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# Gr $\Longrightarrow$ SW
# FROM PSEUDO-HOLOMORPHIC CURVES TO SEIBERG–WITTEN SOLUTIONS

CLIFFORD HENRY TAUBES

The Seiberg-Witten invariants were defined by Witten [24] for any compact, oriented 4-manifold with $b^2_+ > 1$; after the choice of an orientation for a certain determinant line, they consitute an map, SW, from the set $\mathcal{S}$ of $\mathrm{Spin}^{\mathbb{C}}$ structures on $X$ to $\mathbb{Z}$ which depends only on the diffeomorphism type of $X$. Roughly speaking, SW is computed from a weighted count of solutions to a natural, non-linear system of differential equations on $X$. (See [9], [8] and [12].) As remarked in [17], a symplectic 4-manifold has a canonical identification $\mathcal{S} \approx H^2(X;\mathbb{Z})$; and with this identification understood, the Seiberg-Witten invariant on a symplectic $X$ can be thought of as mapping $H^2(X;\mathbb{Z})$ to $\mathbb{Z}$.

Meanwhile, a symplectic 4-manifold has a second natural map, Gr: $H^2(X;\mathbb{Z}) \to \mathbb{Z}$, called the Gromov invariant. The latter invariant is defined in [18]. To a first approximation, Gr assigns to a class $e \in H^2(X;\mathbb{Z})$ a certain weighted count of the symplectic submanifolds of $X$ whose fundamental class is Poincaré dual to $e$. The following theorem was announced in [17]:

**Theorem 1.** *Let $X$ be a compact, symplectic 4-manifold with $b^2_+ > 1$. Use the symplectic structure to orient $X$, to define the Seiberg-Witten invariants of $X$ as a map $SW : H^2(X;\mathbb{Z}) \to \mathbb{Z}$, and also to define the Gromov invariant $Gr : H^2(X;\mathbb{Z}) \to \mathbb{Z}$. Then $Gr = SW$.*

As remarked in [17], there are essentially three parts to the proof of this theorem. The first part appears in [19] where it was shown how

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the existence of solutions to a certain one-parameter family of Seiberg-Witten equations can be used to produce symplectic submanifolds in $X$. This article constitutes the second part of the proof of this Gr=SW theorem. Here, a construction is described which associates solutions of the one-parameter family of Seiberg-Witten equations to certain symplectic submanifolds in $X$ (pseudo-holomorphic ones). The final part of the proof of the Gr=SW equivalence ([22]) uses the constructions in [19], and in this article to prove that the counting for the two invariants yields the same answer.

Thus, what follows in this article is, essentially, an existence theorem for solutions to a certain 1-parameter (indexed by $r \geq 0$) family of Seiberg-Witten equations on a symplectic manifold $X$. Briefly, these equations are defined as follows: Fix a symplectic form $\omega$ on $X$ and a Riemannian metric on $X$ for which $\omega$ is self dual with norm equal to $\sqrt{2}$ everywhere. The metric and a choice of complex line bundle over $X$ define the bundle of positive spinors over $X$ (a $\mathbb{C}^2$ bundle). Then, the Seiberg-Witten equations are equations for a pair $(a, \psi)$, where a is a connection on the chosen complex line bundle over $X$, and $\psi$ is a section of the bundle of positive spinors. The equations read

$$D_{A(a)}\psi = 0 \quad \text{and} \quad P^+F_a = \frac{1}{8}\tau(\psi \otimes \psi^*) - \frac{ir}{8}\omega.$$

Here, $D_{A(a)}$ is the Dirac operator as defined using a certain connection on the bundle of spinors which is canonically determined by $a$. Also, $P^+$ denotes the projection onto self dual forms, and $\tau$ is a certain canonical homomorphism from the spinor endomorphism bundle to the bundle of self dual 2-forms. (These equations are described in more detail in Section 1, below.)

The existence theorem for the preceding equations constructs solutions (for large $r$) from data which consists (in part) of a finite set $\{(C_k, m_k)\}$ whose $k$'th element is a pair consisting of a compact, connected, pseudo-holomorphic submanifold $C_k \subset X$ and a positive integer $m_k$. (The metric and $\omega$ define an endomorphism of $TX$ with square $-1$, which is to say, an almost complex structure. A submanifold of $X$ is said to be pseudo-holomorphic when its tangent space is preserved by this endomorphism.)

This data is further constrained so that the $C_k$'s are pairwise disjoint. Furthermore, the first Chern class of the chosen line bundle should be Poincaré dual to the sum (over $k$) of $m_k$ times the fundamental class of $C_k$. (A pseudo-holomorphic submanifold has a canonical orientation.)

The theorem, below, is an existence theorem for the case when all $m_k = 1$. This theorem follows from the more general constructions in Section 5 of this article. (See Proposition 5.2.)

**Theorem 2.** *Let $X$ be a compact, oriented, 4-dimensional manifold with a symplectic form $\omega$ and a Riemannian metric for which $\omega$ is self-dual and has norm $\sqrt{2}$ everywhere. Let $\{C_k\}$ be a finite set of compact, disjoint pseudo-holomorphic submanifolds of $X$ for which the operator in (1.5) has trivial cokernel. For each $k$, let $e_k \in H^2(X;\mathbb{Z})$ denote the Poincaré dual to the fundamental class of $C_k$. Write the Seiberg-Witten equations as above for the line bundle whose first Chern class is equal to $\Sigma_k e_k$. Then these equations have solutions for all sufficiently large $r$.*

The kernel of the operator in (1.5) gives the deformations of the submanifold which preserve the pseudo-holomorphic condition to the first order . The condition that the operator in (1.5) have trivial cokernel is satisfied if the metric is chosen in a sufficiently generic way. See, e.g. [11] or [18]. Section 5 presents a fairly precise picture of these solutions.

Remark that in the general case (for example, where some of $m_k$ are greater than one), the existence results here find large $r$ solutions of the Seiberg-Witten equations from certain preferred sections of a certain natural fiber bundle over each $C_k$. The fiber bundle in question can be defined abstractly for any complex curve, complex line bundle over the curve and positive integer $m$. Then, the preferred sections are abstractly characterized as solutions to a certain first order, elliptic differential equation on the curve. Solutions to this equation are easy to write down when $m = 1$, but less obvious when $m > 1$.

This article is divided into six sections. Section 1 constitutes an introduction of sorts to the Seiberg-Witten equations on a symplectic manifold, and to the theory of pseudo-holomorphic submanifolds.

Section 2 starts the construction of the Seiberg-Witten solutions. The first part of the section describes a certain canonical fiber bundle over a pseudo-holomorphic submanifold, and the second part of the section describes a map which takes a section of this bundle to approximate solutions of the one-parameter family of Seiberg-Witten equations. The fiber of the bundle in question is the moduli space of solutions to the vortex equations on $\mathbb{C}$ (see, e.g. [5]), the same moduli space which played such a large role in Section 4 of [19]. In some heuristic sense, one should think of a section of this vortex bundle as giving data, $(a, \psi)$, on a tubular neighborhood in $X$ of the pseudo-holomorphic submanifold. And, this $(a, \psi)$ essentially solves the Seiberg-Witten equations in directions tangent to the fibers of the normal bundle, but it is unconstrained in the directions which are orthogonal to the fibers.

Section 3 serves as a digression of sorts. This section introduces a natural, elliptic differential equation on the space of sections of the vortex bundle from Section 2. The solutions of this equation constitute a finite dimensional variety which can be thought of as giving a preferred set of sections to the vortex bundle. This preferred set of sections plays a fundamental role in Section 4 and in the discussions to come in the proof of Gr=SW.

In some heuristic sense, this preferred set of sections constitutes the solutions to an equation which is half way between the Seiberg-Witten equations and the pseudo-holomorphic curve equations. That is, these preferred sections gives data, $(a, \psi)$, which is defined over a tubular neighborhood in $X$ to the pseudo-holomorphic submanifold. This $(a, \psi)$ satisfies Seiberg-Witten type equations along the fibers of the normal bundle to the submanifold (the vortex equations); and they are further constrained to satisfy a sort of Cauchy-Riemann equation in directions which are orthogonal to the fibers of the normal bundle.

Sections 4 and 5 describe a construction which deforms the approximate solutions from Section 2 with the goal of obtaining an honest solution from the deformation. (Section 4 discusses the linear theory, and Section 5 uses the results in Section 4 together with the contraction mapping theorem to bootstrap from the linear approximation to the relevant non-linear equations.) To a very good approximation, the results of Section 5 (and the article itself) can be summarized by the following statement (see Proposition 5.2):

> There is a smooth map from sections of the vortex bundle over a pseudo-holomorphic submanifold to data for the 1-parameter family of Seiberg-Witten equations. And, this map produces a solution precisely when the section of the vortex bundle lies in Section 3's preferred variety.

This statement is literally true where Section 3's variety is nonsingular.

Note that Theorem 2 above is a corollary to Proposition 5.2.

Section 6 discusses the analytic properties of the map which was constructed in Section 5. In particular, Section 6 proves that the map in question is $1-1$ and onto an open subset of solutions to the large $r$ version of the Seiberg-Witten equations.

To the gauge theory cognescenti, this strategy of constructing approximate solutions and then deforming them to honest solutions will, no doubt, seem familiar. (The constructions here also resemble certain constructions in [1].)

## 1. Setting the stage

The purpose of this first section is to set the stage, so to speak, for the constructions that follow. In particular, the basics of the Seiberg-Witten equations and the theory of pseudo-holomorphic submanifolds are reviewed.

### a) The Seiberg-Witten equations

The Seiberg-Witten equations were first introduced by Seiberg and Witten in [14] and [15]; see also [24]. A purely mathematical approach to these equations was first taken in [9]. In any event, what follows is a brief summary of the story.

In this subsection, $X$ is simply a compact, oriented, 4-dimensional manifold. Fix a smooth Riemannian metric on $X$. The metric defines the principle SO(4) bundle of orthornormal frames, $Fr \to X$. Of the various associated bundles to this frame bundle, two in particular play central roles. These are the bundles $\Lambda_+$ of self-dual 2-forms and $\Lambda_-$ of anti-self dual 2-forms. Note that $\Lambda^2 TX \approx \Lambda_+ \oplus \Lambda_-$.

By definition, a $\mathrm{Spin}^{\mathbb{C}}$ structure on $X$ is an equivalence class of lifts of Fr to a principal $\mathrm{Spin}^{\mathbb{C}}(4)$ bundle $F \to X$. In this regard, recall that the group $\mathrm{Spin}^{\mathbb{C}}(4)$ is the group $(SU(2) \times SU(2) \times U(1))/\{\pm 1\}$, this being a central extension of $SO(4) = (SU(2) \times SU(2))/\{\pm 1\}$ by the circle $U(1)$. (The homomorphism $\mathrm{Spin}^{\mathbb{C}} \to (SU(2) \times SU(2))/\{\pm 1\}$ simply forgets the factor of $U(1)$.)

A $\mathrm{Spin}^{\mathbb{C}}$ lift $F$ of $Fr$ has two canoncial associated $\mathbb{C}^2$ bundles, $S_\pm \to X$ which are defined using the two evident homomorphisms of $\mathrm{Spin}^{\mathbb{C}}$ to $U(2) = (SU(2) \times U(1))/\{\pm 1\}$. Note that $S_+$ is distinguished by the fact that its projective bundle is the unit 2-sphere bundle in $\Lambda_+$. (There is, of course, an analagous relationship between $S_-$ and $\Lambda_-$.)

With the preceding understood, the original version of Seiberg and Witten's equations can now be defined. These are equations for a pair $(A, \psi)$, where $A$ is a connection on $\det(S_+)$, and $\psi$ is a section of $S_+$.

The equations read:

$$
D_A\psi = 0, \qquad P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + \mu. \tag{1.1}
$$

In the first line above, $D_A$ is the Dirac operator, a first order differential operator which maps sections of $S_+$ to sections of $S_-$. This $D_A$ is defined as the composition of Clifford multiplication (a homomorphism from $S_+ \otimes T^*X$ to $S_-$) with covariant differentiation using the connection on $S_+$ which comes from the Levi-Civita connection on $Fr$ and the connection $A$ on $\det(S_+)$. In the second line of (1.1), $P_+$ denotes the orthogonal projection from $\Lambda^2 T^*X$ to $\Lambda_+$, and $F_A$ denotes the curvature 2-form of $A$. Meanwhile, $\tau$ is the adjoint of the Clifford multiplication endomorphism from $\Lambda_+ \otimes \mathbb{C}$ into End(S+), and $\mu$ is a fixed, imaginary valued, anti-self dual 2-form on $X$. (Any choice for $\mu$ will do.)

There is a natural action of the group of smooth maps from $X$ to $U(1)$ on the set of solutions to (1.1). The action sends a map $g$ and a pair $(A,\psi)$ to $(A+2gdg^{-1}, g\psi)$. Use $\mathcal{M}$ to denote the set of orbits under this group action. (Typically, notational distinctions will not be made between a pair $(A,\psi)$ and its orbit in $\mathcal{M}$.)

Topologize $\mathcal{M}$ as follows: First, introduce the manifold $\mathrm{Conn}(\det(S_+))$ of Hermitian connections on $\det(S_+)$. This is an affine Frechet manifold modelled on $i\Omega^1(X)$. With $\mathrm{Conn}(\det(S_+))$ understood, introduce the space $\mathrm{Conn}(\det(S_+)) \times C^\infty(S_+)$. The group $C^\infty(X;S^1)$ acts smoothly on the latter (as indicated above), and the space of orbits of this group action, $(\mathrm{Conn}(\det(S_+) \times C^\infty(S_+))/C^\infty(X;S^1)$, is given the quotient topology. The space $\mathcal{M}$ sits in this quotient, and the implicit topology on $\mathcal{M}$ is the subspace topology inherited from the orbit space $(\mathrm{Conn}(\det(S_+) \times C^\infty(S_+))/C^\infty(X;S^1)$.

Here are some basic properties of $\mathcal{M}$ (see, [24] or [9], [8], [12].):

1. $\mathcal{M}$ is always compact.

2. If $b^2_+ > 0$, then there is a Baire set of $\mathcal{U} \subset C^\infty(X; i\Lambda_+)$ of choices for $\mu$ in (1.1) whose corresponding $\mathcal{M}$ has the structure of a smooth, manifold of dimension

$$
2d = -\frac{1}{4}(2\chi + 3\tau) + \frac{1}{4}c_1 \bullet c_1. \tag{1.2}
$$

Here, $\chi$ is the Euler characteristic of $X$ and $\tau$ is the signature of $X$. Also, "$\bullet$" signifies the intersection pairing on $H^2(X;\mathbb{Z})$. Note that when $\mu \in \mathcal{U}$, there are no points in $\mathcal{M}$ where the corresponding $\psi$ is zero.

A Baire set is a countable intersection of open and dense sets and so is dense. The Baire set in question is characterized by the condition that a certain family of first order, elliptic differential operators parameterized by the points in $\mathcal{M}$ has, at each point in $\mathcal{M}$, trivial cokernel.

By the way, the number $2d$ in (1.2) can be even or odd. Its parity is the same as that of $\frac{1}{2}(\chi+\tau) = 1 - b^1 + b^2_+$.

**b) Pseudo-holomorphic submanifolds**

The pseudo-holomorphic submanifold story starts with a review of some of basic properties of symplectic 4-manifolds. This review has six parts.

*Part 1.* A symplectic 4-manifold is, by definition, a pair $(X,\omega)$, where $X$ is a smooth 4-manifold, and $\omega$ is a closed 2-form on $X$ with $\omega\wedge\omega$ nowhere zero. (The characteristic number $\frac{1}{2}(\chi+\tau)$ must be even for $X$ to admit a symplectic form.)

*Part 2.* Every symplectic manifold admits almost complex structures, endomorphisms $J$ of $TX$ with square $-1$. As noted by Gromov [4], one can find almost complex structures with the property that the bilinear form

$$g = \omega(\cdot, J(\cdot)) \tag{1.3}$$

defines a Riemannian metric on $TX$. Such a $J$ will be called *$\omega$-compatible.*

An almost complex structure $J$ decomposes $TX\otimes\mathbb{C} \approx T_{1,0}\oplus T_{0,1}$ into a sum of complex 2-plane bundles such that $J$ has eigenvalue $i$ on the former and $-i$ on the latter. The complexified cotangent bundle decomposes analogously as $T^{1,0}\oplus T^{0,1}$.

*Part 3.* The complex line bundle $K = \det(T^{1,0})$ is called the canonical bundle. Note that the isomorphism class of $K$, and thus its first Chern class $c\in H^2(X;\mathbb{Z})$, is independent of the choice of $\omega$-compatible almost complex structure $J$. Furthermore, this isomorphism class and also $c$ are both unchanged if $\omega$ is changed through a continuous family of symplectic forms. (Note the sign convention here: $c\bullet[\omega] < 0$ when $X = \mathbb{CP}^2$.)

*Part 4.* A submanifold $C$ in $X$ is called pseudo-holomorphic when $J$ preserves $TC$. It follows from the non-degeneracy of (1.3) that $\omega$ is non-degenerate on $TC$ and so orients $C$. Infact, $J$ induces the structure of a complex curve on $C$. Then, the inclusion map of $C$ into $X$ is pseudo-holomorphic in the sense of Gromov [4].

If $C$ is a connected and compact pseudo-holomorphic submanifold, then the genus of $C$ is constrained by the adjunction formula to equal

$$\text{genus} = 1 + \frac{1}{2}(e \bullet e + c \bullet e), \tag{1.4}$$

where $e$ is the Poincaré dual to the fundamental class $[C]$ of $C$.

Henceforth, all pseudo-holomorphic submanifolds in this article should be assumed to be compact unless stated to the contrary.

*Part 5.* Fix a pseudo-holomorphic submanifold $C$. Since $J$ preserves $TC$, it must also preserve the orthogonal compliment in $TX$ of $TC$. The latter is the normal bundle, $N$, of $C$. Thus, $N$ has a natural structure as a complex line bundle over $C$. The metric from $TX$ defines a connection on $N \to C$, and therefore endows $N$ with a holomorphic structure as a bundle over the complex curve $C$. With this understood, one can introduce the associated $d$-bar operator, $\bar\partial$, to map sections of $N$ to sections of $N \otimes T^{0,1}C$, where $T^{0,1}C$ is the usual anti-holomorphic summand of $T^*C \otimes_{\mathbb{R}} \mathbb{C}$.

The first guess is that the kernel of $\bar\partial$ corresponds to the vector space of deformations of $\Sigma$ in $X$ which are pseudo-holomorphic to the first order. However, this guess is wrong, in general. Rather, this vector space corresponds to the kernel of certain canonical zero'th order deformation of $\bar\partial$. This deformation is an $\mathbb{R}$ linear operator, $D$, which also maps sections of $N$ to sections of $N \otimes T^{0,1}C$, and is defined as follows: The 1-jet off of $\Sigma$ of the almost complex structure defines a pair $(\nu, \mu)$ of section of $T^{0,1}C$ and $N^{\otimes 2} \otimes T^{0,1}C$. (See (2.3).) Then

$$Dh = \bar\partial h + \nu h + \mu \bar h. \tag{1.5}$$

*Part 6.* Note that the index of $D$ is given by the Riemann-Roch formula:

$$d = e \bullet e - c \bullet e, \tag{1.6}$$

where $e \in H^2(X;\mathbb{Z})$ is Poincaré dual to $[\Sigma]$. As the index is, by definition, the difference between the dimensions (over $\mathbb{R}$) of the kernel and the cokernel of $D$, a necessary condition for the triviality of cokernel($D$) is that d be non-negative. In general, this condition is not sufficient. However, all pseudo-holomorphic submanifolds are regular if the almost complex structure is chosen from a certain Baire subset of $\omega$ compatible almost complex structures. (This fact is proved in, e.g. [11].)

### c) The Seiberg-Witten equations on a symplectic manifold

This subsection points out some of the special features of the Seiberg-Witten equations on a symplectic manifold.

To begin, remember that a symplectic manifold $X$ has a canonical orientation which is give by $\omega \wedge \omega$; and this orientation will be assumed throughout.

As remarked above, an admissable almost complex structure on $X$ defines a Riemannian metric as in (1.3). The form $\omega$ is self-dual with respect to the splitting of $\Lambda^2 T^*X$ as defined by this metric, and it has everywhere length $\sqrt{2}$. When an $\omega$-compatible $J$ has been specified, the metric in (1.3) will be taken implicitly for the Riemannian metric on $X$.

As observed in [20], a symplectic 4-manifold $X$ also has a canonical $\mathrm{Spin}^{\mathbb{C}}$ structure. With the metric chosen from an $\omega$-compatible $J$, the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure is characterized by the identifications

$$S_+ = \mathbb{I} \oplus K^{-1} \quad \text{and} \quad S_- = T^{0,1}. \tag{1.7}$$

Indeed, the splitting of $S_+$ is defined as follows: Clifford multiplication defines an endomorphism from $\Lambda_+$ into the bundle of skew symmetric endomorphisms of $S_+$. With this understood, the splitting of $\Sigma_+$ in (1.7) is the decomposition of $S_+$ into eigenbundles for the action of $\omega$; here $\omega$ acts with eigenvalue $-2i$ on the trivial summand $\mathbb{I}$, and with eigenvalue $+2i$ on the $K^{-1}$ summand.

Now, for any oriented 4-manifold, the set $\mathcal{S}$ of $\mathrm{Spin}^{\mathbb{C}}$ structures on $X$ is naturally a principal $H^2(X;\mathbb{Z})$ bundle over a point. Thus, the identification in (1.4) of a canonical element in $\mathcal{S}$ identifies

$$\mathcal{S} \approx H^2(X;\mathbb{Z}). \tag{1.8}$$

Under this identification, a class $e \in H^2(X;\mathbb{Z})$ is sent to the $\mathrm{Spin}^{\mathbb{C}}$ structure whose $S_\pm$ bundles are given by

$$S_+ = E \oplus (K^{-1} \otimes E) \quad \text{and} \quad S_- = T^{0,1} \otimes E, \tag{1.9}$$

where $E$ is a complex line bundle whose first Chern class is $e$. Once again, this splitting of $S_+$ is into eigenbundles for the action of $\omega$ on

$S_+$; and the convention is that the bundle where $\omega$ acts as $-2i$ is written first.

Now it turns out that there is a nice way to rewrite (1.1) which exploits the decomposition in (1.9). This rewriting of (1.1) requires a preliminary two-part digression. Part 1 of the digression observes that the bundle $K^{-1}$ comes equipped with a canonical connection (up to the action of $C^\infty(X; U(1))$ (see, e.g. [20])). This connection is defined as follows: In general, fix a $\mathrm{Spin}^{\mathbb{C}}$ structure. As remarked previously, the choice of a connection on $\det(S_+)$ and the Levi-Civita connection on the bundle $Fr$ defines a connection on the $\mathrm{Spin}^{\mathbb{C}}$ lift $F$. Thus, the choice of a connection (say $A$) on $\det(S_+)$ gives a covariant derivative, $\nabla_A$, on sections of $S_+$. Now consider the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure in (1.7). Restriction of $\nabla_A$ to a section of the trivial summand $\mathbb{I}$ and projection of the resulting covariant derivative onto $\mathbb{I} \otimes T^*X$ defines a covariant derivative $\nabla_A$ on the trivial complex line bundle. With the preceding understood, remark that there is a unique choice of connection $A_0$ (up to the afore-mentioned gauge equivalence) on $\det(S_+) = K^{-1}$ for which the corresponding covariant derivative on the trivial line bundle admits a non- trivial, covariantly constant section.

For Part 2 of the digression, consider the general $\mathrm{Spin}^{\mathbb{C}}$ structure in (1.7). Since $\det(S_+) = E^2 \otimes K^{-1}$, the choice of the connection $A_0$ on $K^{-1}$ allows any connection $A$ on $\det(S_+)$ to be written uniquely as

$$A = A_0 + 2a, \tag{1.10}$$

where $a$ is a connection on the complex line bundle $E$. Thus, with $A_0$ chosen, the Seiberg-Witten equations in (1.1) can be thought of as equations for a pair $(a, \psi)$, where $a$ is a connection on $E$, and $\psi$ is a section of $S_+$ in (1.9).

End the digression. With this reinterpretation of (1.1) understood, remark now that it proves useful to "renormalize" the form $\mu$ in (1.1) by writing

$$\mu = -\frac{ir}{4}\omega + P_+F_{A_0} + i\mu_0. \tag{1.11}$$

Here, $r$ can be any non-negative number and $\mu_0$ can be any section of $\Lambda_+$. (In practice, think of $\mu_0$ as being close to 0.) Furthermore, in the case where $r > 0$, it also proves useful to write

$$\psi = r^{1/2}(\alpha, \beta) \tag{1.12}$$

to correspond with the splitting in (1.9). Then, with the preceding understood, the Seiberg-Witten equations in (1.1) read

$$\begin{aligned} &D_A(\alpha,\beta) = 0. \\ (1.13)\qquad &P_+F_a + \frac{ir}{8}(1-|\alpha|^2+|\beta|^2)\omega - \frac{r}{4}(\alpha\beta^* - \alpha^*\beta) - i\mu_0 = 0. \end{aligned}$$

Here, $\alpha\beta^*$ and $\alpha^*\beta$ are sections of $K$ and $K^{-1}$, where the latter are naturally identified as the orthogonal compliment of the span of $\omega$ in $\Lambda_+ \otimes \mathbb{C}$. (Note that this last equation differs from the analogous equations in [17], [19] and [20] in that the $\beta$ used here is -i times that used in the previous papers. The insertion of this factor of $-i$ here avoids numerous factors of $i$ later on.)

As an aside, remark that a fundamental input to the proof of Gr=SW is that solutions to (1.13) for large $r$ (and $\mu_0 = 0$) determine pseudo-holomorphic curves in $X$ with fundamental class Poincaré dual to $e$. The precise statement is proved as Theorem 1.3 in [19]:

**Theorem 1.1.** *Let $X$ be a compact, 4-manifold with symplectic form $\omega$. Fix an $\omega$-compatible almost complex structure on $X$, and use the resulting metric to define the Seiberg-Witten equations. Fix $e \in H^2(X;\mathbb{Z})$ and use $e$ to define the Spin$^{\mathbb{C}}$ structure in (1.9). Also, fix a finite (maybe empty) collection $\{\Omega_k\}$ of closed subsets of $X$. Let $\{r_n\}$ be an increasing and unbounded sequence of positive real numbers, and let $(a_n, (\alpha_n, \beta_n))$ be a sequence of solutions to (1.13) using the Spin$^{\mathbb{C}}$ structure in (1.9), using $r = r_n$ and $\mu_0 = 0$. Suppose that for each $n$ and $k$, the intersection of $\alpha_n^{-1}(0)$ with $\Omega_k$ is not empty. Then there exist a compact, complex curve $C$ (not necessarilly connected), a $J$-pseudo-holomorphic map $\varphi : C \to X$, and a subsequence of $\{(a_n, (\alpha_n, \beta_n))\}_{n=1,\ldots}$ (hence relabled consecutively) with the following properties:*

1. *$\varphi_*[C]$ is Poincaré dual to $e$.*
2. *$\varphi \cap \Omega_k \neq \varnothing$ for all $k$.*
3. *$\lim_{n\to\infty}\{\sup_{x:\alpha_n(x)=0} \operatorname{dist}(x, \varphi(C)) + \sup_{x\in\varphi(C)} \operatorname{dist}(x, \alpha_n^{-1}(0))\} = 0$.*

The goal for the rest of this article is to prove a converse of sorts to the preceding theorem. That is, to use pseudo-holomorphic submanifolds in $X$ to find large $r$ solutions to (1.13). (See the introduction for an outline of the basic strategy.)

## 2. The gluing construction

The purpose of this section is to begin a construction which takes a pseudo-holomorphic submanifold and produces solutions to the large $r$ (and small $\mu_0$) versions of (1.13). This first step is a purely geometric one which constructs approximate solutions to the large $r$ and small $\mu_0$ version of (1.13). The construction of these approximate solutions and the specification of some of their properties occupies this section.

### a) Local geometry

Fix an $\omega$-compatible almost complex structure $J$ for $X$, and use $J$ to define the metric in (1.3). Let $C \subset X$ be a connected, pseudo-holomorphic submanifold. The use of $C$ to build an approximate solution to (1.13) requires a six part digression to describe the pseudo-holomorphic geometry in a neighborhood of $C$. This subsection serves this purpose.

*Part 1.* The normal bundle $\pi : N_C \to C$ is a real, oriented 2-plane bundle. Once identified with the orthogonal compliment of $TC$ in $TX$, the bundle $N_C$ inherits the structure of a complex line bundle. When thought of as a complex line bundle (or when the real structure is immaterial), $N_C$ will be denoted simply by $N$. Note that the Riemannian metric endows $N$ with a hermitian structure; and said metric's Levi-Civita connection endows $N$ with the structure of a holomorphic vector bundle over the complex curve $C$. Here, the fibers and the zero section are holomorphic submanifolds of $N$. (When necessary, $J_0$ will denote the square $-1$ endomorphism of $TN$ which is induced by $N$'s structure of a holomorphic vector bundle. The $J_0$ version of the holomorphic tangent bundle $T_{1,0}N$ naturally splits as $\pi^*T_{1,0}C \oplus \pi^*N$, where the first summand consists of horizontal vectors and the second consists of vectors which are tangent to the fibers of the map $\pi$.)

*Part 2.* Introduce the bundle $\pi^*N \to N$. Let $\theta$ denote the Levi-Civita connection on $N$, and use this symbol as well for the pull-back of said connection to $\pi^*N$. The bundle $\pi^*N$ also has a tautological section, $s$ whose restriction to any fiber of $\pi$ gives a $J_0$-complex coordinate for that fiber. (Another way of saying this is that the covariant derivative $\nabla_\theta s$ restricts to each fiber as a $J_0$-holomorphic 1-form which has unit length as measured with the fiber bundle metric.)

*Part 3.* A *tubular neighborhood* map $\varphi : N \to X$ is a smooth map which maps the zero section to $C$ as the canonical identification, and whose differential along $C$ gives the canonical identification between $N$ and $TX|_C/TC$. The exponential map, $\varphi_0$, for the Riemannian metric is the classical example. For the purposes of the constructions that follow, the tubular neighborhood map described in the next lemma is more useful:

**Lemma 2.1.** *Let $C$ be a compact, connected pseudo-holomorphic submanifold of $X$. Then, there exist $\delta_0 > 0$ and $\zeta > 0$ and a tubular neighborhood map $\varphi : N \to X$ with the following properties:*

1. *$\varphi$ embeds the radius $\delta_0$ disk bundle $N_{(0)} \subset N$ onto its image in $X$.*
2. *The $\varphi$ image of each fiber of $\pi : N_{(0)} \to X$ is a pseudo-holomorphic submanifold.*
3. *The differential of $\varphi$ identifies $J_0$ with $J$ on tangents to the fibers of $\pi$. (That is, $\varphi$ is a pseudo-holomorphic map on each fiber of $\pi$.)*

*Proof of Lemma 2.1.* All of these facts follow using variations of the implicit function theorem given in Lemma 5.5 of [19].

With Lemma 2.1's map $\varphi$ understood, agree to use the map $\varphi$ to implicitly identify the radius $\delta_0$ disk bundle $N_{(0)}$ with its $\varphi$ image in $X$. With this identification understood, the almost complex structure $J$ restricts to an almost complex structure on $N_{(0)}$ and the form $\omega$ pulls back to $N_{(0)}$ as a symplectic form. These identifications are implicit in the next lemma, and throughout this article.

**Lemma 2.2.** *Let $C$ be a compact, connected pseudo-holomorphic submanifold of $X$. The numbers $\delta_0$ and $\zeta$ and the map $\varphi$ in Lemma 2.1 can be chosen so that Lemma 2.1 is true, and so that the following are also true:*

1. *$\nabla_\theta s$ restricts to each fiber of $\pi$ as a $J$-holomorphic 1-form.*
2. *The pull-backs to each fiber of $\pi$ of the symplectic form $\omega$ and $\frac{i}{2}\nabla_\theta s \wedge \nabla_\theta \bar{s}$ are equal to first order along $C$. Put differently, let $\iota$ denote the inclusion map of a fiber of $N_{(0)}$ into $N_{(0)}$. Then $|\iota^*\omega - \iota^*(\frac{i}{2}\nabla_\theta s \wedge \nabla_\theta \bar{s})| \leq \zeta |s|^2$.*

*Proof of Lemma 2.2.* The first assertion is another way of stating Assertion 3 of Lemma 2.1. The proof of the second assertion is an exercise with Taylor's expansions. The point is that if $\varphi$ does not initially satisfy Assertion 2, then it can be changed to do so by precomposing the original by a fiber preserving map of $N_{(0)}$ to itself, which pulls back the section $s$ as $s + h \cdot s^2$; here $h$ is an appropriate section over $C$ of $N^{-1}$. The details are straightforward and omitted.

*Part 4.* The almost complex structure $J$ on $N_{(0)}$ can now be described in the following way: The $J$ version of $T^{1,0}N_{(0)}$ is (locally) spanned by a $g$-orthonormal pair of forms $\{\kappa_0, \kappa_1\}$, where $\kappa_0$ is a section of $\pi^*T^*C \otimes \mathbb{C}$ (which lies in $T^{1,0}C$ along $C$) and

$$\kappa_1 = \varsigma \nabla_\theta s + \sigma. \tag{2.1}$$

Here, $\sigma$ is a section of $\pi^*T^*C \otimes \pi^*N$ which vanishes along $C$, and $\varsigma$ is a function which behaves near $C$ as $\varsigma = 1 + \mathcal{O}(|s|^2)$. Note that

$$\omega = \frac{i}{2}(\kappa_0 \wedge \bar{\kappa}_0 + \kappa_1 \wedge \bar{\kappa}_1). \tag{2.2}$$

*Part 5.* Near $C$, write $\sigma = \sigma_{1,0} + \sigma_{0,1}$ with the former in $T^{1,0}$ and the latter in $T^{0,1}$. These have the Taylor's expansion

$$\sigma_{1,0} = -\bar{\nu}s + \gamma\bar{s} + \mathcal{O}(|s|^2) \quad \text{and} \quad \sigma_{0,1} = \nu s + \mu\bar{s} + \mathcal{O}(|s|^2). \tag{2.3}$$

Here, $\nu$ is a section over $C$ of $T^{0,1}C$, $\gamma$ is a section over $C$ of $T^{1,0} \otimes N^2$, and $\mu$ is a section over $C$ of $T^{0,1}C \otimes N^2$. (The terms proportional to $s$ in the two Taylor's expansions are related when the condition $d\omega = 0$ is imposed.)

*Part 6.* Because $\omega$ is closed and $\sigma$ vanishes along $C$, the first term in (2.2) has the Taylor's expansion

$$\frac{i}{2}\kappa_0 \wedge \bar{\kappa}_0 = \pi^*\omega_C + \mathcal{O}(|s|^2),$$

where $\omega_C$ is the volume form on $C$.

**b) Vortices on $\mathbb{C}$**

This subsection and the next constitute a second digression whose purpose is to describe certain relevant aspects of the vortex equation on $\mathbb{C}$. As remarked in the introduction, approximate solutions to the large $r$ version of (1.13) are constructed with the help of these solutions to the vortex equations. However, this subsection and the subsequent one make no reference to this application. By the way, the vortex equations are discussed at length in [21] and also [5], to which the reader is referred for more details. (See also Section 4 of [19].)

This subsection is broken into seven parts.

*Part 1.* The vortex equation is a system of equations for a pair $(v, \tau)$ of imaginary valued 1-form, $v$, on $\mathbb{C}$ and complex valued function, $\tau$, on $\mathbb{C}$. These equations read

$$\begin{aligned} &1. \quad idv = \frac{1}{4} * (1 - |\tau|^2). \\ &2. \quad \bar{\partial}\tau + v_{0,1}\tau = 0. \\ &3. \quad * (1 - |\tau|^2) \text{ is integrable on } \mathbb{C}. \end{aligned} \tag{2.4}$$

Here, $*$ is the Euclidean Hodge star operator on $\mathbb{C}$, and $v_{0,1}$ is the projection of $v$ onto $T^{0,1}\mathbb{C}$. (One can think of $v$ as defining a connection on the product complex line bundle over $\mathbb{C}$ in which case $v$ defines a holomorphic structure for this bundle. Then, the second equation in (2.4) asserts that $\tau$ is holomorphic with respect to this structure.)

*Part 2.* Here are some facts about the solutions to (2.4):

1. The integral over $\mathbb{C}$ of $\frac{1}{4} * (1 - |\tau|^2)$ is equal to $2\pi m$ for some non-negative integer $m$. This integer is called the "vortex number" of the solution.

2. The integer $m$ above is the same as the number of points (counting multiplicity) at which $\tau$ is zero.

3. In general, $|\tau| < 1$ unless $m = 0$ in which case $|\tau| = 1$ everywhere.

4. $(1 - |\tau|^2)$ and $|\nabla_v \tau|$ decay exponentially fast away from the zero's of $\tau$. To be precise, there exists $\zeta > 0$ such that

$$1 - |\tau|^2 < \exp\left(-\frac{\operatorname{dist}(x, \tau^{-1}(0))}{\zeta}\right);$$

and, given integers $n, k \geq 0$, there exists a constant $\zeta_{m,k}$ with the property that

$$\left|(\nabla_v)^k \nabla_v \tau\right|^2 \leq \zeta_{m,k} \cdot (1 - |\tau|^2). \tag{2.5}$$

*Part 3.* The solution space to (2.4) can be described as follows: Topologize the set of pairs $(v, \tau)$ which solve (2.4) by embedding the latter into $i \cdot \Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C};\mathbb{C}) \times [0,\infty)$ by the map which sends $(v,\tau)$ to the triple $(v, \tau, \int_{\mathbb{C}}(1-|\tau|^2)\cdot\omega_0)$. Now, remark that the set of solutions to (2.4) is invariant under the action of $\mathrm{Maps}(\mathbb{C}; U(1))$ on $i\cdot\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C};\mathbb{C})$ via the action which sends a map $\phi$ and a pair $(v,\tau)$ to $(v - \phi^{-1}d\phi, \phi\cdot\tau)$. Let $\mathfrak{C}$ denote the space of orbits of solutions of (2.4) under this action. Topologize $\mathfrak{C}$ by the quotient topology. Then, the space $\mathfrak{C}$ has naturally the structure of a smooth manifold with components $\{\mathfrak{C}_m\}$ labeled by the integer $m$ which appears in (2.5.1). Furthermore, the component $\mathfrak{C}_m$ is diffeomorphic to $\mathbb{C}^m$.

A useful diffeomorphism from $\mathbb{C}^m$ to $\mathfrak{C}_m$ is defined by a particular embedding of $\mathbb{C}^m$ into $i\Omega^1(C)\times C^\infty(\mathbb{C};\mathbb{C})$ which projects under the action of $\mathrm{Maps}(\mathbb{C};U(1))$ onto $\mathfrak{C}_m \subset (i\Omega^1(\mathbb{C})\times C^\infty(\mathbb{C};\mathbb{C}))/\mathrm{Maps}(\mathbb{C};U(1))$. This embedding of $\mathbb{C}^m$ into $i\Omega^1(\mathbb{C})\times C^\infty(\mathbb{C};\mathbb{C})$ has the following properties:

1. The $m$-tuple $y = (y_1, \ldots, y_m) \in \mathbb{C}^m$ is sent to

$$c(y) = (v,\tau) = (\bar\partial u - \partial u,\ p[y]e^{-u}),$$

where $p[y]$ is the polynomial which sends $\eta \in \mathbb{C}$ to $p[y](\eta) = \eta^m + y_1\eta^{m-1} + \cdots + y_m$, and where $u$ is the unique, real valued function which solves the equation

$$i\partial\bar\partial u = 8^{-1} * (1 - |p|^2 e^{-2u})$$

with the asymptotic condition $u = m\cdot\ln|\eta| + \mathcal{O}(1)$ as $|\eta|\to\infty$.

2. The inverse of this diffeomorphism can be obtained by composing two homeomorphisms. The first sends $\mathfrak{C}_m$ to the space $\mathrm{Sym}^m\mathbb{C}$ of (un-ordered) $m$-tuples of points in $\mathbb{C}^m$ by assigning to $(v,\tau)$ the set $\tau^{-1}(0)$ (counting multiplicities). The second is the inverse of the homeomorphism from $\mathbb{C}^m$ to $\mathrm{Sym}^m\mathbb{C}$ which assigns to each $m$-tuple $(y_1,\ldots,y_m)$ the zeros of $p[y]$.

3. The inverse of the aforementioned diffeomorphism can also be obtained from the observation that when $c(y) = (v,\tau)$ is given by $y$ as in the first point above, then

$$(8\pi)^{-1}\cdot\int_{\mathbb{C}}\eta^q(1-|\tau|^2) = \Sigma_{\lambda:\tau(\lambda)=0}m(\lambda)\cdot\lambda^q,$$

where $m(\lambda)$ is the multiplicity of the zero $\lambda$ of $\tau$. (This holds for any non-negative integer q.)

(2.6)

(The existence and uniqueness assertions for the equation in (2.6.1) follow as in [21]. The integral identity in (2.6.3) can be obtained by using the equation in (2.6.1) with the asymptotic condition from (2.5.4) that $u = \ln|p| + \mathcal{O}(e^{-|\eta|/\zeta})$ as $|\eta| \to \infty$.)

*Part 4.* The group of rotations of $\mathbb{C}$ is $U(1)$, and this group acts naturally on each $\mathfrak{C}_m$ in the following manner: Think of $U(1)$ as the group of unit length, complex numbers. When $\lambda \in U(1)$, introduce $\psi_\lambda : \mathbb{C} \to \mathbb{C}$ to denote the action of rotation which sends a point $\eta$ to $\psi_\lambda(\eta) = \lambda^{-1}\eta$. Then, $\lambda$ acts on $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C};\mathbb{C})$ by pull-back, and hence on the space of solutions to (2.4). That is, $\lambda$ takes a solution $(v, \tau)$ of (2.4) and gives the new solution $(\psi_\lambda^* v, \psi_\lambda^* \tau)$. The induced action on $\mathfrak{C}_m$ has a unique fixed point. This fixed point is the image of the origin in $\mathbb{C}^m$ under (2.6)'s embedding. The latter has the form

$$v_m = u_m(\eta d\bar{\eta} - \bar{\eta} d\eta) \quad \text{and} \quad \tau_m = f_m \cdot \eta^m. \tag{2.7}$$

Here, $u_m$ and $f_m$ are real valued functions of $|\eta|^2$ only, and $f_m$ is non-negative.

Note that the embedding in (2.6) is not preserved by this $U(1)$ action.

*Part 5.* There is a second $U(1)$ action on $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C};\mathbb{C})$ which differs from that in Part 4 by an action of the constants in $\mathrm{Maps}(\mathbb{C}; U(1))$. This modified action does preserve the embedding in (2.6). The modified action sends a unit complex number $\lambda$ and the point $(v, \tau)$ to $(\psi_\lambda^* v, \lambda^m \cdot \psi_\lambda^* \tau)$. As this modified $U(1)$ action preserves the embedding of $\mathbb{C}^m$ in (2.6), it defines a $U(1)$ action on $\mathbb{C}^m$. The latter action sends $\lambda$ and the $m$-tuple $y = (y_1, \ldots, y_m)$ to $(\lambda \cdot y_1, \ldots, \lambda^m \cdot y_m)$.

*Part 6.* The embedding of $\mathfrak{C}_m = \mathbb{C}^m$ into $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C};\mathbb{C})$ from (2.6) leads to less than optimal estimates in the subsequent applications. A less canonical, but more convenient slice for the $\mathrm{Maps}(\mathbb{C}; U(1))$ action is defined by an embedding of $\mathfrak{C}_m$ into $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C};\mathbb{C})$ which has the property that where $|\eta|$ is large. Then

$$\tau = f \cdot \eta^m, \tag{2.8}$$

with $f$ being a positive function. The meaning of the phrase "where $|\eta|$ is large" is as follows: If $c = (v, \tau)$ is defined by $y \in \mathbb{C}^m$ as in (2.6.1), then $|\eta|$ is large when

$$|\eta| > 2 \cdot \Sigma_k (1 + m^2 |y_k|^2)^{1/2k}. \tag{2.9}$$

The next lemma describes the new slice embedding of $\mathfrak{C}_m$ into $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C}; \mathbb{C})$:

**Lemma 2.3.** *There is an embedding of $\mathfrak{C}_m$ into $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C}; \mathbb{C})$ with the following properties:*

1. *Each point in the image obeys (2.4), and the projection back to $\mathfrak{C}_m$ yields the identity map.*
2. *Each point in the image satisfies (2.8) where (2.9) is satisfied.*
3. *The image is mapped to itself by the $U(1)$ on $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C}; \mathbb{C})$ for which $\lambda \in U(1)$ sends $(v, \tau)$ to $(\psi_\lambda^* v, \lambda^m \psi_\lambda^* \tau)$.*
4. *There is a diffeomorphism, $\Upsilon$, from $\mathbb{C}^m$ onto this embedding which sends $y \in \mathbb{C}^m$ to the pair $(\bar\partial\varsigma - \partial\varsigma,\ p(y) \cdot e^{-\varsigma})$, where $\varsigma \in C^\infty(\mathbb{C}; \mathbb{C})$ obeys* $\mathrm{Re}(\varsigma) = u$ *from (2.6.1).*

In later subsections, $\mathfrak{C}_m$ will be implicitly identified with its images in $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C}; \mathbb{C})$ via Lemma 2.3's embedding; and this image will be implicitly identified with $\mathbb{C}^m$ via the map $\Upsilon$ of Assertion 4 of Lemma 2.3.

*Proof of Lemma 2.3.* The first point to observe is that the slice defined by (2.6) obeys Conditions 1 and 3 of the lemma. The new slice will be obtained by translating each $(v, \tau)$ in the slice from (2.6) along the orbit of $\mathrm{Maps}(\mathbb{C}; U(1))$ by a $(v, \tau)$ dependent map. (The map in question will depend smoothly on the pair $(v, \tau)$.)

To prove that the conditions in Lemma 2.3 can be met, introduce $\Omega$ to denote the domain in $\mathbb{C}$ where $|\eta|$ is larger than

$$d_y = \Sigma_k (1 + m^2 |y_k|^2)^{1/(2k)}.$$

The point is that $\tau$ is non-vanishing on $\Omega$ and, furthermore, $\frac{\tau}{|\tau|}$ and $\frac{\eta^m}{|\eta|^m}$ are homotopic as maps from $\Omega$ to the circle. Thus, the first map times the inverse of the second has the form $e^{ih}$ where $h$ is the real valued function which is the argument of $\ln(1 + y_1 \cdot \eta^{-1} + \cdots + y_m \cdot \eta^{-m})$.

With the preceding understood, (2.8) can then be realized by gauge transforming a given $(v, \tau)$ in (2.6) by the map to $U(1)$ defined by $e^{-ixh}$, where $x$ is given by $x(\eta) = w(|\eta|/d_y)$ with $w : [0, \infty) \to [0, 1]$ being a fixed function which is equal to zero on $[0, 1]$ and to 1 on $[2, \infty)$.

The invariance of the slice under the $U(1)$ action is an automatic consequence of the invariance of the embedding in (2.6) under said action.

*Part 7.* The group of dilations of $\mathbb{C}$ which fix the origin is isomorphic to $\mathbb{R}^*$. For the purposes of this paper, this action is defined as follows: A positive number $r$ defines the map $\rho_r : \mathbb{C} \to \mathbb{C}$ which sends a point $\eta$ to $\rho_r(\eta) = \sqrt{r}\eta$. If a pair $(v, \tau)$ solves the vortex equations in (2.4), then $(v', \tau') = (\rho_r^* v, \rho_r^* \tau)$ solves

$$\text{(2.10)} \qquad \begin{aligned} &1. \quad idv' = \frac{r}{4}(1 - |\tau'|^2)\omega_0, \\ &2. \quad \bar\partial\tau' + v'_{0,1}\tau' = 0. \end{aligned}$$

**c) Local properties of the vortex moduli spaces**

The purpose of this subsection is to describe some additional features of $\mathfrak{C}_m$ which are relevant to the subsequent constructions. This subsection has three parts. Part 1 describes the tangent space to the embedding in Lemma 2.3. Parts 2 and 3 concern a certain family of differential operators which are parameterized by points $c \in \mathfrak{C}_m$. Parts 2 and 3 are not used in this section, but find applications in subsequent ones.

