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TWISTING AND NONNEGATIVE CURVATURE METRICS ON VECTOR BUNDLES OVER THE ROUND SPHERE

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Abstract

A complete noncompact manifold M with nonnegative sectional curvature is diffeomorphic to the normal bundle of a compact submanifold S called the soul of M. When S is a round sphere we show that the clutching map of this bundle is restricted; this is used to deduce that there are at most finitely many isomorphism types of such bundles with sectional curvature lying in a fixed interval $[0, \kappa]$. We also examine the opposite question of how the twisting of the bundle limits the type of possible nonnegative curvature metrics on the bundle: It turns out that if the bundle does not admit a nowhere-zero section, then the normal exponential map is necessarily a diffeomorphism onto M, and the ideal boundary of M consists of a single point.

In their paper [3], Cheeger and Gromoll raised the question of which vector bundles over the round sphere admit complete metrics with nonnegative sectional curvature. The significance of this problem is that it attempts to determine to what extent a converse to the Soul theorem holds. Recall that this theorem states that every open (i.e., complete noncompact) manifold M with nonnegative curvature K_M is diffeomorphic to a vector bundle over a compact totally geodesic submanifold S called a soul. A natural question then is whether all such vector bundles admit complete metrics with $K_M \geq 0$.

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In [9], it was shown that when the soul is a Bieberbach manifold, nonnegative curvature metrics force the vanishing of the Euler class of the vector bundle. It follows that among oriented plane bundles over the torus, only the trivial one admits such a metric.

The above case is fairly rigid a priori, however (since for example any bundle with curvature ≥ 0 over such a manifold must also admit a flat metric), and the corresponding question for a simply connected base remains open. An answer could provide insights on the topological structure of compact manifolds with nonnegative curvature, since any open manifold with nonnegative curvature can (after modifying its metric) be isometrically embedded as a convex subset of a compact manifold with the same nonnegative curvature condition [6].

In general, since the metric projection onto the soul $\pi : M \to S$ is a C^2 Riemannian submersion ([10], [7]) the existence of flat planes tangent to the soul has a restrictive effect on its normal holonomy ([8]). Therefore, the most challenging case still corresponds to the soul having positive curvature, and the round sphere is the natural place to start looking for an answer.

The first result in this paper is a negative answer to a modified Cheeger–Gromoll problem in this case:

Theorem A. *Let k, n be positive integers with $n \geq 2$, and κ a positive constant. If M^{n+k} is an open manifold with sectional curvature $0 \leq K_M \leq \kappa$ and soul isometric to the sphere S^n of constant curvature 1, then there are only a finite number of possible isomorphism types for the normal bundle of S^n in M.*

We should point out that the above theorem does not follow from any of the classical finiteness theorems. It is easy to see that the class of Riemannian manifolds satisfying the conditions of Theorem A is $C^{1,\alpha}$ precompact; this does not, however, imply our result, since bundles with diffeomorphic total spaces are not, in general, isomorphic: When $n = 7, 8, 11$ for example, there are nontrivial bundles over S^n that are diffeomorphic to the trivial one; cf.[4].

Moreover, it will be clear from our proof that Theorem 1 follows under much weaker conditions than an upper curvature bound on the whole manifold; in fact, it suffices to assume that the vertical curvatures are bounded above by κ on a ball of fixed radius centered at *just one point of the soul* (see Remark 2.7).

Theorem A has also the following interesting consequence: If there is an infinite number of isomorphism types of rank k vector bundles over

S^n which admit complete metrics of nonnegative curvature (this occurs for instance when $n = k = 2$), then their sectional curvatures grow without bound. Roughly speaking, the curvature increases as the bundle becomes more twisted. It follows from our proof that the curvature would in fact increase along some vertical planes at every point of the soul for the Riemannian submersion π.

