices to estimate the integral

$$\begin{aligned}(Op_\hbar(\chi(\sigma)\sigma)e^{int}, e^{int}) &= \sum_{j,k=1}^{2} (2\pi\hbar)^{-1} \int e^{i(t-s)\tau/\hbar} (\tau - 1) \\ &\qquad \cdot \chi(\tau - 1)\zeta_j(s)\zeta_k(t)e^{in(s-t)} ds d\sigma dt \\ &= \sum_{j,k=1}^{2} (2\pi\hbar)^{-1} \int e^{i(t-s)(\tau - n\hbar)/\hbar} (\tau - n\hbar) \\ &\qquad \cdot \chi(\tau - n\hbar)\zeta_j(s)\zeta_k(t) ds d\tau dt + \mathcal{O}(\hbar),\end{aligned}$$

where, the last line follows from fact that:

$$|n\hbar - 1| \leq C\hbar \tag{183}$$

provided $\hbar$ is sufficiently small. Finally, make an integration by parts in the s variable and apply Calderon-Vaillancourt again to get:

$$(\chi_1^l Op_\hbar(\chi(\sigma)\sigma)u, u) = \mathcal{O}(\hbar). \tag{184}$$

Therefore, $(\chi_1^l Op(q_3^{av})u, u) = \mathcal{O}(\hbar)$ and the theorem follows. q.e.d.

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