*Part 1.* Identify $\mathfrak{C}_m$ as a subspace of $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C}; \mathbb{C})$ as in Lemma 2.3. Consider a point $(v, \tau) \in \mathfrak{C}_m$. The tangent space of $\mathfrak{C}_m$ at $(v, \tau)$ is described below:

**Lemma 2.4.** *Identify $\mathfrak{C}_m$ with a subspace of $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C}; \mathbb{C})$ as in Lemma 2.3. Let $(v, \tau) \in \mathfrak{C}_m$. Then the tangent space to $(v, \tau)$ is a $2m$-dimensional (over $\mathbb{R}$) vector space which consists of pairs $(v^1, \tau^1)$ of imaginary valued 1-form and complex valued function on $\mathbb{C}$ which are square integrable on $\mathbb{C}$ and obey the following constraints:*

1. $idv^1 + \frac{1}{2} * re(\bar\tau\tau^1) = 0.$
2. $\bar\partial\tau^1 + v_{0,1}\tau^1 + v^1_{0,1}\tau = 0.$
3. *If the $L^2$-norm of $|\tau^1|$ and $|v^1|$ are bounded by 1, then their supremum norms are bounded $\zeta \cdot e^{-|\eta|/\zeta}$ where $\zeta$ depends only on $m$ and the maximum of the distances of the points in $\tau^{-1}(0)$ from 0.*

4. *Given the previous $L^2$-norm bound, the norms of the covariant derivatives of $(v^1, \tau^1)$ to order $k$ are bounded by $\zeta_k e^{-|\eta|/\zeta}$ where $\zeta_k$ depends only on $m$, $k$ and the maximum of the distance of the points in $\tau^{-1}(0)$ from 0.*

5. *Where $|\eta| > 2d_y = 2 \cdot \Sigma_k (1 + m^2|y_k|^2)^{1/2k}$, the function $\bar{\tau}\tau^1$ is real valued.*

(2.11)

*Proof of Lemma 2.4.* The first two assertions follow by differentiating (2.4). The fifth assertion follows from (2.8). To prove that $(v^1, \tau^1)$ is square integrable, and to obtain Assertions 3 and 4, one must think of $(v, \tau)$ as depending on the parameters $y \in \mathbb{C}^m$ from (2.6.1). In particular, Assertions 3 and 4 follow from estimates at points $\eta \in \mathbb{C}$ for the $\mathbb{C}^m$ derivatives of the function $\underline{u} = u - xm \ln |p|$. Here, the function $u$ on $\mathbb{C}$ is determined by $y \in \mathbb{C}^m$ in (2.6.1), while $x(\eta) = w(|\eta|/d_y)$ and $d_y = \Sigma_k (1 + m^2|y_k|^2)^{1/2k}$. The pointwise bounds in question are as follows: First of all, the derivative, $\underline{u}'$, of $\underline{u}$ by either the real or imaginary parts of any $y_q$ is square integrable. Second, for any $k \geq 0$ and at any point $\eta \in \mathbb{C}$ one has $|\nabla^k \underline{u}'| \leq \zeta_k e^{-|\eta|/\zeta}$, where $\zeta$ depends only on $d_y \in \mathbb{C}^m$, and $\zeta_k$ depends only on $d_y$ and $k$.

To prove these last assertions, remark first that it follows from the construction in [21] (see also Chapter III in [5]) that $\underline{u}'$ is square integrable with a uniform $L^2$ bound. Furthermore, $\underline{u}'$ obeys an equation of the form $i\partial\bar{\partial}\underline{u}' = *(4^{-1}|\tau|^2\underline{u}' - g')$. Here $|g'|$ is bounded at $\eta \in \mathbb{C}$ by $\zeta e^{-|\eta|/\zeta}$; and the $k$'th derivative of $g'$ is bounded by $\zeta_k e^{-|\eta|/\zeta}$. Also, $\zeta$ depends only on $d_y$, and $\zeta_k$ on $d_y$ and $k$. (This bound for $g'$ follows from (2.5).) The pointwise bounds on $\underline{u}'$ then follow from this last equation by the maximum principle and (2.5.2). The bounds on $\nabla^k \underline{u}'$ follow by differentiating the equation for $\underline{u}'$ and again employing the maximum principle. (In this case, the maximum principle asserts the following: Let $\lambda \geq 0$ and let $f$ be an $L^2_1$ function on $\mathbb{C}$ with the property that $\partial\bar{\partial} f - \lambda \cdot f \geq 0$. Then $f \leq 0$ on $\mathbb{C}$. In the applications at hand, use $f = |\nabla^k \underline{u}'| - \zeta_k e^{-|\eta|/\zeta}$ for an appropriate choice of $\zeta_k$ and $\zeta$.)

*Part 2.* Let $c = (v, \tau)$ be a solution to (2.4). Associated to $c$ in a canonical way is a certain $\mathbb{C}$-linear operator, $\Theta_c$, which maps an ordered pair consisting of a section over $\mathbb{C}$ of $T^{0,1}\mathbb{C}$ and a complex valued function to an ordered pair consisting of a complex valued function and a section over $\mathbb{C}$ of $T^{0,1}\mathbb{C}$. To write $\Theta_c$, agree to trivialize $T^{1,0}\mathbb{C}$ with the form $d\eta$. With the triviliaziation understood, the operator $\Theta_c$ sends the ordered pair $(a, \alpha)$ (of complex valued functions) to

$$\left(\partial a + \frac{1}{2\sqrt{2}}\bar{\tau}\alpha,\ \bar{\partial}_v\alpha + \frac{1}{2\sqrt{2}}\tau a\right). \tag{2.12}$$

Here, $\partial = \frac{\partial}{\partial\eta}$ and $\bar{\partial}_v = \frac{\partial}{\partial\bar{\eta}} + v_{0,1}$, where $v_{0,1}$ is the $(0,1)$ component of the 1-form $v$. The $L^2$ kernel of $\Theta_c$ will be denoted by $V^{0,c}$. (Both $\Theta_c$ and $V^{0,c}$ play key roles in subsequent subsections.)

Note that this operator is naturally equivariant under the action of $\mathrm{Maps}(\mathbb{C}; U(1))$, and so $\Theta_c$ and the vector space $V^{0,c}$ can be canonically associated to the image of $c$ in $\mathfrak{C}_m$ without recourse to any particular slice embedding of $\mathfrak{C}_m$ into $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C}; \mathbb{C})$.

The operator $\Theta_c$ is an elliptic, Fredholm operator (using the $L^2$ inner product) whose index is m. (This was first noticed by [23].) A Bochner-Weitzenboch formula for $\Theta_c$ shows that the cokernel is trivial when $c$ satisfies (2.4). The Weitzenboch formula in question is the first line below:

$$\begin{aligned} &\bullet\ \Theta_c\Theta_c^\dagger = \frac{1}{4}\nabla_v^\dagger\nabla_v + \frac{1}{8}|\tau|^2 + \frac{1}{16}\begin{pmatrix} 0 & 0 \\ 0 & 1-|\tau|^2 \end{pmatrix}, \\ &\bullet\ \Theta_c^\dagger\Theta_c = \frac{1}{4}\nabla_v^\dagger\nabla_v + \frac{1}{8}|\tau|^2 + \frac{1}{16}\begin{pmatrix} 0 & 4\sqrt{2}\bar{\partial}_v\bar{\tau} \\ 4\sqrt{2}\partial_v\tau & 1-|\tau|^2 \end{pmatrix}. \end{aligned} \tag{2.13}$$

Here, $\nabla_v(a, \alpha) = (\nabla a, \nabla\alpha + v \cdot \alpha)$.

The vanishing of the cokernel $\Theta_c$ implies that the $L^2$ kernel of $\Theta_c$ (that is, $V^{0,c}$) defines a complex $m$-dimensional vector space.

*Part 3.* As c varies in $\mathfrak{C}_m$, the vector spaces $V^{0,c}$ fit together to define a vector bundle, $V \to \mathfrak{C}_m$. The fact that $V$ is locally trivial can be proved using the implicit function theorem and the triviality of the cokernel of $\Theta_c$. Here is how: Observe first that when $x$ is an $L^2_1$ section of $iT_{\mathbb{R}}\mathbb{C} \oplus \mathbb{C}$, the operator $\Theta_{c+x}$ then maps $L^2_1(T^{0,1}\mathbb{C} \oplus \mathbb{C})$ to $L^2(\mathbb{C} \oplus T^{0,1}\mathbb{C})$ and differs from $\Theta_c$ by a compact perturbation. Now, let $(V^{0,c})^\perp$ denote the $L^2$ orthogonal compliment in $L^2_1$ to $V^{0,c}$, and define a map $\Phi_c : L^2_1 \times V^{0,c} \times (V^{0,c})^\perp \to L^2$ by the rule $\Phi_c(x, s, w) = \Theta_{c+x}(s+w)$. Because $\Theta_c$ is surjective, the differential of this map at $x = 0$ along the $(V^{0,c})^\perp$ factor is surjective. The implicit function theorem then provides a neighborhood $U_c$ of 0 in $L^2_1$ and a smooth map $p_c : U_c \times V^{0,c} \to (V^{0,c})^\perp$ which is linear on the second factor, vanishes at $0 \in U_c$ and is such that

$\Theta_{c+x}(s + p_c(x,s)) = 0$ for all $s \in V^{0,c}$. Thus, the map $s \to s + p_c(x,s)$ defines a vector bundle isomorphism between $V^{0,c}$ and $V^{0,c+x}$. This trivializes $V$ on a coordinate neighborhood of $c \in \mathfrak{C}_m$.

*Part 4.* The bundle $V \to \mathfrak{C}_m$ as just defined is canonically isomorphic to $T\mathfrak{C}_m$, and hence (via the differential of Lemma 2.3's map $\Upsilon$) to the trivial bundle $\mathfrak{C}_m \times \mathbb{C}^m$. The isomorphism $T\mathfrak{C}_m \approx V$ is obtained as follows: First, when $(v^1, \tau^1) \in T\mathfrak{C}_m$, write $v^1 = (2\sqrt{2})^{-1} \cdot (a^1 d\bar{\eta} - \bar{a}^1 d\eta)$ with $a^1$ in $T^{0,1}\mathbb{C}$ and $\bar{a}^1$ in $T^{1,0}\mathbb{C}$. Then, the isomorphism from $T\mathfrak{C}_m|_c$ to $V^{0,c}$ sends $(v^1, \tau^1)$ to the $L^2$ orthogonal projection of $(a^1, \tau^1)$ onto $V^{0,c}$.

An equivalent view of the isomorphism $T\mathfrak{C}_m \approx V$ notes that the vector space $V^{0,c}$ is canonically isomorphic to the vector space of square integrable solutions to (2.11) which obey the auxiliary condition

$$i * d * v^0 - \frac{1}{4} im(\bar{\tau}\tau^0) = 0. \tag{2.14}$$

Indeed, the identification between these vector spaces procedes as follows: Let $(v^0, \tau^0)$ solve (2.11) and (2.14). Set $a$ to be $2\sqrt{2}$ times the (0,1) component of $v^0$ and set $\alpha = \tau^0$. Then $(a, \alpha)$ solves (2.12). The point here is that the domain of $\Theta_c$ is the space of section of $T^{0,1}\mathbb{C} \oplus \mathbb{C}$ and this is isomorphic to $iT_{\mathbb{R}}\mathbb{C} \oplus \mathbb{C}$ by the $\mathbb{R}$-linear map which sends $(a, \alpha)$ in the former to $(\frac{1}{2\sqrt{2}}(ad\bar{\eta} - \bar{a}d\eta), \alpha)$. This map intertwines $\Theta_c$ with the operator which is defined by the left sides of (2.11) and (2.14). However, when $(a, \alpha) \in V^{0,c}$, the element $(v_0 = (\frac{1}{2\sqrt{2}}(ad\bar{\eta} - \bar{a}d\eta), \tau^0 = \alpha)$ will not, in general, be tangent at $c = (v, \tau)$ to the embedding of $\mathfrak{C}_m$ as described in Lemma 2.3. However, the proof of Lemma 2.3 describes this embedding, and with this description, it is a straightforward task to construct a linear map $\varphi : V^{0,c} \to C^\infty(\mathbb{C})$ so that $(v^1 = v^0 - id\varphi_{(v^0,\tau^0)}, \tau^1 = \tau^0 + i\varphi_{(v^0,\tau^0)}\tau)$ is tangent Lemma 2.3's embedding of $\mathfrak{C}_m$ when $(v^0, \tau^0)$ is in $V^{0,c}$.

### d) The approximate solutions

Let $\{(C_k, m_k)\}$ be a finite set of pairs where each $C_k$ is a compact, connected, pseudo-holomorphic submanifold, and each $m_k$ is a non-negative integer. And, require that the submanifolds $\{C_k\}$ be pairwise disjoint. The goal of this subsection is to construct from this data approximate solutions to the large $r$ and small $\mu_0$ version of (1.13). The construction requires seven steps. In the first six steps, $C$ is a compact, connected, psuedo-holomorphic submanifold of $X$ and $m$ is a positive integer.

*Step 1.* The approximate solutions require the following data: The choice of the parameter $r$ and the choice of a section of a certain auxiliary fiber bundle over $C$. The purpose of this step is to describe this auxilliary fiber bundle. (This is the "vortex bundle" of the introduction.) To begin, introduce $L \subset N$ to denote the unit sphere bundle, a principle $U(1)$ bundle over $C$. With $L$ understood, introduce the associated fiber

$$L \times_{U(1)} \mathfrak{C}_m. \tag{2.15}$$

Here, the action of $U(1)$ on $\mathfrak{C}_m$ is described in Assertion 5 of Lemma 2.3.

By the way, Lemma 2.3's identification $\Upsilon$ between $\mathfrak{C}_m$ and $\mathbb{C}^m$ identifies (2.15) (as a fiber bundle over $C$) with $\oplus_{1 \leq p \leq m} N^m$. This identification will be implicit below.

*Step 2.* Suppose that c is a section of (2.15). Then $c$ assigns to each point $z \in C$ an orbit $c_z = (v_z, \tau_z)$ in $\mathfrak{C}_m$. Thus, one can write $c = (v, \tau)$, where $v$ is a section over $N$ of the bundle of 1-forms along the fibers, and $\tau$ is a section over $N$ of $E$.

It is useful to single out a special section of (2.15). This section is called the *constant* section. With respect to the representation of (2.15) as $\oplus_{1 \leq p \leq m} N^m$, the constant section of (2.15) corresponds to the zero section in each line bundle summand. This section assigns to each $z \in C$ the vortex in (2.7).

*Step 3.* Given $r \geq 1$, introduce $\rho_r^* c$ which assigns to each $z \in C$ the solution $(\rho_r^* v_z, \rho_r^* \tau_z)$ to (2.10). To understand the significance of $\rho_r^* c$, first remember that Lemma 2.3 has identified $\mathfrak{C}_m$ with a submanifold of $i\Omega^1(\mathbb{C}) \times C^\infty(\mathbb{C}; \mathbb{C})$. Next, remark that the assignment of $z$ to $\rho_r^* \tau_z$ defines a section, $\rho_r^* \tau$, of the bundle $(\pi^* N)^m$ over Lemma 2.1's subbundle $N_{(0)}$. This is most easily seen in the case where $c$ is the constant section. In this case, the section $\rho_r^* \tau$ has the form

$$r^{m/2} f_m(r|s|^2) s^m. \tag{2.16}$$

Here, $f_m$ is the non-negative, real valued function on $[0, \infty)$ which appears in (2.7). To define $\rho_r^* \tau$ in the general case, note first that the restriction of $\pi^* N$ to the fiber of $\pi$ over some $z \in C$ is canonically trivial up to an over all factor of $U(1)$. With a choice of such a trivialization for a fiber, the section s of $\pi^* N$ defines a map from the given fiber to $\mathbb{C}$, and, simultaneously, $\tau_z$ defines a complex valued function on $\mathbb{C}$. With this understood, $\tau_z(r^{1/2} s)$ defines a complex valued function on $N_{(0)}|_z$.

And, as $z$ varies through $C$, these functions on the fibers of $\pi$ define the section $\rho_r^*\tau$.

Note that in the general case, where $|s|$ is large, the section $\rho_r^*\tau$ has the form

$$r^{m/2} f_r s^m. \tag{2.17}$$

Here, $f_r$ is a positive, real valued function which is defined where $|s|$ is large. (The term "$|s|$ is large" means that $|s|$ is greater than the supremum over all $z \in C$ of the numbers $r^{-1/2} d_y$; here $y \in \mathbb{C}^m$ defines $(v_z, \tau_z)$ as in Assertion 4 of Lemma 2.3, and $d_y = \Sigma_k (1 + m^2 |y_k|^2)^{1/2k}$.)

Meanwhile, the association of $z \in C$ to $\rho_r^* v_z$ defines a 1-form, $\rho_r^* v$, on $\pi^* N$. To be concrete here, write $v_z = \varsigma_z d\bar{\eta} - \bar{\varsigma}_z d\eta$ where $\varsigma_z$ is a $z$-dependent, complex valued function on $\mathbb{C}$. Then, the assignment of $z$ to $\rho_r^* \varsigma_z$ defines a complex valued function, $\rho_r^* \varsigma$, on $\pi^* N$, and with this understood, the 1-form $\rho_r^* v$ is given by

$$\rho_r^* v = \sqrt{r}((\rho_r^* \varsigma) \nabla_\theta \bar{s} - (\rho_r^* \bar{\varsigma}) \nabla_\theta s). \tag{2.18}$$

*Step 4.* This step makes the choice of a smooth cut-off function $\chi : [0, \infty) \to [0, 1]$ which is equal to 1 on $[0, 1]$ and vanishes on $[2, \infty)$. Fix a constant $\delta > 0$ which is smaller by a factor of $10^3$ than the radius of the disk bundle $N_{(0)}$. Then, promote $\chi$ to a function, $\chi_\delta : X \to [0, 1]$ as follows: Define $\chi_\delta$ to be zero on $X - N_{(0)}$, and on $N_{(0)}$, set

$$\chi_\delta(\cdot) = \chi(|s(\cdot)|/\delta). \tag{2.19}$$

*Step 5.* This step uses the function $\chi_\delta$, as defined above, to extend the section $\rho_r^*\tau$ (which restricts to a section over $N_0$ of $(\pi^* N)^m$) to a section over the whole of $X$ of a complex line bundle whose first Chern class is Poincaré dual to $m$ times the fundamental class of $C$.

To begin, define a complex line bundle $E \to X$ as follows: Let $\delta_0$ denote the fiber radius of the disk bundle $N_{(0)}$, and then introduce $N_{(1)} \subset N_{(0)}$ as the disk bundle of vectors of radius $\frac{\delta_0}{2}$. Write $E$ as the quotient of the disjoint union of two complex line bundles by an equivalence relation. The first line bundle is $(\pi^* N)^m \to N_{(0)}$, and the second is the trivial bundle $(X - N_{(1)}) \times \mathbb{C}$. The equivalence relation identifes a point $(x, f) \in (N_{(0)} - N_{(1)}) \times \mathbb{C}$ with the point $(x, f \frac{\rho_r^*\tau}{|\rho_r^*\tau|}) \in (\pi^* N)^m$. Here, it is assumed that $r$ is chosen large so that the zeros of $\rho_r^*\tau$ all lie in $N_{(1)}$.

With $E$ understood, define the section $\alpha_r$ of $E$ from $\rho_r^*\tau$ as follows: Over $(X - N_{(1)})$ where $E$ is identified with $(X - N_{(1)}) \times \mathbb{C}$, write $\alpha_r = 1$. Over $N_{(0)}$ where $E$ is identified with $(\pi^* N)^m$, write

$$\alpha_r = \frac{\rho_r^*\tau}{(\chi_\delta + (1 - \chi_\delta)|\rho_r^*\tau|)}. \tag{2.20}$$

It is left as an exercise to verify that the first Chern class of $E$ is Poincaré dual to $m$ times the fundamental class of $C$.

*Step 6.* This step uses the function $\chi_\delta$ and the 1-form $\rho_r^* v$ to construct a connection, $a_r$, on the bundle $E$. To begin, define the connection over $X - N_{(1)}$, where $E$ has a canonical trivialization given by $\alpha_r$, so that $\alpha_r$ is $a_r$- covariantly constant. To define $a_r$ over $N_{(0)}$, reintroduce the Levi-Civita connection $\theta$ on the line bundle $N$, and pull this connection up to a connection (still called $\theta$) on $\pi^* N$. Note that $\theta$ induces a unique connection, $\theta_m$, on $(\pi^* N)^m$. With this understood, set

$$a_r = \theta_m - (1 - \chi_\delta)\alpha_r^{-1}\nabla_{\theta_m}\alpha_r + \chi_\delta \rho_r^* v. \tag{2.21}$$

*Step 7.* Let $\{(C_k, m_k)\}$ be a finite of pairs consisting of disjoint, compact, pseudo-holomorphic submanifolds and positive integers. Chose $\delta_0$ small enough so that the tubular neighborhoods of points with distance $\delta_0$ or less about the various $C_k$'s are pairwise disjoint. With such a choice for $\delta_0$, and with $r$ large, the constructions in the preceding steps for the case $(C, m) = (C_k, m_k)$ can be performed simultaneously after the choice of a section of (2.15) for each $k$. (See Step 3, above.) The result is a complex line bundle $E \to X$ with first Chern class Poincaré dual to $\Sigma_k m_k [C_k]$, plus a pair consisting of a connection $a_r$ on $E$ and a section $\alpha_r$ of $E$. Here, $\alpha_r$ trivializes $E$ and is $a_r$ covariantly constant at points of distance $\delta_0$ or greater from each $C_k$. And, near any $C_k$, the pair $(a_r, \alpha_r)$ is given by Steps 1-5 above for the case where $C = C_k$, $m = m_k$.

With the preceding understood, note that the pair $(\alpha = \alpha_r, \beta = 0)$ defines a section of the bundle $S_+$ as given in (1.9), and thus $(a_r, (\alpha_r, 0))$ is data for the Seiberg-Witten equations in (1.13).

**e) How close for the Dirac equation?**

The purpose of this subsection and the next is to estimate just how close the data $(a_r, (\alpha_r, 0))$ comes to solving the Seiberg-Witten equations in (1.13). This section considers the size of $D_{A_r}(\alpha_r, 0)$. Here $A_r$ is the connection on $\det(S_+)$ which is defined by the canonical connection $A_0$ on $K^{-1}$ and the connection $a_r$ on $E$. The following lemma summarizes:

**Proposition 2.5.** *Let $\{(C_k, m_k)\}$ be a finite collection of pairs consisting of a compact, connected, pseudo-holomorphic submanifold and a positive integer. Require that the collection of pseudo-holomorphic submanifolds is pair-wise disjoint. Fix a positive number $d_0$ and a small number $\delta_0$. (The latter is constrained as in Step 6 of the preceding subsection.) Then, there are constants $r_0$ and $\zeta$ which depend on the preceding data and the almost complex structure $J$ with the following significance: For each $k$, choose a section (as defined in Step 3 of Section 2d) of the $C_k$-version of the fiber bundle in (2.15) with the constraint that $d_0$ is greater than the distance of any point in the correponding $\tau^{-1}(0)$ to $C_k$. Use this data to construct, for $r$ large, $(a_r, (\alpha_r, 0))$ as described in the previous subsection. Introduce the function $d = \mathrm{dist}(\cdot, \cup_k C_k)$ which measures the distance to $\cup_k C_k$. If $r > r_0$, then the norm of $D_{A_r}(\alpha_r, 0)$ satisfies the pointwise bound*

$$|D_{A_r}(\alpha_r, 0)| \le \zeta(1 + \delta^{-1})e^{-\sqrt{r}d/\zeta}. \tag{2.22}$$

*(See below for a more precise picture of $D_{A_r}(\alpha_1, 0)$.)*

By the way, if $C$ is not pseudo-holomorphic, then the constructions in the previous subsection can still be made. However, in this case the right side of (2.22) is replaced by $\zeta\sqrt{r}e^{-\sqrt{r}d/\zeta}$, which is $\mathcal{O}(\sqrt{r})$ instead of $\mathcal{O}(1)$.

*Proof of Proposition 2.5.* With $S_-$ as described in (1.9), then the section of $S_-$ given by $D_{A_r}(\alpha_r, 0)$ is the orthogonal projection of $\nabla_{a_r}\alpha_r$ onto $T^{0,1} \otimes E$. By construction, this gives zero where the distance to each $C_k$ is greater than $\delta_0$. Near some $C = C_k$, this projection can be calculated using the form for $a_r$ and $a_r$ in (2.20) and (2.21). With closeness to such a $C$ understood, remark first that

$$\nabla_{a_r}\alpha_r = \chi_\delta(\nabla_{\theta_m}\alpha_r + \rho_r^* v \cdot \alpha_r). \tag{2.23}$$

The subsequent considerations are simplest when $c$ is the constant section of (2.15). In this case,

$$\nabla_{a_r}\alpha_r = \rho_r^*(\partial_v\tau) - \alpha_r\frac{(1-|\rho_r^*\tau|)}{(\chi_\delta+(1-\chi_\delta)|\rho_r^*\tau|)}d\chi_\delta. \tag{2.24}$$

Here, $\rho_r^*\partial_v\tau$ is a certain E-valued 1-form on $N_{(0)}$ which is constructed as in (2.18) but with a replacement for $v$. This replacement assigns, to the fiber of $N_{(0)}$ at a point $z \in C$, the form of type $(1,0)$ on $\mathbb{C}$ which is obtained by taking the $v_z$ covariant derivative of $\tau_z$. (Remember that one of the vortex equations asserts that the $(0,1)$ part of this covariant derivative is zero.) To be more explicit, write $v$ and $\tau$ as in (2.7) and thus introduce the functions $u_m$ and $f_m$. Promote them to functions on $N_{(0)}$ by composing with the map from $N_{(0)}$ to $\mathbb{R}$ which assigns to each point the value of $|s|^2$. Then

$$\partial_v\tau = ((f_m' - q_m f_m)|s|^2 + m f_m)s^{m-1}\nabla_\theta s. \tag{2.25}$$

Here, $f_m'$ denotes the derivative of $f_m$ as a function on $[0,\infty)$.

With (2.24) and (2.25) understood, it remains only to project $\nabla_{a_r}\alpha_r$ onto $T^{0,1}\otimes E$ to obtain the final answer. Using (2.1), one finds that

$$|D_{A_r}(\alpha_r, 0)| \le \zeta(1+\delta^{-1})e^{-\sqrt{r}|s|/\zeta}, \tag{2.26}$$

where $\zeta$ is a constant which depends on $m$ and the almost complex structure $J$.

To understand $\nabla_{a_r}\alpha_r$ in the general case, it is necessary to digress for some additional remarks concerning (2.15). To start the digression, remark that there is a well defined vertical subbundle, Vert, in the tangent bundle to (2.15). These are the tangents to the fiber. Thus, Vert is a $2m$-dimensional bundle whose fiber at a given point $(v,\tau)$ in a fiber of (2.15) is the vector space described in Lemma 2.4. The connection $\theta$ on the bundle $L$ defines a projection from the total tangent bundle of (2.15) onto Vert. Thus, $\theta$ defines a 1-form (call it $\theta_{\mathfrak{C}}$) on (2.15) with values in the subbundle Vert which restricts to Vert as the identity homomorphism. With $\theta_{\mathfrak{C}}$ understood, define the covariant derivative of a section $c$ of (2.15) by the formula $\nabla c = c^*\theta_{\mathfrak{C}}$. This covariant derivative is a section over $C$ of the vector bundle $c^*\,\mathrm{Vert}\otimes T^*C$.

Note that when $\mathfrak{C}_m$ is interpreted as in Lemma 2.3, then $\nabla c$ is a pair $(v^1,\tau^1)$, where $v^1$ is a 1-form along the fibers of $(\pi^*N)^m$ with values in $T^*C$, and $\tau^1$ is a section of $(\pi^*N)^m$ with values in $T^*C$. (Let $U$ be an open set in $C$ over which $L$ has been trivialized. With this trivialization understood the connection $\theta$ is an imaginary valued 1-form on $U$. Also, (2.15) is given as $U \times \mathfrak{C}_m$, and $\tau$ is just a complex function on $U \times \mathbb{C}$. With the preceding undertood, then $\tau^1 = d^C\tau + m\theta\tau - \theta(s\partial^V\tau - \bar{s}\bar{\partial}^V\tau)$, where $d^C$ denotes exterior differentiation in directions tangent to $U$ while $\partial^V$ denotes the holomorphic derivative along $\mathbb{C}$, the fibers of $N$.)

Note that following from Lemma 2.4, $(v^1, \tau^1)$ decay exponentially fast away from $C$.

End the digression.

In the general case, (2.23) is replaced by

$$\nabla_{a_r}\alpha_r = \rho_r^*[\partial_v\tau] + \rho_r^*\tau^1 - \alpha_r\frac{(1 - |\rho_r^*\tau|)}{(\chi_\delta + (1-\chi_\delta)|\rho_r^*\tau|)}d\chi_\delta, \tag{2.27}$$

where $[\partial_v\tau]$ is defined by the equality $\partial_v\tau = [\partial_v\tau]\nabla_\theta s$. Because of (2.4.2), $\rho_r^*[\partial_v\tau]$ restricts to each fiber of $\pi$ to lie in the $J$-version of $T^{1,0}$. Thus, the projection of $\rho_r^*[\partial_v\tau]$ onto $T^{0,1}N$ is given by

$$-\rho_r^*[\partial_v\tau]u^{-1}\sigma_{0,1}, \tag{2.28}$$

where $\sigma_{0,1}$ is the projection of $\sigma$ in (2.1) onto $T^{0,1}N_{(0)}$. Now it follows that (2.22) holds thanks to Lemmas 2.3 and 2.4 plus (2.3).

**f) How close for the curvature equation?**

This subsection estimates the degree to which the data $(a_r, (\alpha_r, 0))$ solves the curvature equation in (1.13). The following lemma summarizes:

**Proposition 2.6.** *Let $\{(C_k, m_k)\}$ be a finite collection of pairs consisting of a compact, connected, pseudo-holomorphic submanifold and a positive integer. Require that the collection of pseudo-holomorphic submanifolds is pair-wise disjoint. Fix a positive number $d_0$ and a small number $\delta_0$. (The latter is constrained as in Step 7 of Subsection 2d.) Then, there are constants $r_0$ and $\zeta$ which depend on the preceding data and the almost complex structure $J$, and which have the following significance: For each $k$, choose a section of the $C_k$-version of the fiber bundle in (2.15) with the constraint that $d_0$ is greater than the distance of any point in the correponding $\tau^{-1}(0)$ to $C_k$. For $r$ large, use this data to construct,$(a_r, (\alpha_r, 0))$ as described in the previous subsection. Introduce the function $d = \mathrm{dist}(\cdot, \cup_k C_k)$ which measures the distance to $\cup_k C_k$. If $r > r_0$, then the norm of $P_+F_{a_r} + \frac{ir}{8}(1 - |\alpha_r|^2)\omega$ satisfies the pointwise bound*

$$\left|P_+F_{a_r} + \frac{ir}{8}(1 - |\alpha_r|^2)\omega\right| \le \zeta\sqrt{r}(1 + \delta^{-1})e^{-\sqrt{r}d/\zeta}. \tag{2.29}$$

*Proof of Proposition 2.6.* First of all, $P_+F_{a_r} + \frac{ir}{8}(1-|\alpha_r|^2)\omega$ vanishes unless the function $d$ is smaller than $\delta_0$, in which case, one can restrict attention to a neighborhood of a fixed $C = C_k$ and assume that $a_r$ has the form given in (2.17). Once again, the discussion is simplest when $c$ is the constant section. With this understood, one finds that

$$\begin{aligned} P_+F_{a_r} =&\chi_\delta\rho_r^*(m - 2q_m|s|^2)F_\theta + d\chi_\delta \wedge \alpha_r^{-1}(\nabla_{\theta_m} + (\rho_r^* v)\alpha_r) \\ &+ 2r\chi_\delta\rho_r^*(q_m'|s|^2 + q_m)\nabla_\theta s \wedge \nabla_\theta \bar{s}. \end{aligned} \tag{2.30}$$

Here, $q_m'$ denotes the derivative of the function $q_m$ as a function on $[0, \infty)$. The argument procedes from here by estimating the various terms in (2.30). To estimate the first term above, first go back to (2.5) to conclude that.

$$\frac{1 - \zeta e^{-|\eta|/\zeta}}{|\eta|^m} \leq |f_m| \leq \frac{1}{|\eta|^m}. \tag{2.31}$$

This last equation (plus (2.5)) implies that

$$|m - 2q_m|\eta|^2| \leq \zeta e^{-\sqrt{r}|s|/\zeta}. \tag{2.32}$$

Thus, the first term in (2.30) is bounded by $\zeta e^{-\sqrt{r}|s|/\zeta}$ for an appropriate choice of $z$.

As for the second term, it is only non-zero where the distance from $C$ is between $\delta/8$ and $\delta/4$. Here, $\alpha_r$ is exponentially close to being $a_r$-covariantly constant. This implies that the second term in (2.30) is bounded by $\zeta\delta^{-1}r^{1/2}e^{-\sqrt{r}|s|/\zeta}$.

Now consider the $P_+$ projection of the last term in (2.30). According to discussion in Parts 4-6 of Section 2a, the self-dual projection of $\nabla_\theta s \wedge \nabla_\theta \bar{s}$ is equal to $-i\omega + \mathcal{O}(|s|)$. Meanwhile, the first equation in (2.4) identifies $2(q_m' + q_m|s|^2)$ with $\frac{1}{8}(1 - |\alpha_r|^2)$. These last remarks imply the lemma in the case where the section $c$ of (2.15) is the constant section.

Now consider the general case where the section c does not assign each point in C the vortex in (2.7). In this case, (2.30) is replaced by

$$\begin{aligned} F_{a_r} &= \chi_\delta\rho_r^*(m - \varsigma\bar{s} - \bar{\varsigma}s)F_\theta + d\chi_\delta \wedge \alpha_r^{-1}(\nabla_{\theta_m}\alpha_r + (\rho_r^* v)\alpha_r) \\ &\quad + r \cdot \chi_\delta\rho_r^* \left(\frac{\partial\varsigma}{\partial\eta} + \frac{\partial\bar{\varsigma}}{\partial\bar{\eta}}\right) \nabla_\theta s \wedge \nabla_\theta \bar{s} \\ &\quad + (1 - \chi_\delta)(\alpha_r^{-1}\rho_r^*(d_{\theta_m}\tau^1) - \alpha_r^{-2}\rho_r^*\tau^1 \wedge \nabla_{\theta_m}\alpha_r) + \chi_\delta\rho_r^* v^1. \end{aligned} \tag{2.33}$$

Here, $v^1$, which is a 1-form on $\pi^*N$ with values in $T^*C$, is interpreted in the obvious way as a 2-form on $\pi^*N$. With (2.33) understood, the proof of Proposition 2.6 procedes by considering, one by one, the various terms in this equation. For the first term, remark that (2.5) plus Lemma 2.3 implies a bound for the first term by $\zeta e^{-\sqrt{r}|s|/\zeta}$. (Here, $m - \varsigma\bar{s} - \bar{\varsigma}s$ decays exponentially fast along the fiber from 0 because, according to (2.5), so does the covariant derivative $\nabla_{v_z}\tau_z$.) The second term in (2.34) is bounded by $\zeta\delta r^{1/2}e^{-\sqrt{r}|s|/\zeta}$ for the same reason. The fourth and fifth terms are bounded by $\zeta r^{1/2}e^{-\sqrt{r}|s|/\zeta}$ courtesy of Lemma 2.4's exponential decay estimates. As for the term in the middle of (2.33), use the first vortex equation in (2.4) to write $(\frac{\partial\varsigma}{\partial\eta} + \frac{\partial\bar{\varsigma}}{\partial\bar{\eta}}) = \frac{1}{8}(1 - |\tau|^2)$. Finally, use the fact that $P_+(\nabla_\theta s \wedge \nabla_\theta \bar{s}) = -i\omega + \mathcal{O}(|s|)$ to complete the argument for Proposition 2.6.

By the way, the largest terms in $P_+F_{a_r} + \frac{ir}{8}(1 - |\alpha_r|^2)\omega$ are $\mathcal{O}(\sqrt{r})$ and

$$
\begin{aligned}
P_+F_{a_r} + &\frac{ir}{8}(1 - |\alpha_r|^2)\omega \\
&= -\frac{r}{8}(1 - |\rho_r^*\tau|^2)(\sigma_{0,1} \wedge \nabla_\theta\bar{s} - \bar{\sigma_{0,1}}) \\
&\quad + \chi_\delta P_+\rho_r^* v^1 + \mathcal{O}(1).
\end{aligned}
\tag{2.34}
$$

## 3. Introduction to $\mathcal{Z}_0$ and $\mathcal{Z}$

This section serves as a digression of sorts to describe a certain finite dimensional variety of the space of sections of the vortex bundle (in (2.15)). This variety is denoted by $\mathcal{Z}_0$ and its non-singular stratum by $\mathcal{Z}$.

The space $\mathcal{Z}_0$ plays a key role in the next section where the approximate solutions (just described) to (1.13) are deformed to honest solutions. Indeed, to a good approximation, the role of $\mathcal{Z}_0$ can be summarized as follows: A large $r$, approximate solution to (1.13) as defined in Section 2 from sections of (2.15) can be deformed to a true solution just when each of the sections lies in the appropriate version of $\mathcal{Z}_0$. This feature of $\mathcal{Z}_0$ is suggested by the discussion in Section 3b and Proposition 3.1. In Section 3b, a modification of the data $(a_r, (\alpha_r, 0))$ is described which results in an order $1/\sqrt{r}$ decrease in the size of the left side of (1.13) precisely when the given sections of (2.15) sit in $\mathcal{Z}_0$.

Except for Section 3b, this section has no applications of $\mathcal{Z}_0$; the goal being to define $\mathcal{Z}_0$ and to describe some of its basic properties. Indeed, except for Section 3b, the discussion below takes place in the context where $C$ is a compact, connected, complex curve of some genus, $g$; and $\pi : N \to C$ is a holomorphic line bundle of degree $n$ with a given hermitian metric. With $C$ and $N$ understood, a positive integer $m$ plus a pair of sections, $\nu$ and $\mu$, of the bundles $T^{0,1}C$ and $T^{0,1}C \otimes N^2$ are sufficient for defining $\mathcal{Z}_0$. (The sections $\nu$ and $\mu$ are provided courtesy of (2.3) when $C$ arises as a pseudo-holomorphic submanifold of a symplectic 4- manifold $X$ with compatible almost complex structure; and when $N$ is the normal bundle to $C$ in $X$.)

**a) The definition of $\mathcal{Z}_0$**

To begin, let $L \subset N$ denote the unit sphere bundle (a principal $U(1)$ bundle). As in the previous section, use $L$ and the tautological section, $s$, of $\pi^*L \to L$ to construct the associated fiber bundle

$$L \times_{U(1)} \mathfrak{C}_m. \tag{3.1}$$

This subsection defines the canonical subset, $\mathcal{Z}_0$, of the space of sections of (3.1). The definition of $\mathcal{Z}_0$ requires ten steps.

*Step 1.* Remember that $N$ is, by assumption, a holomorphic line bundle over $C$ with a Hermitian metric. The hermitian structure and the complex structure together define a unitary connection, $\theta$, on $N$. The connection defines the splitting $TN = \pi^*N \oplus \pi^*TC$ and thus defines a Riemannian metric on $N$. This metric will be used implicitly throughout this section. The splitting above also defines the complex structure $J_0$ on $N$, and $J_0$ will also be used implicitly throughout this section. Note that $J_0$ and the metric restrict to each fiber of $N$ to define a standard complex structure and metric.

*Step 2.* Associated to each vortex solution $c = (v, \tau) \in \mathfrak{C}_m$ is the $m$-complex dimensional vector space $V^{0,c}$ of pairs $(a, \alpha)$ of complex functions on $\mathbb{C}$ which are square integrable and are annihilated by the operator $\Theta_c$ from (2.12). That is,

$$\partial a + \frac{1}{2\sqrt{2}}\bar{\tau}\alpha = 0 \quad \text{and} \quad \bar{\partial}_v \alpha + \frac{1}{2\sqrt{2}}\tau a = 0. \tag{3.2}$$

Here, $\partial = \frac{\partial}{\partial \eta}$ and $\bar{\partial}_v = \frac{\partial}{\partial \bar{\eta}} + v_{0,1}$, where $v_{0,1}$ is the $(0,1)$ component of the 1-form $v$.

*Step 3.* Let $c = (v, \tau)$ be a section of (3.1). This $c$ associates to each $z \in C$ a vortex $(v_z, \tau_z)$ in $\mathfrak{C}_m$. Then, each $z \in C$ has its associated vector space $V_z^{0,c}$ of solutions to (3.2) as defined by $(v_z, t_z)$. As $z$ varies over $C$, these vector spaces fit together to define the vector bundle $V^c \to C$. (The proof that $V^c$ is locally trivial follows from the proof in Section 2c that the bundle $V \to \mathfrak{C}_m$ is locally trivial.) Note that $V^c$ is a rank $m$, complex vector bundle.

*Step 4.* Fix $z \in C$ and a unit length complex parameter $\eta$ for the fiber of $N$ at $z$. Use $d\eta$ to trivialize $T^{1,0}(N|_z)$ and thus its conjugate trivializes $T^{0,1}(N|_z)$. With this parameter $\eta$ fixed, write

$$\partial_v \tau_z = [\partial_v \tau_z] \cdot d\eta.$$

Then, $(\frac{1}{2\sqrt{2}}\eta(1 - |\tau_z|^2)d\bar{\eta}, \eta[\partial_v \tau_z])$ defines a pair consisting of a section of $T^{0,1}(N|_z)$ and a complex valued function on $N|_z$. Likewise, so does $(\frac{1}{2\sqrt{2}}\bar{\eta}(1 - |\tau_z|^2)d\bar{\eta}, \bar{\eta}[\partial_v \tau_z])$.

*Step 5.* Note that the ambiguity in the choice of the parameter $\eta$ cancel out in the expression $(\frac{1}{2\sqrt{2}}\eta(1 - |\tau_z|^2)d\bar{\eta}, \eta[\partial_v \tau_z])$. Thus, with the help of the connection $\theta$, this expression defines over $N$ a section of $T^{0,1}N \oplus E$, where $E = (\pi^* N)^m$. Reintroduce the section $\nu$ of $T^{0,1}C$, and tensor the former section with $\nu$ to obtain a section over $N$ of the bundle $\pi^*(T^{0,1}C \otimes N) \oplus (\pi^* T^{0,1}C \otimes E)$. This last section can be written using the tautological section $s$ as

$$\left(\frac{1}{2\sqrt{2}}\nu s(1 - |\tau|^2)\nabla_\theta \bar{s},\ \nu s[\partial_v \tau]\right). \tag{3.3}$$

*Step 6.* Reintroduce the section $\mu$ of $T^{0,1}C \otimes N^2$. Let $\mu_z$ denote the restriction of $\mu$ to $z$. The choice of the parameter $\eta$ identifes $\mu_z$ as a point in $T^{0,1}C|_z$. With this identification understood, note that the ambiguities in the choice of $\eta$ cancel out when writing

$$\left(\frac{1}{2\sqrt{2}}\mu_z \bar{\eta}(1 - |\tau_z|^2)d\bar{\eta}, \mu_z \bar{\eta}[\partial_v \tau_z]\right).$$

Thus, with the help of the connection $\theta$, the latter defines over the whole of $N$ an unambiguous section of $\pi^*(T^{0,1}C \otimes N) \oplus (\pi^* T^{0,1}C \otimes E)$. Note that this section can also be written using the tautological section $s$ as

$$\left(\frac{1}{2\sqrt{2}}\mu \bar{s}(1 - |\tau|^2)\nabla_\theta \bar{s},\ \mu \bar{s}[\partial_v \tau]\right) \tag{3.4}$$

*Step 7.* As in the previous section, define the covariant derivative $(v^1, \tau^1)$ of the section $c$ of (3.1). One should think of $v^1$ as a 2-form on $N$ and $\tau^1$ as a section over $N$ of the bundle $\pi^* T^* C \otimes E$. Let $v^1_{0,2}$ denote the projection of $v^1$ into $T^{0,2}N = \pi^*(T^{0,1}C \otimes N)$ and let $\tau^1_{0,1}$ denote the projection of $\tau^1$ into $\pi^* T^{0,1} C \otimes E$.

*Step 8.* For each $z \in C$, use $\Pi^c_z$ to denote the $L^2$ orthogonal projection along the fiber of $N$ at $z$ of $L^2(T^{0,1}N|_z) \oplus L^2(E)$ onto the subspace $V^c_z$. As $z$ moves through $C$, the family $\Pi^c = \{\Pi^c_z : z \in C\}$ varies smoothly.

*Step 9.* The canonical section of $V^c \otimes T^{0,1}C$ is defined to be

$$\Pi^c \cdot \Big( -\frac{1}{2\sqrt{2}}(\nu s + \mu \bar{s})(1 - |\tau|^2)\nabla_\theta \bar{s} + 2\sqrt{2} v^1_{0,2}, -(\nu s + \mu \bar{s})[\partial_v \tau] + \tau^1_{0,1} \Big). \tag{3.5}$$

*Step 10.* Define $\mathcal{Z}_0$ to be the set of sections of (3.1) for which (3.5) is identically zero. Topologize $\mathcal{Z}_0$ using the subspace topology.

### b) An application

In this subsection, return to the milieu where $X$ is a compact, symplectic 4-manifold. Let $\{(C_k, m_k)\}$ be as in Section 2, a finite set where each $C_k$ is an embedded, pseudo-holomorphic submanifold in $X$, and each $m_k$ is a positive integer. Assume, in addition, that the $\{C_k\}$ are pairwise disjoint. For each index $k$, fix a section, $c^{(k)}$, of the $(C_k, m_k)$ version of (2.15), and, for $r$ large, construct the data $(a_r, \alpha_r)$ as instructed.

As seen in Section 2, when r is large, the data $(a_r, \alpha_r)$ defines the configuration $(a = a_r, (\alpha = \alpha_r, \beta = 0))$ which solves the $\mu_0 = 0$ version of (1.13) on all of $X$ save for the radius $2\delta$ tubular neighborhoods of the submanifolds from $\{C_k\}$. And, on such a tubular neighborhood, the size of both

(3.6)
- $\frac{1}{\sqrt{r}}\left(P_+ F_a + \frac{ir}{8}(1 - |\alpha|^2 + |\beta|^2)\omega - \frac{r}{4}(\alpha\bar{\beta} - \bar{\alpha}\beta)\right)$
- $D_A(\alpha, \beta)$

are $\mathcal{O}(1)$ in absolute value. With the preceding understood, the purpose of this subsection is to describe a modification of $(a_r, (\alpha_r, 0))$ which, under certain circumstances, makes (3.6) have absolute value $\mathcal{O}(1/\sqrt{r})$. In fact, this size reduction occurs precisely when, for each $k$, the section $c^{(k)}$ lies in the $(C_k, m_k)$ version of $\mathcal{Z}_0$. (See Proposition 3.2, below.)

The modification of $(a_r, (\alpha_r, 0)$ produces data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$. The construction of the latter requires three steps.

*Step 1.* In this step, $C$ is a compact, connected, complex curve, and $\pi : N \to C$ is a holomorphic vector bundle. Fix a non-negative integer $m$, and use the pair $(C, m)$ to define (3.1). Now, let $c = (v, \tau)$ be a section of (3.1), and introduce

$$p = \left( -\frac{1}{2\sqrt{2}}(\nu s + \mu \bar{s})(1 - |\tau|^2)\nabla_\theta \bar{s} \right. \\ \left. + 2\sqrt{2}v^1_{0,2}, -(\nu s + \mu\bar{s})[\partial_v \tau] + \tau^1_{0,1} \right). \tag{3.7}$$

The Fredholm alternative finds a unique, square integrable section $u_1 = (b, \lambda)$ over $N$ of the vector bundle $\pi^* T^{0,1}C \oplus (\pi^*(N \otimes T^{0,1}C) \otimes E)$ whose restriction to each fiber of $N$ obeys the equation

$$\Theta_c^\dagger u_1 = (1 - \Pi^c) \cdot p. \tag{3.8}$$

Note that $c$ is in $\mathcal{Z}_0$ precisely when $\Pi^c p = 0$, so in this case, $u_1$ obeys $\Theta_c^\dagger u_1 = p$.