In the second part of the paper, we examine some of the ways in which the normal holonomy group of the soul interacts with the structure of the bundle. This is exemplified by the following results: First, we show that if a vector bundle over the sphere is sufficiently twisted, then metrics of nonnegative curvature are particularly nice from a metric point of view:

Theorem B. *Let M^{n+k} be an open manifold of nonnegative sectional curvature and soul diffeomorphic to S^n, $n \geq 2$. Suppose that M does not split topologically off an $\mathbb{R}$-factor. Then:*

1. $\exp : \nu(S^n) \to M$ *is a diffeomorphism, and the metric projection $M \to S$ is a C^∞ Riemannian submersion.*
2. *The ideal boundary of M consists of a single point.*

Second, we show that an algebraic condition for the normal curvature R^ν *at just one point* of an arbitrary soul S has rigid global implications for the metric. This interplay between local and global properties seems to be an increasingly key factor in our understanding of nonnegatively curved metrics on open manifolds.

Theorem C. *Let M^{2n} be an open manifold of nonnegative sectional curvature with soul S of dimension n. Suppose that for some p in S, there exists an $x \in T_p(S)$ such that $R^\nu(x, y) : \nu_p \to \nu_p$ is 1-1 for all nonzero $y \in T_p(S)$ orthogonal to x. Then:*

1. $\exp : \nu(S) \to M$ *is a diffeomorphism.*
2. *There is exactly one ray originating from any point outside the soul.*
3. *$M(\infty)$ is a point.*

1. General background and notation

Throughout the paper, M will denote an open manifold with nonnegative curvature and soul S. Recall that M is diffeomorphic to the normal vector bundle of S in M, which we will usually denote by $\nu(S)$. One fundamental fact about this type of spaces is the rigidity theorem of Perelman ([10]) from which we need:

Theorem 1.1[10].

1. *Let* $\gamma : \mathbb{R} \to S$ *be a geodesic with* $p = \gamma(0)$ *and* $u \in \nu(S)_p$. *Extend* u *to a parallel vector field* $U(s)$ *along* γ, *and call* $R : [0, \infty) \times \mathbb{R} \to M$ *the surface obtained by exponentiating* $U(s)$; *i.e,*

$$R(t, s) = \exp_{\gamma(s)} tU(s).$$

   Then R *is a flat, totally geodesic immersed rectangle in* M. *In particular, for each* t_0, $R(t_0, s)$ *is a geodesic.*

2. *If* $\pi : M \to S$ *is the metric projection, then* $\pi(\exp u)$ *is the foot-point of* u; *i.e,* $\pi \circ \exp = \hat{\pi}$, *where* $\hat{\pi} : \nu(S) \to S$ *is the vector bundle projection.*

3. $\pi : M \to S$ *is a* C^1 *Riemannian submersion.*

The last statement was improved to C^2 in [7]. Finally, E will denote an oriented Riemannian vector bundle over S, and $Fr(E)$ the bundle of oriented orthonormal frames of E. Thus, if the rank of E is k, then the frame bundle is a principal $SO(k)$ bundle over S, and E is isomorphic to $Fr(E) \times_{SO(k)} \mathbb{R}^k$. Recall that a Riemannian connection on E is equivalent to an $SO(k)$ connection on its frame bundle: If β is a horizontal curve in $Fr(E)$, then $\rho(\beta, u)$ is horizontal in E for any $u \in \mathbb{R}^k$, where ρ denotes the projection $Fr(E) \times \mathbb{R}^k \to Fr(E) \times_{SO(k)} \mathbb{R}^k$. Conversely, the parallel transport of an oriented orthonormal frame along a curve in S is obtained by parallel translating each individual basis element.

2. Vertizontal comparison

In this section we present the technical argument that is central to our estimates on the "twisting" of the bundle when the soul is a round sphere S^n of radius 1.