*Step 2.* Now, consider the context where $(C, m) \in \{(C_k, m_k)\}$ and $N$ is the normal bundle to $C$ in $X$. Take $c$ to be a section of the $(C, m)$ version of (2.15). (This is the $(C, m, N)$ version of (3.1).) Define $u_1 = (b, \lambda)$ as above.

Given $r \geq 1$, use $b$ to define a section, $b_r$, of $iT^*N_{(0)}$ as follows: Over an open disk $U \subset C$, introduce $\kappa_0$ as in (2.2) and (2.4). Next, introduce $b'_r$ by the formula

$$r^{-1/2}\rho_r^* b = b'_r \pi^*(\bar{\kappa}_0|_C), \tag{3.9}$$

and then write

$$b_r = b'_r \bar{\kappa}_0 - \bar{b}'_r \kappa_0. \tag{3.10}$$

Meanwhile, set

$$\lambda_r = \rho_r^* \lambda. \tag{3.11}$$

*Step 3.* With the preceding understood, the modification to $(a_r, (\alpha_r, 0))$ consists of data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ which is defined as follows:

- Where the distance to any $C_k$ is $2\delta$ or more, then $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r)) = (a_r, (\alpha_r, 0))$.
- Where the distance to $C \in \{C_k\}$ is less than $2 \cdot \delta$, then

$$(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r)) = \left( a_r + \sqrt{\frac{r}{8}} \chi_\delta b_r, (\alpha_r, \chi_\delta \lambda_r) \right). \tag{3.12}$$

The following proposition summarizes the properties of $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ :

**Proposition 3.1.** *Let $\{(C_k, m_k)\}$ be a finite set where each $C_k$ is an embedded, pseudo- holomorphic submanifold in $X$, and each $m_k$ is a positive integer. Assume, in addition, that the $\{C_k\}$ are pairwise disjoint. Let $E \to X$ be a complex line bundle whose first Chern class is Poincaré dual to $\Sigma_k m_k \cdot [C_k]$. For each $k$, fix a submanifold $\mathcal{O}^{(k)}$ in the $(C_k, m_k)$ version of (2.15) with compact closure. There is a constant $\zeta \geq 1$ which depends on $\{\mathcal{O}^{(k)}\}$ and has the following significance: When $r \geq \zeta$, the assignment of $\{c^{(k)}\} \subset \times_k \mathcal{O}^{(k)}$ to $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ (as defined above) defines a smooth map from $\times_k \mathcal{O}^{(k)}$ into* $\mathrm{Conn}(E) \times C^\infty(S_+)$. *In addition, suppose that (3.6) is defined using the data $(a, (\alpha, \beta)) = (\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$. Then:*

- *The resulting expression is zero where the distance to $\cup_k C_k$ is greater than $2\delta$.*
- *Where the distance to some $C \in \{C_k\}$ is less than $2\delta$, (3.6) is bounded pointwise in absolete value by $\zeta \cdot e^{-\sqrt{r} \cdot |s|/\zeta}$.*
- *If, and only if the corresponding $c \in \{c^{(k)}\}$ lies in the $(C, m)$ version of $\mathcal{Z}_0$, then (3.6) is bounded in absolute value by $\zeta r^{-1/2} e^{-\sqrt{r} \cdot |s|/\zeta}$.*

The remainder of this subsection is occupied with the

*Proof of Proposition 3.1.* The proof has three parts. The first part remarks that the fact that the assignment of $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ to $\{c^{(k)}\}$ defines a smooth map to $\mathrm{Conn}(E) \times C^\infty(S_+)$ follows by inspection from the constructions here and in Section 2.

The second and third parts of the proof concern the asserted bounds for (3.6). To start the second part of the proof, introduce

$$u_r = (r^{-1/2} \rho_r^* b,\ \rho_r^* \lambda). \tag{3.13}$$

Define the covariant derivative of $u_r$, $\nabla u_r$, using the Levi-Civita connection on $T^{0,1}C$, the connection $\theta$ on $N$, and the connection $\theta + \rho_r^* v$ on $E$. With $\nabla$ understood, introduce $\nabla^V$ and $\nabla^H$ to denote the respective vertical and horizontal components of $\nabla$.

With the preceding understood, consider that the norm of $u_r$ and of its derivatives to any fixed order satisfy the following apriori bounds: For given non-negative integers $i$ and $j$, there exists $\zeta_{i,j}$ which is independent of $r$, and is such that for all sufficiently large $r$,

$$|(\nabla^H)^i(\nabla^V)^j u_r| \leq \zeta_{i,j} r^{(j-1)/2} e^{-\sqrt{r}\cdot|s|/\zeta}. \tag{3.14}$$

This last estimate is justified shortly.

Granted (3.14), the third part of the proof amounts to a straightforward calculation plus the following observation: Due to (2.3) and (2.28) plus (2.34), the part of (3.6) which is $\mathcal{O}(1)$ instead of $\mathcal{O}(1/\sqrt{r})$ for $(a = a_r, (\alpha = \alpha_r, \beta = 0))$ is after rescaling to $r = 1$, nothing more than $p$ from (3.7).

To justify (3.14), first observe that the estimates for $u_r$ follow from the $r = 1$ version of (3.14) by rescaling the latter. Meanwhile, the $r = 1$ estimates follow by straightforward arguments given two additional set of estimates. The first set asserts that $|(\nabla^H)^i(\nabla^V)^j p| \leq \zeta'_{i,j} e^{-|s|/\zeta}$ for $i, j \geq 0$ and a suitable constant $\zeta_{i,j}$. (See (2.5) and [5]). The second set of estimates bounds the Greens' kernel, $G_c$, for the operator $\Theta_c \Theta_c^\dagger$ in (2.13):

$$(3.15)\qquad \begin{aligned} &\bullet\ |G_c(\eta, \eta')| \leq \zeta \cdot (1 + |\ln|\eta - \eta'||) e^{-|\eta-\eta'|/\zeta}, \\ &\bullet\ |\nabla G_c(\eta, \eta')| \leq \zeta \cdot |\eta - \eta'|^{-1} e^{-|\eta-\eta'|/\zeta}. \end{aligned}$$

Here, $\zeta$ is a suitable constant.

For example, given the preceding estimates, bounds for $u_1$ at a point $\eta$ in the fiber of $N$ over a point $z \in C$ are obtained by using $G_c$ to write

$$u_1(\zeta, \eta) = \int G_c(\eta, \cdot) \cdot p(\cdot), \tag{3.16}$$

where the integral is over $N|_z$. Indeed, the required bounds for $|u_1|$ and the vertical derivative of $u$ follow directly from (3.15) and (3.16) given the asserted bounds for $|p|$.

Bounds on the higher derivatives of $u_1$ are obtained with the help of $G_c$ by differentiating the equation $\Theta_c \Theta_c^\dagger u_1 = \Theta_c p$ and then commuting the derivatives to write the term with the most derivatives of $u_1$ as

$$\Theta_c\Theta_c^\dagger\,(\text{derivatives of } u_1) = \text{fewer derivatives of } u_1 + \text{ derivatives of } p. \tag{3.17}$$

The derivation of (3.15) uses the comparison principle with the Bochner-Weitzenboch formula for $\Theta_c\Theta_c^\dagger$ in (2.13). Indeed, with (2.13) understood, standard asymptotic expansions can be used to estimate $G$ and $\nabla G$ near the diagonal in $\mathbb{C}\times\mathbb{C}$. For example, where $|\eta-\eta'|\leq 1$, these yield

$$\begin{aligned} &\bullet\ |G(\eta,\eta')|\leq \zeta(1+|\ln|\eta-\eta'||), \\ &\bullet\ |\nabla G(\eta,\eta')|\leq \zeta|\eta-\eta'|^{-1}. \end{aligned} \tag{3.18}$$

To bound $|G|$ when $\eta$ is not close to $\eta'$, use (2.13) to obtain the differential inequality: $d^*d|G|+\frac{1}{2}|\tau|^2|G|\leq 0$. The first line of (3.15) then follows from this inequality and the bound in (3.18) using the comparison principle. In this regard, use $(1+|\ln|\eta-\eta'||)e^{-|\eta-\eta'|/\zeta}$ as the comparison function for a suitable choice of $\zeta$. And, remember that $|\tau|>1/2$ on the compliment of some ball about the origin. (See (2.5).)

A similar combination of local estimates and the comparison principle give the estimate for the second line in (3.15). (Note that $\nabla G$ obeys an equation of the form $\Theta_c\Theta_c^\dagger(\nabla G)+\mathcal{R}\cdot\nabla G=0$ where $\eta\neq\eta'$. Here $\mathcal{R}$ is an endomorphism with a bound of the form $|\mathcal{R}|\leq\zeta e^{-|\eta|/\zeta}$.)

### c) The definition of $\mathcal{Z}$

The subspace $\mathcal{Z}\subset\mathcal{Z}_0$ is characterized by the condition that (3.5) vanish in a suitably transversal manner. The precise definition requires three steps

*Step 1.* Introduce p as in (3.7) and then solve (3.8) to obtain $u_1=(b,\lambda)$. Note that when (3.5) vanishes, $(b,\lambda)$ obeys the pair of equations

$$\begin{aligned} &1.\ -\bar\partial b+\frac{1}{2\sqrt{2}}\bar\tau\lambda-\frac{1}{2\sqrt{2}}(\nu s+\mu\bar s)(1-|\tau|^2)+2\sqrt{2}v^1_{0,2}=0, \\ &2.\ -\partial_v\lambda+\frac{1}{2\sqrt{2}}\tau b-(\nu s+\mu\bar s)[\partial_v\tau]+\tau^1_{0,1}=0. \end{aligned} \tag{3.19}$$

*Step 2.* This step introduces a certain $\mathbb{R}$-linear, first order differential operator which maps sections of $V^c$ to sections of $V^c \otimes_{\mathbb{C}} T^{0,1}C$. The operator in question, $\Delta^c$, sends a section $(a, \alpha)$ of $V^c$ to

$$\Delta^c(a,\alpha) = \Pi^c \cdot \left( \bar{\partial}^H a + \mu\bar{a} + \frac{1}{2\sqrt{2}}\lambda\bar{\alpha},\ \bar{\partial}^H \alpha + \nu\alpha + \frac{1}{2\sqrt{2}}(b\alpha + \lambda\bar{a}) \right). \tag{3.20}$$

Here, $\nu$ and $\mu$ are the given sections of $T^{0,1}C$ and $T^{0,1}C \otimes N^2$, respectively. Also, the operator $\bar{\partial}^H$ is the horizontal part of the $\bar{\partial}$ operator on $N$. (To be precise, suppose that $U$ is an open set in $C$ over which $N$ has been trivialized as $U \times \mathbb{C}$, and let $\eta$ denote the complex coordinate on $\mathbb{C}$. This trivialization also trivializes the bundle $E$ over $\pi^{-1}(U)$. Given such a trivialization, both $a$ and $\alpha$ become complex functions on $\pi^{-1}(U)$. With this understood, $\bar{\partial}^H a$ and $\bar{\partial}^H \alpha$ are the respective projections onto $\pi^* T^{0,1}C$ of

$$\begin{aligned} & d^C a + \theta a + \theta \left( \bar{\eta}\frac{\partial}{\partial\bar{\eta}} - \eta\frac{\partial}{\partial\eta} \right) a \\ \text{and} \quad & d^C \alpha + m\theta \cdot \alpha + \theta \left( \bar{\eta}\frac{\partial}{\partial\bar{\eta}} - \eta\frac{\partial}{\partial\eta} \right) \alpha. \end{aligned} \tag{3.21}$$

Here, $d^C$ is the covariant derivative along the $U\times$ constant slices, and $\theta$ is the connection form with respect to the given trivialization.

This operator $\Delta^c$ is described further in the next subsection.

*Step 3.* Let $\mathcal{A}$ denote the space of sections of (3.1). Introduce, as before, $\mathcal{Z}_0 \subset \mathcal{A}$ to denote the subspace of sections of (3.1) for which (3.5) vanishes. Then, let $\mathcal{Z} \subset \mathcal{Z}_0$ denote the subspace of sections $c$ of (3.1) for which the operator $\Delta^c$ in (3.20) has vanishing cokernel.

### d) The structure of $\mathcal{Z}_0$ and $\mathcal{Z}$

Here is a list of some of the salient features of $\mathcal{Z}_0$ and $\mathcal{Z}$ :

**Proposition 3.2.** *Let $C$ be a compact, connected, complex curve of genus $g$, and let $N$ be a holomorphic line bundle over $C$ of degree $n$. Fix a positive integer $m$, and a pair, $\nu$ and $\mu$, of sections of $T^{0,1}C$ and $T^{0,1}C \otimes N^2$, respectively. Use this data to define $\mathcal{Z}_0$ and $\mathcal{Z}$ as above. Then, the following hold:*

1. *$\mathcal{Z}_0$ is locally compact and has the following local structure: Let $c \in \mathcal{Z}_0$. The operator $\Delta^c$ is Fredholm, and so both its kernel and the cokernel of $\Delta^c$ are finite dimensional. More to the point, there is a smooth map from a ball about the origin in the former to the*

*latter whose zero set is homeomorphic to an open neighborhood of $c$ in $\mathcal{Z}_0$.*

2. *$\mathcal{Z}$ is a smooth manifold of even dimension*

$$2d = 2m \cdot (1-g) + m(m+1)n.$$

*(Thus, if $d < 0$, then $\mathcal{Z} = \varnothing$.)*

3. *$\mathcal{Z}$ is orientable with a canonical orientation.*

4. *The diffeomorphism $\Upsilon$ of Lemma 2.3 between $\mathbb{C}^m$ and $\mathfrak{C}_m$ induces an identification (also denoted by $\Upsilon$) between (3.1) and $\oplus_{1\le q\le m} N^q$. Under this identification, the space $\mathcal{Z}_0$ becomes a subspace of $C^\infty(\oplus_{1\le q\le m} N^q)$ which can be described as follows: A point $y \in \oplus_{1\le q\le m} N^q$ yields a point $c = \Upsilon(y) \in \mathcal{Z}_0$ if an only if $y$ solves the equation*

$$\bar\partial y + \nu \aleph y + \mu \mathbb{F}(y) = 0; \tag{3.22 a}$$

*here $\aleph$ is the endomorphism of $\oplus_{1\le q\le m} N^q$ which multiplies the $q$'th summand by $q$, and $\mathbb{F} : \oplus_{1\le q\le m} N^q \to \oplus_{1\le q\le m} N^{q-2}$ is a certain smooth, fiber preserving map. (Note that $\mathbb{F}$ is not holomorphic in the fiber coordinates.)*

5. *When*

$$y = (y_1, \dots , y_q) \in C^\infty(\oplus_{1\le q\le m} N^q),$$

*let*

$$\Upsilon_1 : \oplus_{1\le q\le m} N^q \approx V^c$$

*for $c = \Upsilon(y)$ denote the identification which is induced, via (3.26), by the trivilization of $V \to \mathfrak{C}_m$ in Part 3 of Section 2c. Also use $\Upsilon_1$ to denote the induced identification between*

$$(\oplus_{1\le q\le m} N^q) \otimes T^{0,1}C$$

*and $V^c \otimes T^{0,1}C$. Next, introduce*

$$\Delta_y : \oplus_{1\le q\le m} N^q \to (\oplus_{1\le q\le m} N^q) \otimes T^{0,1}C$$

*to denote the $\mathbb{R}$-linear operator which sends a section $y'$ to*

$$\bar\partial y' + \nu \aleph y' + \mu \mathbb{F}_{*y} y', \tag{3.22 b}$$

*where $F_{*y}$ denotes the differential of $\mathbb{F}$ at $y$. If $c = \Upsilon(y) \in \mathcal{Z}_0$ (and thus $y$ obeys (3.22)), then $\Delta_y = \Upsilon_1^{-1}\Delta^c\Upsilon_1$. In general, $\Delta_y = \Upsilon_1^{-1}\Delta^c\Upsilon_1 + v_y$, where $v_y$ is a linear operator obeying $\|v_y(u)\|_2 \leq \zeta_y\xi_y\|u\|_2$. Here, $\xi_y$ is the supremum norm of the left side of (3.22a) (or, equivalently, of (3.5)), and $\zeta_y$ is bounded independently of $y$ given an apriori bound for $\Sigma_{1\leq q\leq m}(1+m|y_q|^2)^{1/2q}$.*

The equations which define the set $\mathcal{Z}_0$ in $\mathcal{A}$ can be perturbed slightly so that the resulting solution set in $\mathcal{A}$ is a smooth manifold. Of course, the equations can be perturbed by changing $\nu$ and $\mu$. A larger class of perturbations is parameterized by elements in the space of compactly supported sections of $T^{0,2}N$. Indeed, given such a section, $\varepsilon$, introduce $\mathcal{Z}_\varepsilon$ to denote the subspace $c \in \mathcal{A}$ for which

$$\Pi^c\Big(-\frac{1}{2\sqrt{2}}(\nu s + \mu\bar{s})(1-|\tau|^2)\nabla_\theta\bar{s} + 2\sqrt{2}v^1_{0,2} + \varepsilon,\ -(\nu s+\mu\bar{s})[\partial_v\tau] + \tau^1_{0,1}\Big) \tag{3.23}$$

vanishes.

**Proposition 3.3.** *When $m = 1$, and $1 - g + n \geq 0$, there is an open and dense subset in $C^\infty T^{0,1}C \oplus (T^{0,1}C \otimes N^2)$ of pairs $(\nu,\mu)$ for which the corresponding $\mathcal{Z}$ and $\mathcal{Z}_0$ coincide. On the other hand, when $1-g+n < 0$, $\mathcal{Z} = \varnothing$ for all $(\nu,\mu)$. For general $m$, there is a Baire subset of compactly supported sections $\varepsilon$ of $T^{0,2}N$ for which the corresponding space $\mathcal{Z}_\varepsilon$ has, in a natural way, the structure of a smooth submanifold of $\mathcal{A}$ whose dimension is $2d = 2m(1-g) + m(m+1)n$. Furthermore, $\mathcal{Z}_\varepsilon$ is orientable and has a canonical orientation.*

Proposition 3.4, below, gives examples of $\mathcal{Z}_0$. The proposition introduces the constant section of (3.1); the section $c = (v,\tau)$ of (3.1) which is characterized by the property that $\tau = f(|s|^2)\cdot s^m$ where $f$ is a positive, real valued function on $[0,\infty)$. This section is the image under the map $\Upsilon$ of the zero section of $\oplus_{1\leq q\leq m}N^q$.

**Proposition 3.4.** *Identify (3.1) with $\oplus_{1\leq p\leq m}N^p$ as in Proposition 3.2.*

- *When $m = 1$, and with the preceding identification understood, $\mathcal{Z}_0$ is the kernel of the operator $D$ in (1.5). That is, $c = (v,\tau) \in \mathcal{Z}_0$ if and only if the corresponding section $h_c$ of $N$ (whose image is*

*$\tau^{-1}(0)$) obeys $Dh_c = 0$. For example, the constant section of (3.1) lies in $\mathcal{Z}_0$. Furthermore, $\mathcal{Z} = \mathcal{Z}_0$ when cokernel $D = \{0\}$ and $\mathcal{Z} = \varnothing$ otherwise.*

- *Take $m > 1$ and $\mu = 0$ in (3.5). Under the identification above, $\mathcal{Z}_0$ corresponds to the space of sections $(y_1, \ldots, y_m)$ of $\oplus_{1\le p\le m} N^m$ for which $\bar\partial y_q + q \cdot \nu \cdot y_q = 0$ for all $q$. In this case, $\mathcal{Z}_0 = \mathcal{Z}$ when, for each $q$, the cokernel of $\bar\partial + q \cdot \nu$ is trivial. On the other hand, $\mathcal{Z} = \varnothing$ when one of these operators has non-trivial cokernel.*

- *When $m > 1$, and $\mu \neq 0$, the situation is not so simple. For example, the constant section of (3.1) is not in $\mathcal{Z}_0$ when $\mu \neq 0$.*

The remainder of this section is occupied with the proofs of these propositions. The order of proof will be Proposition 3.2 first, Proposition 3.4 second and Proposition 3.3 last. Also, the first three assertions of the Proposition 3.2 follow as a fairly formal consequence of (3.22), so (3.22) will be proved first. With this understood, the proof of Proposition 3.2 is divided into three parts. The first part proves Assertion 4, the second proves Assertion 5 and the third proves Assertions 1-3.

**e) Proof of Assertion 4 of Proposition 3.2**

To begin, remark that the identification via the map $\Upsilon$ in Lemma 2.3 of (3.1) with $\oplus_{1\le q\le m} N^q$ follows from Assertion 4 of Lemma 2.3.

With (3.1) so identified, write $(v, \tau) \in \mathcal{Z}_0$ in terms of a section $y$ of $\oplus_{1\le q\le m} N^q$ as $((\bar\partial^V \varsigma)\nabla_\theta \bar s - (\partial^V \bar\varsigma)\nabla_\theta s, p[y]e^{-\varsigma})$, where $\partial^V$ and $\bar\partial^V$ are the holomorphic and anti-holomorphic derivatives along the fibers of $N$. Note that $\varsigma$ is a complex valued function on $N$ whose restriction to any fiber is as described in Lemma 2.3.

With $(v, \tau)$ so understood, the term in (3.5) with $\Pi^c(2\sqrt{2} v^1_{0,2}, \tau^1_{0,1})$ is equal to $\Pi^c(2\sqrt{2}\bar\partial^H \bar\partial^V \varsigma, (\bar\partial^H p) \cdot e^{-\bar\varsigma} - \bar\partial^H \varsigma \cdot \tau)$. As the horizontal and vertical derivatives on $\varsigma$ can be interchanged here, the resulting terms where $\bar\partial^H \varsigma$ appear have the form $\Theta_c^\dagger x$ on each fiber of $N$. Thus, they vanish when projected onto $V^c$. It then follows that the contribution of the term with $\Pi^c(2\sqrt{2} v^1_{0,2}, \tau^1_{0,1})$ in (3.5) is equal to

$$\Sigma_{1\le q\le m} \bar\partial y_q \Pi^c(0, s^{m-q} e^{-\varsigma}). \tag{3.24}$$

Next, consider the terms in (3.5) which are proportional to $\nu$. In total, these have the form $\nu \cdot \Pi^c(s \cdot ((2\sqrt{2})^{-1}(1 - |\tau|^2), [\partial_v \tau]))$. To analyze the latter, return to the complex plane to see that $(1 - |\tau|^2) = 16 \frac{\partial^2}{\partial\eta\partial\bar\eta} u$ where u is given in (2.6.1). Also, (2.6.1) identifies $[\partial^v \tau] = \Sigma_{1\leq q\leq m} q y_q \eta^{q-1} e^{-\varsigma} - 2(\frac{\partial}{\partial \eta} u)\tau$. Since $(\bar\partial(\eta \cdot \partial u),\ -(2\sqrt{2})^{-1}\tau(\eta\partial u))$ is in the image of $\Theta_c^\dagger$, the terms which are proportional to $\nu$ in (3.5) contribute only

$$\Sigma_{1\leq q\leq m} q y_q \Pi^c(0, s^{m-q} e^{-\varsigma}) \tag{3.25}$$

to (3.5).

Note that (3.22a) follows directly from (3.24) and (3.25) given the trivialization for the bundle $V \to \mathfrak{C}_m$ Part 4 of Section 2c. The latter trivialization identifies $\mathbb{C}^m$ with $V^{0,c(y)}$ via the homomorphism which sends $y' = (y'_1, \ldots, y'_q) \in \mathbb{C}^m$ to

$$\Pi^c(0, p(y') \cdot e^{-\varsigma}). \tag{3.26}$$

### f) Proof of Assertion 5 of Proposition 3.2

The proof of the assertion is arranged in twelve steps.

*Step 1.* The bundle $V \to \mathfrak{C}_m$ whose fiber at $c$ is the vector space $V^{0,c} = \mathrm{kernel}(\Theta_c)$ inherits a natural lift of the $U(1)$ action in Assertion 3 of Lemma 2.3. As such, $V$ defines a vector bundle, $\underline{V} \to L \times_{U(1)} \mathfrak{C}_m$. (If c is a section of (3.1), then $c^*\underline{V} = V^c$.) The homomorphism $\Upsilon_1$ identifies $\underline{V}$ with $\pi^*(\oplus_{1\leq q\leq m} N^q)$. Meanwhile, the bundle Vert of vertical tangent vectors to $L \times_{U(1)} \mathfrak{C}_m$ is also canonically isomorphic to $\underline{V}$; the latter is induced by the isomorphism from Part 4 of Section 2c between $V^{0,c}$ and $T\mathfrak{C}_m|_c$. Thus, $\pi^*(\oplus_{1\leq q\leq m} N^q)$ is canonically isomorphic to Vert. It is left to the reader to verify that the latter isomorphism is given by the differential of the map $\Upsilon$.

*Step 2.* The expression in (3.5) defines a map, $\mathcal{F}$, from the space of smooth sections of (3.1) into the vector space of smooth sections over $N$ of the bundle, $\mathcal{E}$, whose fiber is the tensor product of $\pi^*T^{0,1}C$ with the direct sum of the bundle of $(0,1)$ forms along the fiber and $\pi^*N^m$. Note that a point $w \in V^c \otimes T^{0,1}C$ defines a tautogical section, $\underline{w}$, of $\mathcal{E}|_{\pi(w)}$. With this understood, note that at each $z \in C$, the section $\mathcal{F}(c)$ along $\pi^{-1}(z)$ lies in the subspace $\{\underline{w} : w \in V^c|_z\}$.

The preceding subsection proved that the expression on the left side of (3.22a) is equal to $\Upsilon_1^{-1} \cdot \mathcal{F}(\Upsilon(y)))$.

*Step 3.* If $c$ is a section of (3.1) where $\mathcal{F}$ vanishes, then the differential of $\mathcal{F}$ at $c$ defines a linear map from $c^*\,\mathrm{Vert}$ to $V^c$. Given this last fact, (3.22a) and the remarks in Steps 1 and 2, then the assertion in the case $c \in \mathcal{Z}_0$ follows by demonstrating that the composition of the isomorphism $V^c \approx c^* \mathrm{Vert}$ with the differential of $\mathcal{F}$ yields $\Delta^c$. In fact, the following slight generalization holds:

**Lemma 3.5.** *Fix a bounded section $\varepsilon$ over $N$ of the bundle of $(0,1)$ forms along the fibers of $N$. Then, define $\mathcal{F}_\varepsilon$ as a map from the space of sections of (3.1) to the bundle $\mathcal{E}$ by (3.23). Let $c \in \mathcal{F}_\varepsilon^{-1}(0)$. Define $\Delta^c$ by (3.20) where $(b, \lambda)$ are defined by the modified version of (3.19) which adds $\varepsilon$ to (3.19.1). Then, $\Delta^c$ is equal to the composition of the isomorphism $V^c \approx c^* \mathrm{Vert}$ with the differential of $\mathcal{F}_\varepsilon$.*

*Proof of Lemma 3.5.* The proof of this lemma occupies the Steps 4-11 of Part 2.

*Step 4.* Observe now that the section in (3.23) has the form $\Pi^{(\cdot)} \cdot h(\cdot)$, where $h$ is a smooth map from $\mathcal{A}$ to $C^\infty(C; \Gamma \otimes T^{0,1}C)$. To explore the differential of $\Pi^{(\cdot)} \cdot h$, consider first the projection $\Pi^{(\cdot)}$. Since the projection $\Pi^c$ is defined fiberwise, focus attention on some fixed $z \in C$. Let $\{s_\beta\}$ be an $L^2$ orthonormal basis form $V_z^c$. Then, the differential of $\Pi^{(\cdot)}$ in the direction defined by $t \in C^\infty(C; V^c)$ defines, on the fiber of $N$ at $z$, an operator from $(V_z^c)^\perp$ to $V_z^c$ which sends a given $q$ to

$$\Sigma_\beta\ s_\beta \langle w_\beta, q\rangle_z, \tag{3.27}$$

where $w_\beta$ is the differential (at 0) in the direction $t$ of $p_{c_z}(\cdot, s_\beta)$, thus a certain $t$-dependent element in $(V_z^c)^\perp$. (Here, $\langle , \rangle_z$ denotes the $L^2$ inner product along the fiber of $N$ at $z$.) To be precise, here is $w_\beta$ : Write $s_\beta = (a_\beta, \alpha_\beta)$. Then

$$w_\beta = -\frac{1}{2\sqrt{2}} \Theta_c^{-1}(\bar{x}_0 \alpha_\beta, x_{0,1}\alpha_\beta + x_0 a_\beta), \tag{3.28}$$

(Remember that $(x_{0,1}, x_0)$ depend on $t \in C^\infty(C; V^c)$.)

With the preceding understood, it follows that the differential of $\Pi^{(\cdot)} \cdot h(\cdot)$ in the direction defined by $t$ at the point $c$ has, at a point $z \in C$, the schematic form

$$\Sigma_\beta\ s_\beta \langle w_\beta, h\rangle_z + \Pi_z^c \cdot h', \tag{3.29}$$

where $h = h(c)$ and $h'$ is the differential of $h$ in the direction $t$ at $c$.

*Step 5.* Focus on the first term in (3.29). If $\Pi_z^c h = 0$, then $h = -\Theta_c^\dagger u$. Thus, writing $u = (b, \lambda)$, the first term in (3.29) can be rewritten as:

$$\begin{aligned}(3.30)\qquad &\frac{1}{2\sqrt{2}}\cdot \Sigma_\beta\, s_\beta\langle(\bar{x}_0\alpha_\beta, x_{0,1}\alpha_\beta + x_0 a_\beta), (b,\lambda)\rangle_z \\ &= \frac{1}{2\sqrt{2}}\cdot \Sigma_\beta\, s_\beta\langle(a_\beta,\alpha_\beta), (\lambda\bar{x}_0, bx_0 + \lambda\overline{x_{0,1}})\rangle_z.\end{aligned}$$

This last expression is equal to

$$(3.31)\qquad \Pi^c\cdot\left(\frac{1}{2\sqrt{2}}(\lambda\bar{x}_0, bx_0 + \lambda\overline{x_{0,1}})\right).$$

*Step 6.* Now write $h = h_1 + h_2$, where

$$(3.32)\qquad h_1 = -(\nu s + \mu\bar{s})\cdot\left(\frac{1}{2\sqrt{2}}\nabla_\theta\bar{s}(1-|\tau|^2), [\partial_v\tau]\right).$$

Consider $\Pi^c\cdot h_1'$. The derivative of $h_1$ at $c$ in the direction $t$ is given, on the fiber of $z\in C$, by

$$(3.33)\qquad h_1' = -(\nu s + \mu\bar{s})\left(-\frac{1}{2\sqrt{2}}(\tau\bar{x}_0 + \bar{\tau}x_0), -\frac{1}{2\sqrt{2}}\overline{x_{0,1}}\tau + \partial_v x_0\right).$$

*Step 7.* It is left as an exercise to prove that with $h_2 = (2\sqrt{2}v^1_{0,2}, \tau^1_{0,1})$,

$$(3.34)\qquad \Pi^c\cdot h_2' = \Pi^c\cdot(\bar{\partial}^H x_{0,1}, \bar{\partial}^H x_0).$$

*Step 8.* The purpose of this step is to be more specific about the relationship between $t\in C^\infty(C;V^c)$ and $x\in\mathcal{U}$. For this purpose, write $t=(a,\alpha)$. Then, there is a real valued, smooth, $L^2_2$ function $u$ on $N|_z$ such that

$$(3.35)\qquad x_{0,1} = a + i2\sqrt{2}\bar{\partial}u \quad\text{and}\quad x_0 = \alpha - iu\tau.$$

One can now consider the separate contributions to (3.31), (3.34) and (3.35) from $(a,\alpha)$ and also from $(i2\sqrt{2}u, -iu\tau)$.

*Step 9.* This step considers the contributions of $(a,\alpha)$. Replacing $(x_{0,1}, x_0)$ in (3.31) with $(a,\alpha)$ immediately yields

$$(3.36)\qquad \Pi^c\cdot\left(\frac{1}{2\sqrt{2}}(\lambda\bar{\alpha}, b\alpha + \lambda\bar{a})\right).$$

When replacing $(x_{0,1}, x_0)$ with $(a, \alpha)$ in (3.33), remember that on each fiber of $N$, $\frac{1}{2\sqrt{2}}\tau\bar{\alpha} = -\bar{\partial}\bar{a}$. This means that the $(a, \alpha)$ version of (3.34) on the fiber of $N$ at $z$ is equal to the $\Pi^c_z$ projection of

$$(\nu s + \mu\bar{s}) \cdot \left(-\bar{\partial}\bar{a} + \bar{\tau}a, -\partial_v a + \frac{1}{2\sqrt{2}}\bar{a}\tau\right). \tag{3.37}$$

This last expression can be written concisely as

$$(\nu s + \mu\bar{s}) \cdot \Theta_c^\dagger(\bar{a}, \alpha). \tag{3.38}$$

One can now factor the expression $(\nu s + \mu\bar{s})$ past $\Theta_c^\dagger$ to obtain

$$\Theta_c^\dagger((\nu s + \mu\bar{s}) \cdot (\bar{a}, \alpha)) + (\mu\bar{a}, \nu\alpha). \tag{3.39}$$

Finally, since $\Pi^c$ annihilates the image of $\Theta_c^\dagger$, the $(a, \alpha)$ version of (3.37) is

$$\Pi^c \cdot (\mu\bar{a}, \nu\alpha). \tag{3.40}$$

Meanwhile, the $(a, \alpha)$ version of (3.28) is equal to $\Pi^c(\bar{\partial}^H a, \bar{\partial}^H \alpha)$. By inspection, the sum of the $(a, \alpha)$ versions of (3.31) (see (3.36)) and (3.33) (see (3.40)) and (3.34) give $\Delta^c(a, \alpha)$ from (3.20).

*Step 10.* This next to last step considers (3.31), (3.33) and (3.34) with $(x_{0,1}, x_0)$ replaced by $(i2\sqrt{2}\bar{\partial}u, -iu\tau)$. (Call this the "$u$-version".) The bottom line here is that the contribution here is zero.

To begin, consider (3.31). With the appropriate replacements, this expression can be rewritten as

$$\Pi^c \cdot \{\Theta_c^\dagger((0, i\lambda u)) - i(0, (-\partial_v\lambda + \frac{1}{2\sqrt{2}}\tau b)u)\}.$$

Now, $\Pi^c$ annihilates the image of $\Theta_c^\dagger$. And, by definition,

$$-\partial_v\lambda + \frac{1}{2\sqrt{2}}\tau b = (\nu s + \mu\bar{s})[\partial_v\tau] - \tau^1_{0,1}.$$

Thus, the $u$-version of (3.31) is equal to

$$-i\Pi^c \cdot \{(0, ((\nu s + \mu\bar{s}) \cdot [\partial_v\tau] - \tau^1_{0,1})u)\}. \tag{3.41}$$

By inspection, the u-version of (3.33) contributes

$$-i\Pi^c \cdot \{(0, (\nu s + \mu\bar{s})[\partial_v\tau]u)\} \tag{3.42}$$

to the differential of (3.23).

Finally, consider the $u$-version of (3.34). It is an exercise to verify that the $u$-version of (3.34) is given by

$$\Pi^c \cdot (\Theta_c^{\dagger}(-i2\sqrt{2}\bar{\partial}^H u, 0)) - i\Pi^c \cdot (\tau_{0,1} u). \tag{3.43}$$

*Step 11.* Since the sum of the terms in (3.41)-(3.43) yields zero, the u contribution to the differential of (3.23) at $c$ is seen to vanish. Thus, the differential of (3.23) at a section $c$ where (3.23) vanishes does indeed define the operator $\Delta^c$ as required.

*Step 12.* In the general case where $c$ is not assumed to lie in $\mathcal{Z}_0$, then the identification of $\Upsilon_1^{-1}\Delta^c\Upsilon_1$ still follows from the identification of the left side of (3.22a) with $\Upsilon_1^{-1}\mathcal{F}(\Upsilon(y))$. As $\mathcal{F}$ is not assumed now to vanish, the differential of this expression with respect to $y$ has two extra terms. The first involves the differential with respect to $y$ of the map $\Upsilon_1^{-1}$. The second involves the identification of the differential of $\mathcal{F}$ with $\Delta^c$. In particular, when $\mathcal{F} \neq 0$, there is an extra term on the right side of (3.30) which also involves $\Pi_z^c h$. These two terms together produce the $v_y$ term in the asserted formula for $\Delta_y$. It is a straightforward exercise using the estimates in Lemma 2.4 to verify the stated norm bound for $v_y$.

### g) Proof of Assertions 1-3 of Proposition 3.2

The three assertions are proved consecutively as Steps 1, 2 and 3.

*Step 1.* The assertion that $\Delta^c$ is Fredholm follows from Assertion 5 and (3.22b). The second part of this assertion is proved as follows: To begin, it is convenient to use Assertion 4 to represent $\mathcal{Z}_0$ as the space of sections $y$ of $\oplus_{1\le q\le m}N^q$ where (3.22a) holds. Let $y$ be such a section. Let $Q_y$ denote the $L^2$ orthogonal projection onto the cokernel of $\Delta_y$. Define a map, $\mathcal{T}$, from $C^\infty(\oplus_{1\le q\le m}N^q)$ to

$$(1 - Q_y) \cdot C^\infty((\oplus_{1\le q\le m}N^q) \otimes T^{0,1}C)$$

by associating to $y'$ the section

$$\mathcal{T}(y') = (1 - Q_y) \cdot (\bar{\partial}y' + \nu\aleph y' + \mu(\mathbb{F}(y + y') - \mathbb{F}(y))). \tag{3.44}$$

The differential of $\mathcal{T}$ at $y' = 0$ is

$$\Delta_y : C^\infty(\oplus_{1\le q\le m}N^q) \to (1 - Q_y) \cdot C^\infty((\oplus_{1\le q\le m}N^q) \otimes T^{0,1}C),$$

which is surjective. Thus, the implicit function theorem implies that $\mathcal{T}^{-1}(0)$ intersects a neighborhood of $y$ in $C^\infty(\oplus_{1\le q\le m}N^q)$ as a smooth manifold which is diffeomorphic (by $L^2$-orthogonal projection) to an open ball $B_y \subset \text{kernel}(\Delta_y)$. Use $\varphi_y : B_y \to C^\infty(\oplus_{1\le q\le m}N^q)$ to denote this embedding onto an open set of $\mathcal{T}^{-1}(0)$. Then $\varphi_y$ has the following properties:

- For any $p \ge 0$, there is a constant $\xi_p$ such that

  $$||\varphi(y') - y'||_{2,p} \le \xi_p ||y'||_2.$$

  Here, $||\cdot||_{2,p}$ signifies the Sobolev $L^2_p$ norm.

- Define $\psi : B_y \to \text{cokernel}(\Delta_y)$ by the rule

  $$\psi = Q_y \cdot [\mu(\mathbb{F}(y+\varphi) - \mathbb{F}(y) - \mathbb{F}_{*y}\varphi)).$$

  Then the intersection of $\mathcal{Z}_0$ with a neighborhood of $y$ is homeomorphic (via the map $\varphi$) with $\psi^{-1}(0)$.

(3.45)

(Remark that the first point above follows from the inverse function theorem. The second point follows from the identification of $\mathcal{Z}_0$ with the set of $y$ which obeys (3.22a). Also, note that the application above of the inverse function theorem requires an intermediate step which extends $\mathcal{T}$ as a map from a suitable Sobolev space completion of $C^\infty(\oplus_{1\le q\le m}N^q)$ to a corresponding completion of $C^\infty((\oplus_{1\le q\le m}N^q)\otimes T^{0,1}C)$. This intermediate step is straightforward and left to the reader.)

*Step 2.* This step proves Assertion 2 of Proposition 3.2. The point is that Assertions 4 and 5 identify $\mathcal{Z}$ as the space of sections $y$ of $\oplus_{1\le q\le m}N^q$ where (3.22a) holds and $\Delta_y$ has trivial cokernel. It then follows from (3.45) that $\mathcal{Z}$ is a manifold whose dimension near $y$ is the real index of the operator $\Delta_y$. The latter differs from the $d$-bar operator on $\oplus_{1\le q\le m}N^q$ by zero'th order terms, so the index of $\Delta_y$ is the index of this $d$-bar operator, where the latter is viewed as an $\mathbb{R}$-linear operator. This last index is twice that of the index of the $d$-bar operator as a $\mathbb{C}$-linear operator.

*Step 3.* Use Assertions 4 and 5 to identify $\mathcal{Z}$ as in Step 2, above. In the case where the index of $\Delta_y$ is positive, associate to each $y \in \mathcal{Z}$ the vector space kernel$(\Delta_y)$. As (3.45) indicates, these vector spaces fit together to define a vector bundle over $\mathcal{Z}$ which is canonically isomorphic to $T\mathcal{Z}$. As such, a consistent choice of orientation for kernel$(\Delta_y)$ defines an orientation for $\mathcal{Z}$. Meanwhile, an orientation for the latter is the same as an orientation for the real line $\Lambda^{\text{top}}(\text{kernel}(\Delta_y))$. In the case where the index of $\Delta_y$ is zero (and so $\mathcal{Z}$ is a collection of points), an orientation for $\mathcal{Z}$ consists of a choice of $\pm 1$ at each of $\mathcal{Z}$'s elements.

To consistently orient $\Lambda^{\text{top}}(\text{kernel}(\Delta_y))$ in the index positive case, observe that $\Delta_y$ differs from a $\mathbb{C}$-linear operator (for example, $\bar\partial$) by a zero'th order term. The operator without this term has a natural complex structure on its kernel and cokernel. This means that its determinant line (which is $\Lambda^{\text{top}}(\text{kernel}) \otimes (\Lambda^{\text{top}}(\text{cokernel}))^{-1}$) has a natural orientation. Furthermore, this orientation is preserved by $\mathbb{C}$-linear, zero'th order perturbations of this $\mathbb{C}$-linear cousin of $\Delta_y$.

With the preceding understood, choose a path of zero'th order deformations from $\Delta_y$ to a $\mathbb{C}$-linear perturbation of $\Delta_y$, say one with vanishing cokernel. A generic zero'th order perturbation of the $\mathbb{C}$-linear cousin of $\Delta_y$ will have this property. Furthermore, the cokernel of the operator defined at any time t along the generic path will also be empty. Thus, the determinant lines for $\Delta_y$ and for the $\mathbb{C}$-linear perturbation at the end of the path of deformations are naturally identified. This identification gives kernel$(\Delta_y)$ a natural orientation. (Keep in mind here that the space of paths of zero'th order deformations from $\Delta_y$ to a $\mathbb{C}$-linear perturbation form a contractible space.)

By the way, the orientation of $\mathcal{Z}$ can just as well be defined via a deformation of $\Delta^c$ in (3.20) to a $\mathbb{C}$-linear operator from $V^c$ to $V^c \otimes T^{0,1}C$ because the identification $\Upsilon_1$ between $V^c$ and $\oplus_{1\le q\le m} N^q$ is $\mathbb{C}$-linear.

When the index of $\Delta_y$ is zero, the generic path to a $\mathbb{C}$-linear perturbation will be such that its $\mathbb{C}$-linear endpoint has trivial cokernel (and also kernel). And, there will be a finite number, say $N$, of times $t$ along the generic path for which the resulting operator has non-trivial kernel. Furthermore, these kernels appear in a suitably transverse manner as t passes one of these special points. (Mimic the discussion in Step 2 of Section 2 in [18].) Given the preceding, orient $y$ as $(-1)^N$. This orientation is consistent because the set of $\mathbb{C}$-linear perturbations of $\Delta_y$ (by a zero'th order term) which have trivial kernel form a path connected space.

### h) Proof of Proposition 3.4

As a first step, consider whether the constant section is in $\mathcal{Z}_0$. For this purpose, it is sufficient to exhibit a pair $(b, \lambda)$ which solves (3.20).

With this understood, then $(b, \lambda)$ for the case $m = 1$ is given by

$$(3.46)\qquad \begin{aligned} &1. \qquad b = 2\sqrt{2}\nu s\frac{1}{\tau}[\partial_v \tau], \\ &2. \qquad \lambda = \mu \bar{s}\frac{1}{\tau}(1 - |\tau|^2). \end{aligned}$$

(It is left as an exercise to verify that $\tau^1_{0,1}$ and $v^1_{0,2}$ both vanish when $c$ is a constant section of (3.1).) Note that both $b$ and $\lambda$ are smooth, as can be verified using the explicit form in (2.7) for the symmetric vortex.

In the case where $m > 1$ and $\mu = 0$, take $b$ as in (3.46) and take $\lambda = 0$. Once again, the smoothness of $b$ can be verified using (2.7).

Argue as follows to prove that the constant $m > 1$ section of (3.1) does not make (3.5) vanish when $\mu \neq 0$. First, note that when $\mu$ is non-zero, then $(b, \lambda)$ still solve (3.19) but the expression for $\lambda$ in (3.46.2) is singular at the origin when $m > 1$. In fact, near zero, $\lambda \sim \mu f_m(0)^{-1}(\bar{s})^{-m+1}$. Meanwhile, when $c$ is an $m > 1$, symmetric vortex on $\mathbb{C}$, one can prove that there is a solution $(a, \alpha)$ to (3.2) where $a = q(|\eta|^2)\eta^{m-2}$ and where $q(0) \neq 0$. (The existence of such a solution can be deduced using the identification $\mathfrak{C}_m = \mathrm{Sym}^m(C)$.) In any event, the $L^2$ inner product between $(a, \alpha)$ and $\mu\bar{s}(\frac{1}{2\sqrt{2}}(1 - |\tau|^2), \partial_v \tau)$ can be computed via integration by parts as a residue at the origin, namely $\pi(\bar{s}\bar{\alpha}\lambda)_0 = \pi\mu q(0) f_m(0)^{-1}$.

Now consider the assertions about $\mathcal{Z}_0$ in the $m = 1$ case. The assertions here follow with an identification of $\mathbb{F}(y)$ in (3.22a) with $\bar{y}$. The argument has four steps.

*Step 1.* The milieu for this step is $\mathbb{C}$ and the $m = 1$ vortex moduli space $\mathfrak{C}_1$. The main observation here is that the vortex equations are invariant under the group of translations of $\mathbb{C}$, and so this group acts on any $\mathfrak{C}_m$. This action is free (since a non-trivial translation moves $\tau^{-1}(0)$). Furthermore, in the case where $m = 1$, every point in $\mathfrak{C}_1$ is obtained by such a translation from the $m = 1$ vortex $c_0 = (v_0, \tau_0)$ which has $\tau_0^{-1}(0) = 0$. The latter vortex solution is invariant under rotations of $\mathbb{C}$ which fix the origin. Thus, the vortex with $\tau^{-1}(0) = \lambda$ is invariant under rotations of $\mathbb{C}$ which fix the point $\lambda$.

*Step 2.* This step identifies the vector space of solutions to (3.2) in the $m = 1$ case. This vector space is 1 dimensional (over $\mathbb{C}$) and the reader can check (using the vortex equations (2.4)) that when $c = (v, \tau)$, then the solution space to (3.2) is the span of

$$w_c = \left(\frac{1}{2\sqrt{2}}(1 - |\tau|^2), \partial_v \tau\right). \tag{3.47}$$

Remark here that the vector $w_c$ does not have $L^2$ norm equal to 1. Rather, integration by parts (plus (2.5.1)) can be used to establish that

$$\int_{\mathbb{C}} |w_c|^2 = \pi. \tag{3.48}$$

For future reference, note that in the case where $(v, \tau)$ is an $m > 1$ vortex, then $w_c$ is still a solution to (3.2), although the square of the $L^2$ norm in this case is equal to $m \cdot \pi$.

*Step 3.* When $c = (v, \tau)$ is the $S^1$-invariant vortex with vortex number $m$ (the vortex for which $\tau^{-1}(0)$ is the origin as described in (2.6)), then

$$\int_{\mathbb{C}} f \cdot |w_c|^2 = 0, \tag{3.49}$$

where $f(\eta) = \eta^p$ and $f(\eta) = \bar{\eta}^p$ for $p > 0$. (This is because $|w_c|^2$ is an $S^1$-invariant function.) It follows from (3.49) that when $m = 1$ and $\tau^{-1}(0) = \lambda$, then, with $f$ as above,

$$\int_{\mathbb{C}} f \cdot |w_c|^2 = m \cdot \pi \cdot f(\lambda). \tag{3.50}$$

*Step 4.* In the $m = 1$ case, the map $\Upsilon_1$ at $y$ takes $y' \in N|_{\pi(y)}$ to $\pi^{-1} w_c \langle w_c, (0, e^{-\varsigma})\rangle_2$, where $c = c(y)$ and $\langle , \rangle_2$ denotes the $L^2$ norm along the fiber $\pi^{-1}(y)$. The $L^2$ inner product here is independent of $y$, and so can be done for the case $y = 0$. In this case, the integral in question is equal to

$$\int_{\mathbb{C}} \bar{\partial}[\eta^{-1}(1 - |\tau|^2)]. \tag{3.51}$$

(Remember that $e^{-\varsigma} = \eta^{-1}\tau$, and that $\partial_v \tau = 0$.) The latter integral equals $\pi$. (Integrate by parts in (3.51), taking care to account for a residue at the origin.)

It follows from (3.5) and this last calculation that $\mathbb{F} = \pi^{-1}\langle w_c, -\bar{\eta} w_c\rangle_2$. And, this last expression is equal to $\bar{y}$ because of (3.50).

Finally remark that the $m > 1$ and $\mu = 0$ assertions of Proposition 4.3 follow directly from (4.22) and Assertions 4 and 5.

### i) Proof of Proposition 3.3

Consider first the $m = 1$ assertions. The proof of Proposition 3.4 demonstrated that the $m = 1$ identification in (3.27) between $V^c$ and $N$ identifies the operator $\Delta^c$ with the operator $D$ in (1.5). Meanwhile, Lemma 3.1 in [18] asserts that there is a Baire subspace of pairs $(\nu, \mu)$ in the Frechet space $C^\infty(T^{0,1} \oplus (N^2 \otimes T^{0,1}C))$ for which cokernel$(D) = \{0\}$ when $1 - g + n \geq 0$. Standard perturbation theory (as in [6]) can be used to prove that this Baire set is actually open and dense. On the other hand, when $1 - g + n$ is negative, the index of $D$ (which is twice this number) is negative, so cokernel$(D) \neq \{0\}$ no matter what.

Consider now the assertion about $\mathcal{Z}_\varepsilon$ for generic, compactly supported $\varepsilon$. The proof is based on an argument using the Sard-Smale theorem [16] which is due to Uhlenbeck. (See the proof of Theorem 3.17 in [3].) The first step is to fix $k \geq 2$ and introduce the Banach space $\mathcal{B}$ of $C^k$ sections over $N$ of $T^{0,2}N$ which is obtained by completing the space of smooth, compactly supported sections using the norm

$$\sup_N e^{|s|} \Sigma_{0 \leq j \leq k} |\nabla^j \varepsilon|. \tag{3.52}$$

Let $\mathcal{A}$ denote the Banach space of Sobolev class $L^2_2$ sections of $\oplus_{1 \leq q \leq m} N^q$. Using the map $\Upsilon$ from Assertion 4 of Proposition 3.2, this space is identified with the space of Sobolev class $L^2_2$ sections of (3.1). With this identification understood, introduce the universal moduli space $\underline{\mathcal{Z}} \subset \mathcal{A} \times \mathcal{B}$ which consists of pairs $(y, \varepsilon)$ for which (3.23) vanishes when $c = \Upsilon(y)$. This last condition can be written (in analogy with (3.22a)) as

$$\bar\partial y + \nu \aleph y + \mu \mathbb{F}(y) + \Upsilon_1^{-1} \Pi^c(\varepsilon, 0) = 0. \tag{3.53}$$

It is an exercise to generalize the arguments for Assertion 1 of Proposition 3.2 so as to prove that $\underline{\mathcal{Z}}$ is a smooth submanifold of $\mathcal{A} \times \mathcal{B}$ such that the projection induced map to $\mathcal{B}$ has everywhere Fredholm differential with index equal to $d$. With the preceding understood, the Sard-Smale theorem [16] can be invoked to find a Baire subset of $\mathcal{B}$ which are regular values for the projection induced map $\underline{\mathcal{Z}} \to \mathcal{B}$. Note that if $\varepsilon$ is a regular value of such a map, then the corresponding $\mathcal{Z}_\varepsilon$ is a smooth submanifold of $\mathcal{A}$ of the required dimension.

It remains still to prove that the Frechet space of smooth, compactly supported sections of $T^{0,2}N$ contains a Baire set of regular values for the map from $\underline{\mathcal{Z}}$ to $\mathcal{B}$. In this regard, fix a positive integer $R$ and consider the space $\underline{\mathcal{Z}}^R$ consisting of solutions $(y, \varepsilon)$ for which the $L^2_2$ norm of $y$ is bounded by $R$. It is an exercise to check that the map from $\underline{\mathcal{Z}}^R$ to $\mathcal{B}$ is proper. With this understood, the set of regular values of the latter map form an open and dense subset of $\mathcal{B}$. The preceding open and dense subset intersects the Frechet space of smooth, compactly supported sections of $T^{0,2}N$ as an open and dense subset, $\mathcal{O}^R$. Note that the set $\cap_{0<R\in\mathbb{Z}}\mathcal{O}^R$ is a Baire subset of the space of smooth, compactly supported sections of $T^{0,2}N$. And, each point in this set is a regular value for the map from $\underline{\mathcal{Z}}$ to $\mathcal{B}$ since the $\underline{\mathcal{Z}}^R$'s are nested and their union is $\underline{\mathcal{Z}}$.

Note that the Baire set here can be characterized by the following condition: A section $\varepsilon$ is in the Baire set when the following is true at each $c \in \mathcal{Z}_\varepsilon$ : Define the first order operator, $\Delta^c : C^\infty(V^c) \to C^\infty(V^c \otimes T^{0,1}C)$ as in (3.20) where $(b, \lambda)$ solves the modified version of (3.19) which adds $\varepsilon$ to the left-hand side of (3.19.1). Then cokernel$(\Delta^c) = \{0\}$. (This implies that the tangent space to $\mathcal{Z}_\varepsilon$ at c is naturally isomorphic to the kernel of this very same operator $\Delta^c$.)

With the tangent space to $\mathcal{Z}_\varepsilon$ identified as above, the discussion of orientations for $\mathcal{Z}_\varepsilon$ is the same as that for $\mathcal{Z}$ in Assertion 3 of Proposition 3.2 and in Subsection 3g.

## 4. From almost solutions to true solutions, I

The first purpose of this section is to define a deformation map which takes certain of the approximate solutions from Sections 2 and 3b of the large $r$ version of (1.13) to honest solutions.

A brief digression is required in order to make a precise statement. To start the digression, suppose that $E \to X$ is a complex line bundle with first Chern class $e$. Now fix a finite set $\{(C_k, m_k)\}_{1\le k\le n}$ of pairs, where $\{C_k\}$ is a pair wise disjoint collection of connected, pseudo-holomorphic submanifolds, and the set $\{m_k\}$ consists of positive integers. These are constrained so that $\Sigma_k m_k[C_k]$ is Poincaré dual to $e$.

For each $k$, the data $(C_k, m_k)$ specifies a fiber bundle as in (2.15) (and (3.1)). With this understood, introduce, for each $k$, the space $\mathcal{Z}_0$ and the manifold $\mathcal{Z}$ from Section 3c; and fix an open set $\mathcal{K}^{(k)}$ in the $k'$th version of $\mathcal{Z}$ which has compact closure in $\mathcal{Z}$.

Notice that a point in $\times_k \mathcal{K}^{(k)}$ along with a fixed choice of small, but positive $\delta$ gives, for all sufficiently large $r$, the data $(a_r, \alpha_r)$ as dictated in Section 2. Here, $a_r$ is a connection on the bundle $E$ and $\alpha_r$ is a section of $E$. The number $\delta$ should be fixed once and for all, and should be chosen so that $10^3 \cdot \delta$ is much smaller than the minimum distance between any two members of $\{C_k\}$. With $(a_r, \alpha_r)$ constructed, define $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ as in (3.12). Here, $a_r$ is a connection on $E$, and $(\underline{\alpha}_r, \underline{\beta}_r)$ is a section of the bundle $S_+$ as defined in (1.9) using the line bundle $E$.

Use the Levi-Civita connection on $T^*X$ and the connection $a_r$ (plus the connection $A_0$ on $K$) to define a covariant derivative, $\nabla$, on sections of $iT^*X \oplus S_+$. Finally, use $||\cdot||_2$ to denote the $L^2$ norm for sections of Hermitian vector bundles over $X$.

End the digression.

**Proposition 4.1.** *Fix a complex line bundle $E \to X$, and then fix $\{(C_k, m_k)\}$ and $\{\mathcal{K}^{(k)}\}$ as above. This data determines $\zeta \geq 1$. Then, given $r \geq \zeta$, let $\mathcal{M}^{(r)}$ denote the moduli space of solutions of the $\mu_0 = 0$ version* (1.13) *for the* $\mathrm{Spin}^{\mathbb{C}}$ *structure given by* (1.9). *There exists a continuous map $\Psi_r : \times_k \mathcal{K}^{(k)} \to \mathcal{M}^{(r)}$ which has the following form: Construct the connection $\underline{a}_r$ on $E$ and the section $(\underline{\alpha}_r, \underline{\beta}_r)$ of $S_+$ from the data given by a point $q$ in the domain of $\Psi_r$ as dictated in Sections 2 and 3b using some fixed $\delta < \zeta^{-1}$. Then*

$$\Psi_r(q) = \left(\underline{a}_r + \sqrt{\frac{r}{8}}a', (\underline{\alpha}_r + \alpha', \underline{\beta}_r + \beta')\right),$$

*where $a' \in i \cdot \Omega^1$ and $(\alpha', \beta') \in C^\infty(S_+)$ obey the following:*

$$\begin{aligned} &1) \quad ||\nabla(a', (\alpha', \beta')||_2 + r^{1/2}||(a', (\alpha', \beta')||_2 \leq \zeta r^{-1/2}. \\ &2) \quad \sup_X |(a', (\alpha', \beta')| \leq \zeta r^{-1/2}. \\ &3) \quad \sup_X |\nabla(a', (\alpha', \beta')| \leq \zeta. \end{aligned} \tag{4.1}$$

Note that Theorem 1 in the introduction is an immediate corollary to Propositions 3.4 and 4.1.

There is a generalization of Proposition 4.1 which is valid when the gluing data (the set of $n$ sections of the $n$ different versions of (2.15)) is required only to sit in the n versions of $\mathcal{Z}_0$. To fully appreciate the statement of the more general result, one must keep in mind the identification via the map $\Upsilon$ in Proposition 3.2 of $\mathcal{Z}_0$ with the space of sections of $\oplus_{1\le q\le m} N^q$ which obey (3.22a). Also keep in mind that a neighborhood in $\mathcal{Z}_0$ of $y \in \mathcal{Z}_0$ is homeomorphic to the zero set of a smooth map, $\psi$, from a ball about the origin in kernel$(\Delta_y)$ to cokernel$(\Delta_y)$. Here, $\Delta_y$ is given by (3.22). Finally, remember that both kernel$(\Delta_y)$ and cokernel$(\Delta_y)$ are naturally normed vector spaces, where the norm comes from the $L^2$ norm on the complex curve in question.

Here, and in what follows, the map $\Upsilon$ of Proposition 3.2 is used to implicitly identify (2.15) with $\oplus_{1\le q\le m} N^q$ and to identify $\mathcal{Z}_0$ with the space of sections of $\oplus_{1\le q\le m} N^q$ which satisfy (3.22a).

The generalization below of Proposition 4.1 also re-introduces the space Conn$(E)$ of $C^\infty$ connections on $E$ (toplogized as an affine Frechet space modelled on $i\,\Omega^1(X)$). Let $C^\infty(S_+)$ denote the Frechet space of smooth sections of $S_+$ and let $C^\infty(X;S^1)$ denote the Frechet manifold of smooth maps from $X$ to $S^1$. Note that the latter acts continuously on the product Conn$(E) \times C^\infty(S_+)$. Endow the orbit space

$$(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X;S^1)$$

with the quotient topology.

Here is the statement of this more general result:

**Proposition 4.2.** *Fix a complex line bundle $E \to X$ and $\{(C_k, m_k)\}$ as above. For each $k$, fix a point, $y_0^k$, in the $k$'th version of $\mathcal{Z}_0$ (as defined using $C_k$ and $m_k$). Use $\mathcal{K}_0^{(k)}$ and $\mathcal{K}_1^{(k)}$ to denote, respectively,* kernel$(\Delta_y)$ *and* cokernel$(\Delta_y)$ *when $y = y_0^k$. Then, there is, for each $k$, a ball $B^{(k)}$, about the origin in $\mathcal{K}_0^{(k)}$, and, for all large $r$, there are*

- *a smooth map $\psi_r : \times_k B^{(k)} \to \times_k \mathcal{K}_1^{(k)}$ and*
- *a continuous map*

$$\Psi_r : \times_k B^{(k)} \to (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X;S^1)$$

*which have the following properties:*

1. *$\Psi_r$ maps $\psi_r^{-1}(0)$ to $\mathcal{M}_r$.*
2. *The map $\Psi_r$ has the following form: When*

$$y = (y^1, \dots, y^m) \in \times_k B^{(k)},$$

*then $y^k$ defines a section $c^{(k)} = \Upsilon(y^k)$ of the $k$'th version of (2.15) lying close to $\Upsilon(y_0^k)$. Construct from $\{c^{(k)}\}$ the connection $\underline{a}_r$ on $E$*

*and the section* $(\underline{\alpha}_r, \underline{\beta}_r)$ *of* $S_+$ *(as dictated in Sections 2 and 3b). There exist an imaginary valued 1-form* $a'$ *and section* $(\alpha', \beta')$ *of* $S_+$ *such that*

$$\Psi_r(\{y^k\}) = \left(\underline{a}_r + \sqrt{\frac{r}{8}}a', (\underline{\alpha}_r + \alpha'\underline{\beta}_r + \beta')\right).$$

3. *The data* $(a', (\alpha', \beta'))$ *obey (4.1) where* $\zeta$ *depends on* $\{y_0^k\}$*, but is independent of* $r$ *and* $\{y^k\}$*.*

4. *The map* $\psi_r$ *has the following property: For each* $k$*, let* $\psi^k$ *denote the map* $\psi$ *as defined above with* $y_0^k$*. Then,* $|\psi_r - \times_k \psi^k| \leq \zeta r^{-1/2}$*. Here, the norm is the product norm from that on each of the spaces* $\mathcal{K}_1^{(k)}$*.*

Remark that $\Psi_r$ in both Propositions 4.1 and 4.2 is a homeomorphism. Furthermore, $\Psi_r$ from Proposition 4.1 maps onto a non-degenerate, manifold part of $\mathcal{M}^{(r)}$ as a diffeomorphism. These facts are stated more formally and proved in Section 6.

These two propositions are proved by exhibiting the required maps $\Psi_r$ and $\psi_r$. The remainder of this section is occupied with the linear aspects of the construction, while the next section completes the job. Thus, the proofs are completed in the last subsection of Section 5. (Various analytic aspects of these maps are discussed in Section 6.)

### a) The formal structure of the proof

The construction of the map $\Psi_r$ employs a strategy which is briefly outlined below. (See Section 5.1 for a more detailed outline.)

*Step 1.* Search for a solution to (1.13) of the form $(\underline{a}_r + \frac{\sqrt{r}}{2\sqrt{2}}a', (\underline{\alpha}_r + \alpha', \underline{\beta}_r + \beta'))$ where the imaginary valued 1-form $a'$ and the spinor $(\alpha', \beta')$ are small. The Seiberg-Witten equations in (1.13) can be written as equations for the triple $q' = (a', (\alpha', \beta'))$ in the following schematic form:

$$Lq' + \sqrt{r} \cdot \varpi(q', q') + \text{err} = 0, \tag{4.2}$$

where $L$ is a first order differential operator with canonical symbol (the zero'th order part depends explicitly on $r$ and on $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$. In (4.2), $\varpi$ is a certain canonical ($r$-independent) vector bundle homomorphism from the bundle $\otimes_2(iT^* \otimes S_+)$ to the bundle $i\Lambda_+ \oplus S_-$. In (4.2), the term denoted by "err" is determined by the failure of $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ to satisfy (1.13). The latter is given by (3.6) where $a = \underline{a}_r$, $\alpha = \underline{\alpha}_r$, and $\beta = \underline{\beta}_r$.

(Remark that the factor of $\frac{\sqrt{r}}{2\sqrt{2}}$ in the definition of $a'$ is necessary in order to make $\varpi$ in (4.2) independent of $r$.)

The operator $L$ sends

$$q' = (a', (\alpha', \beta')) \in i\Omega^1 \oplus C^\infty(S_+)$$

to $i\Omega^{2+} \oplus C^\infty(S_-)$, where the two components of $Lq'$ are, respectively,

$$P_+ da' - i\frac{\sqrt{r}}{2\sqrt{2}} re(\bar{\alpha}\alpha' - \bar{\beta}\beta')\omega + \frac{\sqrt{r}}{\sqrt{2}}(\bar{\alpha}\beta' + \bar{\alpha}'\beta - \alpha\bar{\beta}' - \alpha'\bar{\beta}),$$

$$\bar{\partial}_a \alpha' - (\bar{\partial}_{A_0+a})^* \beta' + \frac{\sqrt{r}}{2\sqrt{2}}\alpha a'_{01} + \frac{\sqrt{r}}{2\sqrt{2}}\beta \bar{a}'_{01} \tag{4.3}$$

Here, $(a, (\alpha, \beta)) = (\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$. In this last equation, $\bar{\partial}$ is the projection of the covariant derivative onto $T^{0,1}X$, and $\bar{\partial}^*$ is the formal $L^2$-adjoint of the projection of the covariant derivative onto $\Lambda^2 T^{0,1}X = K^{-1}$. Also, $a'_{0,1}$ is the projection of $a'$ onto $T^{0,1}N$.

Note that the operator in (4.3) is not elliptic. (This is a relic of the gauge invariance of the Seiberg-Witten equation.) However, there is a natural extension of the operator to an elliptic operator which maps the space $i\Omega^1 \oplus C^\infty(S_+)$ to $i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$. This extended operator will also be called $L$, and (4.2) should be interpreted with this extended $L$. The projection of (the now extended) $L$ into $i\Omega^{2+} \oplus C^\infty(S_-)$ gives (4.3), while $L$'s projection into $i\Omega^0$ gives

$$*d * a' + i\frac{\sqrt{r}}{\sqrt{2}} im((\bar{\alpha}\alpha' + \bar{\beta}\beta'), \tag{4.4}$$

where $(a, (\alpha, \beta)) = (\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$.

With the preceding understood, (4.2) will henceforth be interpreted as the condition for the vanishing a certain section (as defined from $q'$) of $i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$.

*Step 3.* The contraction mapping theorem provides the basic tool for proving existence and uniqueness assertions for an equation such as (4.2). Here is the statement of the version that will be employed below (see, e.g. [7]):

**Contraction Mapping Theorem.** *Suppose that $\mathcal{H}$ is a Banach space with norm $||\cdot||$. Let $\mathcal{B} \subset \mathcal{H}$ be a closed subspace, and let $Y : \mathcal{B} \to \mathcal{B}$ be a smooth map for which there exists $\varepsilon > 0$ such that*

$$||Y(u) - Y(v)|| \leq (1 - \varepsilon) \cdot ||u - v||$$

*for all pairs $(u, v) \in \mathcal{B}$. (Such a map will be called a contraction mapping.) Then there is a unique fixed point $u_0 \in \mathcal{B}$ of the map $Y$. More generally, suppose that $\mathcal{T}$ is a smooth manifold and $Y : \mathcal{B} \times \mathcal{T} \to \mathcal{B}$ is a smooth map such that for each $t \in \mathcal{T}, Y(\cdot, t)$ is a contraction mapping. Then, there exists a smooth map $u : \mathcal{T} \to \mathcal{B}$ such that for each $t, u(t)$ is the unique fixed point of $Y(\cdot, t)$. Furthermore, if $T\mathcal{T}$ is endowed with a norm (for example, if $\mathcal{T}$ is a Banach manifold), then the norm of the differential of $u(\cdot)$ at a point $t$ is bounded by $\varepsilon^{-1}$ times that of the differential of the map $Y(u(t), \cdot) : \mathcal{T} \to \mathcal{B}$ at $t$.*

The application of the contraction mapping theorem to (4.2) requires the operator $L$ to be invertible on a suitable domain, in which case (4.2) is implied by the fixed point equation $q' = Y(q')$ where $Y(\cdot)$ has the following schematic form:

$$Y(q') = -\underline{L}^{-1}(\sqrt{r}\varpi(q', q') + \mathrm{err}).$$

Here, $\underline{L}^{-1}$ is a partial inverse to $L$ in that $\underline{L}$ is, in a suitable sense, an inverse to $L$ on the compliment of a certain finite dimensional subspace of its range. The contraction mapping theorem will be used to solve this last equation for $q'$ as an implicit function of the data $\{c^{(k)}\}$. The map $\Psi_r$ in both Propositions 4.1 and 4.2 is determined explicitly in terms of the data $q'$ in (4.1). In the case where the domain of $\underline{L}^{-1}$ is not the whole of the range of $L$, there are finitely many components of (4.2) which are not spoken for in the preceding equation. These components then define a finite system of equations on the data $\{c^{(k)}\}$. The latter give the map $\psi_r$ in Proposition 4.2. (This strategy, introduced to geometers by Kuranishi [10], is now well known in gauge theory circles.)

**b) A strategy for the equation $\mathbf{Lq' = g}$.**

The use of the contraction mapping theorem puts the onus on finding useful estimates for the operator $L$ in (4.3) and (4.4). With this understood, the remainder of this section is occupied with the study of the operator $L$ and the construction of an inverse. The full non-linear problem, (4.2), is taken up in the next section. This subsection discusses the strategy which is used for analyzing $L$. (The details of the strategy are worked out in the subsequent subsections.)

As will be demonstrated below, when $r$ is large, the operator $L$ in (4.2) is nicely invertible over a vast amount of $X$, but has problematic inverse near each $C_k$. The region where $L$ has such an inverse consists of those points where the distance to any $C_k$ is greater than $2\delta$. On the other hand, the behavior of $L$ on sections with support near some $C_k$ is rather complicated. This suggests a strategy for analyzing the equation

$$Lq' = g \tag{4.5}$$

which isolates separate contributions from a neighborhood of each $C_k$. The purpose of this subsection is to describe, in a very general way, how this isolation is obtained.

To effect this isolation strategy, introduce, for each $k$, the bump function $\chi_{\delta,k}$ which is the $C_k$ version of the function in (2.19). Now, search for $q'$ with the following form:

$$q' = \Pi_k(1 - \chi_{4\delta,k})q^0 + \Sigma_k \chi_{100\delta,k} q^k. \tag{4.6}$$

Here, $q^0 = (a^0, (\alpha^0, \beta^0))$ consists of an imaginary valued 1-form $a^0$ and a section $(\alpha^0, \beta^0)$ of the plus spin bundle $S_{+,0}$ of the *canonical* Spin$^{\mathbb{C}}$ structure on $X$ (as defined in (1.7)). Meanwhile, each $q^k$ consists of a triple $(a^k, (\alpha^k, \beta^k))$ consisting of:

(4.7)

1. an imaginary valued 1-form $a^k$ on the normal bundle $\pi : N_k \to C_k$,
2. a section, $\alpha^k$, over this normal bundle of $E$,
3. a section, $\beta^k$, over this normal bundle of $E \otimes \pi^*(N_k \otimes T_{1,0}C)$.

Implicit in the use of (4.6) in (4.7) is, for each $k$, an identification via the exponential map from Lemma 2.1 of a disk subbundle $N_{(0)k}$ in $N_k$ with a diameter $\delta_0 > 10^3\delta$ tubular neighborhoood of $C_k$ in $X$. Also, implicit here are:

(4.8)

- An identification of $S_+$ with $S_{+,0}$ on the support of the function $\Pi_k(1 - \chi_{4\delta,k})$.
- An identification of $iT^*N_k \oplus E \oplus (E \oplus \pi^*(N_k \otimes T_{1,0}C))$ with $iT^*X \oplus S_+$ on the support of $\chi_{100\delta,k}$.

Suppose as well that a decomposition of $g$ has been given:

$$g = \Pi_k(1 - \chi_{4\delta,k})g^0 + \Sigma_k \chi_{100\delta,k} g^k, \tag{4.9}$$

where

$$\begin{aligned} &\bullet\ g^0 = \Pi_k(1 - \chi_{25\delta,k})g, \\ &\bullet\ g^k = \chi_{25\delta,k} g. \end{aligned} \tag{4.10}$$

With (4.6) and (4.9) understood, then the equation $Lq' = g$ is implied by the following set of equations:

$$\begin{aligned} &\bullet\ L_0 q^0 + \Sigma_k \wp(d\chi_{100\delta,k}, q^k) = g^0, \\ &\bullet\ L_k q^k - \wp(d\chi_{4\delta,k}, q^0) = g^k. \end{aligned} \tag{4.11}$$

Here, $L^0$ is the operator which is given by (4.3) and (4.4) but for the case where the Spin$^{\mathbb{C}}$ structure is canonical, the connection $\underline{a}_r$ is replaced by the product connection on the bundle $\varepsilon_{\mathbb{C}} = X \times \mathbb{C}$, and $(\underline{\alpha}_r, \underline{\beta}_r)$ is replaced by (1, 0). In particular, with (4.8) implicitly understood, $L_0$ and $L$ agree on the support of the function $\Pi_k(1 - \chi_{4\delta,k})$. Meanwhile, for each $k$, the operator $L_k$ is a specific operator on the normal bundle $N_k$ which agrees with $L$ on the support of $\chi_{100\delta,k}$. (Here again, (4.8) is implicit.) In both lines of (4.10), $\wp$ denotes the principle symbol of the operator $L$. Furthermore, (4.8) must be used to interpret $(g^0, \{g^k\})$ in (4.11).

Note that (4.11) implies (4.2). Indeed, multiply the $q^0$ equation by $\Pi_k(1 - \chi_{4\delta,k})$ and multiply each $q^k$ equation by $\chi_{100\delta,k}$. Then add the resulting equations together to obtain (4.2). Thus, (4.10) rewrites the one equation (4.2) as a finite number of coupled equations as indicated schematically in (4.11).

By the way, given a solution $q'$ to (4.2), there exists data $(q^0, \{q^k\})$ so that both (4.6) and (4.11) are satisfied. Indeed, this data is found in terms of data $\{f^k\}$, where each $f^k$ is a section over the correponding normal bundle $N_k$ of $iT^*N_k \oplus \varepsilon_{\mathbb{C}} \oplus \pi^*(N_k \otimes T_{1,0}C)$. (Here, $\varepsilon_{\mathbb{C}}$ denotes the trivial complex line bundle.) Here is the formula for $(q^0, \{q^k\})$ in terms of $q'$ and $f^k$ :

$$\begin{aligned} &\bullet\ q^0 = \Pi_k(1 - \chi_{25\delta,k})q' + \Sigma_k \chi_{100\delta,k} f^k. \\ &\bullet\ q^k = \chi_{25\delta,k} q' - (1 - \chi_{4\delta,k}) f^k. \end{aligned} \tag{4.12}$$

(Implicit in this last equation is an appropriate identification between $E$ and the trivial bundle where the distance to $C_k$ is greater than $2 \cdot \delta$.)

No matter the choice for $\{f^k\}$, this last equation implies (4.6). However, (4.11) requires that each $f^k$ be given implicitly in terms of $q'$ as a solution to the equation

$$L_k^0 f^k - \wp(d\chi_{25\delta,k})q' = 0. \tag{4.13}$$

Here, $L_k^0$ is the same as $L_k$ but for the replacement of the connection $\underline{a}_r$ on $E$ by the product connection on $\varepsilon_{\mathbb{C}}$ and for the replacement of $(\underline{\alpha}_r, \underline{\beta}_r)$ by $(1, 0)$. As shall be demonstrated below, (4.13) has a unique solution. (Remark that if $q'$ is given apriori by (4.6), then the solution to (4.13) is $f^k = \chi_{25\delta,k}q^0 - (1 - \chi_{25\delta,k})q^k$.)

With the schematics of the strategy understood, the remainder of this section considers, in turn, the operator $L^0$, then the operators $\{L^k\}$, and finally, the coupled system in (4.11).

Note that the author learned of the method of separating one equation on a manifold (in this case, (4.2)) into a number of localized equations (i.e., (4.7)) from Donaldson's approach in [2] to the problem of constructing anti- self dual connections on connected sums of 4-manifolds.

### c) The operator $\mathbf{L}_0$

The purpose of this subsection is to describe and analyze the operator $L_0$ in (4.11). To begin, remark that $L_0$ maps sections of $iT^* \oplus S_{+,0}$ to $i(\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_{-,0}$, where $\varepsilon_{\mathbb{R}}$ is the trivial, real line bundle, and $S_{\pm,0}$ come from the canonical Spin$^{\mathbb{C}}$ structure in (1.7). Introduce the canonical connection $A_0$ on $K^{-1}$. Then, the operator $L_0$ is given by (4.3, 4) with $\underline{a}_r = 0$ and $\underline{\alpha}_r = 1$ and $\underline{\beta}_r = 0$.

With regard to (4.11), note that there is a canonical identification between $S_{\pm,0}$ (the bundles in (1.7)) and the spinor bundles $S_{\pm}$ from (1.9) on the support of $\Pi_k(1 - \chi_{4\delta,k})$ and a corresponding identification between the operators $L_0$ and $L$. Indeed, where the distance to any $C_k$ is greater than $2\delta$, the bundle $E$ in (1.9) is trivial with the section $\underline{\alpha}_r$ defining the trivialization. Furthermore, the connection $\underline{a}_r$ is trivial here, and $\underline{\alpha}_r$ is $a_r$-covariantly constant. Finally, $\underline{\beta}_r = 0$ where the distance to $\cup_k C_k$ is greater than $2\delta$.

In the lemma below, use the Levi-Civita connection and the canonical connection on $K^{-1}$ to define covariant derivatives on sections of tensor bundles and of the bundles $S_{\pm,0}$. Use these covariant derivatives when defining the Sobolev $L_k^2$ norms, the various $C_k$ norms and Hölder norms.

**Lemma 4.3.** *There is a constant $\zeta$ and which has the following significance: Use $r \geq 1$ to define the operator $L_0$. Then $L_0$ defines a continuous operator from $i\Omega^1 \oplus C^\infty(S_{+,0})$, a Frechet space, to $i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_{-,0})$ with continuous inverse. Furthermore, let $g^0 \in i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_{-,0})$, and $q^0 = L_0^{-1} g^0$. Then the following hold:*

- $||\nabla q^0||_2 + \sqrt{r} \cdot ||q^0||_2 \leq \zeta ||g^0||_2$,
- $||\nabla q^0||_2 \leq \zeta r^{-1/2} ||\nabla g^0||_2$,
- $\sup_X |q^0| \leq \zeta r^{-1/2} \sup_X |g^0|$,
- *For any $k > 0$, there is an $r$ independent constant $\zeta_k$ with the property that the $C^{k,1/2}$ Hölder norm of $q^0$ is bounded by $\zeta_k$ times the $C^{k-1,1/2}$ Hölder norm of $g^0$.*

*Proof of Lemma 4.3.* Introduce $L_0^\dagger$ to denote the formal, $L^2$ adjoint of $L_0$. It is an exercise to compute $L_0 L_0^\dagger$. The result has the following schematic form:

$$L_0 L_0^\dagger = \frac{1}{4}\nabla^\dagger\nabla + \mathcal{R}_0 + \sqrt{r} \cdot \mathcal{R}_1 + \frac{r}{8}, \tag{4.14}$$

where $\mathcal{R}_{0,1}$ are $r$-independent endomorphisms of $i(\varepsilon_\mathbb{R} \oplus \Lambda_+) \oplus S_{-,0}$, $\varepsilon_\mathbb{R}$ being the trivial real line bundle over $X$. In (4.14), $\nabla$ is the covariant derivative on $i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_{-,0})$ which comes from the metric and the canonical connection $A_0$ on $K^{-1}$. Standard local estimates for the "Laplacian" $\frac{1}{4}\nabla^\dagger\nabla + \frac{r}{8}$ show that when $r$ is large, (4.14) defines a continuous operator on the Frechet space $i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_{-,0})$ with continuous inverse. With this understood, the inverse of $L_0$ is $L_0^\dagger \cdot (L_0 L_0^\dagger)^{-1}$.

The estimates for $L_0^{-1}$ follow $\mathcal{Z} = \varnothing$. from estimates for $(L_0 L_0^\dagger)^{-1}$. For example, the $L^2$ and $L^2_1$ estimates in the lemma all follow from the fact that for sufficiently large $r$, the smallest eigenvalue of the operator in (4.14) is larger than $r/16$. As for the $C^0$ and Hölder estimates, these follow from standard estimates for the Green's function of "Laplacian" $\frac{1}{4}\nabla^\dagger\nabla + \frac{r}{8}$.

**Note.** In the preceding, and below, the Greek letter $\zeta$ will represent a constant of size larger than 1 whose precise value may change from appearance to appearance. Unless noted explicitly, the precise value of this constant is immaterial to subsequenct discussions. What is important is its lack of dependence on data such as $r, (g^0, \{g^k\})$. This independence should be assumed if not stated explicitly.

**d) The operator $\mathbf{L}_k$ and an operator $\mathbf{L}'$**

As Lemma 4.3 illustrates, the analysis for $L_0$ is straightforward. However, the story is nowhere near as simple for the operator $L_k$ in (4.11). The analysis of this operator occupies this and the subsequent four subsections.

To set the stage for this business, fix a compact, pseudo-holomorphic submanifold $C \subset X$ and a positive integer $m$. Let $\pi : N \to C$ denote the normal bundle to $C$ in $X$. This subsection will describe the operator $L_k$ in (4.8) when $C = C_k$. The description is in terms of an auxilliary operator, $L'$, which is, essentially, the first order Taylor's expansion of $L$ off of $C$. The idea here is that the operator $L'$ reflects the structure of $N$ as a vector bundle over $C$, while this is only true of $L$ to leading order in the distance to $C$. The fact that $L'$ sees the vector bundle structure on $N$ simplifies the analysis in that a version of the method of separation of variables can be employed for $L'$. However, be forwarned that the ultimate story is not as simple as "$L'$ plus perturbation theory", since the difference between $L'$ and $L_k$ is not relatively small on a certain subspace of the domain.

The description of $L'$, its relation to $L$ and the definition of $L_k$ occupy this subsection. The description requires the six steps that follow. In these steps, the existence of a chosen, section, c, of (2.15) will be assumed. Here, the vortex number $m$ is assumed non-negative. (Thus, $m = 0$ is permissable.)

*Step 1.* This step introduces four important first order differential operators on $N$. The first two operators are $\partial^H$ and $\bar\partial^H$. The former takes a function on $N$ to a section of $\pi^*T^{1,0}C$; it is simply the horizontal projection of the $J_0$-holomorphic part of the exterior derivative. Thus, if $\alpha$ is a function on $N$, then $\partial^H\alpha$ is obtained by taking the (1,0) part of the $m = 0$ version of the right most expression in (3.21). The operator $\bar\partial^H$ is the complex conjugate operator which takes a function and gives the section of $\pi^*T^{0,1}C$ which is the horizontal projection of the $J_0$-version of the d-bar operator on $N$. Thus, $\bar\partial^H\alpha$ is obtained from the $m = 0$ version of the right most expression in (3.21) by taking only the (0, 1) part.

The remaining pair of operators are denoted by $\partial^V$ and $\bar\partial^V$, respectively. These differentiate along the fibers of $N$. To be precise, note that the tautological section $s$ is defined over $N_{(0)}$ as a section of $\pi^*N$. The section s restricts to each fiber of $N$ as a complex parameter. Use $\partial^V$ to denote the resulting derivative; that is,

$$\partial^V = \frac{\partial}{\partial s}. \tag{4.15}$$

Note that these four operators can be extended to differentiate sections of any vector bundle with connection over $N$. For example, when differentiating sections of $\pi^* N$, the connection in $\partial^V$ is trivial along the fiber. (Think of using the connection $\theta$.) When differentiating sections of $E = (\pi^* N)^m$, the connection for use in $\partial^V$ is the trivial connection plus the 1-form $\rho_r^* v$. (Thus, for large $r$, this connection on $N_{(0)}$ is very close to $a_r$ from (2.21).) In any event, be forewarned that the presence of the connection will not be noted explicitly.

*Step 2.* This step introduces an operator, $L'$, which is essentially the first order Taylor's expansion of $L$ off of $C$. The operator is defined on $N$ where it takes a section of

$$\mathcal{V}_0 = \pi^* N \oplus E \oplus \pi^* T^{0,1} C \oplus (E \otimes \pi^* N \otimes \pi^* T^{0,1} C) \tag{4.16 a}$$

to a section of

$$\mathcal{V}_1 = \varepsilon_{\mathbb{C}} \oplus (E \otimes \pi^* N) \oplus (\pi^* N \otimes \pi^* T^{0,1} C) \oplus (E \otimes \pi^* T^{0,1} C). \tag{4.16 b}$$

Here, $E = (\pi^* N)^m$, and $\varepsilon_{\mathbb{C}}$ is the trivial complex line bundle. Let $t = (a_V, \alpha', a_C, \beta') \in C^\infty(\mathcal{V}_0)$. Then, the four components of $L't$ in $C^\infty(\mathcal{V}_1)$ are:

$$\begin{aligned}
&1.\quad \partial^V a_V + \frac{\sqrt{r}}{2\sqrt{2}} \rho_r^* \bar{\tau} \alpha' + \frac{1}{2} \langle \partial^H a_C, i\omega_C \rangle. \\
&2.\quad \bar{\partial}^V \alpha' + \frac{\sqrt{r}}{2\sqrt{2}} \rho_r^* \tau a_V + \partial^H \beta' - \bar{\nu} \beta'. \\
&3.\quad \bar{\partial}^H a_V + \nu a_V + \mu \bar{a}_C - \bar{\partial}^V a_C + \frac{\sqrt{r}}{2\sqrt{2}} \rho_r^* \bar{\tau} \beta'. \\
&4.\quad \bar{\partial}^H \alpha' - \partial^V \beta' + \sqrt{r} 2\sqrt{2} \rho_r^* \tau a_C.
\end{aligned} \tag{4.17}$$

Here, $\omega_C$ is the volume form on $C$, and the brackets, $\langle\, ,\, \rangle$, denotes the inner product with respect to the metric $g_0$. Be aware that in (4.17), the covariant derivatives use the connections $\theta$ on $\pi^* N$ and $\theta_m + \rho_r^* v$ on $E$.

*Step 3.* This step introduces operators $k$ and $\mathbb{Q}$ as follows: First, define $k$ to be the operator $k = (\bar\nu\bar s + \bar\mu s)\bar\partial^V + (-\bar\nu s + \gamma\bar s)\partial^V$, where $\nu, \mu$, and $\gamma$ are defined in (2.3). Here, the covariant derivatives in question use the connection $\theta_m + \rho_r^* v$ on $E$. Meanwhile, the operator $\mathbb{Q}$ maps $C^\infty(\mathcal{V}_0)$ to $C^\infty(\mathcal{V}_1)$ by the following rule: The four components of $\mathbb{Q}\cdot t$ are:

$$(4.18)\qquad \begin{aligned} &1. \quad -\frac{1}{2}\langle ka_C,\ i\omega_C\rangle. \\ &2. \quad -k\beta'. \\ &3. \quad -\bar k a_V + \iota_+ a_C + \iota_- \bar a_C. \\ &4. \quad -\bar k\alpha'. \end{aligned}$$

Here, $\iota_\pm$ are certain specific $r$ and $(v,\tau)$-independent bundle homomorphisms whose precise form is irrelevent to the subsequent discussions.

*Step 4.* Introduce $u_r$ as in (3.13) where $(b,\lambda)$ are as described in Step 1 of Section 3c. Consider $u_r$ simultaneously as the section $(0, 0, r^{-1/2}\rho_r^* b, \rho_r^*\lambda)$ of $\mathcal{V}_0$ in (4.16a).

*Step 5.* Let $N_{(0)} \subset N$ denote the radius $\delta_0$ disk bundle. Remember that this disk bundle has an implicit identification with a tubular neighborhood of $C$ in $X$. As such, the operator $L$ in (4.3, 4) induces an operator on $N_{(0)}$. The latter operator maps sections of $iT^{0,1}N_{(0)} \oplus S_+$ to sections of $i\varepsilon \oplus i\Lambda_+ \oplus S_-$, where $\varepsilon \to N$ is the trivial real line bundle. Here, $T^{0,1}N_{(0)}$ refers to the $J$- version. Likewise, $S_+$ is given by (1.9) where $K^{-1}$ is the $J$-version of the canonical bundle of $N_{(0)}$, and $E = (\pi^* N)^m$ as a vector bundle over $N_{(0)}$. By the way, as an operator over $N_{(0)}$, the covariant derivatives in (4.3) are defined using $\underline{a}_r$ in (3.12), while $\underline{\alpha}_r$ and $\underline{\beta}_r$ in (4.3) and (4.4) are also given by (3.12).

The following lemma describes the relationship between $L$ and $L'$ :

**Lemma 4.4.** *If $r$ is sufficiently large, then there are natural isomorphisms over $N_{(0)}$ which identify the bundle $iT^{0,1}N_{(0)} \oplus S_+$ and $\mathcal{V}_0$ (see (4.16a)) and identify $i(\varepsilon \oplus \Lambda_+) \oplus S_-$ and $\mathcal{V}_1$ (see (4.16b)). And, with these identifications, $L$ on $N_{(0)}$ takes $t = (a_V, \alpha', a_C, \beta') \in C^\infty(\mathcal{V}_0)$ to*

$$Lt = L't + \mathbb{Q}t + 2\sqrt{r}\varpi(u_r, t) + \operatorname{Rem}(t)$$

*where $|\operatorname{Rem}(t)| < \zeta|s|^2(|\partial^V t| + |\bar\partial^V t|) + \zeta|s|(|\partial^H t| + |\bar\partial^H t| + |t|)$, $\zeta$ being independent of $r$, and the section $t$.*

This lemma is proved below.

*Step 6.* The operator $L_k$ maps sections of $\mathcal{V}_0$ to sections of $\mathcal{V}_1$ according to the rule:

$$L_k t = L' t + \chi_{400\delta,k}(\mathbb{Q}t + 2\sqrt{r} \cdot \varpi(u_r, t) + \mathrm{Rem}(t)), \tag{4.19}$$

where Rem is described in Lemma 4.4.

*Proof of Lemma 4.4.* The first observation is that the connection $\theta_m + \rho_r^* v$ which is used for covariant derivatives in $L'$ is, up too an exponentially small factor (the exponent is proportional to $-\sqrt{r}$), simply the connection $a_r$ (from (2.21) which is used to define $\underline{a}_r$ via (3.12). (Remember that the latter defines, with $\theta$, the covariant derivatives in $L$.) Also, the section $\rho_r^* \tau$ of $E$ is similarly close to $\alpha_r$ from (2.20).

Next, note that a section $f$ of the $J$ version of $T^{0,1}N$ can be written as

$$f_V \bar{\kappa}_1 + f_C, \tag{4.20}$$

where $f_V$ is a section of $\pi^* N$, and $f_C$ annihilates tangents to the fibers of $N$ and so is a section of $\pi^* T^* C$. Here, $\bar{\kappa}_1$ is the complex conjugate of $\kappa_1$ from (2.1). Note that $f_C$ lies in the subbundle generated by the complex conjugate of $\kappa_0$ in (2.2). That is, $f_C$ lies in a subbundle of $\pi^* T^* C$ which is isomorphic to $\pi^* T^{0,1} C$ and equals the latter on $C$.

More generally, an imaginary valued 1-form $a'$ on $N$ can be decomposed as

$$a' = a_V \bar{\kappa}_1 - \bar{a}_V \kappa_1 + a_C - \bar{a}_C, \tag{4.21}$$

where $a_V$ is a section of $\pi^* N$,and $a_C$ is a section of $\pi^* T^* C$ which lies in $T^{0,1}N$. The assignment of $a'$ in (4.21) to the pair $(a_V, a_C)$ defines the identification between $iT^* N_{(0)}$ and $\pi^* N \oplus \pi^* T^{0,1} C$ and thus completes the identification between the domain of $L$ and the domain of $L'$.

The identification between $i \cdot (\varepsilon \oplus \Lambda_+) \oplus S_-$ and $\mathcal{V}_1$ comes about as follows: First of all, use (1.9) to identify $S_-$ with $T^{0,1} N_{(0)}$, and then use (4.20) to identify the latter with a direct sum of line bundles. This explains the second and fourth summands in (4.16b). The first and third summands are obtained with the identification of $i(\varepsilon \oplus \Lambda_+)$ with the complex bundle $\varepsilon_{\mathbb{C}} \oplus K^{-1}$. Here, the real part of $\varepsilon_{\mathbb{C}}$ is identified with $-i\,\mathrm{span}(\omega)$, and the imaginary part with $i\varepsilon$. Also, the identification of $T^{0,1} N_{(0)}$ with $\pi^* N \oplus \pi^* T^{0,1} C$ induces the identification of the $J$-version of $K^{-1}$ with $\pi^* N \oplus \pi^* T^{0,1} C$.

Given the preceding, the proof of (4.19) is obtained as follows: First, the term $2\sqrt{r}\varpi(u_r,\cdot)$ in (4.19) appears when $(\underline{a}_r,(\underline{\alpha}_r,\underline{\beta}_r))$ is written explicitly as in (3.12) in terms of $(a_r,\alpha_r)$ and $u_r$. With the $u_r$ contribution to $L$ written explicitly, consider now the operator in (4.3) and (4.4) as defined with $a = a_r$, $\alpha = \alpha_r$ and $\beta = 0$. Let $\nabla$ denote the covariant derivative as defined by the connection $a_r$. This $\nabla$-covariant derivative of a section $\alpha'$ of the bundle over $N$ can be written as

$$\nabla\alpha' = \varsigma^{-1}(\bar\partial'^V\alpha')\bar\kappa_1 + \varsigma^{-1}(\partial'^V\alpha')\kappa_1 + \bar\partial'^H\alpha' + \partial'^H\alpha'. \tag{4.22}$$

Here, $\bar\partial'^H$ is the projection of the $J$-anti-holomorphic part of the covariant derivative along $\pi^*T^*C$, while $\partial'^H$ is the projection of the $J$-holomorphic part along $\pi^*T^*C$. Also, $\varsigma$ is the function which appears in (2.3). (Thus, $\varsigma = 1 + \mathcal{O}(|s|^2)$.) Finally, the operator $\partial'^V$ is defined like $\partial^V$ but with the connection on $E$ taken to be $a_r$ instead of $\theta_m + \rho_r^*v$.

With the preceding understood, note that the bottom component of the $(a,(\alpha,\beta)) = (a_r,(\alpha_r,0))$ version of (4.3), can be decomposed into two equations by projecting onto the spans of the forms $\kappa_0$ and $\kappa_1$ in (2.1) and (2.2). The resulting two equations have the following form:

$$\begin{aligned} &1.\quad \bar\partial'^H\alpha' - \varsigma^{-1}\partial'^V\beta' + \frac{\sqrt{r}}{2\sqrt{2}}\alpha_r a_C,\\ &2.\quad \varsigma^{-1}\bar\partial'^V\alpha' + \frac{\sqrt{r}}{2\sqrt{2}}\alpha_r a_V + \partial'^H\beta'. \end{aligned} \tag{4.23}$$

The second and fourth lines of (4.17) follow from (4.23) by introducing the Taylor's expansion of the function $\varsigma$ in (2.1), and by writing the operators $\partial'^H$ and $\partial'^V$ in terms of $\partial^H$, $\partial^V$, their complex conjugates and Taylor's expanding the coefficients. On sections of $E$, one finds that $\partial'^H = \partial^H - k$ plus terms which contribute to Rem. On sections of $E\otimes\pi^*N\otimes\pi^*T^{0,1}C$, one finds that $\partial'^H = \partial^H - k - \bar\nu$ plus terms which contribute to Rem. Meanwhile, $\partial'^V$ and $\partial^V$ are exponentially close at large $r$ (the exponent is proportional to $-\sqrt{r}$).