Let p and q denote antipodal points of S^n. Let $\Sigma_p = S^{n-1}$ denote the collection of unit vectors in T_pS^n. For each $x \in \Sigma_p$, the geodesic $\gamma_x : [0, \pi] \to S^n$ in direction x connects p and q. We can use any vector $u \in E_p$ to construct a map $\Phi^u : \Sigma_p \to E_q$ defined as $\Phi^u(x) = P_{\gamma_x} u$, where P_{γ_x} is the parallel transport of u to q along γ_x.

Notice that Φ^u is smooth and usually dependent on the u with which we started. Observe also that, since parallel transport is an isometry, $\|\Phi^u(x)\| = \|u\|$, so the image of Φ^u is contained in some sphere in E_q. After normalizing, Φ^u becomes a map from S^{n-1} to the unit sphere S^{k-1} in E_q.

Our purpose is to use the curvature conditions to bound the differential of $\Phi^u : \Sigma_p \to S^{k-1}$. This amounts to controlling the local Lipschitz constant of Φ^u at every $x \in \Sigma_p$.

Lemma 2.1. *Let M be an open manifold with soul S^n, the sectional curvature of which satisfies $0 \leq K_M \leq \kappa$. Then for a good choice of $\|u\|$, the derivative of the map $\Phi^u : \Sigma_p \to S^{k-1}$ constructed above satisfies*

$$\|\Phi^u_*\| \leq C(\kappa), \tag{2.2}$$

where $C(\kappa)$ is a constant depending only on κ.

Remark. The argument below shows that $C(\kappa)$ may be taken to be $\pi\sqrt{\kappa}$; this will in general, of course, not be the optimal value.

Proof. Let $x, y \in \Sigma_p$, and $\alpha = \angle(x, y)$ the angle between x and y. Since $\pi : M \to S^n$ is a Riemannian submersion, we can lift γ_x, γ_y to the point $p = \exp(u)$ on the fiber of E over p. These lifts are geodesics $\bar{\gamma_x}$, $\bar{\gamma_y}$ forming an angle α at p by (1) and (3) of Theorem 1.1.

The hinge version of Toponogov's comparison theorem ([2]) for nonnegative curvature implies that

$$d := d_M(\bar{\gamma_x}(\pi), \bar{\gamma_y}(\pi)) \leq 2\pi \sin\frac{\alpha}{2}, \tag{2.3}$$

where d_M is the distance function on M induced by the Riemannian metric.

On the other hand, there is a second geodesic hinge ending at $\bar{\gamma_x}(\pi)$ and $\bar{\gamma_y}(\pi)$: it is vertical and formed by the two radial geodesics tangent to $\Phi^u(x)$ and $\Phi^u(y)$ respectively. Under an upper curvature bound, and if the original u is taken with $\|u\|$ small enough, the angle between $\Phi^u(x)$ and $\Phi^u(y)$ will be bounded in terms of the distance between its endpoints.

More precisely, let $\beta = \angle(\Phi^u(x), \Phi^u(y))$. By [12, Theorem 4], the injectivity radius of M is at least as large as $\pi/\sqrt{\kappa}$, where for simplicity we are assuming that $\kappa > 1$. Convexity radius estimates (cf. [11, Ch.6, Theorem 3.6]) show that the ball $B_r(q)$ is convex, where $r := \frac{1}{2}\pi/\sqrt{\kappa}$. Then, for any $\|u\|$ slightly smaller than r, we can use Corollary 1.30 in [2] to conclude that there exists a comparison triangle to $\Delta(q, \bar{\gamma}_x(\pi), \bar{\gamma}_y(\pi))$ in the space form of curvature κ, S^2_κ, for which the corresponding angle $\hat{\beta}$ satisfies $\beta \le \hat{\beta}$.

Setting $a := \|u\|$, the usual spherical trigonometric formula in the comparison triangle yields

$$\cos(\sqrt{\kappa}d) = \cos^2(\sqrt{\kappa}a) + \sin^2(\sqrt{\kappa}a)\cos\hat{\beta}. \tag{2.4}$$

Now, recall that $\hat{\beta}$ and d are both functions of α. Differentiating (2.4) twice and evaluating at zero, we obtain

$$\kappa d'(0)^2 = \sin^2(\sqrt{\kappa}a)\,\hat{\beta}'(0)^2. \tag{2.5}$$

But since $0 \le d(\alpha) \le 2\pi \sin\frac{\alpha}{2}$ for all α, we must have that $d'(0) \le \pi$; since we may take a to be arbitrarily close to $\frac{1}{2}\pi/\sqrt{\kappa}$, it follows from (2.5) that

$$|\hat{\beta}'(0)| \le \sqrt{\kappa}\pi, \tag{2.6}$$

which establishes the lemma. q.e.d.

Remark 2.7. It is worth noticing that the lemma holds under considerably weaker conditions. In fact, after adapting the injectivity radius estimate of [12], the proof shows that the upper curvature bound is only needed on a small ball of fixed radius around one point of the soul. In this situation $\beta'(0)$ will be bounded in terms of κ and the radius of such a ball.

3. Bounding the clutching map

In this section we will use a connection on a Riemannian vector bundle over S^n to give a geometric description of its clutching map, where the latter denotes the restriction to the equator of the transition charts of the bundle. Then we will see how the curvature conditions restrict its homotopy class.

Let E be a vector bundle over S^n with structure group $SO(k)$. Denote by p and q a pair of antipodal points, and by U_+ and U_- their complements in S^n. Since U_+, U_- are contractible, there are trivializations

$$\phi_+ : U_+ \times \mathbb{R}^k \to \pi^{-1}(U_+), \qquad \phi_- : U_- \times \mathbb{R}^k \to \pi^{-1}(U_-). \tag{3.1}$$

Restricting to the equator, we have a map

$$\phi_-^{-1} \circ \phi_+ : S^{n-1} \times \mathbb{R}^k \to S^{n-1} \times \mathbb{R}^k$$

sending $(p, u) \to (p, g(p)u)$ with $g : S^{n-1} \to SO(k)$. g is called the *clutching map* of E, and its importance resides in that free homotopy classes of such maps classify vector bundles over S^n up to isomorphism type.

When E has a Riemannian connection ∇, the above trivializations can be constructed by means of parallel transport: Let b_p be an oriented orthonormal basis of the fiber of E at p, i.e., an element of the fiber over p of the frame bundle $Fr(E)$. b_p may be viewed as a linear isometry $\mathbb{R}^k \to E_p$, and any other frame at p can be written as $b_p \circ h$ for some $h \in SO(k)$. If γ_r is the minimal geodesic connecting p to some point $r \in U_+$, and P_{γ_r} denotes parallel transport along it, then the map $U_+ \times SO(k) \to \pi^{-1}(U_+)$ given by $(r, h) \to P_{\gamma_r}(b_p \circ h)$ is a trivialization ϕ_+ of $Fr(E)$ over U_+. Obviously, a similar construction can be carried out over U_-.

For these trivializations, the clutching map is actually related to the function Φ studied in the last section, since if $\phi_-^{-1} \circ \phi_+(r, h) = (r, \bar{h})$, then

$$P_{\gamma_1}(b_p h) = P_{\gamma_2}(b_q \bar{h}), \tag{3.2}$$

where γ_1 and γ_2 are the corresponding geodesics connecting p and q to r. Clearly, $\gamma_1 \cup -\gamma_2$ is then a geodesic γ_x connecting p to q for some $x \in \Sigma_p$, and thus, the clutching map is given by

$$g(x) = b_q^{-1} \circ P_{\gamma_x} \circ b_p, \qquad x \in \Sigma_p = S^{n-1}. \tag{3.3}$$

But since transporting an orthonormal frame along γ_x consists of transporting each element in the frame, g is actually defined in terms of Φ. Recall that Φ depended on a choice of $u \in E_p$; denote by Φ^u the result. Then, if we define $\hat{\Phi} : \Sigma_p \to Fr(E_q)$ by

$$\hat{\Phi}(x) = (\Phi^{u_1}(x), \ldots, \Phi^{u_k}(x)), \tag{3.4}$$

where $\{u_1, \ldots, u_n\}$ denotes the b_p-image of the standard basis in Euclidean space $\mathbb{R}^k$, our clutching map will be given by $g = b_q^{-1} \circ \hat{\Phi}$.