The first and third equations in (4.17) come from the top equation in the $(a_r,(\alpha_r,0))$ version of (4.3) and (4.4). To see this, first write $a'$ as in (4.21). Then, the exterior derivative $da'$ has the expansion

$$\begin{aligned} da' =&\varsigma^{-1}(\partial'^V a_V + \bar\partial'^V\bar a_V)\kappa_1\wedge\bar\kappa_1 + (\partial'^H a_V + \varsigma^{-1}\bar\partial'^V\bar a_C)\wedge\bar\kappa_1\\ &+ (\bar\partial'^H a_V - \varsigma^{-1}\bar\partial'^V a_C)\wedge\bar\kappa_1 + a_V d\bar\kappa_1\\ &- \bar a_V d\kappa_1 + (\partial'^H a_C - \bar\partial'^H\bar a_C) - (\partial'^H\bar a_V - \varsigma^{-1}\partial'^V\bar a_C)\wedge\kappa_1\\ &- (\bar\partial'^H\bar a_V + \varsigma^{-1}\partial'^V a_C)\wedge\kappa_1. \end{aligned} \tag{4.24}$$

Here, the covariant derivatives of the section $a_V$ of $\pi^*N$ and those of its complex conjugate are defined using the connection $\theta$.

It follows from (4.24) that the top component of (4.3) can be written as two equations. The first of these equations is obtained by projecting onto $\Lambda^2 T^{0,1}$ and gives the third equation in (4.17) plus terms which contribute to $\mathbb{Q}$ and to Rem. The first equation is obtained by projecting along the symplectic form $\omega$. This last equation plus the appropriate multiple of (4.4) gives the first equation in (4.17) plus terms which contribute to $\mathbb{Q}$ and to Rem. (Remark that the $\iota_\pm$ operators in (4.18) arise from the fact that the identification of $\pi^*T^{0,1}C$ with a summand in $T^{0,1}N$ does not usually identify $\partial^V$ with $\partial'^V$.)

**e) The vertical operator**

The description above for the operator $L_k$ facilitates its description via a two part strategy. The first part (just completed) writes $L_k = L' + \chi_{400\delta,k}(\mathbb{Q} + 2\cdot\sqrt{r}\cdot\varpi(u_r,\cdot) + \mathrm{Rem})$. This subsection focuses attention on $L'$. The vector bundle structure of $N$ will be used to decompose $L'$ into what will be called its vertical and horizontal pieces. (This second part of the strategy is, essentially, separation of variables.) The vertical piece of $L'$ differentiates only along the fiber of $N$ and thus defines a family of operators on $\mathbb{C}$ which are parameterized by the curve $C$. The horizontal piece of $L'$ contains all of the derivatives along horizontal directions in $N$.

With the preceding understood, the purpose of this section is to analyze the vertical piece of $L'$. This vertical part will be denoted by $M$. It has the same domain and range as does $L'$ and is defined as follows: Let $t = (a_C, \alpha', a_C, \beta') \in C^\infty(\mathcal{V}_0)$. Then, the four components of $M \cdot t$ in $C^\infty(\mathcal{V}_1)$ are

$$
(4.25)\qquad
\begin{aligned}
&1. \qquad \partial^V a_V + \frac{\sqrt{r}}{2\sqrt{2}}\rho_r^*\bar{\tau}\alpha',\\
&2. \qquad \bar{\partial}^V \alpha' + \frac{\sqrt{r}}{2\sqrt{2}}\rho_r^*\tau a_V,\\
&3. \qquad -\bar{\partial}^V a_C + \frac{\sqrt{r}}{2\sqrt{2}}\rho_r^*\bar{\tau}\beta',\\
&4. \qquad -\partial^V \beta' + \frac{\sqrt{r}}{2\sqrt{2}}\rho_r^*\tau a_C.
\end{aligned}
$$

Note that this operator, as advertised, differentiates only along the fibers of $N$.

Use the hermitian metric on $N$ (as a vector bundle over $C$) to define a Riemannian metric for $N$. (Use the connection $\theta$ on $N$ to split $TN = \pi^*TC \oplus \pi^*N$ as a sum of 2-plane bundles with metrics. Note that this metric restricts to each fiber as the standard Euclidean metric on $\mathbb{C}$.) Use the preceding metric to define the formal $L^2$ adjoint of $M$. This adjoint is the operator $M^\dagger : C^\infty(\mathcal{V}_1) \to C^\infty(\mathcal{V}_0)$ which sends $h = (b_0, \lambda_0, b_1, \lambda_1)$ to

$$
(4.26)\qquad
\begin{aligned}
&1. \qquad -\bar{\partial}^V b_0 + \frac{\sqrt{r}}{2\sqrt{2}}\rho_r^*\bar{\tau}\lambda_0,\\
&2. \qquad -\partial^V \lambda_0 + \frac{\sqrt{r}}{2\sqrt{2}}\rho_r^*\tau b_0,\\
&3. \qquad \partial^V b_1 + \frac{\sqrt{r}}{2\sqrt{2}}\rho_r^*\bar{\tau}\lambda_1,\\
&4. \qquad \bar{\partial}^V \lambda_1 + \frac{\sqrt{r}}{2\sqrt{2}}\rho_r^*\tau b_1.
\end{aligned}
$$

The following three lemmas describe the operators $M$ and $M^\dagger$. In these lemmas and subsequently, the kernel of $M$ and $M^\dagger$ on a particular fiber of $N$ consists always of square integrable data along the fiber. As above, the Riemannian metric for $N$ is that which is induced by the hermitian structure of $N$ as a vector bundle over $C$.

**Lemma 4.5** *Suppose that $r \geq 1$. Then, for each $z \in C$, the kernel of $M|_z$ is an $m$-dimensional complex vector space $K_{0,z}$ of elements of the form $(a_V, \alpha', 0, 0)$. Meanwhile, the kernel of $M^\dagger|_z$ is an $m$-dimensional vector space, $K_{1,z}$, consisting of elements of the form $(0, 0, b_1, \lambda_1)$. Furthermore, as $z$ varies in $C$, these vector spaces fit together to define the vector bundles $K_0 \to C$ and $K_1 \to C$ which are naturally isomorphic to Section 3's bundles $V^c$ and $V^c \otimes T^{0,1}C$, respectively.*

Note that a point in $K_0$ has a dual interpretation. On the one hand, a point in $K_0$ is simply a point in a vector bundle over $C$. On the other hand, by definition, such a point is a particular section of (4.16a) over a fiber of $\pi : N \to C$. To distinguish these two roles, the section of (4.16a) which corresponds to the point $w \in K_0$ will be denoted by $\underline{w}$. Similarly, a point $w \in K_1$ defines a particular section, denoted by $\underline{w}$, over a fiber of $N$ of (4.16b).

The next lemma requires the introduction of the covariant derivative $\nabla^V$ along the fiber of $N$. (This is defined using the connection $\theta_m + \rho_r^* v$

for sections of $E$, and the Euclidean Levi-Civita connection on the fiber of $N$ for sections of $T^*N$.)

**Lemma 4.6.** *There is a constant $\zeta > 0$ which is independent of $r \geq 1$ and has the following significance: Let $z \in C$. Then*

$$||M^\dagger(h)||_z \geq \zeta(||\nabla^V h||_z + r^{1/2}||h||_z) \tag{4.27}$$

*whenever $h$ is $L^2$ orthogonal to all $\underline{w}$ coming from points $w \in K_1|_z$. Likewise, $||M(h)||_z$ is greater than $\zeta(||\nabla^V h||_z + r^{1/2}||h||_z)$ whenever $h$ is $L^2$ orthogonal to all $\underline{w}$ coming from points $w \in K_0|_z$.*

**Lemma 4.7.** *There is a constant $\zeta > 0$ which is independent of $r \geq 1$ and has the following significance: Let $z \in C$. The operator $M|_z$ is invertible on those square integrable $h$ which are $L^2$ orthogonal to all $\underline{w}$ coming from $w \in K_1|_z$. Furthermore, this inverse has the following property:*

1. $||\nabla^V(M|_z)^{-1}(h)||_z^2 + r||(M|_z)^{-1}(h)||_z^2 \leq \zeta||h||_z^2$.
2. *The $C^0$ norm of $(M|_z)^{-1}(h)$ is bounded by $\zeta r^{-1/2}(\sup|h| + r^{1/2}||h||_z)$.*
3. *Suppose that $|h| \leq \xi e^{-\sqrt{r}|s|/\xi'}$ where $\xi \geq 0$ and $\xi' \geq 1$ are constants. Then $|(M|_x)^{-1}(h)|$ is bounded by $\zeta\xi r^{-1/2}e^{-\sqrt{r}|s|/\zeta'}$ at each point of $N$, where $\zeta' = \zeta\max(1,\xi)$.*

This subsection ends with the proofs of these lemmas:

*Proof of Lemma 4.5.* The operator $M|_z$ is block diagonal in the sense that it does not mix elements of the form $(a_V, \alpha', 0, 0)$ with those of the form $(0, 0, a_C, \beta')$; and it will some times be written in the 2 by 2 block diagonal form

$$M = \begin{pmatrix} \Theta & 0 \\ 0 & \Theta^\dagger \end{pmatrix}. \tag{4.28}$$

Here, $\Theta$ on each fiber is the recaled version of the operator $\Theta_c$ that was introduced in Sections 2c and 3a. (When $(b, \lambda)$ is an ordered pair of (0,1) form on $\mathbb{C}$ and complex valued function on $\mathbb{C}$, then $\Theta_c(b, \lambda) = (\partial b + \frac{1}{2\sqrt{2}}\bar{\tau}\lambda, \bar{\partial}_v\lambda + \frac{1}{2\sqrt{2}}\tau b) = (\lambda', b')$ is an ordered pair of complex valued function on $\mathbb{C}$ and (0, 1) form on $\mathbb{C}$. With this understood, then $(r^{1/2}\rho_r^* b', \rho_r^*\lambda') = \Theta(r^{-1/2}\rho_r^* b, \rho_r^*\lambda)$.) Meanwhile, $\Theta^\dagger$ is, on each fiber, the formal $L^2$ adjoint of $\Theta$ and is a rescaled version of the operator $\Theta_c^\dagger$

from the previous section. Thus, the kernel of $\Theta$ on each fiber of $N$ is the suitably rescaled kernel of the operator $\Theta_c$, and the operator $\Theta^\dagger$ has no $L^2$ kernel on any fiber.

With the preceding understood, note for future reference that an appropriate rescaling of the Weitzenboch formulae for the operators $\Theta_c$ and $\Theta_c^\dagger$ (see (2.13)) gives the following Weitzenboch formulae for $\Theta$ and $\Theta^\dagger$ on the fiber of $N$ at any $z \in C$ :

$$
(4.29)\quad
\begin{aligned}
&1.\quad ||\Theta h||_z^2 \geq \frac{1}{4}||\nabla^V h||_z^2 + \frac{r}{8}||\rho_r^*\tau_z h||_z^2 - \zeta r||(1-\rho_1^*\tau_z)^{1/2}h||_z^2.\\
&2.\quad ||\Theta^\dagger h||_z^2 \geq \frac{1}{4}||\nabla^V h||_z^2 + \frac{r}{8}||\rho_r^*\tau_z \cdot h||_z^2.
\end{aligned}
$$

Here, the given section of $c$ of (2.15) has been written as $c = (v, \tau)$. (These last equations are valid when $||\nabla h||_z + ||h||_z$ is finite.)

*Proof of Lemma 4.6.* Note first that $M^\dagger$ is block diagonal in the sense that it does not mix elements of the form $(b_0, \lambda_0, 0, 0)$ with those of the form $(0, 0, b_1, \lambda_1)$. On the former, the operator $M^t$ acts as $\Theta^\dagger$. On the latter, it acts as $\Theta$. With this understood, the inequality in Assertion 1 follows from (4.29) and the fact that $|\tau|$ is almost equal to 1 at large distances from the origin. The inequality in the second assertion follows by a similar argument.

*Proof of Lemma 4.7.* The inverse of $M$ has the form $M^{-1} = M^\dagger(MM^\dagger)^{-1}$. The invertibility of the operator $MM^\dagger$ on the $L^2$ orthogonal compliment of $K_{1,z}$ follows from (4.29) by standard arguments, as does the $L_1^2$ estimate in the lemma. Given the $C^0$ estimate for the $r = 1$ case, the $r \geq 1$ estimates follow by rescaling the $r = 1$ estimates. Meanwhile, the $r = 1$ estimates follow by standard techniques.

### f) $\mathbf{L'}$ on $\mathbf{K_0}$'s compliment

With $M$ understood, consider now the question of inverting the operator $L'$. The lemma below summarizes what this subsection has to say about this issue. However, the statement of the lemma requires a five part digression to discuss various conventions.

*Part 1.* Below, in the statement of the lemma, and in the subsequent proof, the metric on $N$ is defined by the structure on $N$ of a hermitian vector bundle over $C$. This metric is used implicitly in the definition of the function space norms which appear, and specifically in the definition of the $L^2$ norm on $N$. (The latter is denoted by $|| \cdot ||_2$.)

*Part 2.* Covariant derivatives of sections of $E$ are defined with the aid of the connection $\theta_m + \rho_r^* v$. Covariant derivatives of $\pi^* N$ and tensor bundles are defined using $\theta$ and the metric's Levi-Civita connection respectively.

*Part 3.* Lemma 4.6 introduces the vector bundle $K_1 \to C$. By definition an element of $K_1$ at some $z \in C$ is an element in the kernel of the adjoint of the operator $M$. Similarly, a point in the vector bundle $K_0 \to C$ over some $z \in C$ is also (by definition) an element of the kernel of the operator $M$ on the fiber of $N$ at $z$. As remarked previously, this dual personality for points $w \in K_1$ (and, likewise, $K_0$) will be noted explicitly by underlining to distinguish $w$ (a point in a vector bundle over $C$) from its incarnation, $\underline{w}$, as a section over a fiber of $N$ in the kernel of some operator (either $M^\dagger$ or $M$).

*Part 4.* Vector bundle metrics are required for $K_0$ and also for $K_1$. The metric is obtained by polarizing the following fiber norm: Let $z \in C$ and suppose that $w \in K_{0,z}$ or $K_{1,z}$. Then as remarked, $w$ represents $\underline{w}$, which is a pair of 1-form and complex valued function along the fiber of $N$ at $z$. With this understood, define

$$|w| = r^{1/2} \cdot ||\underline{w}||_z, \tag{4.30}$$

the former is the norm of $w$ as a point in a vector bundle over $C$, and where the latter is the $L^2$ norm of $\underline{w}$ over the fiber of $N$ at $z$. (This norm has the property that a uniform multiple of $|w|$ bounds the supremum norm of $\underline{w}$ along the fiber of $N$ at $z$. This uniform factor is independent of $r$, but it does depend on the given section of (2.15).)

As remarked above, when $w$ is a section of $K_0$ or $K_1$, $\underline{w}$ is a section of $\mathcal{V}_0$ or $\mathcal{V}_1$, as the case may be. And, the $L^2$ norm of $w$ (as computed by an integral over $C$) is equal to $r^{1/2}$ times that $L^2$ norm of $\underline{w}$ (as computed by an integral over $N$.)

*Part 5.* Introduce $L^2(\mathcal{V}_0)$ and $L^2(\mathcal{V}_1)$ to denote the $L^2$ completions of the spaces of sections over $N$ of the vector bundles $\mathcal{V}_0$ and $\mathcal{V}_1$. Now, introduce $L^2(\mathcal{V}_0; K_0)$ to denote the (closed) subspace of $\mathcal{V}_0$ whose elements are $L^2$ orthogonal to all sections of $\mathcal{V}_0$ of the form $\underline{v}$ for $v$ a smooth section of $K_0$. Define $L^2(\mathcal{V}_1; K_1)$ similarly.

**Lemma 4.8.** *There are constants $\zeta$, $r_0 \geq 1$ which have the following significance: Suppose that $r \geq r_0$. The operator $L'$ has a partial inverse on $L^2(\mathcal{V}_1; K_1)$ in the following sense: There is a bounded operator $P$ : $L^2(\mathcal{V}_1; K_1) \to L^2_1(\mathcal{V}_0) \cap L^2(\mathcal{V}_0; K_0)$ which is an isomorphism, and is* *such that* $L'P(h) - h = \underline{w}$ *where* $w$ *is a square integrable section of* $K_1$. *Furthermore, this* $P$ *has the following properties:*

$$(4.31)\qquad \begin{array}{ll} 1. & \zeta^{-1}||h||_2^2 \leq ||\nabla P(h)||_2^2 + r||P(h)||_2^2 \leq \zeta||h||_2^2, \\ 2. & ||L'P(h) - h||_2^2 \leq \zeta r^{-1}||h||_2^2. \end{array}$$

The remainder of this subsection is occupied with the proof of Lemma 4.8.

*Proof of Lemma 4.8.* First, let $\Pi' : L^2(\mathcal{V}_0) \to L^2(\mathcal{V}_0)$ denote the $L^2$ orthogonal projection whose image is the space $\{\underline{w} : w \in K_0\}$. Thus, $\Pi'$ is defined by the family of projections $\{\Pi'_z : z \in C\}$, where $\Pi'_z$ is the finite rank projection onto the kernel of the operator $\Theta|_z$ from (4.28).

Now, fix a smooth section $h$ of $\mathcal{V}_1$. Then $P(h)$ will have the form $(1-\Pi')L'^{\dagger}u_0$ where $u_0 \in L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$. The construction of $P(h)$ plus the verification of (4.31) requires six steps.

*Step 1.* Consider minimizing the following functional on $L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$ :

$$(4.32)\qquad f(u) = 2^{-1}||(1 - \Pi')L'^{\dagger}u||_2^2 - \langle u, h\rangle_2.$$

Here, $\langle , \rangle_2$ denotes the $L^2$ pairing on $L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$. A minimum, $u_0$, of this functional in $L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$ is a weak (that is, $L^2_1$) solution of the condition that $L(1-\Pi')L^{\dagger}u_0 - h$ defines a section of $K_1$. With this understood, then $(1 - \Pi')L^{\dagger}u_0$ is a candidate for $P(h)$.

The functional $f$ is evidently bounded. Indeed, an exercise with the triangle inequality yields

$$(4.33)\qquad f(u) \leq \zeta(||\nabla u||_2^2 + r||u||_2^2 + r^{-1}||h||_2^2),$$

where $\zeta$ is independent of the choice of $u$ in $L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$. The functional $f$ is also convex, so if it has a minimum, then said minimum is unique. The existence of a minimum for $f$ then follows with the establishment of a coercive lower bound. This is a bound of the form

$$f \geq \zeta_1(||\nabla u||_2^2 + ||u||_2^2) - \zeta_2||h||_2^2$$

with $\zeta_1$ positive and with both $\zeta_{1,2}$ independent of the choice for

$$u \in L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1).$$

*Step 2.* Write $L' = T + M$ and $L'^{\dagger} = T^{\dagger} + M^{\dagger}$. The key to the required coercive lower bound for $f$ is a certain algebraic identity which is satisfied by the principal symbols of $T$ and $M$ : These symbols ($\sigma(T)$ and $\sigma(M)$, respectively) obey

$$\sigma(T)\sigma(M)^{\dagger} + \sigma(M)\sigma(T)^{t} = 0. \tag{4.34}$$

Indeed, if $M$ is written in the block diagonal form of (4.28), then the operator $T$ has the 2 by 2 block form

$$T = \begin{pmatrix} 0 & \Delta \\ -\Delta^{t} + \varphi & 0 \end{pmatrix}. \tag{4.35}$$

Here, $\Delta$ is a complex linear operator which involves horizontal differentiation, and $\Delta^{\dagger}$ is the formal $L^2$ adjoint of $\Delta$. In (4.35), the operator $\varphi$ is complex anti-linear and sends $(a_V, \alpha')$ to $(\mu \underline{a}_V, 0)$. With (4.35) understood, one can use integration by parts to establishes the existence of a constant $\zeta$ such that for all $u \in L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$,

$$||(1-\Pi')L'^{\dagger}u||_2^2 \geq ||T^{\dagger}u||_2^2 + 2^{-1}||M^{\dagger}u||_2^2 - \zeta\sqrt{r}||u||_2^2. \tag{4.36}$$

Here, $r \geq 1$ is assumed. (Note that $\zeta$ here is independent of $u$ and $r$, but depends on $c$.)

By the way, the derivation of (4.36) uses the fact that the restriction of $u$ to almost every fiber of $N$ is $L^2$ orthogonal to elements in the cokernel of $M$. Indeed, write $u = \binom{u_0}{u_1}$ corresponding to the block diagonal form in (4.35). Likewise, write $\underline{v}$ from a section $v$ of $K_0$ in block form. The latter has only a top component, $v_0$, which is annihilated on each fiber of $N$ by $\Theta$. Then, the inner product between $\underline{v}$ and $L'^{\dagger}u$ is equal to that between $v_0$ and $(-\Delta + \varphi^{\dagger})u_1$. On the other hand, $u_1$ is, by assumption, orthogonal on almost every fiber of $N$ to the kernel of $\Theta$, so $\langle v_0, (-\Delta + \varphi^{\dagger})u\rangle = \langle(-\Delta^{\dagger} + \varphi)v_0, u\rangle$.

To progress further, note that because $u$ is $L^2$ orthogonal to the sections of $K_1$, it follows that for almost all $z \in C$, $u$ must be $L^2$ orthogonal to $K_{1,z}$ along $\pi^{-1}(z)$. This implies (see Lemma 4.6) that

$$||(1-\Pi')L'^{\dagger}u||_2^2 \geq ||T^{\dagger}u||_2^2 + \zeta^{-1}(||\nabla^V u||_2^2 + r||u||_2^2), \tag{4.37}$$

when $r$ is large. The size here is independent of the particulars of $u$. Also, $\zeta > 1$ is a constant (which is different from that used previously) which is independent of $u$ and of $r$, and again, depends only on $c$. Further integration by parts in the $T^\dagger u$ term above finds a different constant $\zeta$ which is independent of $r$ such that when $r$ is large, then

$$||(1-\Pi')L'^\dagger u||_2^2 \geq \zeta^{-1}(||\nabla^H u||_2^2 + ||\nabla^V u||_2^2 + r||u||_2^2) \tag{4.38}$$

when $u \in L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$. This last inequality and the triangle inequality imply that for large $r$, there is a constant $\zeta$ such that

$$f(u) \geq \zeta^{-1}(||\nabla^H u||_2^2 + ||\nabla^V u||_2^2 + r||u||_2^2) - \zeta r^{-1}||h||_2^2, \tag{4.39}$$

when $u \in L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$. Equation (4.39) is the required coercive bound. Here, $\zeta$ is independent of $u$, and depends on $c$.

It follows from (4.39) that $f$ has a unique minimum in $L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$.

*Step 3.* Let $u \in L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$ denote the unique minimum of the functional $f$. This $u_0$ is characterized by the fact that

$$\langle L'^\dagger u, (1-\Pi')L'^\dagger u_0\rangle_2 - \langle u, h\rangle_2 = 0 \tag{4.40}$$

for all $u \in L^2(\mathcal{V}_1; K_1) \cap L^2_1(\mathcal{V}_1)$. This last equation can be used to estimate the $L^2_1$ norm of $u_0$ by choosing $u = u_0$. With this choice, (4.39) and (4.40) plus the triangle inequality imply that

$$||\nabla u_0||_2^2 + r||u_0||_2^2 \leq \zeta r^{-1}||h||_2^2. \tag{4.41}$$

*Step 4.* Equation (4.40) asserts that the projection of $L'(1-\Pi')L'^\dagger u_0$ into the space $L^2(\mathcal{V}_1; K_1)$ is equal to $h$. However,this does not yet imply that $L'(1-\Pi')L'^\dagger u_0$ is itself square integrable, as (4.40) says nothing about the projection into $L^2(K_1)$. It is the purpose of this step to prove that $L'(1-\Pi')L'^\dagger u_0$ is square integrable. In this regard, remark that, what with (4.40), it is sufficient to establish an apriori bound on the absolute value of the quantity $\langle L'^\dagger u, (1-\Pi')L'^\dagger u_0\rangle_2$ by a multiple of $||u||_2$ in the case where $u$ is a (smooth) section of $K_1$.

To consider $\langle L'^\dagger u, (1-\Pi')L'^\dagger u_0\rangle_2$ when $u$ is a section of $K_1$, remark first that, $M^t$ annihilates $u$, and so $L'^\dagger u = T^\dagger u$. Second, introduce, for each $z \in C$, the $L^2$ projection $\Pi_z$ onto $K_{1,z}$. (This is $L^2$ projection on $\pi^{-1}(z)$ from $L^2(\mathcal{V}_1|_z)$ to $K_{1,z}$.) Note that $\Pi_z u_0 = 0$ and $\Pi_z u = u$ for each $z \in C$. With the preceding understood, write

$$\begin{aligned}\langle L'^\dagger u, (1-\Pi')L'^\dagger u_0\rangle_2 &= \langle T^\dagger u, (1-\Pi')L'^\dagger u_0\rangle_2 \\ &= \langle T^\dagger \Pi_{(\cdot)} u, (1-\Pi')(T^\dagger u_0 + M^\dagger u_0)\rangle_2.\end{aligned} \tag{4.42}$$

One can then compute the commutator of $\Pi_{(\cdot)}$ with the operator $T^\dagger$. (Note that $T^\dagger$ only differentiates in horizontal directions.) The latter is an exercise, and the end result (plus (4.34) and the triangle inequality) yields:

$$
\begin{aligned}
|\langle L'^\dagger u, (1-\Pi')L'^\dagger u_0\rangle_2| &\le \zeta ||u||_2(||\nabla u_0||_2^2 + r||u_0||_2^2)^{1/2} \\
&\le \zeta' r^{-1/2}||u||_2 \cdot ||h||_2.
\end{aligned} \tag{4.43}
$$

Here, $\zeta'$ is independent of $u$ and $u_0$, and it depends only on $c$ through the maximum distance from $\tau^{-1}(0)$ to $C$. This last equation plus (4.38) imply $L'L'^\dagger u_0$ is square integrable with $L^2$ norm bounded by an $r$-independent multiple of the $L^2$ norm of $h$.

*Step 5.* Set $P(h) = (1-\Pi')L'^\dagger u_0$. It then follows from (4.38) that $P(h)$ defines an $L^2$ section of the vector bundle $\mathcal{V}_0$. One can further conclude from the previous step that $P(h)$ is in the domain of $L'$. The latter implies that $P(h)$ is an $L^2_1$ section of $\mathcal{V}_0$.

The left-hand inequality in (4.31.1) follows from the equation $(1-\Pi)L'P(h) = h$. Meanwhile, the bound by $\zeta||h||_2$ of the $L^2$ norm of $L'P(h)$ implies (with some integrating by parts and use of the triangle inequality) that the $L^2_1$ norm of $P(h)$ also obeys the righthand inequality in (4.31.1).

The inequality in (4.31.2) follows from (4.41).

*Step 6.* This last step proves that the operator $P$ is onto. For this purpose, suppose that $q \in L^2_1(\mathcal{V}_0) \cap L^2(\mathcal{V}_0; K_0)$. The claim is that $q = P((1-\Pi)L'q)$ when $r$ is suffiently large (independent of $q$). For this purpose, set $u = q - P((1-\Pi)L'q)$, and note that $u$ is in $L^2_1(\mathcal{V}_0) \cap L^2(\mathcal{V}_0; K_0)$ whereas $L'u \in L^2(K_1)$. To obtain a contradiction, one can switch the roles of $L'$ and $L'^\dagger$ in the derivation of (4.36) to find

$$0 \ge ||Tu||_2^2 + 2^{-1}||Mu||_2^2 - \zeta\sqrt{r}||u||_2^2. \tag{4.44}$$

Then, quote Lemma 4.6 to conclude that $u = 0$ when $r$ is large.

**g) $\mathbf{L}_k$ and $\Delta^c$ and sections of $\mathbf{K}_0$.**

According to Lemma 4.8 an $L^2_1$ section $q$ of $\mathcal{V}_0$ has an $L^2$ orthogonal decomposition as

$$q = P(h) + \underline{w}. \tag{4.45}$$

Here, $\underline{w}$ comes from a section, $w$, of $K_0 \to C$ and $h \in L^2(\mathcal{V}_1; K_1)$. The purpose of this subsection is to examine $L_k\underline{w}$. Lemma 4.9, below, summarizes. However, the statement of the lemma requires a preliminary, four part digression.

*Part 1.* This part concerns the fact that the bundle $K_0$ depends on $r$ as well as the chosen section $c$ of (2.15). However, the dependence on r is straightforward, as there is a fiberwise rescaling which identifies $K_0$ with the $r$-independent vector bundle $V^c \to C$ as defined in the previous subsection. (The fiber of $V^c$ at $z \in C$ consists of the space of $L^2$ solutions to (3.2) on $N|_z$.) This identification comes about as follows:

- A point $w_r$ of $K_0$ is a section $\underline{w}_r = (e_r, \gamma_r, 0, 0)$ of $\mathcal{V}_0$ over a fiber of $N$ which is annihilated by the operator $M$ in (4.28).
- $(e_r, \gamma_r)$ can be written uniquely as $(r^{-1/2}\rho_r^* e, \rho_r^*\gamma)$ where $\underline{w} = (e, \gamma)$ solves (3.2) over the same fiber, and thus defines a point, $w \in V^c$.
- If the norm of the point $w \in V^c$ over $z \in C$ is defined to be the $L^2$ norm over $N|_z$ of the corresponding $\underline{w}$, then this isomorphism between $V^c$ and $K_0$ is an isometric one. (Here, use the norm in (4.30) for $K_0$.)
- Note that a similar scaling isomorphism canonically identifies the bundle $K_1 \to C$ with the $r$-independent bundle $V^c \otimes T^{0,1}C$.

(4.46)

(And, remember that $\Upsilon_1$ from Proposition 3.2 identifies $V^c$ with $\oplus_{1\le q\le m} N^q$.)

*Part 2.* The $L^2_1$ norm on a section $w$ of $V^c$ is defined using a connection, $\nabla$, which is obtained as follows: First introduce the covariant derivative of $\underline{w} = (b, \lambda)$ by using the connections $\theta$ and $\theta_m + v$ to define the respective covariant derivatives of $b$ and $\lambda$. Second, take the horizontal projection of $\nabla\underline{w}$, and then, on each fiber of $N$, use the $L^2$ orthogonal projection to project the latter onto the kernel of $\Theta^c$. The result is a section over $C$ of $V^c \otimes T^*C$ which is, by definition, $\nabla w$. Use $||w||_{1,2}$ to denote $(||\nabla w||_2^2 + ||w||_2^2)^{1/2}$. This is equivalent to the norm obtained by identifying $V^c$ with $\oplus_{1\le q\le m} N^q$ and using the Hermitian connection on $\oplus_{1\le q\le m} N^q$ to define the norm.

*Part 3.* Remember that the norm of a section of $K_0$ is given by (4.30). Also, the $L^2_1$ norm on a section of $\mathcal{V}_0$ is defined using the connections $\theta$ and $\theta_m + \rho_r^* v$ on the appropriate summands in (4.16a).

*Part 4.* Let $c$ be a section of (2.15). Let $p$ be as in (3.7), and define $u_1 = (b, \lambda)$ by means of (3.8). Then, use the pair $(b, \lambda)$ in (3.20) to define the operator $\Delta^c$.

**Lemma 4.9.** *Let $c$ be a section of (2.15) and introduce the operator $\Delta^c$ as described above. There is a constant $\zeta > 1$ which depends on $c$, and has the following significance: Suppose that $r \geq \zeta$. Let $w$ be an $L_1^2$ section of $V^c$. Then*

$$\begin{aligned}
&\bullet\ \Pi(L'\underline{w} + \mathbb{Q}\underline{w} + 2\sqrt{r}\varpi(u_r, \underline{w})) = \underline{\Delta^c w}, \\
(4.47)\qquad &\bullet\ ||\nabla^V(1-\Pi)L_k\underline{w}||_2 + \sqrt{r}||\nabla^H(1-\Pi)L_k\underline{w}||_2 \\
&\qquad\qquad + \sqrt{r}||(1-\Pi)L_k\underline{w}||_2 \leq \zeta||w||_{1,2}.
\end{aligned}$$

*Proof of Lemma 4.9.* Both lines are obtained by direct computations. For the first line, remark that the $(b, \lambda)$ terms in (3.20) come from the term $2\sqrt{r}\varpi(u_r, \cdot)$. The remaining terms in (3.20) are derived from the left side of the first line in (4.47) with the help of the following observations: First, remember that $w$ is a section of $V^c$, which means that when $w$ is written as $(a, \alpha)$, (3.2) holds on each fiber of $N$. Second, remember that the projection $\Pi$ projects onto elements of the form $(0, 0, a, \alpha)$ in $\mathcal{V}_1$, where $(a, \alpha)$ also satisfy (3.2) on each fiber of $N$. This means that $(a, \alpha)$ is $L^2$ orthogonal on each fiber to sections which are in the image of $\Theta_c^\dagger$.

To obtain the second line of (4.47), note that the operator $(1-\Pi)L'$ acts as a zero'th order operator on sections $\underline{w}$ of $\mathcal{V}_0$ which come from $w \in V^c$. And, remember that the $L^2$ norms of $\underline{w}$ and $\nabla^H\underline{w}$ are equal to $r^{-1/2}||w||_2$ and $r^{-1/2}||\nabla w||_2$, respectively.

#### h) $\mathbf{L}_k$ reconsidered

Let $C \subset X$ be a pseudo-holomorphic submanifold, and let $m$ be a non-negative integer. Fix a section $c$ of the $(C, m)$ version of (2.15) and then, for some large value of $r$, use $c$ to define the operator $L_k$ as in Lemma 4.4. (The data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ from Sections 2 and 3b are determined completely by $c$ near $C$.) The task here is to solve the equation $L_k q^k = g^k$ for $q^k \in L_1^2(\mathcal{V}_0)$ given $g^k \in L^2(\mathcal{V}_1)$. The lemma below reduces this task to that of inverting a certain perturbation of the operator $\Delta^c$ in the case $c = c^{(k)}$.

**Lemma 4.10.** *Fix a compact set in the space of smooth sections of (2.15) for some $m \geq 0$ and there exists $\zeta > 1$ with the following significance: Choose $c$ from the given compact set and use $c$ and $r > \zeta$ to define the data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ as instructed in Sections 2 and 3b using a fixed value of $\delta < \zeta^{-1}$. Use this data to define $L_k$. Then, there exist* *three linear maps,*

$$\gamma_0 : L^2_1(V^c) \to L^2(V^c \otimes T^{0,1}C)$$

*and*

$$x, \gamma_1 : L^2(\mathcal{V}_1) \to L^2(V^c \otimes T^{0,1}C),$$

*which obey:*

- $||\gamma_0 w||_2 \leq \zeta r^{-1/2} ||w||_{1,2}$.
- $x(g)$ *is defined by the condition that* $\underline{x(g)} = \Pi g$.
- $\gamma_1$ *factors through* $(1-\Pi)$ *and* $||\gamma_1 g||_2 \leq \zeta ||(1-\Pi)g||_2$.
- *The equation* $L_k q = g$ *is solvable if and only if the equation*

$$\Delta^c w + \gamma_0(w) = x(g) + \gamma_1(g) \tag{4.48 a}$$

*is solvable. In this case, the solution* $q$ *of* $L_k q = g$ *has the form* $q = p(h) + \underline{w}$, *where* $w$ *solves (4.48). Here,* $h \in L^2(\mathcal{V}_1; K_1)$ *is a bilinear functional of* $g$ *and* $w$ *which obeys*

$$||h||_2 \leq \zeta(||(1-\Pi)g||_2 + r^{-1/2}||w||_{1,2}). \tag{4.48 b}$$

- *Write* $c = \Upsilon(y)$ *and* $w = \Upsilon_1 u$ *for* $y, u \in C^\infty(\oplus_{1\leq q\leq m} N^q)$. *Then, (4.48a) is equivalent to the following equation for* $u$ :

$$\Delta_y u + \upsilon_y(u) + \Upsilon_1^{-1}\gamma_0(\Upsilon_1 u) = \Upsilon_1^{-1}x(g) + \Upsilon_1^{-1}\gamma_1(g). \tag{4.48 c}$$

*Here,* $\Delta_y$ *is given by (3.22b), and* $\upsilon_y$ *is a bounded operator from* $L^2(\oplus_{1\leq q\leq m} N^q)$ *to* $L^2((\oplus_{1\leq q\leq m} N^q) \otimes T^{0,1}C)$ *which obeys* $||\upsilon_y(u)||_2 \leq \xi_y \cdot ||u||_2$ *where* $\xi_y$ *is proportional to the sup norm over* $C$ *of the expression on the left side of (3.22a) (or, equivalently, of (3.5).)*

The remainder of this subsection is occupied with the

*Proof of Lemma 4.10.* The equation $L_k q = g$ will be analyzed by writing $q = P(h) + \underline{w}$, where $h \in L^2(\mathcal{V}_1; K_1)$ and $\underline{w}$ comes from $w \in L^2_1(V^c)$. Then $h$ and $w$ are found as solutions of the coupled equations

$$\begin{aligned} &\bullet\ h + (1-\Pi)[\chi_{400\delta,k}(\mathbb{Q}P(h) + 2\sqrt{r}\varpi(u_r, P(h)) \\ &\qquad\qquad + \mathrm{Rem}(P(h))) + L_k\underline{w}] = (1-\Pi)g, \\ &\bullet\ \underline{\Delta^c w} + \Pi(\mathrm{Rem}(\underline{w}) + L_k P(h)) = \Pi g. \end{aligned} \tag{4.49}$$

The contraction mapping theorem will be used to find solutions to (4.49).

To consider the first line in (4.49). Introduce the self map $T$ of $L^2(\mathcal{V}_1, K_1)$ which sends $h$ to

$$T(h) = -(1-\Pi)[\chi_{400\delta,k}(\mathbb{Q}\cdot P(h) + 2\sqrt{r}\varpi(u_r, P(h)) + \mathrm{Rem}(P(h))) + L_k\underline{w}] + (1-\Pi)g. \tag{4.50}$$

Note that $T$ is an affine map and obeys

$$||T(h-h')||_2 \leq \zeta(\delta + r^{-1/2})||h||_2, \tag{4.51}$$

where $\zeta$ is determined by $c$, but is independent of $\delta$ and $r$. (This follows from Lemma 4.8.) Thus, with $c$ fixed, and then $\delta$ and $r$ taken small, the map $T$ is a contraction mapping. In particular, the first line in (4.49) has a unique solution $h = h(w, g)$ which varies in $L^2(\mathcal{V}_0, K_0) \cap L^2_1(\mathcal{V}_0)$ as a smooth function of $w \in L^2(V^c)$ and $g \in L^2(\mathcal{V}_1)$. Furthermore, this $h$ obeys (4.48b). (The term in (4.48b) with $||w||_{1,2}$ is obtained with the help of Lemma 4.9.)

With $h$ and the first line of (4.49) understood, then the first four assertions of Lemma 4.10 and (4.48a) follow from Lemmas 4.8 and 4.9 with (4.48b). Here, one must call upon the following facts:

- Given $o \in L^2(\mathcal{V}_1)$, let $x[o] \in L^2(V^c \otimes T^{0,1}C)$ be such that $\underline{x} = \Pi o$. Then rescaling identifies $||x||_2 = r^{1/2}||\Pi o||_2$, and the latter is no larger than $r^{-1/2}||o||_2$.

- Consider the previous remark with $o = L_k P(h)$. According to Lemmas 4.4 and 4.8 and (4.48b),
$$||x||_2 \leq \zeta||h||_2 \leq \zeta(||g||_2 + r^{-1/2}||w||_{1,2}).$$
Note that the arguments here use implicitly the observation that every $\underline{x}$ from $x \in V^c \otimes T^{0,1}C$ has uniform exponential decay along the fibers of $N$ (see Lemma 2.4). Thus, explicit factors of $s$ in $\mathbb{Q}$ and Rem can be traded for factors of $r^{-1/2}$ when estimating the size of $\Pi(\mathbb{Q} + \mathrm{Rem})(P(h))$.

- Consider the first remark with $o = \mathrm{Rem}(\underline{w})$. Here,
$$||x||_2 \leq \zeta r^{-1/2}||w||_2$$
which can be seen by trading the explicit factors of $|s|$ in Lemma 4.4's estimate for Rem for factors of $r^{-1/2}$ when calculating the $L^2$ norm of $\Pi\,\mathrm{Rem}(w)$.

(4.52)

The final assertion of Lemma 4.10 follows from Assertion 5 of Proposition 3.2.

### i) The operator L

This section combines the results in Subsections 4c and 4h to describe the behavior of the operator $L$ in (4.3) and (4.4) over the whole of $X$. This description is given in Lemma 4.12, below. This is to say that the lemma describes solutions $q \in i\Omega^1 \oplus C^\infty(S_+)$ to the equation $Lq = g'$ where $g'$ is specified in advance in $i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$. More generally, suppose that $\varsigma : L^2(iT^* \oplus S_+) \to L^2(i(\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-))$ is a bounded map. Lemma 4.12 also considers solutions to the equation $Lq + \varsigma q = g'$.

First comes a digression to summarize the context.

To start the digression, let $\{C_k, m_k\}$ be a finite set, where each $(C_k, m_k)$ is a pair consisting of an embedded, pseudo-holomorphic submanifold $C_k$ and a positive integer $m_k$. Here, the set $\{C_k\}$ must be pairwise disjoint. For each $k$, let $e_k$ denote the Poincaré dual to the fundamental class of $C_k$, and set $e = \Sigma_k m_k \cdot e_k$. Use $e$ to define the $\mathrm{Spin}^{\mathbb{C}}$ structure in (1.9). For each $k$, fix a compact set of sections of the $(C_k, m_k)$ version of (2.15).

**Lemma 4.11.** *Let $\varsigma$ be as described above, and write*

$$\zeta_0 = \sup_{q \neq 0} ||q||_2^{-1} ||\varsigma q||_2.$$

*The compact sets chosen above determine a constant $\zeta' > 0$, and together with $\zeta_0$, they determine a constant $\zeta > 1$ having the following significance: For each $k$, fix a section, $c^{(k)}$, of the $(C_k, m_k)$ version of (2.15) in the given compact set. Fix $\delta < \zeta'$, and then use $c^{(k)}$ and $r > \zeta$ to construct $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r)$ as in Sections 2 and 3b. Use $(a = \underline{a}_r, (\alpha = \underline{\alpha}_r, \beta = \underline{\beta}_r))$ to define the operator $L$ in (4.3) and (4.4). Then, there exist, for each $k$, linear maps $\gamma_0^k : L_1^2(\times_{k'} V^{c_{k'}}) \to L^2(V^{c_k} \otimes T^{0,1} C_k)$ and $\gamma_1^k : L^2(i(\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-) \to L^2(V^{c_k} \otimes T^{0,1} C_k)$ which obey*

- *Write $w \in \times_{k'} V^{c_{k'}}$ as $w = (w^1, \dots, w^n)$. Then*
$$||\gamma_0^k w||_2 \leq \zeta r^{-1/2} \Sigma_{k'} ||w^{k'}||_{1,2}.$$
- $||\gamma_1^k g'||_2 \leq \zeta(||(1-\Pi) g'^k||_2 + r^{-1/2} ||g'^0||_2 + r^{-1} \Sigma_{k'} ||(1-\Pi) g'^{k'}||_2)$. *Here, define $(g'^0, \{g'^k\}) \equiv (\Pi_k (1 - \chi_{25\delta,k}) g', \{\chi_{25\delta,k} g'\})$.*

- *The equation $Lq + \varsigma q = g'$ is solvable if and only if, for each $k$,*

$$\Delta_{c_k} w^k + \gamma_0^k(w) + \varsigma_k(w) = x(g'^k) + \gamma_1^k(g'). \tag{4.53 a}$$

*Here, $\varsigma_k(v) \equiv \Pi(\chi_{25\delta,k}\varsigma(\Sigma_{k'}\chi_{100\delta,k'}\underline{v}^{k'}))$.*

- *Furthermore, if $Lq + \varsigma q = g'$ is solvable, then $q$ is given by (4.6) where*

  *a) $q^0$ obeys*

$$\begin{aligned} ||\nabla q^0||_2 + &\sqrt{r}||q^0||_2 \\ &\leq \zeta(||g'^0||_2 + \Sigma_k(r^{-1/2}||w^k||_{1,2} \\ &\quad + r^{-1/2}||(1-\Pi)g'^k||_2)). \end{aligned} \tag{4.53b}$$

  *b) $q^k = P(h^k) + \underline{w}^k$ where $w^k$ solves (4.53a), and $h^k$ is in the $C_k$ version of $L^2(\mathcal{V}_1; K_1)$ and obeys*

$$||h^k||_2 \leq \zeta(||(1-\Pi)g'^k||_2 + r^{-1/2}||w^k||_{1,2} + ||q^0||_2). \tag{4.53 c}$$

  *Here, (4.53b) can be used to bound $||q_0||_2$.*

Note: When $c^{(k)} = \Upsilon(y^k)$, $w^k$ in (4.53a) can be written as $w^k = \Upsilon_1 u^k$, in which case (4.53a) is equivalent to an equation for $u^k$ which is the obvious analog of (4.49c).

The remainder of this subsection is occupied with the proof of this last lemma.

*Proof of Lemma 4.11.* The proof requires three steps. The first step proves that the equation $Lq + \varsigma q = g'$ has a solution $q$ if and only if it has a solution of the form in (4.6) where $(q^0, \{q^k\})$ obeys (4.11) with $(g^0 = \Pi_k(1-\chi_{25\delta,k})(g' - \varsigma q)$, $\{g^k = \chi_{25\delta,k}(g' - \varsigma' q)\})$. The second and third steps use Lemmas 4.4 and 4.10 plus the contraction mapping theorem to analyze (4.11).

*Step 1.* The fact that (4.11) with (4.6) implies (4.5) follows by construction. To prove that a solution $q'$ to (4.5) has the asserted form, it is sufficient to find the data $\{f^k\}$ which solve (4.13). In this regard, the existence of a unique, square integrable $f^k$ follows from the $m = 0$

case of Lemma 4.10. (When $m = 0$, $V^c$ is the zero dimensional vector bundle, and so $f^k$ can be written as $P(h)$ with $h$ given by Lemma 4.8.)

*Step 2.* This step considers the $q^0$ equation in (4.11) with the data $\{q^k\}$ as parameters. In this regard, the contraction mapping theorem (with the help of Lemma 4.3) finds a unique solution, $q^0$, which obeys (4.53b). This $q^0$ varies as bilinear function of $g^0$ and $\{q^k\}$. In fact, the derivative, $q^{0\prime}$, of $q^0$ with respect to some $q^{k\prime}$ obeys the estimate

$$||\nabla q^{0\prime}||_2 + \sqrt{r}||q^{0\prime}||_2 \leq \zeta\delta^{-1}||q^{k\prime}||_2. \tag{4.54}$$

*Step 3.* With $q^0$ now considered as a functional of the data $\{q^k\}$, turn to the second equation in (4.11). This equation will be analyzed by writing $q^k = P(h^k) + \underline{w}^k$, where $h^k$ is in the $(C_k, m_k)$ version of $L^2(\mathcal{V}_1; K_1)$, and $\underline{w}^k$ comes from $w^k$ in the $c = c^{(k)}$ version of $L^2_1(V^c)$. Then, $\{h^k\}$ and $\{w^k\}$ are found as solutions of coupled equations which are given by (4.49) with the modification that $g$ is replaced by $g'^k + \wp(d\chi_{4\delta,k})q^o - \chi_{25\delta,k}\zeta q$, and $q^0$ is considered to be an implicit function of the data $\{h^{k\prime}, w^{k'}\}$ via the analysis in Step 2. The equation for $h^k$ is analyzed with the data $\{w^{k\prime}\}$ fixed as parameters. The result (using Lemma 4.8) gives $\{h^k\}$ as an implicit function of the data $\{w^k\}$ and $g'$, and yields (4.53c).

With $\{h^k\}$ and $q^0$ now understood as functions of $\{w^k\}$ and $g'$, the derivation of (4.53a) follows as in the proof of Lemma 4.10 and is left to the reader.