Lemma 3.5. *Let M be an open nonnegatively curved $(n + k)$-dimensional manifold with soul S^n. If the sectional curvature of M is* *bounded above by a positive number κ, then there is a constant $\bar{C} = \bar{C}(k,\kappa)$ such that*

$$\|g_*\| \leq \bar{C}.$$

Proof. This is a simple consequence of the above discussion together with Lemma 2.1. In fact, $\bar{C} = kC(\kappa)$ would work. q.e.d.

Proof of Theorem A. If $n = 2$, we only need to consider plane bundles, since there are exactly 2 bundles of rank ≥ 2 over the 2-sphere. This special case will be considered in Section 4.1. So suppose the base has dimension $n \geq 3$. We may clearly assume that $k \geq 3$, since any plane bundle over S^n is trivial in this case. Recall from [5] that *the dilation of a map* $f : X \to Y$ between metric spaces is defined as

$$\operatorname{dil} f = \sup \frac{d(f(x), f(y))}{d(x,y)},$$

where the supremum is taken over pairs with $x \neq y$. When X, Y are compact Riemannian manifolds and f is differentiable, then the dilation coincides with the maximum value of $\|f_{*x}\|$, as x ranges over X. As a consequence of Theorem 7.10 in [5], we know that there are only a finite number of homotopy classes of maps from Σ_p to $Spin(k)$ with dilation less than the constant $\bar{C}$ appearing in the last lemma. Since $k > 2$, it follows that the same is true for maps from Σ_p to $SO(k)$. But recall that each isomorphism type of vector bundles over S^n with sectional curvature between 0 and κ produces one such homotopy class, thus proving the Main Theorem in this case. q.e.d.

4. Twisting versus normal holonomy

In the preceding sections, we examined the relation between the twisting of a bundle and the existence of a nonnegatively curved metric whose curvature is bounded above. Our purpose in this section is to show that even when the upper bound condition is removed, the twisting (as reflected in the holonomy) of the bundle severely restricts the type of such a metric.

4.1. The Euler class $e(E)$

Let E denote a rank n oriented vector bundle over S^n (where we now allow any metric with nonnegative curvature on the sphere). One way to compute the Euler class of E is the following: Let p, q be two different points of S^n. If ϕ is a diffeomorphism with the round sphere that maps the north and south pole to p and q, we can send the meridians to a set of paths $\{\gamma_x\}$ from p to q covering S^n, with $x \in \Sigma_p$. Choose $u \in E_p$, and extend u to the section U of E obtained by parallel transporting u along the $\{\gamma_x\}$. U will usually be multivalued at q. As in Section 2, this induces a map $\Phi : \Sigma_p \to E_q^1$. It is straightforward to check that the Euler class of E equals, up to sign, the degree of this map; cf. [1, Theorem 11.17].

For example, if E is an oriented plane bundle over the 2-sphere, then the Euler number of E must satisfy

$$|e(E)| = |\deg \Phi| \leq \max_{x \in \Sigma_p} \|\Phi_{*x}\|^2$$

by [5, Proposition 2.11]. But these bundles are determined, up to isomorphism, by their Euler class. Together with Lemma 2.1, this establishes the remaining case in the proof of Theorem A.