### m) Pointwise estimates

The purpose of this subsection is to establish certain pointwise estimates for a solution $q$ to the equation $Lq = g$ on $X$. In particular, consider:

**Lemma 4.12.** *The conclusions of Lemma 4.11 can be amended to include the following: Suppose that $q$ solves the equation $Lq = g$, where $g \in i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$. Then*

$$\sup_X |q| \leq \zeta(r||q||_2 + r^{-1/2} \sup_X |g|). \tag{4.55}$$

Note that if $g$ obeys an exponential decay estimate away from $\cup_k C_k$ of the form $|g(x)| \leq \xi_1 \exp(-\sqrt{r}\,\mathrm{dist}(x, \cup_k C_k)/\xi_2)$ where $\xi_{1,2} > 0$, then a solution $q$ to $Lq = g$ will obey a similar estimate (with $\xi_2$ replaced by $\zeta\sup(1, \xi_2)$) and with the constant $\xi_1$ replaced as well. These sorts of estimates are proved using maximum principle arguments such as those which appear in Section 2 of [19]. The details here are omitted.

This subsection ends with the

*Proof of Lemma 4.12.* The estimate in (4.55) is obtained by considering the equation $Lq = g$ in a ball of radius $2r^{-1/2}$ about a point $x \in X$. Use Gaussian coordinates centered at $x$ to rescale the ball to the radius 2 ball about the origin in $\mathbb{R}^4$. After rescaling appropriately, $q$ and $g$ define data $\underline{q}$ and $\underline{g}$ which obey an equation of the form $\underline{Lq} = \underline{g}$ in the radius 2 ball. Here, $\underline{L}$ is an elliptic, first order operator whose coefficients and their derivatives have smooth limits as $r$ tends to $\infty$. Thus, standard estimates for the supremum norm of $\underline{q}$ in the concentric, radius 1 ball can be obtained in terms of the $L^2$ norm of $\underline{q}$ over the radius 2 ball and in terms of the supremum norm of $\underline{g}$. Rescaling back translates the bounds for $|\underline{q}|$ into bounds for $|q|$.

## 5. From almost solutions to true solutions, II

The purpose of this section is to complete the proofs of Propositions 4.1 and 4.2 by constructing the deformation map which takes certain of the approximate solutions from Sections 2 and 3b of the large $r$ version of (1.13) to honest solutions. The map $\Psi_r$ in Propositions 4.1 and 4.2 is this deformation map.

The first subsection below constitutes a digression which introduces some notions which unify the treatment of Propositions 4.1 and 4.2. (The results in this first subsection are also used in subsequent applications.)

### a) Embedding $\mathcal{Z}_0$ in a manifold

To start this subsection, return to the milieu of Section 3 where $C$ is a compact, complex curve, and $m$ is a non-negative integer. Suppose that $\pi : N \to C$ is a holomorphic, complex line bundle. Also, suppose that a pair $(\nu, \mu)$ of sections of $T^{0,1}C \oplus (N^2 \otimes T^{0,1}C)$ have been specified. This is precisely the data required for the definition of the subspace $\mathcal{Z}_0$ of (3.1). In this subsection, it proves convenient to explicitly consider $\mathcal{Z}_0$ as the subspace of sections of $\oplus_{1 \le q \le m} N^q$ which obey (3.22a). (The map $\Upsilon$ from Proposition 3.2 is used here.)

The following lemma describes certain natural embeddings of compact subspaces of $\mathcal{Z}_0$ into finite dimensional submanifolds of

$C^\infty(\oplus_{1\leq q\leq m}N^q)$. The lemma introduces the integer

$$d = 2m\cdot(n+1-g) + m\cdot(m-1),$$

where $n = \text{degree}(N)$ and $g = \text{genus}(C)$.

**Lemma 5.1.** *Let $(C, m, N, \nu, \mu)$ be as described above. Let $\mathcal{K} \subset \mathcal{Z}_0$ be a subspace with compact closure. Then, there exists a vector subspace $\Lambda \subset C^\infty((\oplus_{1\leq q\leq m}N^q)\otimes T^{0,1}C)$ which is finite dimensional such that for all $y \in \mathcal{K}$, the tautological projection of $\Lambda$ onto cokernel$(\Delta_y)$ is surjective. Furthermore, suppose that $\Lambda$ is any such subspace with this last property. Let $Q_\Lambda$ denote the associated, $L^2$-orthogonal projection. Then, there is a smooth, $d + \dim(\Lambda)$ dimensional submanifold $\mathcal{K}_\Lambda \subset C^\infty(\oplus_{1\leq q\leq m}N^q)$ with the following properties:*

- *If $y \in \mathcal{K}_\Lambda$, then*

$$(1 - Q_\Lambda)(\bar\partial y + \nu\aleph y + \mu\mathbb{F}(y)) = 0. \tag{5.1 a}$$

- *$\mathcal{K}$ embeds in $\mathcal{K}_\Lambda$ as the zero set of the map $\psi_\Lambda : \mathcal{K}_\Lambda \to \Lambda$ which sends $y$ to*

$$\psi_\Lambda(y) = Q_\Lambda(\bar\partial y + \nu\aleph y + \mu\mathbb{F}(y)). \tag{5.1 b}$$

  *$\mathcal{K}_\Lambda$ has compact closure in $C^\infty(\oplus_{1\leq q\leq m}N^q)$.*

Here are some examples: First, suppose that $\mathcal{K}$ is an open subset of $\mathcal{Z}$ with compact closure. In this case, one can take $\Lambda = \{0\}$ and then $\mathcal{K}_\Lambda = \mathcal{K}$. For a second example, fix a point $y \in \mathcal{Z}_0$ and let $\mathcal{K}$ be some sufficiently small neighborhood of $y$. Take $\Lambda = \text{cokernel}(\Delta_y)$. In this case, $\mathcal{K}_\Lambda$ contains (as an open set) an open neighborhood of $y$ in $\mathcal{T}^{-1}(0)$, where $\mathcal{T}$ is given in (3.44). Furthermore, the map $\psi_\Lambda$ in this case coincides on this neighborhood of $y$ with the map $\psi$ in (3.45).

The proof of Lemma 5.1 is given below. Consider first the following generalization of Propositions 4.1 and 4.2:

**Proposition 5.2.** *Let $E \to X$ be a complex line bundle with first Chern class $e$. Fix a finite set $\{(C_k, m_k)\}_{1\leq k\leq n}$ of pairs, where $\{C_k\}$ is a pair-wise disjoint collection of connected, pseudo-holomorphic submanifolds, and $\{m_k\}$ consists of positive integers. These are constrained so that $\Sigma_k m_k\cdot[C_k]$ is Poincaré dual to $e$. For each $k$, choose a subspace $\mathcal{K}^{(k)}$ in the $(C_k, m_k)$ version of $\mathcal{Z}_0$ which has compact closure. For each*

*$k$, choose a subspace $\Lambda_k$ in the $(C_k, m_k)$ version of $C^\infty(\oplus_{1\le q\le m} N^q)$ with the property that the projection $\Lambda_k$ onto cokernel$(\Delta_y)$ is surjective for each $y \in \mathcal{K}^{(k)}$. Then, the following hold :*

- *For each $k$, there is a submanifold $\mathcal{K}^{(k)}_\Lambda$ as in (5.1);*
- *For all large $r$, there is a smooth map $\psi_r : \times_k \mathcal{K}^{(k)}_\Lambda \to \times_k \Lambda_k$;*
- *For all large $r$, there is a smooth map*

$$\Upsilon_r : \times_k \mathcal{K}^{(k)}_\Lambda \to (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1).$$

*These have the following properties:*

1. *$\Psi_r$ maps $\psi_r^{-1}(0)$ to $\mathcal{M}^r$.*
2. *The map $\Psi_r$ is the image via the tautological projection of a map (bearing the same name) into* $\mathrm{Conn}(E) \times C^\infty(S_+)$. *The latter $\Psi_r$ is described as follows: For $y \in \times_k \mathcal{K}^{(k)}_\Lambda$, let $c = \Upsilon(y)$ denote the corresponding section of (2.15). Then, use $c$ to define the data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ as in Sections 2 and 3b. There exists an imaginary valued 1-form $a'$ and a section $(\alpha', \beta')$ of $S_+$ such that*

$$\Psi_r(y) = \left(\underline{a}_r + \frac{\sqrt{r}}{2\sqrt{2}} a', (\underline{\alpha}_r + \alpha', \underline{\beta}_r + \beta')\right). \tag{5.2}$$

3. *The data $(a', (\alpha', \beta'))$ obeys (4.1), where the constant $\zeta$ depends on $\{\Lambda_k\}$, $\{\mathcal{K}^{(k)}\}$, but not on $r$.*
4. *The map $\psi_r$ has the following property: For each $k$, let $\psi^k_\Lambda$ denote the map in (5.1b) as defined by $\Lambda = \Lambda_k$. Then $|\psi^r - \times_k \psi^k_\Lambda| + |d(\psi^r - \times_k \psi^k_\Lambda)| \le \zeta \cdot r^{-1/2}$, where $\zeta$ is as above. Here, the norm on $\times_k \Lambda_k$ is the product of the $L^2$ norms on each of the factors.*

Remark that the map $\Psi_r$ here is actually a smooth embedding. This is proved in the next section.

Note that both Propositions 4.1 and 4.2 are special cases of Proposition 5.2. The remaining subsections of Section 5 are occupied with the proof. The remainder of this first subsection contains the

*Proof of Lemma 5.1.* The existence of the required vector space $\Lambda$ follows from the assumed compactness of the closure of $\mathcal{K}$. (Note that if $\Lambda$ projects surjectively onto the cokernel of $\Delta_y$, then perturbation theory as in [6] insures that such will be the case for all $y'$ in a neighborhood of $y$.) The construction of $\mathcal{K}_\Lambda$ uses the inverse function theorem as in the proof of Assertion 1 of Proposition 3.2 as given in Section 3g. The point is that the left side of (5.1a) defines a map, $\mathcal{T}$, from $C^\infty(\oplus_{1\le q\le m}N^q)$ to $(1-Q_\Lambda)C^\infty((\oplus_{1\le q\le m}N^q)\otimes T^{0,1}C)$ whose differential is, by construction, surjective at all points of $\mathcal{K}$. This last fact (and the implicit function theorem) implies that $\mathcal{T}^{-1}(0)$ is a smooth manifold of the asserted dimension near $\mathcal{K}$. (Argue here as in the proof of Assertion 1 of Proposition 3.2.) By construction, $\mathcal{K}\subset\mathcal{K}_\Lambda$ where it is given by the zero set of $\psi_\Lambda$.

The assertion that $\mathcal{K}_\Lambda$ can be chosen to have compact closure in $C^\infty(\oplus_{1\le q\le m}N^q)$ follows from the fact that $\mathcal{K}$ is compact and that points in $\mathcal{K}_\Lambda$ obey an elliptic equation. Indeed, take $\mathcal{K}_\Lambda$ to be the intersection of the set of solutions of (5.1a) with a suitably small, open neighborhood of $\mathcal{K}$.

**b) Strategy**

The condition that the orbit in (5.2) solve the large $r$ and $\mu_0=0$ version of (1.13) translates into a non-linear, elliptic equation for $q'=(a',(\alpha',\beta'))$ which has the schematic form given in (4.2).

The strategy for solving the non-linear equation in (4.2) is modeled on that which was used to analyze the linear equation $Lq=g$ as described in Section 4b and in the subsequent parts of Section 4. In particular, the search will be for a solution $q'$ which is given by (4.6). Given such a decomposition, then (4.2) is implied by a coupled system for the data $(q^0,\{q^k\})$ which is a non-linear analog of the system in (4.11). These equations are as follow:

$$(5.3)\qquad \begin{aligned} &\bullet\ L_0q^0+\sqrt{r}\Pi_k(1-\chi_{4\delta,k})\varpi(q^0,q^0)+\Sigma_k\wp(d\chi_{100\delta,k},q^k)=0.\\ &\bullet\ L_kq^k+\chi_{100\delta,k}\sqrt{r}\varpi(q^k,q^k)+\sqrt{r}2\chi_{400\delta,k}(1-\chi_{4\delta,k})\varpi(q^k,q^0)\\ &\qquad\qquad\qquad -\wp(d\chi_{4\delta,k},q^0)+\mathrm{err}^k=0.\end{aligned}$$

In the $k'$th version of the bottom equation, the operator $L_k$ and the various bundles which are implicit in its definition are defined by the section $c=c^{(k)}$ of the $(C_k,m_k)$ version of (2.15).

Note that (4.2) is recovered from (5.3) by summing (over $k$) the product of $\chi_{100\delta,k}$ with the $k$-version of the bottom line in (5.2) and then adding $(\Pi_k(1-\chi_{4\delta,k}))$ times the top line to the result.

### c) The $\mathbf{L}_0$ equation

The purpose of this subsection is to analyze the top line in (5.3). In particular, Lemma 4.3 can be used to solve the top line of (5.3) for $q^0$ as a function of a given collection of $\{q^k\}_{k>0}$. Consider:

**Lemma 5.3.** *There exists $\delta_0$, $\varepsilon_0 > 0$ and $\zeta > 1$ with the following significance: Fix $\delta < \delta_0$ and then fix $r > \zeta$. For each $k$, let $N_{(0)k}$ denote the radius $10^3\delta$ tubular neighborhood of $C_k$. For each $k$, fix $q^k \in C^\infty(N_{(0)k}; iT^* \oplus S_+)$ with the $L^2$ norm of the section $\chi_{400\delta,k}(1-\chi_{4\delta,k})q_k$ being less than $\varepsilon_0\delta r^{-1/2}$. Then,*

1. *The top line in (5.3) has a unique solution $q^0$ which has $L^2_1$ norm less than $\zeta\varepsilon_0 r^{-1/2}$.*
2. *Let $||\cdot||_2$ indicate the $L^2$ norm. The solution $q^0$ has*

$$||\nabla q^0||_2 + \sqrt{r}||q^0||_2 \leq \zeta r^{-1/2}\delta^{-1}\Sigma_k||\chi_{400\delta,k}(1-\chi_{4\delta,k})q^k||_2, \tag{5.4}$$

3. *If $\varepsilon_n$ bounds the $C^n$ norm of each $q^k$. Then, the $C^n$ norm of $q^0$ is bounded apriori by $\zeta_n(1+\delta^{-1})^n\varepsilon_n r^{-1/2}$. Here, $\zeta_n$ is a constant which is indendent of $\{q^k\}$.*
4. *The solution $q^0$ depends smoothly on the data $\{q^k\}$. Furthermore, the directional derivative, $v^0$, of $q^0$ in the direction of a tangent vector, $v^k$, to $q^k$ obeys*

$$||\nabla v^0||_2 + \sqrt{r}||v^0||_2 \leq \zeta r^{-1/2}\delta^{-1}\Sigma_k||\chi_{200\delta,k}(1-\chi_{\delta,k})v^k||_2, \tag{5.5}$$

*Proof of Lemma 5.3.* Observe first that $q^0$ solves the top line of (5.3) if and only if $q^0$ is a fixed point of the map, $I$, from $i\Omega^1 \oplus C^\infty(S_+)$ to itself which sends $q$ to

$$I(q) = -L_0^{-1}(\sqrt{r}\Pi_k(1-\chi_{2\delta,k})\varpi(q,q) + \Sigma_k\wp(d\chi_{100\delta,k}, q^k)). \tag{5.6}$$

According to Lemma 4.3, if the $L^2_1$ norm of $q$ is bounded by, say $d$, then the $L^2_1$ norm of the image of $q$ by the map in (5.6) is bounded by

$$\zeta(r^{1/2}d^2 + \varepsilon_0 r^{-1/2}). \tag{5.7}$$

Here, $\zeta$ is a constant which is independent of $r$ and the data $\{q^k\}$ and comes via the estimate in Lemma 4.3.

It follows from (5.7) that there exists $\varepsilon_0$ such that when $d < \varepsilon_0 r^{-1/2}$, then the $L^2_1$ norm of $I(q)$ is bounded by $\varepsilon_0 r^{-1/2}$ as well. Thus, with this choice of $\varepsilon_0$, $I$ maps the ball in $i\Omega^1 \oplus C^\infty(S_+)$ of $L^2_1$ radius $\varepsilon_0 r^{-1/2}$ to itself. A similar estimate shows that $I$ is a contraction mapping on this ball. With the preceding understood, an appeal to the contraction mapping theorem to conclude that $I$ has a unique fixed point, $q^0$, in the radius $\varepsilon_0 r^{-1/2}$ ball in the Hilbert space $L^2_1(iT^* \oplus S_+)$. Straight forward elliptic regularity arguments can be employed to prove that $q^0$ is smooth and thus satisfies the first line of (5.3).

The asserted apriori estimates for the derivatives of $q^0$ follow readily from Lemma 4.3 by differentiating (5.6).

The assignment of $q^0$ to the data $\{q^k\}$ defines a map from the evident domain in the product $(\times_k C^\infty(N_{(0)k}; iT^* \oplus S_+))$ to $i\Omega^1 \oplus C^\infty(S_+)$. The fact that this map is smooth (with the $L^2_1$ topology or the $C^\infty$ topology) is a standard consequence of the contraction mapping theorem and the afore-mentioned apriori estimates for the higher derivatives of $q^0$.

**Note.** In this section, as in the previous one, the Greek letter $\zeta$ will represent a constant of size larger than 1 whose precise value may change from line to line. Unless noted explicitly, the precise value of this constant is immaterial to subsequenct discussions. What is important is its lack of dependence on data such as $r$, $\{q^k\}$. This independence should be assumed if not stated explicitly.

### d) Splitting the space of sections of $\mathcal{V}_0$

This subsection starts the analysis of the bottom line in (5.3). For this purpose, fix attention for the time being on one particular value of k, and then on the corresponding version of the bottom line in (5.3). Note that the definition of the bottom line in (5.3) requires the specification of a section, $c^{(k)}$, of the $(C_k, m_k)$ version of (2.15).

Note that $c = c^{(k)}$ defines the vector bundles such as $\mathcal{V}_{0,1}$ and $K_{0,1}$, as well as the projections $\Pi$ and $\Pi'$ which are used below. Furthermore, $c$ also defines the bundles $V^c$ and $V^c \otimes T^{0,1}C$ (for $C = C_k$), as well as the operator $\Delta^c$. In this regard, remember that $c$ is determined by a section $y$ of $\oplus_{1\le q\le m} N^q$, and from this point of view, the bundle $V^c$ is equivalent via $\Upsilon_1$ with $\oplus_{1\le q\le m} N^q$, and the operator $\Delta^c$ is equivalent to the operator $\Delta_y$ in (3.22).

The strategy for the bottom line in (5.3) decomposes the latter using the projection $\Pi$ into a pair of equations which correspond to the projection along $(1 - \Pi)$ and to the projection along $\Pi$. The resulting two equations are subsequently analyzed with the help of an analog of the decomposition $q = P(h) + \underline{w}$ in (4.45). (The analysis of (5.3) here is a non-linear analog of the decomposition in (4.49).)

The purpose of this subsection is to describe the replacement for (4.45). This is described in Lemma 5.4, below. However, a short digression is required prior to said lemma.

The digression returns to the milieu of Sections 2 and 3. To begin, fix a section $c$ of (2.15) and let $y$ denote the corresponding point in $\oplus_{1\le q\le m} N^q$. When $x \in \oplus_{1\le q\le m} N^q$, write

$$\Upsilon(y+x) = \Upsilon(y) + ((2\sqrt{2})^{-1}(b-\bar{b}), \lambda).$$

Then, introduce

$$t(x) = (r^{-1/2}\rho_r^* b, \rho_r^* \lambda, 0, 0), \tag{5.8}$$

a section of the vector bundle $\mathcal{V}_0 \to N$.

For future reference, note that when $|x| < 1$, the difference between $(b, \lambda)$ and $\Upsilon_1 x$ is bounded at any point $\eta$ in a fiber of $N$ by $\zeta_p |x|^2 e^{-|\eta|/\zeta}$. (This follows from the relationship between $\Upsilon_1$ and the differential of $\Upsilon$.) This implies that

$$|t(x) - \underline{\Upsilon}_1 \underline{x}| \le \zeta |x|^2 e^{-\sqrt{r}|s|/\zeta} \tag{5.9}$$

everywhere on $N$.

End the digression.

**Lemma 5.4.** *Given a compact set $\mathcal{N}$ of sections of $\oplus_{1\le q\le m} N^q$, there exists $\delta_0 > 0$, and given $\delta$ and less than $10^{-3}\delta_0$, there are constants $\zeta > 1$ and $\varepsilon > 0$, which have the following significance: Suppose that $y \in \mathcal{N}$ and let $c = \Upsilon(y)$ denote the resulting section of (2.15). Suppose that $r \ge \zeta$. Let $q$ be an $L^2_1$ section of $\mathcal{V}_0$. Let $w$ denote the section of $V^c$ for which the corresponding $\underline{w}$ is equal to $\Pi' q$. Suppose that $|w|$ is pointwise bounded by $\varepsilon$. Then, there is a unique pair $(x, h)$ in $(\oplus_{1\le q\le m} N^q) \times L^2(\mathcal{V}_1; K_1)$ which obeys*

$$q = P(h) + t(x). \tag{5.10}$$

*Furthermore, $x$ and $h$ obey:*

$$\begin{aligned} &\bullet\ \sup_C |x| \le \zeta \sup_C |w|, \\ &\bullet\ ||h||_2 + ||x||_2 + r^{-1/2}||\nabla x||2 \le \zeta(||\nabla q||_2 + \sqrt{r}||q||_2). \end{aligned} \tag{5.11}$$

*Proof of Lemma 5.4.* Since the composition of the differential of $\Upsilon$ with fiberwise orthogonal projection onto $V^c$ is the map $\Upsilon_1$ (an isomorphism), the implicit function theorem finds $\varepsilon > 0$ such that when $w$ is a section of $V^c$ with $|w| < \varepsilon$, there is a unique section $x$ of $\oplus_{1\le q\le m} N^q$ with $|x| \le \zeta\varepsilon$ and $\Pi^c(\Upsilon(y+x) - \Upsilon(y)) = w$. Note that this $x$ obeys $|x| \le \zeta|w|$.

With the preceding understood, write $\Pi' \cdot q = \underline{w}$ for $w \in L^2_1(V^c)$. Find the small $x$ solving $\Pi^c(\Upsilon(y+x) - \Upsilon(y)) = w$. Then $q - t(x) \in L^2_1(\mathcal{V}_0; K_0)$, so can be written uniquely as $P(h)$ according to Lemma 4.8. The estimats in the second line of (5.11) follows from those in Lemma 4.8.

**e) The non-linear problem on the compliment of $\mathbf{K}_1$**

For each $k$, fix a compact set $\mathcal{N}^{(k)}$ of the $(C_k, m_k)$ version of $\oplus_{1\le q\le m} N^q$, and then fix $y^k \in \mathcal{N}^{(k)}$. Use $y^k$ to define the section $c^{(k)}$ of the $(C_k, m_k)$ version of (2.15). When considering the $k$ version of the bottom line in (5.3), the plan will be to search for a solution $q^k$ having the form of (5.10) with $h^k \in L^2(\mathcal{V}_1; K_1)$ and $t^k = t(x^k)$ defined, as in Lemma 5.4, from a section $x^k$ of the $(C_k, m_k)$ version of $\oplus_{1\le q\le m} N^q$ which satisfies $|x^k| < \varepsilon = \varepsilon(y^k)$. (This is required to make $t(x^k)$ well defined.) Given that $q^k$ has this form, the projection of the bottom line in (5.3) along $(1-\Pi)$ will be considered as an equation that determines $h^k$ as a function of the data $\{x^k\}$.

In particular, with $q^k$ given by (5.10), the $(1-\Pi)$ projection of the bottom line of (5.3) reads as follows:

$$
\begin{aligned}
&h^k + (1-\Pi)(\chi_{400\delta,k}(\mathbb{Q}P(h) + 2\sqrt{r}\varpi(u_r, P(h)) + \mathrm{Rem}(P(h)))\\
&+ (1-\Pi)(Tt^k + \chi_{400\delta,k}(\mathbb{Q} + \mathrm{Rem})(t^k)\\
&+ 2\sqrt{r}\varpi(u_r, t^k)(1-\chi_{100\delta,k})Mt^k)\\
&+ (1-\Pi)(\chi_{100\delta,k}\sqrt{r}[2\varpi(t^k, P(h^k)) + \varpi(P(h^k), P(h^k))])\\
&+ (1-\Pi)(\sqrt{r}2\chi_{400\delta,k}(1-\chi_{4\delta,k})\varpi(t^k + P(h^k), q^0)\\
&\qquad - \wp(d\chi_{4\delta,k}, q^0) + \mathrm{err}^k) = 0.
\end{aligned}
\tag{5.12}
$$

Remember here that $q_0$ is a function of $\{q^{k>0}\}$. It is important to note that there is no term in (5.12) with $\chi_{100\delta,k}Mt^k$, nor is there a term with $\chi_{100\delta,k}\sqrt{r}\varpi(t^k, t^k)$. Indeed, these two terms cancel because of $\Upsilon(y+x)$ is automatically a section of (2.15). In fact, (5.10) with $t(x)$ is used instead of (4.45) with $\underline{w}$ for the sole purpose of cancelling these two terms. (The reader should compare (5.12) with its linear version in the top line of (4.49).)

For the time being, (5.12) should be viewed as a fixed point equation to determine $\{h^k\}$ as a function of the data $\{\mathrm{err}^k\}$ and $\{x^k\}$. In this regard, remark that for each $k$, there is a version of (5.12), and these different versions are coupled through the appearance of $q^0$.

The main result of this subsection is Lemma 5.5, below, which is an existence and uniqueness result for (5.12). In the statement of this lemma, the norm $||x||_{1,2}$ on a section $x$ of $\oplus_{1\leq q\leq m}N^q \to C$ is the $L^2_1$ norm, $||x||_{1,2} = (||\nabla x||_2^2 + ||x||_2^2)^{1/2}$. Here, $\nabla$ is the connection on $\oplus_{1\leq q\leq m}N^q$ which is defined by $\theta$.

**Lemma 5.5.** *For each $k$, fix a compact set $\mathcal{N}^{(k)}$ in the $(C_k, m_k)$ version of $\oplus_{1\leq q\leq m}N^q$. This data determines $\delta_0$, and given $\delta \in (0, 10^{-3}\cdot \delta_0)$, there are constants $\varepsilon_0 > 0$ and $\zeta > 1$ with the following significance: For each $k$, fix $y^k \in \mathcal{N}^{(k)}$ and set $c^{(k)} = \Upsilon(y^k)$. Fix $r \geq \zeta$ and suppose that for each $k$, $(1-\Pi)\,\mathrm{err}^k$ is a smooth section over the normal bundle of $C_k$ of the $k$'th version of $\mathcal{V}_1$ which has $L^2$ norm less than $\varepsilon_0 r^{-1/2}$. And, suppose that for each $k$, $x^k$ is a section of the $(C_k, m_k)$ version $\oplus_{1\leq q\leq m}N^q$ with the property that $\sup|x^k| + ||x^k||_{1,2} < \varepsilon_0$. Then, there exists a unique set $\{h^k\}$, where each $h^k$ is in the $(C_k, m_k)$ version of $L^2(\mathcal{V}_1; K_1)$, and the following two conditions are satisfied:*

- *Equation (5.12) is solved for each $k$.*
- *For each $k$, $||h^k||_2 < \zeta\varepsilon_0 r^{-1/2}$.*

*Furthermore, each $h^k$ obeys*

$$\begin{aligned} ||h^k||_2 \leq{}& \zeta(r^{-1/2}||x^k||_{1,2} + ||(1-\Pi)\,\mathrm{err}^k\,||_2 \\ &+ r^{-1}\delta^{-2}\Sigma_{k'}(r^{-1/2}||x^{k'}||_{1,2} + ||(1-\Pi)\,\mathrm{err}^{k'}\,||_2)). \end{aligned} \tag{5.13}$$

*Finally, each $h^k$ varies smoothly in the $k$'th version of $L^2(\mathcal{V}_1; K_1)$ as a function of the data $\{\mathrm{err}^{k'}\}$ and $\{x^k\}$. In fact, the derivative $h'^k$ of $h^k$ in the direction of $x'^k$ satisfies*

$$||h'^k||_2 \leq \zeta(r^{-1/2}||x'^k||_{1,2} + r^{-1}\delta^{-2}\Sigma_{k'}r^{-1/2}||x'^{k'}||_{1,2}). \tag{5.14}$$

Remark that $\mathrm{err}^k$ is given by restricting (3.6) to where the distance to $C_k$ is less than $4\cdot\delta$. In particular, because of (3.8),

$$||(1-\Pi)\cdot \mathrm{err}^k\,||_2 \leq \zeta r^{-1}, \tag{5.15}$$

where $\zeta$ depends on $\delta$ and on the compact set $\mathcal{N}^{(k)}$. Note that the $L^2$ norm of $\mathrm{err}^k$ may be only $\mathcal{O}(r^{-1/2})$.

*Proof of Lemma 5.5.* Look at the $k$'th version of (5.12), and consider the size of the various terms. To begin, note that if $h^k$ and $h'^k$ are in $L^2(\mathcal{V}_1; K_1)$, then the $L^2$ norm of $\sqrt{r}\varpi(P(h^k), P(h'^k))$ is bounded by $\sqrt{r}$ times the product of the $L^4$ norms of $P(h^k)$ and $P(h'^k)$. Meanwhile, the $L^4$ norm of $P(h^k)$ is bounded (using a well known Sobolev inequality) by an $r$-independent multiple of $||\nabla P(h^k)||_2 + ||h^k||_2$. (The fact that $N$ is non-compact has no bearing on this particular Sobolev inequality.) This last norm is bounded (courtesy of Lemma 4.8) by a uniform multiple of $||h^k||_2$. Meanwhile, the $L^2$ norms of $\mathbb{Q}P(h^k)$ and $\mathrm{Rem}(P(h^k))$ are bounded (courtesy of Lemma 4.8) by $\zeta\delta||h^k||_2$.

The bounds on the terms with $\{t^k\}$ are straightforward and left to the reader.

As for the $q^0$ term, remember that, Lemma 5.3 controls its behavior as a function of the data $\{h^{k'}\}$.

With these last points understood, the contraction mapping theorem provides $r$-independent constants $\zeta$, $\delta_0$ and $\varepsilon_0$ and simultaneous solutions $\{h^k\}$ to the $n$ versions of (5.12) (all as described by the lemma) when, for each $k$, both $||(1-\Pi)\,\mathrm{err}^k\,||_2 < \varepsilon_0 r^{-1/2}$ and also $\sup_c |x^k| + ||x^k||_{1,2} < \varepsilon_0$. The remaining assertions of the lemma also follow as consequences of the contraction mapping theorem using Lemmas 4.8, 5.3 and 5.4.

### f) The appearance of $\mathcal{Z}_0$

Suppose now that $\{c^{(k)}\}$, $\{x^k\}$ and $\mathrm{err}^k$ obey the conditions of Lemma 5.5 for all sufficiently large $r$. As instructed by Lemma 5.5, use this data to construct the set $\{h^k\}$. Thus, each $h^k$ is in the $(C_k, m_k)$ version of $L^2(\mathcal{V}_1; K_1)$, and the set $\{h^k\}$ satisfies (5.12) and (5.13) for each $k$. And, with $q^0$ given by Lemma 5.3, the first line of (5.3) is also satisfied.

If $\{q^k = t^k + P(h^k)\}$ is further required to satisfy each of the versions of bottom line in (5.3), then, for each $k$, the following additional equation must hold:

$$
\begin{aligned}
&\Pi(Tt^k + \mathbb{Q}t^k + \sqrt{r}2\varpi(u_r, t^k) + \mathrm{Rem}(t^k) + L_k P(h^k)) \\
&\qquad + \Pi\sqrt{r}(\chi_{100\delta,k}[2\varpi(t^k, P(h^k)) + \varpi(P(h^k), P(h^k))] \\
&\qquad\qquad + 2\chi_{400\delta,k}(1-\chi_{4\delta,k})\varpi(t^k + P(h^k), q^0)) \\
&\qquad + \Pi(-\wp(d\chi_{4\delta,k}, q^0) + \mathrm{err}^k) = 0
\end{aligned}
\tag{5.16}
$$

In the problem at hand, consider the data $\mathrm{err}^k$ to be a function of the section $y^k$ of $\oplus_{1\le q\le m}N^q$ (or, alternatively, as a function of the corresponding $c^{(k)}$.) Then, the data $\{h^k\}$ is implicitly a function of $\{y^k\}$ and $\{x^k\}$. Thus, (5.16) defines an equation which constrains the possible choices for the data $\{y^k\}$ and $\{x^k\}$.

The purpose of this subsection is to point out that the subvariety $\mathcal{Z}_0$ of Section 3 appears through (5.16) through an analysis of the sizes of the various terms. Indeed, the conclusion of the discussion in Sections 2 and 3b indicates that up to terms which have $L^2$ norm bounded by $\zeta r^{-1}$, the term $\Pi\,\mathrm{err}^k$ is equal to the following section of $K_1$ :

$$\Pi\bigg( -\frac{\sqrt{r}}{2\sqrt{2}}(\nu s+\mu\bar{s})(1-|\rho_r^*\tau|^2)\nabla_\theta s \\ + 2\sqrt{2}\rho_r^* v_{0,2}^1, -\sqrt{r}(\nu s+\mu\bar{s})\rho_r^*[\partial_v\tau] + \rho_r^*\tau_{0,1}^1\bigg). \tag{5.17}$$

Here, $(v,\tau) = c^{(k)}$ as defined by $y^k$.

Use (4.30) to define the $L^2$ norm of a section of $K_1 \to C$, and it follows from Lemma 5.5 that with $y^k$ and thus $c^{(k)}$ fixed, all terms in (5.16) save that from (5.17) have $L^2$ norm which is bounded for large $r$ by $\zeta\cdot(\varepsilon_0 + r^{-1/2})$. Here, $\varepsilon_0$ comes via the bound in the assumptions of Lemma 5.5 on the $L_1^2$ norm of $x^k$. Meanwhile, the $L^2$ norm of the expression in (5.17) is independent of $r$. In fact, as remarked previously (see (4.46)), the bundle $K_1$ as defined by a given section $c$ of (2.15) and $r \ge 1$ is canonically isometric to the $r$-independent bundle $V^c \otimes T^{0,1}C$, and under this isomorphism, the section of $K_1$ in (5.17) corresponds to the section in (3.5) of $V^c\otimes T^{0,1}C$. Thus, if (5.16) is to have solutions near a given $y^k$ for large $r$, it is necessary that the corresponding $c^{(k)}$ lies close to Definition 3.1's subvariety $\mathcal{Z}_0$.

### g) Rewriting the $\mathbf{K}_1$ equation

The purpose of this subsection is to begin a more detailed analysis of (5.16) by rewriting (5.16) as pair of equations using the projection $Q_\Lambda$. This rewriting of (5.16) is a five-part procedure.

*Part 1.* With the use of the rescaling isomorphism in (4.46), the left side of (5.16) defines, for each $k$, a section of the bundle $V^c\otimes T^{0,1}C_k$ for $c = c^{(k)}$. As remarked earlier, this rescaling isomorphism identifies the contribution of (5.17) with the expression in (3.5). Then, use the isomorphism $\Upsilon_1$ from Proposition 3.2 to identify the left side of (5.16) as a section, $\vartheta^k$, of the $(C_k, m_k)$ version of the bundle $\oplus_{1\le q\le m}N^q$.

*Part 2.* For each $k$, fix a subspace $\mathcal{K}^{(k)}$ in the $(C_k, m_k)$ version of $\mathcal{Z}_0$ (thought of as a subspace in the space of sections of $\oplus_{1\leq q\leq m} N^q$) with compact closure. Let $\Lambda_k$ be a finite dimensional subvector space in the $(C_k, m_k)$ version of $C^\infty(\oplus_{1\leq q\leq m} N^q)$ with the property that for all $y \in \mathcal{K}^{(k)}$, the projection of $\Lambda_k$ onto cokernel$(\Delta_y)$ is surjective. Then, let $\mathcal{K}_\Lambda^{(k)} \subset C^\infty(\oplus_{1\leq q\leq m} N^q)$ denote the submanifold which satisfies (5.1). Given $\varepsilon_1 > 0$, one can require that each $y \in \mathcal{K}_\Lambda^{(k)}$, there exists $y_0 \in \mathcal{Z}_0$ such that

$$||y - y_0||_2 < \varepsilon_1. \tag{5.18}$$

Note that elements in $\mathcal{K}_\Lambda^{(k)}$ satisfy an elliptic equation, and thus, if $\varepsilon_0$ is small (though positive), then the bound in (5.18) determines bounds for derivatives of all orders of $y - y_0$. That is, one can assume the following with no loss of generality: Given an integer $p \geq 0$, there is a constant $\zeta_p$ such that $\sup|\nabla^p(y - y_0| \leq \zeta_p \cdot \varepsilon_1$. Furthermore, $\zeta_p$ can be assumed to be independent of $y$ and $y_0$. (This last assumption exploits the fact that $\mathcal{K}^{(k)}$ is assumed to have compact closure.) An $r$-independent choice for $\varepsilon_1$ will be described below.

By a suitable choice for $\varepsilon_1$, one can assume, without loss of generality, that for each $y \in \mathcal{K}_\Lambda^{(k)}$, the operator $(1 - Q_\Lambda)\Delta_y$ is surjective onto $(1 - Q_\Lambda)C^\infty((\oplus_{1\leq q\leq m} N^q) \otimes T^{0,1}C)$ because the latter is an open condition which holds for each y in $\mathcal{K}^{(k)}$.

*Part 3.* For each $k$ and point $y \in \mathcal{K}_\Lambda^{(k)}$, let $\mathcal{L}_y^{(k)} \subset C^\infty(\oplus_{1\leq q\leq m} N^q)$ denote the kernel of the operator $(1 - Q_\Lambda)\Delta_y$, and let $(\mathcal{L}_y^{(k)})^\perp$ denote the $L^2$-orthogonal compliment of $\mathcal{L}_y^{(k)}$. Note that $\mathcal{L}_y^{(k)}$ is naturally isomorphic to the tangent space at $y$ to $\mathcal{K}_\Lambda^{(k)}$, while $(\mathcal{L}_y^{(k)})^\perp$ is naturally isomorphic to the normal bundle fiber to $\mathcal{K}_\Lambda^{(k)}$ at $y$.

*Part 4.* Given the preceding, think of the condition $\vartheta^k = 0$ (that is, Equation (5.16)) as an equation for pairs $\{(y^k, x^k)\}$, where $y^k \in \mathcal{K}_\Lambda^{(k)}$ and $x^k \in (\mathcal{L}_y^{(k)})^\perp$ for $y = y^k$. That is, use $c^{(k)} = \Upsilon(y^k)$ and $t^k = t(x^k)$ in defining (5.16) and thus $\vartheta^k$. In this regard, note that a bound of the form

$$\sup|x^k| + ||x^k||_{1,2} < \varepsilon_0, \tag{5.19}$$

for suitably small $\varepsilon_0 > 0$ must be imposed in order to make (5.16) well defined. An $r$ independent choice of $\varepsilon_0$ will be described below. (A bound as in (5.19) is required so that $h^k$ in (5.16) can be defined via Lemma 5.5.)

With the preceding understood, the condition $\vartheta^k = 0$ (which is the $k$'th version of (5.16) viewed in $\oplus_{1\leq q\leq m} N^q$) defines a pair of equations which are obtained by taking the respective $L^2$ orthogonal projections using first $(1 - Q_\Lambda)$ and then $Q_\Lambda$ with $\Lambda = \Lambda^k$. It follows from Lemma 4.9 and (3.22a,b) that the former has the schematic form

$$(1 - Q_\Lambda)\Delta_y x^k + (1 - Q_\Lambda)\mathcal{R}^k = 0, \tag{5.20}$$

where $\mathcal{R}^k$ is small (see below). Meanwhile, Lemma 4.9 implies that the $Q_\Lambda$ projection of $\vartheta^k$ has the schematic form

$$\psi_\Lambda(y^k) + Q_\Lambda(\Delta_y x^k + \mathcal{R}^k) = 0, \tag{5.21}$$

where $\psi_\Lambda(\cdot)$ is given by the left side of (5.1b). The term with $\mathcal{R}^k$ is seen below to be a small remainder.

To derive (5.20) and (5.21), remember that (5.17) translates to (3.5), and (3.5) translates to (3.22a) via the inverse of the map $\Upsilon$. Remember as well from (5.1a) that the condition that $y^k$ lie in $\mathcal{K}_\Lambda$ is the vanishing of the $(1 - Q_\Lambda)$ part of the right side of (3.22a). Also, it is important to note that the isomorphism from $K_0$ through $V^c$ to $\oplus_{1\leq q\leq m} N^q$ sends $\Pi' t(x)$ to $x + \mathcal{O}(|x|^2)$. This last fact follows from (5.10). The $\underline{\Upsilon}_1 \cdot \underline{x}^k$ part produces that $\Delta_y x^k$ contributions to (5.20) and (5.21).

**h) The equations for $\{x^k\}$**

The purpose of this subsection is to analyze (5.20) as an equation for the data $\{x^k\}$ as functions of the data $\{y^k\}$. The results of this analysis will be fed back into (5.21) to find the data $\{y^k\}$ and complete the proof of Proposition 5.2. The latter step is deferred to the next subsection. The analysis of (5.20) has three parts.

*Part 1.* This part summarizes the results of the analysis with Lemma 5.6, below.

**Lemma 5.6.** *For each $k$, fix a compact set $\mathcal{K}^{(k)}$ lying in the $(C_k, m_k)$ version of $\mathcal{Z}_0$ and fix a vector space $\Lambda_k$ so that for each $y \in \mathcal{K}^{(k)}$, the projection from $\Lambda_k$ onto cokernel$(\Delta_y)$ is surjective. There exists $\delta_0 > 0$ (which depends only on $\mathcal{K}^{(k)}$), and given $\delta \in (0, 10^{-3}\cdot\delta_0)$, there exist $\varepsilon_0 > 0$, $\varepsilon_1 > 0$ and $\zeta \geq 1$ with the following significance: For each $k$, construct the submanifold $\mathcal{K}_\Lambda^{(k)}$ so that (5.18) holds, and then fix $y^k$ in $\mathcal{K}_\Lambda^{(k)}$. Use $\{y^{k'}\}$, and $r > \zeta^2$ to define the terms in (5.20). Then the n* *versions of (5.20) have a unique solution, $x = \{x^{k'}\}$, with the following properties:*

$$\begin{aligned} &\bullet \quad \textit{Each } x^k \in (\mathcal{L}_\Lambda^{(k)})^\perp. \\ &\bullet \quad \sup_C |x^k| + ||x^k||_{1,2} < \varepsilon_0 \textit{ for all } k. \end{aligned} \tag{5.22}$$

*Furthermore, for each $k, x^k$ obeys*

$$\sup_C |x^k| + ||x^k||_{1,2} \leq \zeta r^{-1/2}. \tag{5.23}$$

*In addition, the data $\{x^k\}$ varies smoothly as a function of the data $\{y^k\}$; indeed, the $L_1^2$ norm of the differential of each $x^k$ as a function of $\{y^{k''}\}$ is bounded by $\zeta r^{-1/2}$ also.*

The proof of this lemma occupies Parts 2 and 3 of this subsection. Part 2 (which is lengthy) contains a proof of the assertion that (5.20) has at least one solution with the required properties. Then Part 3 (which is short) considers the uniqueness assertion in Lemma 5.6.

*Part 2.* The existence assertions of Lemma 5.6 are established below by rewriting (5.20) as the fixed point condition $x = (x^1, \ldots, x^n) = T(x)$ for a map $T = (T^1, \ldots, T^n)$ on a particular closed subset, $\mathcal{B}$, in a Banach space which is described below. The existence of a fixed point will be established using the contraction mapping theorem.

Note that Part 2 is subdivided into nine steps. There are seven steps to the existence argument; there is an eighth step which establishes that the solution from the previous steps satisfies (5.21); and there is a ninth step which discusses the behavior of this solution as a function of $\{y^k\}$.

*Step 1.* The Banach space in question will be a product $\times_k \mathcal{H}^k$, one for each $C_k$. To define the Banach space $\mathcal{H}^k$, let $y = y^k$. The space $\mathcal{H}^k$ is the vector space of $L_1^2$ sections of $\oplus_{1 \leq q \leq m} N^q$ which are $L^2$ orthogonal to kernel$((1 - Q_\Lambda)\Delta_y)$. The norm $||\cdot||_{1,2}$ defines the Banach space structure on $\mathcal{H}^k$. When $x = (x^1, \ldots, x^k) \in \times_k \mathcal{H}^k$, set $||x||_{1,2} = (\Sigma_k ||x^k||_{1,2}^2)^{1/2}$.

*Step 2.* This step describes the components, $\{T^k\}$, of the map $T$. In particular, note that $T^k$ is given by

$$T^k = -((1 - Q_\Lambda)\Delta_y)^{-1}((1 - Q_\Lambda)\mathcal{R}^k), \tag{5.24}$$

where $\mathcal{R}^k$ is considered as an implicit function of $\{x^{k'}\}$. Of course, the implicit assumption here (and below) is that $T$ is defined on the subspace of points $x \in \times_k \mathcal{H}^k$ which satisfy (5.19) for each $k$.

For future reference, note that $\mathcal{R}^k$ has the following schematic form:

$$\mathcal{R}^k = f^k(x^k)\nabla x^k + g^1 + g^2 + g^3 + g^4, \tag{5.25}$$

where the notation is as follows:

- The term $f^k(x^k)\nabla x^k$ corresponds to the term $T(t^k - \underline{\Upsilon}_1\underline{x}^k)$ and that part of $\mathrm{Rem}t^k$ which involves horizontal derivatives on $t^k$. (Remember that $t^k(x)$ differs from $\underline{\Upsilon}_1\underline{x}^k$ by $\mathcal{O}(|x^k|^2)$ as described in (5.10).) The part of $Tt^k$ which is linear in $x^k$ contributes to the $\Delta_y x$ term in (5.20). Note that $f^k$ is a fiber preserving map from the radius $\varepsilon_1$ ball in $\oplus_{1\le q\le m}N^q$ to $(\oplus_{1\le q\le m}N^q)\otimes T^{0,1}C$ which is smooth in its argument, and has a smooth limit as $r\to\infty$. In particular,

$$|f^k(x)| \le \zeta(|x| + r^{-1/2}) \quad\text{and}\quad |f^k_*|_x\lambda| \le \zeta|\lambda|,$$

  where $\zeta$ is independent of $x$ (with $|x| < \varepsilon_1$) and $r$. Here, $f^k_*$ denotes the differential of $f^k$.

- The term $g_1$ contains the contributions from the following terms in (5.14): First, from the term $\mathbb{Q}(t^k - \underline{\Upsilon}_1\underline{x}^k) + \sqrt{r}\varpi(u_r, t^k - \underline{\Upsilon}_1\underline{x}^k)$; second, from the terms in $\mathrm{Rem}(t^k)$ that lack horizontal derivatives on $t^k$; third, from the term $2\sqrt{r}\chi_{400\delta,k}(1-\chi_{4\delta,k})\varpi(t^k, q^0)$; and finally, from $\Upsilon_1^{-1}\Delta^c(\Upsilon_1 x) - \Delta_v x = -\upsilon_y x$, where $\upsilon_y$ is described in Proposition 3.2.

- The term $g_2$ contains the contribution from $\sqrt{r}\chi_{100\delta,k}\varpi(t^k, P(h^k))$.

- The term $g_3$ contains the contribution from $\sqrt{r}\chi_{100\delta,k}\varpi(P(h^k), P(h^k))$.

- The term $g_4$ contains all of the remaining contributions to (5.14). In particular, this term contains the terms in (5.14) with $L_k P(h^k)$, with $2\sqrt{r}\chi_{400\delta,k}(1-\chi_{4\delta,k})\varpi(P(h^k), q^0)$, with $-\wp(d\chi_{4\delta,k}, q^0)$, and with $\mathrm{err}^k$.

(5.26)

*Step 3.* The map T as defined above has the following important property:

**Lemma 5.7.** *For each $k$, fix a compact set $\mathcal{K}^{(k)}$ lying in the $(C_k, m_k)$ version of $\mathcal{Z}_0$, and fix a vector space $\Lambda_k$ so that for each $y\in\mathcal{K}^{(k)}$, the* *projection from $\Lambda_k$ onto cokernel$(\Delta_y)$ is surjective. There exists $\delta_0 > 0$ (which depends only on $\mathcal{K}^{(k)}$), and given $\delta \in (0, 10^{-3} \cdot \delta_0)$, there exist $\varepsilon > 0$, $\varepsilon_1 > 0$ and $\zeta \geq 1$ with the following significance: For each $k$, construct the submanifold $\mathcal{K}_\Lambda^{(k)}$ so that (5.18) holds, and then fix $y^k$ in $\mathcal{K}_\Lambda^{(k)}$. Use $\{y^{k'}\}$, and $r > \zeta^2$ to define the map $T$. Let $x \in \times_k \mathcal{H}^k$ obey*

- $||x||_{1,2} < \varepsilon$,
- $\sup_C |x^k| < \varepsilon$ *for each* $k$.

*Then* $||T(x)||_{1,2} \leq \zeta(\varepsilon + r^{-1/2})||x||_{1,2} + \zeta r^{-1/2}$.

*Proof of Lemma 5.7.* It is important to remember that $\Delta_y$ is elliptic (it equals $\bar{\partial}+$ zero'th order term). Thus, when $y = y^k$, there is a constant $\zeta_1 > 0$ such that if $p$ is a section of $\oplus_{1 \leq q \leq m} N^q$ which is $L^2$ orthogonal to the $\mathcal{L}_\Lambda^{(k)}$, then $||p||_{1,2} \leq \zeta_1||(1 - Q_\Lambda)\Delta_y p||_2$. This last fact implies that $||T^k(x^k)||_{1,2} \leq \zeta_1||(1 - Q_\Lambda)\mathcal{R}^k||_2$ for each $k$.

The following observations will be used to bound $||(1 - Q_\Lambda)\mathcal{R}^k||_2$ :

- The $L^2$ norm of a section $p$ of $(\oplus_{1 \leq q \leq m} N^q) \otimes T^{0,1}C$ is no greater than $\zeta r^{1/2}$ times the $L^2$ norm of the corresponding section $\underline{\Upsilon_1 p}$ of the vector bundle $\mathcal{V}_1 \to N$. (The latter bundle is defined in (4.16).)
- According to a standard Sobolev inequality, the $L^4$ norm of a section of the vector bundle $\mathcal{V}_0 \to N$ is bounded by a uniform constant times its $L^2_1$ norm.
- $||\nabla P(h^k)||_2 + \sqrt{r}||P(h^k)||_2 \leq \zeta||h^k||_2$.
- $||h^k||_2 \leq \zeta(r^{-1/2}||x||_{1,2} + r^{-1})$.
- $||q^0||_{1,2} \leq \zeta(r^{-3/2}||x||_{1,2} + r^{-2})$.
- If $w$ is a section of $V^c$ or $V^c \otimes T^{0,1}$, then
$$(1 + r^{-1/2}|s| + r^{-1}|s|^2)(|\underline{w}| + r^{-1/2}|\nabla^V \underline{w}|) \leq \zeta e^{-\sqrt{r}|s|/\zeta}$$
at points in $N$.

(5.27)

(The first point follows from (4.30). The third is a consequence of Lemma 4.8, and the fourth follows from Lemma 5.5 and (5.15). The fifth point follows from Lemma 5.3 and (5.4) with the help of the previous two points. The sixth point follows from the fact that elements in the kernel of the operator $\Theta_c$ in (2.12) decay expontentially fast to zero as $|\eta| \to \infty$ in $C$.)

With (5.27) understood, the contribution of the various terms on the right side of (5.25) to $||(1 - Q_\Lambda)\mathcal{R}^k||_2$ are as follows:

- The term $f^k(x^k)\nabla x^k$ contributes no more than $\zeta\varepsilon||x||_{1,2}$ when $\sup_C |x^k| \leq \varepsilon$.
- The term $g_1$ in (5.25) contributes no more than

$$\zeta(\varepsilon + \varepsilon_1 + r^{-1/2})||x||_{1,2}.$$

Here are the reasons: First, the contributions from the terms with

$$\mathbb{Q}(t^k - \underline{\Upsilon}_1\underline{x}^k) + \sqrt{r}\varpi(u_r, t^k - \underline{\Upsilon}_1\underline{x}^k)$$

are no more than $\zeta||x^k||_4^2 \leq \zeta\varepsilon||x^k||_{1,2}$ since $|t^k - \underline{\Upsilon}_1\underline{x}^k|$ is bounded by $|x^k|^2 e^{-\sqrt{r}|s|/\zeta}$. (Invoke the last point in (5.27)). Second, the contribution here from the term from $\mathrm{Rem}(t^k)$ is no greater than $\zeta r^{-1/2}||x||_{1,2}$ since the norm of $\mathrm{Rem}(t^k)$ is nowhere greater than $r^{-1/2}|x^k|e^{-\sqrt{r}|s|/\zeta}$. (Invoke the last point in (5.27).) Third, the contribution of the term with $q^0$ is no greater than $e^{-r/\zeta}||x||_{1,2}$. This is because $\sqrt{r}\varpi(t^k, q^0)$ is at no point ever greater than $\chi_{400\delta,k}(1 - \chi_{4\delta,k})e^{-\sqrt{r}\zeta}|x^k||q^0|$. (Invoke Points 2 and 5 of (5.27).) Fourth, because of (5.18) and Proposition 3.2, the contribution of $\upsilon_y x^y$ is no greater than $\zeta\varepsilon_1||x^k||_2$.

- The term $g_2$ contributes no more than $\zeta\varepsilon(||x||_{1,2} + r^{-1/2})$. This is because $\sqrt{r}$ times $L^2$ norm of $\sqrt{r}\varpi(t^k, P(h^k))$ is bounded by $r\sup_C(|x^k|) \cdot ||P(h^k)||_2$. And, this last expression is bounded using Points 3 and 4 of (5.27).
- The term $g_3$ in (5.25) contributes no more than

$$\zeta r||P(h^k)||_4^2 \leq \zeta r||h^k||_2^2.$$

And, this last expression is no greater than $\zeta(\varepsilon||x||_{1,2} + r^{-1})$. (Use Points 3 and 4 of (5.27).)

- The term $g_4$ in (5.25) contributes no more than

$$\zeta(r^{-1/2}||x||_{1,2} + r^{-1/2}).$$

Indeed, $\Pi \cdot L_k P(h^k)$ has $L^2$ norm bounded by $\zeta r^{-1/2}||h^k||$; the latter follows from Points 3 and 6 in (5.27) after an integration by parts. (Remember that $\Pi$ annihilates the image of the operator $M$ in (4.28). Also remember that $P(h^k)$ is $L^2$ orthogonal to the kernel of $M$ on each fiber.) Then, use Point 4 of (5.27). Meanwhile, the contribution from the term with $q^0$ can be bounded using Points 2-5 of (5.27). Finally, the contribution from the term with $\mathrm{err}^k$ is no greater than $\zeta r^{-1/2}$ because of (5.1a).

*Step 5.* This step establishes that T satisfies a certain contracting property:

**Lemma 5.8.** *For each k, fix a compact set $\mathcal{K}^{(k)}$ lying in the $(C_k, m_k)$ version of $\mathcal{Z}_0$ and fix a vector space $\Lambda_k$ so that for each $y \in \mathcal{K}^{(k)}$, the projection from $\Lambda_k$ onto cokernel$(\Delta_y)$ is surjective. There exists $\delta_0 > 0$ (which depends only on $\mathcal{K}^{(k)}$), and given $\delta \in (0, 10^{-3} \cdot \delta_0)$, then the constants $\varepsilon > 0$, $\varepsilon_1 > 0$ and $\zeta \geq 1$ in Lemma 5.7 can be chosen so that the following additional assertion can be added to those of Lemma 5.7: For each k, construct the submanifold $\mathcal{K}_\Lambda^{(k)}$ so that (5.18) holds and then fix $y^k$ in $\mathcal{K}_\Lambda^{(k)}$. Use $\{y^{k'}\}$, and $r > \zeta^2$ to define the map T. Let $x \in \times_k \mathcal{H}^k$ obey $||x||_{1,2} < \varepsilon$ and $\sup_C |x^k| < \varepsilon$ for each k. Let $x' \in \times_k \mathcal{H}^k$ obey*

- $||x'||_{1,2} < r^{-1/3}$.
- *For each k,* $\sup_C |x'^k| < \varepsilon_1$.
- *For each k, and for each $r > 0$ and for each ball $B \subset C_k$ of radius $\rho$,*

$$\int_B |\nabla x'^k|^2 \leq \varepsilon^2 \rho^{1/50}. \tag{5.28}$$

*Then* $||T(x) - T(x')||_{1,2} \leq 2^{-1}||x - x'||_{1,2}$.

Note that $x$ and $x'$ are not treated symmetrically here.

*Proof of Lemma 5.8.* As $T(x)$ and $T(x')$ are $L^2$ orthogonal to the kernel of $(1 - Q_\Lambda)\Delta_y$, the size of $||T(x) - T(x')||_2$ is no greater than $\zeta||(1 - Q_\Lambda)(\mathcal{R}^k(x) - \mathcal{R}^k(x'))||_2$. The size of the latter will be bounded by considering the contributions of the various terms in (5.25).

The first remark is that the contribution to

$$||(1 - Q_\Lambda)(\mathcal{R}^k(x) - \mathcal{R}^k(x'))||_2$$

from the terms $g_1$, $g_3$ and $g_4$ in (5.25) is no greater than

$$\zeta(\varepsilon + r^{-1/2})||x - x'||_{1,2}.$$

(With the use of (5.5), (5.14) and the chain rule, the discussion here is similar in all essential respects to the discussion above concerning the contribution of these same terms to $||(1 - Q_\Lambda)\mathcal{R}^k||_2$.)

A bound for the contribution to $||(1 - Q_\Lambda)(\mathcal{R}^k(x) - \mathcal{R}^k(x'))||_2$ from the term $f^k(x^k)\nabla x^k$ in (5.25) requires the $L^2$ norm constraint for $\nabla x'^k$ over balls in (5.28). A bound is obtained as follows: To begin, observe that

$$\begin{aligned} f^k(x^k)\nabla x^k - &f^k(x'^k)\nabla x'^k \\ &= f^k(x^k)\nabla(x^k - x'^k) \\ &\quad + (f^k(x^k) - f^k(x'^k))\nabla x'^k. \end{aligned} \tag{5.29}$$

Set $w = x^k - x'^k$, and then (5.29) and the first line of (5.26) imply that

$$\begin{aligned} &||f^k(x^k)\nabla x^k - f^k(x'^k)\nabla x'^k||_2 \\ &\qquad \le \zeta|| \, |w||\nabla x'^k| \, ||_2 + \zeta \sup_C(|x^k|)||\nabla w||_2. \end{aligned} \tag{5.30}$$

Thus, as $|x^k| < \varepsilon$ by assumption, the last term on the right side of (5.30) is bounded apriori by $\zeta\varepsilon||w||_{1,2}$.

A bound in terms of $||w||_{1,2}$ on the first term on the right side of (5.30) requires the following result from Morrey [13]:

**Lemma 5.9.** *Let $f$ be an $L^1$ function on a compact, connected, Riemann surface $C$ and let $\kappa, \lambda > 0$ be constants with the following property: Let $B \subset C$ be any geodesic ball of radius $\rho > 0$,*

$$\int_B |f| \le \kappa\rho^\lambda. \tag{5.31}$$

*Then, the assignment of $w \in L^2_1(C)$ to $\int_C |y|u^2$ defines a bounded functional on $L^2_1(C)$ which obeys*

$$\int_C |f|w^2 \le \zeta\kappa||w||^2_{1,2}. \tag{5.32}$$

*Furthermore, if $w$ has support in a ball of radius $\rho > 0$, then*

$$\int_C |f|w^2 \le \zeta\kappa\rho^{\lambda/2}||w||^2_{1,2}. \tag{5.33}$$

*Here, $\zeta$ depends only on the geometry of $C$.*

*Proof of Lemma 5.9.* This follows immediately from Lemma 5.4.1 in [13].

By this last lemma, the first term on the right side of (5.30) is also bounded by $\zeta\varepsilon||w||_{1,2}$; and therefore

$$||f^k(x^k)\nabla x^k - f^k(x'^k)\nabla x'^k||_2 \le \zeta\varepsilon||x^k - x'^k||_{1,2}. \tag{5.34}$$

Now consider the contribution of the $g_2$ term in (5.25) to the bound for $||(1-Q_\Lambda)(\mathcal{R}^k(x) - \mathcal{R}^k(x'))||_2$. It is at this point where the bound by $r^{-1/3}$ of $||x'||_2$ enters. Indeed, the $g_2$ term contributes no more than

$$\zeta r\Big[\int_N (|x^k - x'^k|^2 e^{-\sqrt{r}|s|/\zeta}|P(h^k(x'))|^2 + |x^k|^2|P(h^k(x) - h^k(x'))|^2)\Big]^{1/2} \tag{5.35}$$

to the size of $||(1-Q_\Lambda)(\mathcal{R}^k(x) - \mathcal{R}^k(x'))||_2$. Here, the dependence of Lemma 5.5's $h$ on $x$ is noted explicitly. The second term in (5.35) can be bounded using (5.14) and Lemma 4.8; and because $|x^k| < \varepsilon$, the resulting bound implies that this second term contributes no more than $\zeta\varepsilon||x^k - x'^k||_{1,2}$ to $||(1-Q_\Lambda)(\mathcal{R}^k(x) - \mathcal{R}^k(x'))||_2$.

Meanwhile, the first term in (5.35) can be bounded by

$$\begin{aligned}\zeta r^{3/4}||x^k - x'^k||_4||P(h^k(x'))||_4 \\ \le \zeta r^{1/4}||x^k - x'^k||_{1,2}(||x'||_{1,2} + r^{-1/2}).\end{aligned} \tag{5.36}$$

This last estimate uses the $L^1_2$ to $L^4$ Sobolev embeddings in dimension 2 (for the $x^k - x'^k$ part) and in dimension 4 (for the $P(h^k(x'))$ part). Points 3 and 4 of (5.27) have also been invoked.

Under the assumption that $||x'||_{1,2} \le r^{-1/3}$, the right-hand side of (5.36) is bounded by $\zeta r^{-1/12}||x^k - x'^k||_{1,2}$.

*Step 6.* This step states and proves Lemma 5.10, below, which explains when the map T preserves the conditions which are imposed in (5.28).

**Lemma 5.10.** *For each $k$, fix a compact set $\mathcal{K}^{(k)}$ lying in the $(C_k, m_k)$ version of $\mathcal{Z}_0$, and fix a vector space $\Lambda_k$ so that for each $y \in \mathcal{K}^{(k)}$, the projection from $\Lambda_k$ onto cokernel($\Delta_y$) is surjective. There exists $\delta_0 > 0$ (which depends only on $\mathcal{K}^{(k)}$), and given $\delta \in (0, 10^{-3}\cdot\delta_0)$, then the constants $\varepsilon > 0$, $\varepsilon_1 > 0$ and $\zeta \ge 1$ in Lemmas 5.7 and 5.8* *can be chosen so that the following additional assertion can be added to those of Lemmas 5.7 and 5.8: For each* $k$, *construct the submanifold* $\mathcal{K}_\Lambda^{(k)}$ *so that (5.18) holds, and then fix* $y^k$ *in* $\mathcal{K}_\Lambda^{(k)}$. *Use* $\{y^{k'}\}$, *and* $r > \zeta^2$ *to define the map* $T$. *Let* $\mathcal{B} \subset \times_k \mathcal{H}^k$ *consists of the set of* $x'$ *which obey (5.28). Then* $T$ *maps* $\mathcal{B}$ *to itself.*

*Proof of Lemma 5.10.* Lemma 5.7 asserts that $T$ maps $\mathcal{B}$ to the subset of $\times_k \mathcal{H}^k$ where the first point of (5.28) holds. Thus, it remains only to establish that $x' = T(x)$ obeys the second and third points of (5.28) when $x \in \mathcal{B}$. In this regard, note that the second point of (5.28) follows from the first and third points if $r$ is large relative to $\varepsilon_1$. To see that such is the case, (and for other reasons), it is convenient to first digress to introduce a new norm for the set of $x' \in \times_k \mathcal{H}^k$ where the third line of (5.28) holds. This norm assigns to $x'$ the number

$$p(x') = \left( \Sigma_k \sup_{\{(z,\rho) \in C \times (0,1]\}} \rho^{-1/50} \int_{B(z,\rho)} |\nabla x'^k|^2 \right)^{1/2}; \tag{5.37}$$

here $B(z, \rho) \subset C$ is the ball of radius $\rho$ and center $z$.

With the digression ended, appeal now to Theorem 3.5.2 in [13] to conclude that $x'$ which satisfies the third point in (5.28) is Hölder continuous with exponent $1/100$ and

$$\sup_{z \neq z' \in C} |x'^k(z) - x'^k(z')| \leq \zeta p(x') \operatorname{dist}(z, z')^{1/100} \tag{5.38}$$

for each $k$. In this last equation, $x'^k(z)$ is compared to $x'^k(z')$ by an implicit parallel transport of the latter from $z'$ to $z$ along a shortest geodesic using the connection $\theta$. It then follows from (5.38) that

$$\sup_C |x'^k| \leq \zeta(||x'^k||_2 p(x')^{100})^{1/101}. \tag{5.39}$$

To derive (5.39), note first that if $|x'^k(z)| > b$, then (5.38) insures that $|x'^k|$ will be greater than $b/2$ on the ball of radius $\rho = \zeta(b/p(x'))^{100}$. This implies that the $L^2$ norm of $x'^k$ will be at least as large as $\zeta b(b/p(x'))^{100}$. Equation (5.39) amounts to a rewording of this last sentence.

In the situation at hand, $||x'^k||_2 \leq r^{-1/3}$ and $p(x') < \varepsilon$ by assumption, so (5.39) implies that

$$\sup_C |x'^k| \leq \zeta(r^{-1/3}\varepsilon^{100})^{1/101} \tag{5.40}$$

for all $k$ when $x'$ obeys the first and third points in (5.26). Thus, given $\varepsilon_1$, every $x'$ in $\mathcal{B}$ will have each $x'^k$ bounded by $\varepsilon_1$ when $r$ is sufficiently large or $\varepsilon$ sufficiently small.

The fact that $x' = T(x)$ satisfies the third point of (5.28) is a direct corollary to the next lemma.

**Lemma 5.11.** *For each $k$, fix a compact set $\mathcal{K}^{(k)}$ lying in the $(C_k, m_k)$ version of $\mathcal{Z}_0$, and fix a vector space $\Lambda_k$ so that for each $y \in \mathcal{K}^{(k)}$, the projection from $\Lambda_k$ onto* cokernel$(\Delta_y)$ *is surjective. There exists $\delta_0 > 0$ (which depends only on $\mathcal{K}^{(k)}$), and given $\delta \in (0, 10^{-3}\delta_0)$, then the constants $\varepsilon > 0$, $\varepsilon_1 > 0$ and $\zeta \geq 1$ in Lemmas 5.7 and 5.8 can be chosen so that the following additional assertion can be added to those of Lemmas 5.7 and 5.8: For each $k$, construct the submanifold $\mathcal{K}^{(k)}_\Lambda$ so that (5.18) holds and then fix $y^k$ in $\mathcal{K}^{(k)}_\Lambda$. Use $\{y^{k'}\}$, and $r > \zeta^2$ to define the map $T$. Let $\mathcal{B} \subset \times_k \mathcal{H}^k$ consist of the set of $x'$ which obeys (5.28). If $x \in \mathcal{B}$, then $p(T(x)) \leq 2^{-1}p(x) + \zeta(||x||_{1,2} + r^{-1/2})$.*

*Proof of Lemma 5.11.* The proof of Lemma 5.11 uses a special case of Theorem 5.4.1 in Morrey's book [13] to obtain estimates of $p(T(x))$ from estimates of the $L^2$ norm of the function $|(1 - Q_\Lambda)\mathcal{R}^k|$ over various size balls in $C$. Here is Morrey's result:

**Lemma 5.12.** *Let $C$ be a compact, connected Riemann surface and let $B_0 \subset C$ be a geodesic ball of radius $\rho_0 > 0$ which is much smaller than the injectivity radius of $C$. Suppose that $w$ is an $L^2_1$ section over $B_0$ of $\oplus_{1\leq q\leq m} N^q$ and $\kappa, \lambda \in (0,1)$ are constants such that the following is true:*

- *The $L^2_1$ norm of $w$ over $B_0$ is bounded by $\kappa$.*
- *When $B \subset B_0$ is a concentric ball of radius $\rho < \rho_0/2$, then $\int_B |\Delta_y w|^2 \leq \kappa^2 \rho^{2\lambda}$.*

*Then, $\int_{B'} |\nabla w|^2 \leq \zeta\kappa^2\rho^{2\lambda}$ when ever $B' \subset B$ is a concentric ball of radius $\rho < \rho_0/4$. Here, $\zeta$ is a constant which depends on $\lambda$, but not on $B_0$, $w$ nor $\kappa$.*

*Proof of Lemma 5.12.* As noted, this lemma is a case of Theorem 5.4.1 in [13]. To apply Morrey's theorem, remark that Theorem 5.4.1 in [13] does not refer directly to systems of equations. However, the extension of Morrey's result to this particular system is absolutely straightforward. (Simply write the system out in components.) Remark also that Morrey's Theorem 5.4.1 refers to Equation (5.1.1) in [13]. The relevant version of Morrey's Equation (5.1.1) reads $\int \langle \Delta_y \xi, \Delta_y w + e \rangle = 0$ for all compactly supported sections $\xi$ of $V^c$ over $B_0$. Here $e = -\Delta^c w$.

It follows from Lemmas 5.7 and 5.12 that

$$p(T(x)) \leq \Sigma_k b_k + \zeta(||x||_{1,2} + r^{-1/2}), \tag{5.41}$$

where $b_k^2 = \sup_{\{(z,\rho)\in C\times(0,1]\}} \rho^{-1/50} \int_{B(z,\rho)} |(1-Q_\Lambda)\mathcal{R}^k|^2$. When deriving (5.41), write the expression $(1-Q_\Lambda)\Delta_y T$ as $\Delta_y T + \Sigma_\alpha o_\alpha \langle \Delta_y^\dagger o_\alpha, T\rangle_2$, where $\{o_\alpha\}$ is an $L^2$-orthonormal basis for $\Lambda_k$, and $\langle , \rangle$ signifies the $L^2$ norm. Then, control the contribution of this term with $\{o_\alpha\}$ using Lemma 5.7 and the fact that the basis $\{o_\alpha\}$ is a finite set of smooth sections.

The task now is to bound, for each $k$, the number $b_k$ in terms of $p(x)$ and $||x||_{1,2}$. For this purpose, fix attention on a ball $B \subset C$ of some radius $\rho > 0$. Let $\{g_j\}_{j=0,\dots,4}$ denote the various terms in (5.25). Here, $g_{i=1,\dots,4}$ are as described, while $g_0 = f^k(x^k)\nabla x^k$. Clearly,

$$\int_B |(1-Q_\Lambda)\mathcal{R}^k(x)|^2 \leq \zeta \Sigma_j \int_B |(1-Q_\Lambda)g_j|^2. \tag{5.42}$$

In all cases, one finds that

$$\rho^{-1/50}\int_B |(1-Q_\Lambda)g_i|^2 \leq \zeta(r^{-1}+\varepsilon)^{1/100}(p(x)+||x||_{1,2}+r^{-1/2})^2. \tag{5.43}$$

The derivation of (5.43) mimics the proof of Lemma 5.7 in almost all essential respects except for the case of $g_3$ (see below). Thus, the derivation is left to the reader except for the following remarks:

- The contribution to the $g_4$ term from $\mathrm{err}^k$ is handled as follows: Let $\underline{\Upsilon}_1\underline{w} = \Pi\,\mathrm{err}^k$ and let $w$ denote the corresponding section of $(\oplus_{1\leq q\leq m} N^q) \otimes T^{0,1}C$. Then $(1-Q_\Lambda)w$ is smooth and has pointwise norm which is bounded by $\zeta r^{-1/2}$. (This follows from (5.16).) Thus, the square of the $L^2$ norm of this term over a ball $B$ of radius $\rho$ is bounded by $\zeta\rho^2 r^{-1}$.

- For the rest of $g_4$, and $g_{j\neq 4}$, the derivation of (5.43) uses the fact that $Q_\Lambda$ is a finite rank projection operator whose image consists of smooth sections to conclude that

$$\int_B |(1-Q_\Lambda)g_i|^2 \leq \zeta \int_B |g_i|^2 + \zeta\rho^2||g_i||_1^2. \tag{5.44}$$

- The $L^1$ norm of $g_j$ is bounded by
$$\zeta||g_j||^2 \le \zeta(r^{-1}+\varepsilon)^{-1/100}(||x||_{1,2}+r^{-1/2}).$$
(The latter bound follows directly the arguments which prove Lemma 5.7.)

- In the case of $g_0$,
$$\begin{aligned}\int_B |f(x^k)\nabla x^k|^2 &\le \zeta\left(\sup_C |x^k|\right)^2\int_B|\nabla x^k|^2 \\ &\le \zeta\left(\sup_C |x^k|\right)^2 p(x)^2\rho^{1/50}.\end{aligned}\tag{5.45}$$
For $r>\zeta$, the latter is less than $(r^{-1}+\varepsilon)^{1/100}p(x)^2\rho^{1/50}$ because of (5.40).

- In the case of $g_2$, the integral in question is bounded by
$$\begin{aligned}&\zeta r^2\left(\sup_C |x^k|\right)^2\int_{\pi^{-1}(B)} e^{-\sqrt{r}|s|/\zeta}|P(h^k)|^2 \\ &\le \zeta\left(\sup_C |x^k|\right)^2 r^{3/2}\rho||P(h^k)||_4^2 \\ &\le \zeta\left(\sup_C |x^k|\right)^2 r^{1/2}\cdot\rho(||x||_{1,2}^2+r^{-1}).\end{aligned}\tag{5.46}$$
(The derivation of (5.46) uses (5.28).) By assumption, $||x||_{1,2}<r^{-1/3}$, so it follows from (5.39) that the far right side of (5.46) is bounded by $\zeta p(x)^{200/101}r^{-1/6}\rho$ which is consistent with (5.43).

To apply (5.44) for the case of $g_3$, it is necessary to introduce some new technology in order to obtain a useful bound for the $L^2$ norm of $g_3$ over balls. (Direct appeal to Lemma 5.5 will produce a bound on the square of the $L^2$ norm of $g_3$ over a ball $B$ by $r||P(h^k)||_4^2$, which is $\mathcal{O}(||x||_{1,2}^2+r^{-2})$. Though small, this lacks any obvious $\rho$ dependence.) A digression is required to introduce the extra technology to use for the $g_3$ case.

To start the digression, consider:

**Lemma 5.13.** *Let $C$ be a compact, connected, Riemann surface, let $W\to C$ be a Riemannian vector bundle with metric compatible connection $\nabla$. Given $\lambda\in(0,1)$, there is a constant $\zeta$ with the following* *significance: Let $g$ be an $L^2$ section of $W$ and let $x$ be a constant such that $||g||_2 \leq \kappa$ and that for any $\rho > 0$ and any radius $\rho$ ball $B \subset C$,*

$$\int_B |\nabla g| \leq \kappa \rho^\lambda. \tag{5.47}$$

*Then for any $\rho > 0$ and ball $B \subset C$ of radius $\rho$, $\int_B |g|^2 \leq \zeta \kappa^2 \rho^{2\lambda}$.*

*Proof of Lemma 5.11.* The proof starts with the following claim: There is a constant $\zeta$ which is independent of $g$ such that for any $\rho > 0$ and ball $B \subset C$ of radius $\rho$, $\int_B |g| \leq \zeta \kappa \rho^{1+\lambda}$. Given this claim, the result follows from a standard Sobolev inequality. (See, for example Theorem 3.5.3 in [13].) The claim is proved via an integration by parts. (Note that for any $z \in C$, the function $|\nabla g| \cdot \operatorname{dist}(\cdot, z)^{-\lambda}$ is integrable.)

End the digression. In the case at hand (where $g = g_3$),

$$\begin{aligned} \int_B |\nabla g_3| &\leq \zeta r^{3/2} \int_{\pi^{-1}(B)} e^{-\sqrt{r}|s|/\zeta} |\nabla P(h^k)||P(h^k)| \\ &\leq \zeta r^{3/2} ||\nabla P(h^k)||_2 \left[ \int_{\pi^{-1}(B)} e^{-\sqrt{r}|s|/\zeta} |P(h^k)|^2 \right]^{1/2}, \end{aligned} \tag{5.48}$$

and thus

$$\begin{aligned} \int_B |\nabla g| &\leq \zeta r^{5/4} \rho^{1/2} ||\nabla P(h^k)||_2 ||P(h^k)||_4 \\ &\leq \zeta r^{5/4} \rho^{1/2} ||h^k||_2^2 \\ &\leq \zeta r^{1/4} \rho^{1/2} (||x^k||_{1,2}^2 + r^{-1}). \end{aligned} \tag{5.49}$$

Because $||x||_{1,2} < r^{-1/3}$, the latter is therefore bounded by $\zeta \rho^{1/2} r^{-5/12}$.

*Step 7.* This step completes the proof of the existence assertion of Lemma 5.6. To begin the argument, reintroduce the domain $\mathcal{B}$ in the Banach space $\times_k \mathcal{H}^k$. It is an exercise to check that $\mathcal{B}$ is closed. According to Lemma 5.10, the map $T$ sends $\mathcal{B}$ to itself. According to Lemma 5.8, the map $T$ is a contraction mapping on $\mathcal{B}$. Thus, the contraction mapping theorem insures that there is a unique point $x \in \mathcal{B}$ which is a fixed point of $T$. That is, there is a unique solution $x \in \mathcal{B}$ to (5.20).

The fact that $x$ satisfies (5.22) follows from the definition of $\mathcal{B}$ and from (5.40).

*Step 8.* This step establishes that the fixed point solution $x \in \mathcal{B}$ to (5.20) satisfies the apriori estimate in (5.23). In this regard, remark that the bound on $||x||_{1,2}$ by $\zeta r^{-1/2}$ follows directly from Lemma 5.7. Meanwhile, the sup norm bound on $x^k$ can be obtained from (5.39) given that $||x||_{1,2} \leq \zeta r^{-1/2}$ and that $p(x) \leq \zeta r^{-1/2}$. This last bound on $p(x)$ follows from Lemma 5.10 in light of the established bound for $||x||_{1,2}$ by $\zeta r^{-1/2}$.

*Step 9.* This last step considers (briefly) the behavior of the solution in $\mathcal{B}$ to (5.20) as a function of the data $\{y^k\}$. In particular, the smooth dependence of $x$ on the data $\{y^k\}$ is a direct consequence of the contraction mapping theorem. (See Subsection 4a.) The apriori bound on the differential of $x$ also follows from the contraction mapping theorem and Lemma 5.8 after using the chain rule to describe the change in the map $T$ with respect to changing the data $\{x^k\}$. The latter task is tedious, but straightforward and will be omitted.

*Part 3.* The goal here is to establish the uniqueness assertion of Lemma 5.6. In this regard, suppose that the various constants $\delta$, $\varepsilon_0 = \varepsilon$, $\varepsilon_1$ and $\zeta$ are as given in Lemma 5.11. Then, Part 3 has established that there exists a solution, $x'$, to (5.20) which sits in the space $\mathcal{B}$ and which obeys the apriori estimates in (5.23). Furthermore, the norm $p(x')$ is bounded by $\zeta r^{-1/2}$. Now, let $x$ be a second solution to (5.20) which obeys the conditions in (5.22). Then, $x$ is also a fixed point of the map $T$. And, according to Lemma 5.8, the $L^2_1$ norm of $x - x'$ is no greater than half itself. This implies that $x = x'$.

### i) Proof of Proposition 5.2

There are five steps involved in the proof.

*Step 1.* For all $r$ sufficiently large, the solution $\{x^k\}$ to (5.20) from Lemma 5.6 varies smoothly as a function of the data $\{y^k\}$ as the latter varies in $\times_k \mathcal{K}^{(k)}_\Lambda$. Thus, the left side of (5.21) defines, for $r$ large, a smooth map $\psi_r : \times_k \mathcal{K}^{(k)}_\Lambda \to \times_k \Lambda_k$. By Lemmas 5.3, 5.5 and 5.6 and Proposition 3.1,

$$|\psi_r - \times_k \psi^k_\Lambda| \leq \zeta r^{-1/2}, \tag{5.50}$$

where $\psi^k_\Lambda$ is the map in (5.1b) whose zero set is homeomorphic to $\mathcal{K}^{(k)}$. The analogous bound for $|d(\psi^r - \times_k \psi^k_\Lambda)|$ by $\zeta r^{-1/2}$ follows using the Chain rule with (5.5), (5.14) and the last remark in Lemma 5.6.

*Step 2.* Meanwhile, the preceding constructions define a map $\Psi_r$ of $\times_k \mathcal{K}^{(k)}_\Lambda$ into the quotient by $C^\infty(X; S^1)$ of the space of pairs consisting of a connection on the line bundle $E$ and a section of the bundle $S_+$

as defined by $E$ in (1.9). This map is defined by first taking, for each $k$, the section $c^{(k)}$ of (2.15) to be equal to $\Upsilon(y^k)$. One then defines the image of $\Psi_r$ as $(\underline{a}_r + \frac{r}{2\sqrt{2}}a', (\underline{\alpha}_r + \alpha', \underline{\beta}_r + \beta'))$, where $q' = (a', (\alpha', \beta'))$ is given in terms of $q^0$ and $\{q^k\}$ by (4.6). Then, $q^0$ becomes an implicit function of $\{q^k\}$ through Lemma 5.3. Meanwhile, each $q^k$ is written as in $P(h^k) + t^k$, where $t^k$ is a function of $x^k$ (as described in (5.8).) Lemma 5.5 then defines $h^k$ as an implicit function of $\{y^{k'}\}$ and $\{x^{k'}\}$, and Lemma 5.6 defines $x^k$ as an implicit function of $\{y^{k'}\}$.

By construction, $\Psi_r$ maps $\{y^k\}$ into the moduli space $\mathcal{M}^{(r)}$ of $\mu_0 = 0$ solutions to (1.13) precisely when $\psi_r(\{y^k\}) = 0$.

*Step 3.* Except for (4.1.2) and (4.1.3), the assertions in Proposition 5.2 follow directly from the preceding remarks and the various estimates in Lemmas 5.3, 5.5 and 5.6 plus (5.50). (The assertion that the restriction of $\Psi_r$ is continuous follows in a straightforward manner from the properties of the contraction mapping constructions.)

*Step 4.* The last step proves the estimates in (4.1.2) and (4.1.3). The arguments here occupy the final subsection, below.

### j) Pointwise estimates

The purpose of this subsection is to establish the estimates in (4.1.2) and (4.1.3) which concern the supremum norm for $(a', (\alpha', \beta'))$ and $\nabla(a', (\alpha', \beta'))$. The proof requires three steps.

*Step 1.* Let $C$ denote one of the submanifolds from the set $\{C^k\}$, and let $m$ denote one of the corresponding $m_k$. The goal is to estimate the $L^2$ norm of the corresponding $q^k$ over balls of radius $r^{-1/2}$. In this regard, remember that $q^k = P(h^k) + t^k$, where $P(h^k)$ is obtained from Lemma 5.5, and $t^k$ is defined via Lemma 5.44 using $x^k$ from Lemma 5.5.

To estimate the $L^2$ norm of $q^k$, note first that the sup norm of $x^k$ from Lemma 5.5 is bounded by $\zeta r^{-1/2}$. This implies that the supremum norm of $t^k$ is bounded by $\zeta r^{-1/2}$, and that its $L^2$ norm is bounded apriori by $\zeta r^{-1}$. Thus, the $L^2$ norm of $t^k$ over any ball of radius $r^{-1/2}$ is bounded by $\zeta r^{-3/2}$.

To estimate the $L^2$ norm of $P(h^k)$ over radius $r^{-1/2}$ balls, first observe that the $L^2$ norm of $h^k$ is bounded by $\zeta r^{-1}$. This follows from (5.12), (5.15) and (5.22). Then by Lemma 5.5, the $L^2$ norm of $P(h^k)$ is bounded apriori by $\zeta r^{-3/2}$.

Moreover the remarks in the preceding two paragraphs imply that $||q^k||_2 \leq \zeta r^{-1}$, and that the square of the $L^2$ norm of $q^k$ in any ball $B$

of radius $r^{-1/2}$ is apriori bounded by

$$\int_B |q^k|^2 \le \zeta r^{-3}. \tag{5.51}$$

*Step 2.* Construct $q'$ from $(q^0, \{q^k\})$ as indicated in (4.6). The preceding equations plus Lemma 5.3 imply that $||q'||_2 \le \zeta r^{-1}$. And, (5.51) plus Lemma 5.3 imply that the $L^2$ norm of $q'$ over any ball $B \subset X$ of radius $r^{-1/2}$ is bounded apriori by $\zeta r^{-3/2}$. That is, (5.51) holds with $q'$ replacing $q^k$.

*Step 3.* If the data $\{y^k\}$ lies in the zero set of the map $\psi_r$, then $q'$ obeys (4.2). In the general case, $q'$ obeys an equation of the form

$$Lq' + \sqrt{r} \cdot \varpi(q', q') + \mathrm{err}', \tag{5.52}$$

where $\mathrm{err}' = \Sigma_k \chi_{100\delta,k}\, \mathrm{err}'^k$. Here, $\mathrm{err}'^k$ is pointwise bounded by $\zeta \varepsilon e^{-\sqrt{r}|s|/\zeta}$.

Since the operator $L$ is elliptic, and elements in $\Lambda_k$ satisfy apriori derivative bounds to any desired order, (5.52) can be used to obtain pointwise bounds on the derivatives of $q'$ in terms of the preceding $L^2$ bound of $q'$ over radius $r^{-1/2}$ balls. In this regard, standard elliptic estimates (see, [13], Chapter 6) give (4.1.2) and (4.1.3).

Indeed, to obtain (4.1.2) and (4.13) from (5.52), first fix a geodesic ball $B \subset X$ of radius $r^{-1/2}$ and let $B' \subset B$ denote the concentric ball with radius $2^{-1}r^{-1/2}$. Now, dilate $B$ to have unit radius. The dilated $q'$ obeys an equation such as (5.52), but where

- the dilated version of $L$ has irrelevant $r$ factors,
- there is no $\sqrt{r}$ factor before the $\varpi$ term, and
- the $\mathrm{err}'$ factor is suitably dilated; infact, after dilating, its pointwise norm is bounded by $\zeta \varepsilon r^{-1/2}$;
- the dilated $q'$ has $L^2$ norm in the unit radius ball which is bounded by $\zeta \cdot r^{-1/2}$.

Standard regularity theorems using the dilated equation bound the supremum norm of the dilated $q'$ in the radius $1/2$ ball by $\zeta r^{-1/2}$; and this is the bound in $B'$ for $q'$. Bootstrapping these theorems produces a bound of $\zeta r^{-1/2}$ on the supremum norm of the covariant derivative of the dilated $q'$ in this same radius $1/2$ ball. After undoing the dilation, the latter bound produces a bound of $\zeta$ for $\nabla q'$ in $B'$.

## 6. Analytic structures

The purpose of this section is to consider the map $\Psi_r$ from Proposition 5.2 in greater detail, especially with regard to the topological and smooth structure of $\mathcal{M}^{(r)}$.

The precise statement of the first result requires a brief digression to review the topology on $\mathcal{M}^{(r)}$. The digression starts by remarking that $\mathrm{Conn}(E)$ (the space of smooth, Hermitian connections on $E$) has the structure of an affine Frechet manifold which is modelled on $i\Omega^1(X)$. Meanwhile, $C^\infty(S_+)$ is a smooth Frechet vector space, so, $\mathrm{Conn}(E)\times C^\infty(S_+)$ has the structure of a smooth Frechet manifold. The group $C^\infty(X;S^1)$ acts smoothly on the latter, with quotient $(\mathrm{Conn}(E)\times C^\infty(S_+))/C^\infty(X;S^1)$. This last set is given the quotient topology. The space $\mathcal{M}^{(r)}$ sits in this quotient, and, as stated in Section 1, the topology on $\mathcal{M}^{(r)}$ is the subspace topology which is inherited from its inclusion inside $(\mathrm{Conn}(E)\times C^\infty(S_+))/C^\infty(X;S^1)$.

**Proposition 6.1.** *Let $E \to X$ be a complex line bundle with first Chern class e. Fix a finite set $\{(C_k, m_k)\}_{1\le k\le n}$ of pairs, where $\{C_k\}$ is a pair-wise disjoint collection of connected, pseudo-holomorphic submanifolds, and where $\{m_k\}$ consists of positive integers. These are constrained so that $\Sigma_k m_k[C_k]$ is Poincaré dual to e. For each k, choose a subspace $\mathcal{K}^{(k)}$ in the $(C_k, m_k)$ version of $\mathcal{Z}_0$ which has compact closure; and choose a subspace $\Lambda_k$ in the $(C_k, m_k)$ version of $C^\infty(\oplus_{1\le q\le m}N^q)$ with the property that the projection of $\Lambda_k$ onto cokernel$(\Delta_y)$ is surjective for each $y \in \mathcal{K}^{(k)}$. Then $\mathcal{K}_\Lambda^{(k)}$ in Proposition 5.2 can be chosen so that the following are true:*

- *When $r \ge 1$, introduce*

$$\Psi_r : \times_k \mathcal{K}_\Lambda^{(k)} \to (\mathrm{Conn}(E)\times C^\infty(S_+))/C^\infty(X;S^1)$$

  *as in Proposition 5.2. If r is sufficiently large, then this map is an embedding.*

- *Introduce $\psi_r : \times_k \mathcal{K}_\Lambda^{(k)} \to \times_k \Lambda_k$ as in Proposition 5.2. When r is sufficiently large, $\Psi_r$ maps $\psi_r^{-1}(0)$ homeomorphically onto an open subset of $\mathcal{M}^{(r)}$.*

The action of $C^\infty(X;S^1)$ on $\mathrm{Conn}(E)\times C^\infty(X;S^1)$ is free at points $(a,(\alpha,\beta))$ where the second pair is not identically zero. (Note that $\mathcal{M}^{(r)}$ misses this set when $r$ is large.) Away from this set, $(\mathrm{Conn}(E)\times$

$C^\infty(S_+))/C^\infty(X;S^1)$ has the structure of a smooth Frechet manifold. Indeed, the tangent space at a point $\Xi = (a,(\alpha,\beta))$ consists of the vector space $\mathcal{T}_\Xi$ of elements $(b,(\eta,\lambda)) \in i\Omega^1 \oplus C^\infty(S_+)$ which satisfy the slice condition

$$*d*b + \frac{i\sqrt{r}}{\sqrt{2}} im(\bar{\alpha}\eta + \bar{\beta}\lambda) = 0. \tag{6.1}$$

Furthermore, a small radius ball $\mathcal{B} \subset \mathcal{T}_\Xi$ about 0 gives local coordinates near $\Xi$ via the map which sends $(b,(\eta,\lambda))$ to the orbit of $(a + \frac{\sqrt{r}}{2\sqrt{2}}b, (\alpha + \eta, \beta + \lambda))$. (This $\mathcal{B}$ can be defined using the $L^2_2$ metric.)

Remark that it is customary in gauge theory circles to define a manifold structure on such an orbit space by completing the latter using a Sobolev space topology. Add some standard elliptic regularity results to the proof of the slice theorem for the Sobolev topology to obtain the $C^\infty$ topology slice theorem used here.

The analytic structure of $\mathcal{M}^{(r)}$ is essentially that of the zero set of a smooth section of a finite rank vector bundle over a finite dimensional manifold. In order to make this notion precise, re-introduce, for each point $\Xi = (a,(\alpha,\beta)) \in \mathcal{M}^{(r)}$, the operator

$$L = L_\Xi : i\Omega^1(X) \oplus C^\infty(S_+) \to i(\Omega^0(X) \oplus \Omega^{2+}) \oplus C^\infty(S_-) \tag{6.2}$$

from (4.3) and (4.4). As the next proposition demonstrates, the behavior of $L$ determines the local structure of $\mathcal{M}^{(r)}$.

**Proposition 6.2.** *Suppose that $r$ is large. Fix $\Xi \in \mathcal{M}^{(r)}$. There are non-empty balls $B \subset$ kernel$(L)$ and $\mathcal{B} \subset \mathcal{T}_\Xi$ with 0 as center; and there are smooth maps $\varphi : B \to$ cokernel$(L)$ and $\Phi : B \to \mathcal{B}$ with the following properties:*

- $\varphi(0) = 0$.
- $\Phi - 1$ *maps into the $L^2$-orthogonal compliment of* kernel$(L)$.
- *The ball $\mathcal{B}$ can be identified, as described above, with a neighborhood of $\Xi$ in the orbit space* $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X;S^1)$. *After doing so, the map $\Phi$ defines a homeomorphism from the subset $\varphi^{-1}(0) \subset B$ onto an open neighborhood of $\Xi$ in $\mathcal{M}^{(r)}$.*
- $||\Phi(v) - v|| \le \xi \cdot ||v||^2$, *where the former norm is any $C^k$ or Sobolev norm, and the latter is the $L^2$ norm. Here, the constant $\xi$ depends on the norm in question (and on $r$), but not on the point $v \in B$.*

- *If* cokernel$(L) = \varnothing$, *then* $\mathcal{M}^{(r)}$ *is a smooth near* $\Xi$, *and* $\Phi$ *gives a coordinate chart for* $\mathcal{M}^{(r)}$ *near* $\Xi$. *In particular, at such a point,* kernel$(L) \subset \mathcal{T}_\Xi$ *is naturally isomorphic to the tangent space to* $\mathcal{M}^{(r)}$ *at* $\Xi$.

Given the preceding proposition, say that $\Xi \in \mathcal{M}^{(r)}$ is a *smooth* point when the operator $L$ as defined from this data has no cokernel. Perturbation theory insures that the set of smooth points of $\mathcal{M}^{(r)}$ is open; and it follows from Proposition 6.2 that this set has a natural smooth manifold structure.

The following proposition describes a relationship between the analytic structures of $\mathcal{M}^{(r)}$ and of $\{\mathcal{K}_\Lambda^{(k)}\}$ :

**Proposition 6.3.** *Make the same assumptions as in Proposition 6.1, but assume here that each* $\mathcal{K}^{(k)}$ *is an open set. Then the submanifolds* $\{\mathcal{K}_\Lambda^{(k)}\}$ *from Proposition 5.2 can be chosen so that for all* $r$ *sufficiently large, the following is true: When* $\Xi \in \Psi_r(\psi_r^{-1}(0))$, *the kernel of* $L_\Xi$ *is tangent to the image of* $\Psi_r$. *In fact,* kernel$(L_\Xi)$ *at such* $\Xi$ *is the image by the differential of* $\Psi_r$ *of the kernel of the differential of* $\psi_r$. *In particular, if each* $\mathcal{K}^{(k)}$ *is in the* $(C_k, m_k)$ *version of* $\mathcal{Z}$, *then* $\Psi_r$ *maps* $\times_k \mathcal{K}^{(k)}$ *onto an open subset of smooth points in* $\mathcal{M}^{(r)}$ *as a diffeomorphism onto its image.*

The image of $\times_k \mathcal{K}_\Lambda^{(k)}$ by $\Psi_r$ gives an example of a submanifold $Y \subset (\mathrm{Conn}(E) \times C^\infty(X; S_+))/C^\infty(X; S^1)$ with the following special property:

$$\text{(6.3)} \qquad \begin{array}{l} \bullet Y \text{ contains a nonempty open subset } \mathcal{N} \subset \mathcal{M}^{(r)}. \\ \bullet \text{ If } \Xi \in \mathcal{N}, \text{ then kernel}(L_\Xi) \subset TY \text{ (as a subspace of } \mathcal{T}_\Xi). \end{array}$$

Such a submanifold $Y$ of $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$ will be called a "Kuranishi model" for $\mathcal{N}$. It turns out that the local and also global analytic structure of $\mathcal{N}$ can be described in terms which are intrinsic to the Kuranishi model $Y$. In particular, the sequel to this article [22] explains how to compute the Seiberg-Witten invariants as the Euler class of a certain section over $Y$ of a certain finite dimensional vector bundle. (The sequel also proves that $\mathcal{M}^{(r)}$ always has a Kuranishi model.)