Unfortunately, the Euler class of a rank k bundle over S^n has no significance unless $n = k$ is an even integer. The map above, however, may be constructed for any values of n and k, and thus its homotopy type generalizes in a sense the Euler class. In the following argument, it will be convenient to use a slightly different, albeit equivalent formula for this map: Fix a vector $u \in S^{k-1}$, and define a map $f : S^{n-1} \to S^{k-1}$ by

$$f(x) = g(x)u, \qquad x \in S^{n-1}. \tag{4.1.1}$$

Theorem 4.1.2. *Let E denote the total space of an oriented rank-k vector bundle over S^n. If the bundle does not admit a nowhere-zero cross-section, then the holonomy group of any Riemannian connection on the bundle acts transitively on E.*

Proof. Suppose, to the contrary, that this action is not transitive. Then the map f of (4.1.1) cannot be onto S^{k-1}, since the image of f, after identification of the fiber over q with $\mathbb{R}^k$, consists of the parallel translates of a single vector along different geodesics to q. But a sphere with a point deleted is contractible, so that f is an inessential map. Now, an oriented vector bundle over a sphere admits a nowhere-zero cross-section iff the map f is homotopically trivial. To see this, we proceed as follows:

Let $\rho : SO(k) \to S^{k-1}$ denote the map given by $\rho(h) = h(u)$, $h \in SO(k)$. ρ is the projection of a principal bundle

$$SO(k-1) \to SO(k) \to S^{k-1},$$

and one has the commutative diagram:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{g} & SO(k) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \rho} \\ S^{k-1} & = & S^{k-1} \end{array}$$

Since f is inessential, the homotopy class of g must belong to the kernel of the homomorphism

$$\rho_* : \pi_{n-1}(SO(k)) \to \pi_{n-1}(S^{k-1})$$

induced by ρ. By the long exact homotopy sequence of the fibration

$$\cdots \to \pi_{n-1}(SO(k-1)) \xrightarrow{i_*} \pi_{n-1}(SO(k)) \xrightarrow{\rho_*} \pi_{n-1}(S^{k-1}) \to \cdots$$

the homotopy class of g belongs to the image of the homomorphism induced by the inclusion $i : SO(k-1) \to SO(k)$. But this implies that the structure group of the bundle is reducible to $SO(k-1)$; in other words, the bundle admits a nowhere-zero section, which contradicts the hypothesis. Thus, the holonomy acts transitively on E. q.e.d.

Theorem B. *Let M^{n+k} be an open manifold with nonnegative sectional curvature and soul S diffeomorphic to S^n. If M does not split topologically off an $\mathbb{R}$-factor, then:*

1. $\exp : \nu(S) \to M$ *is a diffeomorphism, where $\nu(S)$ is the normal bundle of S in M.*
2. *The metric projection $M \to S$ is a C^∞ Riemannian submersion.*
3. *The ideal boundary of M consists of a single point.*

Proof. Recall that if γ is a ray originating at the soul, and u is the parallel transport of $\dot\gamma(0)$ along a curve c in the soul, then $t \mapsto \gamma_1(t) := \exp(tu)$ is again a ray, and the distance between $\gamma(t)$ and $\gamma_1(t)$ is no larger than the length of c for all t [15]. Since in an open manifold there is at least one ray through every point, every normal direction to the soul has to be minimizing in the interval $[0, \infty)$, thus establishing 1. 2 follows from 1 together with Perelman's result [10], since the metric projection π equals in this case $\pi_\nu \circ \exp^{-1}$, where π_ν denotes the projection of the normal bundle onto the soul:

$$\begin{array}{ccc} \nu(S) & \xrightarrow{\exp} & M \\ {\scriptstyle \pi_\nu}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ S & = & S \end{array}$$

Finally for 3, let u, v be unit vectors in some fiber, and $\alpha : [0, L] \to S$ a path so that $P_\alpha u = v$. Then $d(\gamma_u(t), \gamma_v(t)) \le \text{length}(\alpha)$, so that γ_u and γ_v must represent the same point in the ideal boundary $M(\infty)$. q.e.d.

4.2. Global effects of pointwise conditions

Given an open manifold M with nonnegative curvature and soul S, there are simple conditions on the curvature tensor of the normal bundle at one point of the soul that have even stronger consequences than Theorem B. The results in this section apply to any type of soul, not only to spheres.

We first explore how the amount of rays in M bounds the topology of the normal bundle; roughly speaking, the more rays, the more flat the bundle. To make this more precise, let n denote the dimension of S, $n + k$ that of M. For $p \in M$, define $l(p)$ to be the maximal number of linearly independent ray directions at p, and set

$$l := \max_{p \in M \setminus S} l(p).$$

The curvature tensor of the normal bundle will be denoted by R^ν.

Lemma 4.2.1. *Suppose that $l > k - n + 1$. Then for any x tangent to the soul, there exists a nonzero y orthogonal to x such that $R^\nu(x, y) : \nu(S) \to \nu(S)$ has nontrivial kernel.*

Corollary 4.2.2. *Suppose that at some $p \in S$, there is an $x \in T_p(S)$ such that $R^\nu(x, y)$ is 1-1 for all nonzero y orthogonal to x. If the soul S has dimension ≥ 2, then the number l of linearly independent ray directions at any point outside the soul satisfies $l \le k - n + 1$, and in particular, l is strictly less than the codimension of S.*

Example 4.2.3. Let M^4 be an open manifold of nonnegative curvature with soul a 2-sphere S. If at some point outside the soul there is more than one ray direction, then M splits metrically as $S \times P^2$, where

P^2 denotes $\mathbb{R}^2$ with a metric of nonnegative curvature: This follows from [13], since the normal bundle of S in M must be flat by Corollary 4.2.2.

Proof of 4.2.1. Let $x \in T_p(S)$. By assumption, there exists a point outside the soul from which $l > k - n + 1$ linearly independent rays emanate. By [7], since ray vectors (even for rays not originating at the soul) are vertical and preserved under parallel translation along geodesics which are horizontal for the Riemannian submersion $M \to S$, there exists a point q on the fiber over p from which also $l > k - n + 1$ linearly independent rays emanate. Furthermore, since

$$< A_{z_1} z_2, u >= - < z_2, \nabla_{z_1} U >$$

for any horizontal vectors z_1, z_2, and vertical u with U a vertical extension, these l ray directions are orthogonal to the image of the O'Neill tensor A. Thus, the rank of A at q is no larger than $k - l < n - 1$, and there exists a nonzero $y \in T_p(S)$ orthogonal to x such that $A_X Y(q) = 0$. Here, X and Y denote the basic lifts of x and y along the fiber. It follows from [8] that $A_X Y \equiv 0$ along the Sharafutdinov line connecting q to p. But $A_X Y$ is a Jacobi field along any radial geodesic γ_u in direction u starting at the soul, with $(A_X Y \circ \gamma_u)(0) = 0$, and $(A_X Y \circ \gamma_u)'(0) = -2R^\nu(x, y)u$, cf. [14]. Hence $R^\nu(x, y)u = 0$ for any $u \in \nu_p$ for which $\gamma_u(t_0)$ (for small t_0) lies in the Sharafutdinov line between q and p. q.e.d.

Theorem C. *Let M^{2n} be an open manifold of nonnegative sectional curvature with soul S of dimension n. Suppose that for some p in S, there exists an $x \in T_p(S)$ such that $R^\nu(x, y) : \nu_p \to \nu_p$ is 1-1 for all nonzero $y \in T_p(S)$ orthogonal to x. Then:*

1. $\exp : \nu(S) \to M$ *is a diffeomorphism.*
2. *There is exactly one ray originating from any point outside the soul.*
3. $M(\infty)$ *is a point.*

Proof. To establish 1, suppose to the contrary that the exponential map is not a diffeomorphism; i.e., there is some p in S, and $u \in \nu_p$ of unit norm, such that $t \mapsto \gamma_u(t) = \exp(tu)$ is not a ray. Since limits of rays are rays, there are arbitrarily small values $t_0 > 0$ such that the translated geodesics $t \mapsto \gamma_u(t_0 + t)$ are not rays either. Choosing such a t_0 smaller than the injectivity radius of the normal exponential map, and considering a ray direction (which is necessarily linearly independent from $\dot{\gamma}_u(t_0))$ at $\gamma_u(t_0)$, it follows as in the proof of Lemma 4.2.1 that A_X has rank less than $n-1$ at that point, and thus, $R^\nu(x,y)u = 0$ for some nonzero y orthogonal to x.

3 is a consequence of 2, and 2 can be deduced from Lemma 4.2.1 as follows: By hypothesis, the number l of linearly independent ray directions at any point satisfies $l \leq k-n+1 = 1$. It remains to rule out the possibility that there are two rays pointing in opposite directions emanating from some point p outside the soul. If this were the case, then by 1, one of these rays would have to be a minimal connection to S. The basic soul construction at p would then provide a soul S' disjoint from S. It is well known that this can only happen if the normal bundle of S admits a parallel section, see e.g. [16]; but this would contradict the hypothesis on the curvature tensor of the normal bundle.

As a final remark, we point out that Theorem C implies that if the normal exponential map is not a diffeomorphism, then the nullity of the curvature tensor R^ν at any point p in the soul is essentially invariant under local deformations of the metric around p that preserve nonnegative curvature: For if after such a deformation, $R^\nu(x,y)(p)$ were to become 1-1 as in Theorem C, then the exponential map would be a diffeomorphism on every fiber, which is impossible since the deformation was only local.

References

[1] R. Bott & L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Math. **82** Springer, 1982.

[2] J. Cheeger & D. Ebin, *Comparison theorems in Riemannian geometry*, North Holland, 1975.

[3] J. Cheeger & D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972) 413–443.

[4] R. De Sapio & G. Walschap, *Diffeomorphism of total spaces and equivalence of bundles*, Topology, to appear.

[5] M. Gromov, *Structures métriques pour les variétés Riemanniennes*, Cedic–F. Nathan, 1981.

[6] L. Guijarro, *Improving the metric in an open manifold with nonnegative curvature*, Proc. Amer. Math. Soc. **126** (1998) 1541–1545.

[7] ———, *On the metric structure of open manifolds with nonnegative curvature*, Pacific J. Math., to appear.

[8] L. Guijarro & G. Walschap, *The metric projection onto the soul*, Trans. Amer. Math. Soc., to appear.

[9] M. Ozaydin & G. Walschap, *Vector bundles with no soul*, Proc. Amer. Math. Soc. **120** (1994) 565–567.

[10] G. Perelman, *Proof of the soul conjecture of Cheeger and Gromoll*, J. Differential Geom. **40** (1994) 209–212.

[11] P. Petersen, *Riemannian geometry*, Graduate Texts in Math. 171, Springer, 1998.

[12] V. A. Sharafutdinov, *The Pogorelov-Klingenberg theorem for manifolds homeomorphic to* $\mathbb{R}^n$, Sibirsk. Math. Zh. **18** (1977) 915–925.

[13] M. Strake, *A splitting theorem for open nonnegatively curved manifolds*, Manuscripta Math. **61** (1988) 315–325.

[14] M. Strake & G. Walschap, Σ*-flat manifolds and Riemannian submersions*, Manuscripta Math. **64** (1989) 213–226.

[15] G. Walschap, *Nonnegatively curved manifolds with souls of codimension 2*, J. Differential Geom. **27** (1988) 525–537.

[16] J. Yim, *Space of souls in a complete open manifold of nonnegative curvature*, J. Differential Geom. **32** (1990) 429–455.

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