The remainder of this section is occupied with the proofs of these propositions.

### a) Proof of Proposition 6.1: $\Psi_r$ as an embedding

Given the first assertion (that the map $\Psi_r$ is an embedding), the second assertion follows with a demonstration that $\Psi_r$ maps $\psi_r^{-1}(0)$ onto an open set. This subsection demonstrates that $\Psi_r$ is an embedding, and the next subsection considers the image of $\Psi_r$ on $\psi_r^{-1}(0)$.

The argument that $\Psi_r$ is an embedding starts with the definition of a map, $Y$, from $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$ into a finite dimensional, complex Euclidean space. The argument then establishes that the composition of $Y$ with $\Psi_r$ defines an embedding. This implies that $\Psi_r$ itself must define an embedding.

The argument that $Y \bullet \Psi_r$ embeds procedes by composing the map $Y$ with an auxilliary map, $\Psi_r^0$. Here, $\Psi_r^0$ sends $y \in \times_k \mathcal{K}_\Lambda^{(k)}$ to the orbit of $(\underline{a}_r(y), (\underline{\alpha}_r(y)\underline{\beta}_r(y)))$, where the latter is defined as in Section 3b using $\{c^{(k)} = \Upsilon(y^k)\}$. The map $Y$ will be constructed so that its composition with $\Psi_r^0$ gives an embedding whose differential is uniformly large. Then, the proof that $Y$ composes with $\Psi_r$ to give an embedding reduces to what is an essentially perturbative calculation.

The details of the argument are given in the six steps that follow.

*Step 1.* As $q$ ranges from 1 to $m$, the integral in (2.6.3) exhibits a smooth map, $H$, from the vortex moduli space $\mathfrak{C}_m$ to $\mathbb{C}^m$. As remarked in (2.6.3) the $q$'th component of $H$ can be written in terms of the zeros of $\tau$ with their multiplicities: $H^q = \Sigma_{\lambda:\tau(\lambda)=0} m(\lambda) \cdot \lambda^q$. Parameterize $\mathfrak{C}_m$ in the usual way by $\mathbb{C}^m$ (as in (2.6.1)) and then $H$ defines a diffeomorphism from $\mathbb{C}^m$ to $\mathbb{C}^m$. Indeed, it is a well known fact in the theory of symmetric functions that if $y = (y_1, \dots, y_m) \in \mathbb{C}^m$, then $H^q(y) = -q \cdot y_q + g_q$, where $g_q$ is a universal polynomial in $(y_1, \dots, y_{q-1})$. In particular, it follows that the differential of the map $H : \mathbb{C}^m \to \mathbb{C}^m$ is upper triangular (in the obvious sense), and its determinant is equal to $(-1)^m \cdot m!$.

For the purposes below, the map $H$ is unwieldy because its definition involves an integral over the whole of the complex plane. However, the map $H$ can be approximated on compact subsets of $\mathbb{C}^m$ as follows: Fix $R$ large and consider the map $H_R : \mathbb{C}^m \to \mathbb{C}^m$ whose $q$'th component sends $y$ to

$$H_R^q = (8\pi)^{-1} \int_{\mathbb{C}} \eta^q \chi_R (1 - |\tau|^2). \tag{6.4}$$

Here, $\chi_R$ is the cut-off function on $\mathbb{C}$ which sends $\eta$ to $\chi(|\eta|/R)$. (Remember that $\chi_R$ is defined in Step 4 of Section 2d.) The relevant properties of the map $H_R$ are summarized by

**Lemma 6.4.** *Given $\varepsilon > 0$ and a compact set $K \subset \mathbb{C}^m$, there exists $R_0$ such that for all $R > R_0$, the map $H_R$ obeys*

$$|H_R(y) - H(y)| + |dH_R|_y - dH|_y| < \varepsilon \quad \textit{when} \quad y \in K. \tag{6.5}$$

*Proof of Lemma 6.4.* This follows from (2.5) and Lemma 2.3.

*Step 2.* This step defines a prototype for the map $Y$. The map defined here requires the choice of the index $k$, a point $z \in C_k$, and $R \geq 1$. The prototype will be denoted by $Y_{(k,z,R)}$. Given this data, choose a complex, hermitian identification of the normal bundle fiber at $z$ with $\mathbb{C}$. Also, set $m = m_k$ below.

Now characterize

$$Y_{(k,z,R)} : (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1) \to \oplus C^m$$

by the condition that it send $(a, (\alpha, \beta))$ to the point $q$'th coordinate is

$$Y^q_{(k,z,R)} = (8\pi)^{-1} \int_{\mathbb{C}} \eta^q \chi_R \rho^*_{1/r}(1 - |\alpha|^2). \tag{6.6}$$

Here, $(1 - |\alpha|^2)$ is implicitly restricted to where $|s| < 2Rr^{-1/2}$ on the fiber of the normal bundle to $C_k$ at $z$ and the latter is identified with $\mathbb{C}$ as described above.

*Step 3.* This step defines the map $Y$. The building blocks for the construction are the maps $Y_{(k,z,R)}$ of the previous step. The idea here is to first choose a suitably large $R$, and subsequently choose, for each $k$, a finite set $U(k)$ of points $z$ in $C_k$. Then set

$$Y = \oplus_{(k,z)} Y_{(k,z,R)}. \tag{6.7}$$

To define $R$, remark that for each $k$ and each $z \in C_k$, restriction to the normal fiber of $C_k$ at $z$ (together with a complex hermitian identification of this fiber with $\mathbb{C}$) maps $\mathcal{K}^{(k)}_\Lambda$ to a set in $\mathbb{C}^m$ whose closure, $K$, is compact. Choose $R$ so that (6.5) holds for this set $K$ and $\varepsilon = 10^{-3}$. Because each $\mathcal{K}^{(k)}_\Lambda$ is compact, a choice of $R$ can be made which works for all $k$ and all $z \in C_k$.

The choice of the points for a given index $k$ will be made as follows: Choose a finite set, $U[k] \subset C$, of points which has the property that the evaluation map

$$\text{ev} : \mathcal{K}_\Lambda^{(k)} \to \oplus_{z \in U[k]} (\oplus_{1 \le q \le m} N^q|_z) \tag{6.8}$$

defines an embedding. (Such a finite set exists because $\mathcal{K}_\Lambda^{(k)}$ has compact closure in $C^\infty(\oplus_{1 \le q \le m} N^q)$.) Use the data $\{z \in U[k]\}$ for the definition of the sets $Y_{(k,z,R)}$ in (6.6).

The following lemma summarizes the properties of the map $Y$. In the statement of this lemma, $|| \cdot ||_2$ denotes the norm on $T(\times_k \mathcal{K}_\Lambda^{(k)})$ which is obtained by taking the direct sum of the $L^2$ norms on the various components. In this regard, remark that any given $\mathcal{K}_\Lambda$ is a submanifold in the space of sections of $\oplus_{1 \le q \le m} N^q$, and so its tangent space inherits the $L^2$ inner product from that on $C^\infty(\oplus_{1 \le q \le m} N^q)$. Furthermore, since $\mathcal{K}_\Lambda$ has compact closure (by assumption), the $L^2$ topology on this space is equivalent to the $C^\infty$ topology.

**Lemma 6.5.** *The manifolds $\{\mathcal{K}_\Lambda^{(k)}\}$ in Proposition 5.2 can be chosen so that there is a constant $\zeta \ge 1$ with the following properties: Construct $Y$ as just indicated. Reintroduce the map $\Psi_r : \times_k \mathcal{K}_\Lambda^{(k)} \to (\text{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$ from Proposition 5.2, and let $\Psi_r^0$ denote the map between the same two spaces which assigns to $y$ in the domain the orbit of the point $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ in the range. When $r$ is large, then*

$$|(Y \circ \Psi_r)_* p - (Y \circ \Psi_r^0)_* p| < \zeta r^{-1/2} ||p||_2. \tag{6.9}$$

*at all $y \in \times_k \mathcal{K}_\Lambda^{(k)}$ and for all $p \in T(\times_k \mathcal{K}_\Lambda^{(k)})|_y$.*

The proof of this lemma starts in Step 5, below.

*Step 4.* This step uses Lemma 6.5 to complete the proof of Proposition 6.1's assertion that the map $\Psi_r$ is an embedding. To begin, remark that the fundamental theorem of calculus plus (6.9) imply

$$\begin{aligned} |(Y \circ \Psi_r)(y) - (Y \circ \Psi_r)(y')| \ge & |(Y \circ \Psi_r^0)(y) - (Y \circ \Psi_r^0)(y')| \\ & - \zeta r^{-1/2} ||y - y'||_2 \end{aligned} \tag{6.10}$$

for any two points $y, y' \in \times_k \mathcal{K}_\Lambda^{(k)}$. Here, $\zeta$ is independent of $r$ and the chosen points $(y, y')$.

Meanwhile, note that the range of $Y$ decomposes in the obvious way as a direct sum of vector spaces indexed by $k$ and $z \in U[k]$. And, the

$(k, z)$ summand of $Y \circ \Psi_r^0$ is equal to the composition of restricting $\mathcal{K}_\Lambda$ to the fiber of the normal bundle of $C_k$, and then, after a complex, hermitian identification of this fiber with $\mathbb{C}$, applying the map $H_R$. Thus, given the properties of the set $\{z \in U[k]\}$, it follow from Lemma 6.6 that

$$|(Y \circ \Psi_r^0)(y) - (Y \circ \Psi_r^0)(y')| \geq \zeta^{-1} \cdot ||y - y'||_2, \tag{6.11}$$

where $\zeta$ is independent of $r$ and of $y$ and $y'$. (This $\zeta$ will depend on the initial choice for $\{\mathcal{K}^{(k)}\}$.)

When $r$ is large, (6.10) and (6.11) imply that $y$ and $y'$ are mapped to the same point only if $y = y'$. This implies that $\Psi_r$ is one-to-one. The fact that the differential is injective follows from (6.10) and (6.11) by letting $y'$ approach $y$.

*Step 5.* This step contains the

*Proof of Lemma 6.5.* The first thing to understand is that both $\Psi_r$ and $\Psi_r^0$ are defined in terms of maps (with the same names) from $\times_k \mathcal{K}_\Lambda^{(k)}$ into $\mathrm{Conn}(E) \times C^\infty(S_+)$. (See Propositions 3.1 and 5.2.) And, (5.2) describes $\Psi_r$ in terms of $\Psi_r^0$ and $q' = (a', (\alpha', \beta'))$; where the latter defines a map from $\times_k \mathcal{K}_\Lambda^{(k)}$ into $i\Omega^1 \oplus C^\infty(S_+)$. With this understood, it follows from (6.5) that

$$|(Y \circ \Psi_r)_* p - (Y \circ \Psi_r^0)_* p| \leq \zeta R^2 \sup_X |q'_* p|. \tag{6.12}$$

Here, $q'_*$ denotes the directional derivate of $q'$ in the direction of $p$. Thus, Lemma 6.5 follows from

**Lemma 6.6.** *The assertions of Proposition 5.2 can be amended with the following addition: As described above, think of $q' = (a', (\alpha', \beta'))$ as a smooth map from $\times_k \mathcal{K}_\Lambda^{(k)}$ into $i\Omega^1 \oplus C^\infty(S_+)$. Then*

$$\sup_X |q'_* p| \leq \zeta r^{-1/2} ||p||_2. \tag{6.13}$$

The remainder of this step is occupied with the

*Proof of Lemma 6.6.* A slight modification of the argument in Section 5j gives the bounds for $q'_* p$ given a proof of the following claim: Let $B \subset X$ be a ball of radius $2r^{-1/2}$. Then, the $L^2$ norm of $q'_* p$ over $B$ is bounded by $\zeta r^{-3/2} ||p||$, where $\zeta$ is independent of $r$ (when large), $y$ and $p$.

The proof of this claim procedes as follows: Remember that $q'$ is a sum of parts (as in (4.6)), so there is a corresponding sum for $q'_* p$ which has terms involving the directional derivatives of $(q^0, \{q^k\})$. The latter will be denoted by $(q^0_p, \{q^k_p\})$. The subsequent arguments will bound $||q^0_p||_2$ and $[\int_B |q^k_p|^2]^{1/2}$ by $\zeta r^{-3/2}||p||$.

To bound $q^0_p$, suppose first that the bound $||q^k_p||_2 \le \zeta r^{-1/2}||p||$ has been established. Then apply (5.5) with the Chain rule to prove that $||q^0_p||_2 \le \zeta r^{-3/2}||p||$.

The key to $L^2$ norm bounds for $q^k_p$ is an estimate for the norm of the directional derivative, $x^k_p$ of $x^k$. In this regard, Lemma 5.6 gives the bound $||x^k_p||_{1,2} \le \zeta r^{-1/2}||p||$. With the preceding understood, write $q^k = t^k + P(h^k)$. The directional derivative of $t^k$ will be denoted by $t^k_p$. The latter can be computed (using the Chain rule) in terms of the directional derivative of $x^k$, and this computation yields

$$||\nabla t^k_p||_2 + \sqrt{r}||t^k_p||_2 \le \zeta ||x^k_p||_{1,2} \le \zeta r^{-1/2}||p||. \tag{6.14}$$

This last equation (with the $L^2_1 \to L^4$ Sobolev inequality) implies that

$$\left[\int_B |t^k_p|^2\right]^{1/2} \le \zeta r^{-1}||t^k_p||_4 \le \zeta r^{-3/2}||p||. \tag{6.15}$$

Next, consider the contribution to $q^k_p$ from the directional derivative of $P(h^k)$. The $L^2$ norm of the latter can be bounded using the Chain rule with (5.14). The result is a bound by $\zeta r^{-3/2}||p||$ of its $L^2$ norm over the whole of the normal bundle to $C_k$. With (6.15), this last point implies that $[\int_B |q^k_p|^2]^{1/2} \le \zeta r^{-3/2}||p||$ and that $||q^k||_2 \le \zeta r^{1/2}||p||$.

**b) Proof of Proposition 6.1: The image by $\Psi_r$ of $\psi_r^{-1}(0)$**

This subsection proves that the map $\Psi_r$ of Proposition 5.2 maps $\psi_r^{-1}(0)$ onto an open set. The arguments in this subsection are arranged in twelve steps.

*Step 1.* Fix a point $y_0 = \{y^k_0\} \in \times_k \mathcal{K}^{(k)}_\Lambda$ which is in $\psi_r^{-1}(0)$. Consider a point $\Xi \in \mathcal{M}^{(r)}$ which is close to $\Psi_r(y)$. The goal will be to find a point $y = \{y^k\} \in \times_k \mathcal{K}^{(k)}_\Lambda$ which is close to $y_0$ and is mapped by $\Psi_r$ to $\Xi$. Note that given any $\varepsilon > 0$ and positive integer $p$, it is sufficient for the purposes of Proposition 6.1 to restrict attention to the case where $\Xi$ has distance $\varepsilon$ or less from $\Psi_r(y_0)$ in the $C^p$ topology. This freedom to take $\Xi$ as close as desired to $\Psi_r(y_0)$ is used (often without comment) extensively below.

Here are some parenthetical remarks concerning the distance from $y$ to $y_0$ and from $\Xi$ to $\Psi_r(y_0)$ : First, in measuring the closeness of $y$ to $y_0$, notice that it is sufficient to measure closeness in any $L^2_{k\geq 0}$ topology, and this will imply closeness in the $C^p$ topology for any $p$. Thus, the distance between $y$ and $y_0$ will be denoted by $||y-y_0||$, without specific mention of the particulars of the norm.

Second, remember that the Seiberg-Witten equations are elliptic equations for the gauge orbit point. This implies, in particular, that "close" in some Sobolev topology, say $L^2_2$, implies "close" in any given $C^p$ topology. Thus, in the discussion below, the phrase "$\Xi$ is close to $\Psi_r(y_0)$" will be used without reference to the precise topology involved.

Third, remark that in previous sections, it was of critical import to develop estimates which were controlled with respect to the parameter $r$ in (1.13). This kind of control is, for the most part, unnecessary here because $r$ is going to be fixed at some large value, and then the distance from $\Xi$ to $\Psi_r(y_0)$ will be assumed to be small.

*Step 2.* Think of $\Psi_r$ as a map into $\mathrm{Conn}(E) \times C^\infty(S_+)$. (See Proposition 5.2.) Then, $\Psi_r(y_0)$ can be written as

$$\left(\underline{a}_r(y_0) + \frac{\sqrt{r}}{2\sqrt{2}}a'(y_0), (\underline{\alpha}_r(y_0) + \alpha'(y_0), \underline{\beta}_r(y_0) + \beta'(y_0))\right). \tag{6.16}$$

Here, $(\underline{a}_r(y_0), (\underline{\alpha}_r(y_0), \underline{\beta}_r(y_0))$ are defined from $y_0$ as specified in Proposition 5.2 and Sections 2 and 3b. Also, $q'(y_0) = (a'(y_0), (\alpha'(y_0), \beta'(y_0)))$ is given by Proposition 5.2.

Fix any point $y \in \times_k \mathcal{K}_\Lambda^{(k)}$ which is close to $y_0$. The point $y$ will act, for the time being, as a parameter in the ensuing discussion. A particular value for $y$ (determined by $\Xi$) is fixed at the end. Given $y$ as above, a point on a gauge orbit $\Xi \in \mathcal{M}^{(r)}$ can be written as

$$\Xi = \left(\underline{a}_r(y) + \frac{\sqrt{r}}{2\sqrt{2}}a', (\underline{\alpha}_r(y) + \alpha', \underline{\beta}_r(y) + \beta')\right). \tag{6.17}$$

Note that the right side of (6.17) differs from the right side of (6.16) by a term of the form $(\frac{\sqrt{r}}{2\sqrt{2}}a'', (\alpha'', \beta''))$ ,where $(a'', (\alpha'', \beta''))$ can be assumed small in any apriori chosen $C^p$ norm by choosing $\Xi$ sufficiently close to $\Psi_r(y_0)$.

With $y \in \times_k \mathcal{K}_\Lambda^{(k)}$ chosen in a small diameter neighborhood of $y_0$, a preliminary goal is to find a point on the gauge orbit of $\Xi$ so that the resulting $q' = (a', (\alpha', \beta'))$ in (6.17) decomposes as in (4.6) in terms of data $(q^0, \{q^k\})$ where $q^0$ satisfies the first line in (5.3) and, for each index $k$, $q^k$ satisfies the second line in (5.3). The operator $L$ and the $\{\mathrm{err}^k\}$ terms for (5.3) will be defined, using the point $y \in \times_k \mathcal{K}_\Lambda^{(k)}$ to define $(\underline{a}_r(y), (\underline{\alpha}_r(y), \underline{\beta}_r(y))$. Note that the interest here will be in finding $(q^0, \{q^k\})$ which have suitably small norm.

The construction of $(q^0, q^k)$ requires a digression that occupies Steps 3 through 9.

*Step 3.* Remember that $q'(y_0) = (a'(y_0), (\alpha'(y_0), \beta'(y_0)))$ has, by construction, a decomposition as in (4.6) where (5.3) are obeyed in the case where $L$ and $\{\mathrm{err}^k\}$ are defined using the point $y_0 \in \times_k \mathcal{K}_\Lambda^{(k)}$. This means that the failure for a given point on the gauge orbit $\Xi$ to satisfy such a decomposition is small when $y$ is in a small diameter neighborhood of $y_0$ and the difference between (6.16) and (6.17) is small; that is, when $\Xi$ is in a small diameter neighborhood of $\Psi_r(y_0)$.

*Step 4.* This step argues that a point on the gauge orbit of $\Xi$ can be chosen (when $y$ is in a certain neighborhood of $y_0$, and $\Xi$ is in a certain neighborhood of $\Psi_r(y_0)$) so that the resulting $q' = (a', (\alpha', \beta'))$ from (6.17) obeys the slice (gauge fixing) condition that (4.4) should vanish; here, (4.4) uses the data $(a, (\alpha, \beta)) = (\underline{a}_r(y), (\underline{\alpha}_r(y), \underline{\beta}_r(y)))$. Also, the resulting $q'$ should be close to $q'(y_0) = (a'(y_0), (\alpha'(y_0), \beta'(y_0)))$ when $\Xi$ is close to $\Psi_r(y_0)$.

Here is an argument for the existence of such a point: To begin, pick any point on the gauge orbit of $\Xi$ and let $q'_{\mathrm{old}} = (a'_{\mathrm{old}}, (\alpha'_{\mathrm{old}}, \beta'_{\mathrm{old}}))$ denote the data from (6.17) for this initial choice. Of course, if $\Xi$ is close to $\Psi_r(y_0)$ and $y$ is close to $y_0$, then this $q'_{\mathrm{old}}$ can be assumed to be close to $q'(y_0)$ in the sense that the $C^p$ norm of the difference is less than $\varepsilon$. Here, $p$ and $\varepsilon > 0$ can be predetermined in advance. Given the initial choice of gauge orbit point, the final choice will be determined with the help of a function $u$ on $\Xi$ as follows:

$$(6.18) \quad (a', \alpha', \beta') = \Big( a'_{\mathrm{old}} - \frac{i2\sqrt{2}}{\sqrt{r}} du, (e^{i \cdot u} \alpha'_{\mathrm{old}} + e^{i \cdot u} \underline{\alpha}_r(y) - \underline{\alpha}_r(y), e^{i \cdot u} \beta'_{\mathrm{old}} + e^{i \cdot u} \underline{\beta}_r(y) - \underline{\beta}_r(y)) \Big).$$

The function $u$ will be found by considering the vanishing of (4.4) as an equation for $u$ on $\Xi$. The latter equation is equivalent to an equation of the form:

$$(6.19) \qquad - * d * du + 2^{-1/2} r (|\alpha_r(w_0)|^2 + |\underline{\beta}_r(w_0)|^2) u + Z(u) = 0,$$

where $Z(u)$ is a non-linear function of $u$. For the present purposes, it is suffient to know that $Z(u) = \gamma_1 \cdot u + \gamma_2 + \mathcal{O}(u^2)$, where $\gamma_1$ and $\gamma_2$ are functions of $q'_{\text{old}} - q'(y_0)$ and $y - y_0$. In particular, their $C^p$ norms are bounded by the $C^{p+1}$ norms of the latter. (The point here is that (4.4) vanishes when modified by replacing $y$ by $y_0$ and $q'_{\text{old}}$ by $q'(y_0)$.)

With the preceding understood, it becomes an exercise with the contraction mapping theorem to find a small solution $u$ to (6.19) when $\Xi$ is in a certain neighborhood of $\Psi_r(y_0)$ and $y$ is in a certain neighborhood of $y_0$. For such a solution, standard elliptic regularity results bound its $C^p$ norm by a uniform (i.e., $\Xi$-independent) multiple of $||y - y_0||$ and the $C^p$ norm of $q'_{\text{old}} - q'(y_0)$.

*Step 5.* Given that (4.4) vanishes for the case

$$(a, (\alpha, \beta)) = (\underline{a}_r(y), (\underline{\alpha}_r(y), \underline{\beta}_r(y))),$$

the condition that $\Xi \in \mathcal{M}^{(r)}$ implies that $q' = (a', (\alpha', \beta'))$ obeys (4.2) for the case where $y$ defines the term err, and defines $L$ by (4.3) and (4.4) using $(a, (\alpha, \beta))$ as above.

With the preceding understood, the purpose of this step is to define $(q^0, \{q^k\})$ so that (4.6) holds. In this regard, note that the data $(q^0, \{q^k\})$ are defined from $q'$ using some auxiliary data, $\{f^k\}$. Here, $f^k$ is a section over the normal bundle of the $C = C_k$ and trivial $E$ version of the bundle $\mathcal{V}_0$ from (4.16a). The data $\{f^k\}$ is specified below. However, given this data, here is $(q^0, \{q^k\})$ :

(6.20)
- $q^0 = \Pi_k(1 - \chi_{25\delta,k})q' + \Sigma_k \chi_{100\delta,k} f^k.$
- $q^k = \chi_{25\delta,k} q' - (1 - \chi_{4\delta,k}) f^k.$

Note that (4.6) holds, as required. When interpreting the first line in (6.20), use Lemma 4.4 to interpret $\chi_{100\delta,k} f^k$ as a section of $iT^*X \oplus S_{+,0}$, where $S_{+,0}$ is the plus spin bundle for the canonical $\text{Spin}^{\mathbb{C}}$ structure as given in (1.7). In this regard, remember that $q^0$ should be interpreted as a section of $iT^*X \oplus S_{+,0}$ also. The latter interpretation requires the identification of the given $S_+$ as in (1.9), and it requires the identification using $\alpha_r$ of $E$ with the trivial line bundle where $\alpha_r \neq 0$. The interpretation of the second line in (6.20) also uses Lemma 4.4 and the identification via $\alpha_r$ of $E$ with the trivial bundle where $\alpha_r \neq 0$.

*Step 6.* The data $\{f^k\}$ in (6.20) must be constrained by the requirement that (5.3) hold. Given that (4.2) holds, this constraint is implied by the requirement that $f^k$ obey

$$L_k f^k + \sqrt{r}(1-\chi_{2\delta,k})\chi_{200\delta,k}[\varpi(q', f^k) + \varpi(q^k, q^0)] - \wp(d\chi_{25\delta,k})q' = 0 \tag{6.21}$$

on the normal bundle $N$ of $C_k$. In (6.21), $L_k$ is as defined in Section 4 but with the trivial vortex ($v = 0, \tau = 1$) replacing the given section of the $(C_k, m_k)$ version of (2.15). This is to say that $L_k$ is defined using the $m_k = 0$ version of (2.15). Note that when viewing (6.21) as an equation to determine $f^k$ from $q'$, one should look at $q^0$ and $q^k$ as functionals of $q'$ and $f^k$ through (6.20). In this way, (6.21) reads:

$$\begin{aligned} L_k f^k - \sqrt{r}(1-\chi_{4\delta,k})\chi_{100\delta,k}\varpi(f^k, f^k) \\ + \sqrt{r}(1-\chi_{25\delta,k})\chi_{25\delta,k}\varpi(q', q') - \wp(d\chi_{25\delta,k})q' = 0. \end{aligned} \tag{6.22}$$

It is left as an exercise for the reader to verify that (6.21) (or, equivalently, (6.22)) insures that $(q^0, \{q^{(k)}\})$ solves (5.3).

*Step 7.* This step and the next verify that (6.22) has a small solution when $y$ is in a certain neighborhood of $y_0$, and $\Xi$ is in a certain neighborhood of $\Psi_r(y_0)$. In this regard, there are two key points to keep in mind: First, there is a small solution to (6.22) when $q'$ is replaced by $q'(y_0)$, this being

$$f^k = \chi_{25\delta,k} q^0(y_0) - (1-\chi_{25\delta,k})q^k(y_0). \tag{6.23}$$

Here, $q^0(y_0)$ and $\{q^k(y_0)\}$ are the data from (4.6) which are produced by the construction of $\Psi_r(y_0)$ from $y_0$.

The second key point to remember is that both $L_k$ and its formal $L^2$-adjoint are robustly invertible. Indeed, appeal to the $m = 0$ case of Lemma 4.10 finds, for large $r$,

$$\begin{aligned} &\bullet\ \|L_k b\|_2 \geq \zeta^{-1}(\|\nabla b\|_2 + \sqrt{r}\|b\|_2), \\ &\bullet\ \|L_k^\dagger b\|_2 \geq \zeta^{-1}(\|\nabla b\|_2 + \sqrt{r}\|b\|_2). \end{aligned} \tag{6.24}$$

In both lines, $\zeta \geq 1$ is independent of $b$ and $r$. (The top line holds for all smooth sections $b$ of $\mathcal{V}_0$ with compact support, and the second line holds for all smooth, compactly supported sections of the bundle $\mathcal{V}_1$ from (4.16b).) With regard to applying the analysis in Lemma 4.10, note that neither $V^c$ nor $K_1$ arise in this case because the vortex which defines $L_k$ has vortex number zero.

With the preceding understood, appeal to the $m = 0$ case of Lemma 4.10 to prove that $L_k$ has a bounded inverse which maps $L_2(\mathcal{V}_1)$ to $L_1^2(\mathcal{V}_0)$. Furthermore, this inverse obeys the following apriori bound: $||\nabla L_k^{-1}(h)||_2 + \sqrt{r}||L_k^{-1}(h)||_2 \leq \zeta||h||_2$. Here, $\zeta$ is independent of $r$ and $h$.

*Step 8.* Write (6.22) as the condition for a fixed point of the map $T$ on $L^2(\mathcal{V}_1)$ to itself which sends $h$ to $L_k^{-1}(\sqrt{r}(1 - \chi_{4\delta,k})\chi_{100\delta,k}\varpi(h,h) + \Phi(q'))$. The existence of a constant $\zeta \geq 1$ and a unique fixed point, $h_0$, of $T$ with $L^2$ norm bounded by $\zeta^{-1}r^{-1/2}$ follows from standard dimension 4 Sobolev inequalities when $||y - y_0||$ is assumed small, and when $q' - q'(y_0)$ is assumed to have very small $C_0$ norm.

*Step 9.* Pointwise estimates for $f^k = P(h_0)$ can be obtained by employing the $L_1^2$ estimate for $f^k$ from the previous step with standard elliptic "bootstrapping" arguments. The result is a bound on the derivatives of $f^k$ to any desired order given bounds for $||y - y_0||$ and bounds on the appropriate derivatives of $q' - q'(y_0)$. (The bootstrapping arguments are of the sort given in Chapter 6 of [13].)

*Step 10.* The following has now been established: There are neighborhoods, $U$, of $y_0$ in $\times_k\mathcal{K}_\Lambda^{(k)}$ and, $\mathcal{U}$, of $\Psi_r(y_0)$ in $\mathcal{M}^{(r)}$ such that when $y \in U$, each $\Xi \in \mathcal{U}$ has a point on its gauge orbit as in (6.4) where $q' = (a', (\alpha', \beta'))$ is given by (4.6) and where the data $(q^0, \{q^k\})$ satisfies (5.3). Furthermore, the norms of $(q^0 - q^0(y_0), \{q^k - q^k(y_0)\})$ and their derivatives to any given order can be assumed as small as desired by restricting the neighborhoods of $y_0$ and $\Psi_r(y_0)$ so that points in these neighborhoods have small distance to $y_0$ and $\Psi_r(y_0)$, respectively.

To complete the argument that $\Psi_r$ maps $\psi_r^{-1}(0)$ onto an open set, it remains still to verify that there is some choice of $y \in \times_k B^k$ near $y_0$ for which the data $(q^0, \{q^k\})$ is constructed as described in Section 5. (That is, for which $q^0 = q^0(y)$ and $q^k = q^k(y)$, where $(q^0(y), \{q^k(y)\})$ is given by the constructions in Section 5 which start from the point $y$.)

When $\Xi$ is in a certain neighborhood of $\Psi_r(y_0)$, the existence of such a point $y = y(\Xi)$ follows directly from the uniqueness aspects of the existence assertions in Lemmas 5.3-5.6 with an extra application of the inverse function theorem. The details of this argument occupy the remaining two steps.

*Step 11.* Because the $C^p$ norms of $q_0 - q_0(y_0)$ and $\{q^k - q^k(y_0)\}$ can be made small by restricting to small diameter neighborhoods of $y_0$ and $\Psi_r(y_0)$, the uniqueness assertion of Lemma 5.3 can be safely invoked to insure that $q^0$ from (6.20) is the function of $q^k$ (from (6.20)) and $y$ as described by Lemma 5.3. Likewise, the uniqueness assertions from Lemmas 5.4 and 5.5 can also be invoked to insure that $q^k$ from (6.20) has the form $q^k = P(h^k) + t(x^k)$, where $h^k$ comes from Lemma 5.5. Here, $x^k$ is a section of the bundle $\oplus_{1\leq q\leq m} N^q$. If $x^k$ is $L^2$ orthogonal to the kernel of the operator $(1 - Q_\Lambda)\Delta_{y^k}$, then the uniqueness assertion of Lemma 5.6 can be invoked to conclude that $x^k$ as defined above from $q^k$ is that which is given by Lemma 5.6. In this case, the point $y$ must solve (5.21) and so $\Xi = \Psi_r(y)$.

*Step 12.* Thus, the issue comes down to this: Given $\Xi$ close to $\Psi_r(y_0)$, is there a point $y$ (which is close to $y_0$) so that the corresponding $x^k$ as defined above is $L^2$ orthogonal to the kernel of the operator $(1 - Q_\Lambda)\Delta_y$?

The answer here is yes. Indeed, the existence of a (unique) such point is a consequence of the implicit function theorem. To see why, consider first the case where the point $y$ is chosen to equal $y_0$. In this case, the projection of the resulting $x^k$ onto the kernel of $(1 - Q_\Lambda)\Delta_{y_0}$ is bounded by some uniform multiple of some $C^p$ measure of distance between $\Xi$ and $\Psi_r(y_0)$. (Remember that this projection vanishes when $y = y_0$ and $X = \Psi_r(y_0)$ because then $(q^0, \{q^k\})$ and $(q^0(y_0), \{q^k(y_0)\})$ are the same.) Now, if $y$ is changed to $y = y_0 + v$, with $|v|$ small, then (with $\Xi$ fixed), the resulting $x^k$ differs from the $v = 0$ version by $v + o\cdot v$ where $o$ is also uniformly bounded by some $C^p$ measure of the distance from $\Xi$ to $\Psi_r(y_0)$. Here, one should think of $\Xi$ as being fixed, and changing the reference point y in some small neighborhood of $y_0$. In any event, this last estimate, with the inverse function theorem, finds the desired point $y$ when $\Xi$ is close to $\Psi_r(y_0)$.

### c) Proof of Proposition 6.2

Propositions of this sort are standard in gauge theory circles. The reader is referred to [12] or the forthcoming [8]. However, here is an outline of the construction of $\varphi$ and $\Phi$ : Write $\Xi = (a,(\alpha,\beta))$ and suppose that $w = (a',(\alpha',\beta')) \in i\Omega^1 \oplus C^\infty(X;S_+)$. Then the point $(a + \frac{\sqrt{r}}{2\sqrt{2}}a', (\alpha+\alpha', \beta+\beta'))$ is in $\mathcal{M}^{(r)}$ if w satisfies the equation:

$$Lw + \sqrt{r}\varpi(w,w) = 0. \tag{6.25}$$

Here, $y$ is defined as in (4.2).

To find solutions to (6.25) with $w$ small, write $w = v + \Phi_0$, where $v \in \text{kernel}(L)$, and $\Phi_0$ is in the $L^2$ orthogonal compliment to kernel$(L)$.

Let $\Pi_L$ denote the $L^2$ orthogonal projection onto the cokernel of $L$. Then $\Phi_0$ is found as a function of $v \in \text{kernel}(L)$ by solving the equation

$$L\Phi_0 + \sqrt{r}(1 - \Pi_L)\varpi(v + \Phi_0, v + \Phi_0) = 0. \tag{6.26}$$

The contraction mapping theorem (with some Sobolev inequalities and the Fredholm alternative) can be used to show that (6.30) has a unique, small solution, $\Phi_0(v)$ whenever $v$ has small norm. Here, the norm on $v$ can be taken to be its $L^2$ norm, and the norm on $\Phi_0$ its $L^2_1$ norm. (The upper bound on the norm of $v$ will be $r$ dependent.) When $v$ has small norm, (6.26) has a unique, small solution, $\Phi_0(v)$, which depends smoothly (in fact, analytically) on $v$. (The small solution to (6.26) is found as a fixed point of the map which sends $\Phi$ to $-\sqrt{r}L^{-1}(1 - \Pi_L)\varpi(v + \Phi, v + \Phi)$. This map sends the $L^2_1$ completion of the $L^2$-orthogonal compliment of kernel$(L)$ to itself.)

With $\Phi_0(v)$ understood, then the map $\varphi$ for Proposition 6.2 sends $v$ to

$$\varphi(v) = \Pi_L(\varpi(v + \Phi_0(v), v + \Phi_0(v))). \tag{6.27}$$

### d) Proof of Proposition 6.3

Remark first that Proposition 6.1 insures that $\{\mathcal{K}^{(k)}_\Lambda\}$ can be chosen so that $\Psi_r$ embeds $\times_k \mathcal{K}^{(k)}_\Lambda$ as a submanifold, $Y$, of $(\text{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$ which contains a neighborhood of the $\Psi_r$ image of $\psi_r^{-1}(0)$. Now, suppose that

$$\text{kernel}(L_{\Psi_r,(y)}) = (\Psi_r)_* \cdot \text{kernel}(\psi_{r*}|_y) \tag{6.28}$$

has been established for each $y \in \psi_r^{-1}(0)$. Then, the proof of Proposition 6.3 is completed as follows: It is sufficient, after (6.28), to prove that the index of $L$ is equal to the sum of the indices of the operators $\{\Delta_{y^k}\}$. (The argument here is as follows: If the dimension of the kernel of $L$ is equal to its index, then cokernel$(L)$ is trivial and the point in question is a smooth point. The dimension of the kernel of $L$ is, according to (6.28), that of the kernel of the differential of $\psi_r$. As no $\Delta_y$ has cokernel when $y \in \mathcal{Z}$, Assertion 4 of Proposition 5.2 insures that $\psi_r^{-1}(0)$ is a submanifold (when $r$ is large) which is diffeomorphic to $\times_k \mathcal{K}^{(k)}$. Assertion 4 of Proposition 5.2 also insures that $\psi_r$ vanishes transversally along this submanifold. In particular, this implies that the dimension of the kernel of the differential of $\psi_r$ at a point in $\psi_r^{-1}(0)$ is the sum of the dimensions of the submanifolds $\mathcal{K}^{(k)}$. The latter sum is equal to the sum of the dimensions of the kernels of the operators $\{\Delta_{y^k}\}$. And this last sum is the same as the sum of the indices of the same set $\{\Delta_{y^k}\}$ because each $\Delta_{y^k}$ has trivial cokernel. Thus, if $\operatorname{index}(L)$ is also equal to the sum of the indices of $\{\Delta_{y^k}\}$, then $\dim(\operatorname{kernel}(L)) = \operatorname{index}(L)$.

To prove that $\operatorname{index}(L) = \Sigma_k \operatorname{index}(\Delta_{y^k})$, note first that the index of L is equal to $2d = e \bullet e - c \bullet e$. Since the set $\{C_k\}$ consists of pairwise disjoint submanifolds, this number $2d$ is equal to $\Sigma_k 2 \cdot d_k$, where $2 \cdot d_k = m_k^2 \cdot e_k \bullet e_k - m_k \cdot c \bullet e_k$. And, the number $2 \cdot d_k$ is the index of $\Delta_{y^k}$. (In Proposition 3.2, the integer n is equal to $e_k \bullet e_k$. And, by the adjunction formula, $c \bullet e_k = 2 \cdot (\operatorname{genus}(C_k) - 1) - e_k \bullet e_k$.)

The proof of (6.28) is deferred to the next subsection. The remainder of this subsection contains a necessary discussion of the operator in (4.3) and (4.4) as defined by $(a, (\alpha, \beta)) = \Psi_r(y)$. Lemma 6.7, below, summarizes. In the statement of the lemma, use the notation $N^{(k)}$ to denote the $C = C_k$ and $m = m_k$ version of the vector bundle $\oplus_{1 \leq q \leq m} N^q$.

**Lemma 6.7.** *The sets $\{\mathcal{K}_\Lambda^{(k)}\}$ in Proposition 5.2 and a constant $\zeta \geq 1$ can be chosen with the following property: Suppose that $r > \zeta$ and that $y \in \times_k \mathcal{K}_\Lambda^{(k)}$. There exist a pair of homomorphisms*

$$\phi_0^k : \oplus_{k'} L_1^2(N^{(k')}) \to L^2(N^{(k)} \otimes T^{0,1} C_k)$$

*and*

$$\phi_1^k : L^2(i(\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-) \to L^2(N^{(k)} \otimes T^{0,1} C_k)$$

*which obey*

- $||\phi_0^k u||_2 \leq \zeta(\xi_y ||u^k||_2 + r^{-1/4} \Sigma_{k'} ||u^{k'}||_{1,2})$, *where $\xi_y$ is bounded by the distance from $y^k$ to $\mathcal{Z}_0$. In particular, given $\varepsilon$, then $\{\mathcal{K}_\Lambda^{(k)}\}$ can be chosen so that $\xi_y < \varepsilon$.*
- $||\phi_1^k g'||_2 \leq \zeta \cdot (||(1 - \Pi) g'^k||_2 + r^{-1/2} ||g'^0||_2 + r^{-1} \Sigma_{k'} ||(1 - \Pi) g'^{k'}||_2)$.
- *Let $L_{\Psi_r(y)}$ denote the operator in (4.3) and (4.4) in the case where $(a, (\alpha, \beta)) = \Psi_r(y)$. Let $g' \in i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$. Then the equation $L_{\Psi_r(y)} p = g'$ has an $L_1^2$ solution $p$ if and only if there exists $u = (u^1, \ldots, u^k) \in \oplus_k L_1^2(N^{(k)})$ for which*

$$\Delta_y u^k + \phi_0^k(u) = \Upsilon_1^{-1} x(g'^k) + \phi_1^k(g') \tag{6.29 a}$$

*holds for each $k$. Here, while $x(g'^k)$ is defined by the condition that $\underline{x} = \Pi \cdot g'^k$.*

- *In addition, if $L_{\Psi_r(y)}p = g'$ has a solution $p$, then*

$$p = \Pi_k(1 - \chi_{4\delta,k})p^0 + \Sigma_k \chi_{100\delta,k} p^k,$$

*where*

*a) $p^0$ obeys*

$$\begin{aligned} ||\nabla p^0||_2 + \sqrt{r}||p^0||_2 & \\ \leq \zeta \cdot (||g'^0||_2 + \Sigma_k(r^{-1/2}||u^k||_{1,2} & \\ + r^{-1/2}||(1-\Pi)g'^k||_2)); & \end{aligned} \tag{6.29b}$$

*b) $p^k = P(h'^k) + \underline{\Upsilon}_1 \underline{u}^k$; here $u^k$ solves (4.62) while $h'^k$ is in the $c = c_k$ version of $L^2(\mathcal{V}_1; K_1)$ and obeys*

$$||h'^k||_2 \leq \zeta(||(1-\Pi)g'^k||_2 + r^{-1/2}||u^k||_{1,2} + ||p^0||_2), \tag{6.29 c}$$

*where (6.29b) can be used to bound $||p^0||_2$.*

The remainder of this subsection is occupied with the

*Proof of Lemma 6.7.* There are three steps to the proof.

*Step 1.* Write $\Psi_r(y)$ as in Proposition 5.2 in terms of data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ from Sections 2 and 3b and in terms of $q' = (a', (\alpha', \beta'))$. Agree now to use $L$ to denote the operator which is given by (4.3) and (4.4) for $(a, (\alpha, \beta)) = (\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$. With this notation change understood, the version of (4.3-4) for $(a, (\alpha, \beta)) = \Psi_r(y)$ has the schematic form

$$L_{\Psi_r(y)} = L + \sqrt{r} 2\varpi'(q', \cdot), \tag{6.30}$$

where $\varpi'(\cdot, \cdot)$ is a certain natural, $r$-independent homomorphism from $\otimes_2(iT^* \oplus S_+)$ to $i(\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-$. In this regard, it is important to note that the component of $\varpi'$ in $i\Lambda_+ \oplus S_-$ is equal to the endomorphism $\varpi$ which appears in (4.2).

*Step 2.* Now suppose that $p \in i\Omega^1 \oplus C^\infty(S_+)$ obeys $L_{\Psi_r(y)}p = g'$. Then $p$ satisfies an equation of the form $Lp + \varsigma p = g'$, where $\varsigma p =$

$\sqrt{r}2\varpi'(q', p)$. Since $|q'| \leq \zeta r^{-1/2}$, the homomorphism $\varsigma$ has a pointwise norm bound which is independent of $r$. This means, in particular, that one can invoke Lemma 4.11 with $q$ there set to equal $p$ here. Then, (6.29b) and (6.29c) follow from (4.53b,c) using $w^k = \Upsilon_1 u^k$. Also, (4.53a) for $w^k$ implies (6.29a) for $u^k$ if one sets $\phi_0^k(u) = \Upsilon_1^{-1}\gamma_0^k(\Upsilon_1 u) + 2\sqrt{r}\Upsilon_1^{-1}x(\Pi\chi_{25\delta,k}\varpi'(q', \underline{\Upsilon}_1\underline{u}^k)) + v_y(u^k)$ where $v_y$ is described by Assertion 5 of Proposition 3.2. (See also (4.49c).) Also, to obtain (6.29a) from (4.53a), it is necessary to set $\phi_1^k = \Upsilon_1^{-1}\gamma_1^k$. Thus, Lemma 6.7 follows with an appropriate bound for the $L^2$ norm of

$$2\sqrt{r}\Upsilon_1^{-1}x(\Pi\chi_{25\delta,k}\varpi'(q', \underline{\Upsilon}_1\underline{u}^k)). \tag{6.31}$$

*Step 3.* The first remark is that $2\sqrt{r}\Upsilon_1^{-1}x(\Pi\chi_{25\delta,k}\varpi'(q^k, \underline{\Upsilon}_1\underline{u}^k))$ differs from (6.31) by a term whose $L^2$ norm is bounded by $\zeta e^{-\sqrt{r}/\zeta}||u^k||_2$. This is because the $q^0$ contribution to (6.31) occurs where the projection $\Pi$ is $\mathcal{O}(e^{-\sqrt{r}/\zeta})$. (See the last line in (5.27).)

Furthermore, the algebraic structure of $\Pi$ demands that

$$\Pi\varpi'(q^k, \underline{\Upsilon}_1\underline{u}^k) = \Pi\varpi(P(h^k), \underline{\Upsilon}_1\underline{u}^k), \tag{6.32}$$

where $h^k$ is given by Lemma 5.5. (This last point is absolutely crucial to the argument.) These last remarks imply that the $L^2$ norm of (6.31) is bounded from above by

$$\zeta(r|||\underline{\Upsilon}_1\underline{u}^k||P(h^k)|||_2 + r^{-1/2}\Sigma_{k'}||u^{k'}||_2'). \tag{6.33}$$

Now, use Hölder's inequality to bound (6.33) by

$$\zeta(r||\underline{\Upsilon}_1\underline{u}^k||_4||P(h^k)||_4 + r^{-1/2}\Sigma_{k'}||u^{k'}||_2'). \tag{6.34}$$

Next, note that $||P(h^k)||_4 \leq \zeta r^{-1}$ using Lemma 5.6 and the second, third and fourth points of (5.27). And, note that $||\underline{\Upsilon}_1\underline{u}^k||_4 \leq \zeta r^{-1/4}||u^k||_4$ because of the last point in (5.27). Finally, use the fact that $||u^k||_4 \leq \zeta||u^k||_{1,2}$ to deduce Lemma 6.7's bound on $||\phi_0^k||_2$ from (6.34).

**e) Proof of (6.28)**

There are six steps to the proof. The first four steps prove that the kernel of $L$ at the $\Psi_r$ image of any point in $\psi_r^{-1}(0)$ is tangent to $\Psi_r(\times_k\mathcal{K}_\Lambda^{(k)})$.

*Step 1.* Let $v \in T(\times_k\mathcal{K}_\Lambda^{(k)})$, and let $p_v$ denote the image under the differential of $\Psi_r$ of $v$. Then $p_v$ satisfies an equation of the form

$L_{\Psi_r(y)} p_v = g'_v$. Here, $g'_v = \Sigma_k \chi_{100\delta,k} \lambda^k v$ and $\lambda^k v$ depends linearly on $v \in T(\times_k \mathcal{K}_\Lambda^{(k)})$.

Concerning this term $\lambda^k v$, remark that it is apriori a section of the bundle $\mathcal{V}_1$ in (4.16). However, the following observations follow from the fact that $\psi_r(y) = 0$ (and thus $\Psi_r(y) \in \mathcal{M}^{(r)}$):

- The projection $(1 - \Pi)$ as defined by $c = \Upsilon(y^k)$ annihilates $\lambda^k v$.
- Likewise, the projection $(1 - Q_\Lambda)$ for $\Lambda = \Lambda_k$ annihilates $\Upsilon_1^{-1} x(\lambda^k v)$.

(6.35)

Thus, $\lambda^k v = \underline{\Upsilon}_1 \underline{x}_v^k$, where $x_v^k \in \Lambda_k$.

Here is the argument for (6.35): When $y \in \times_k \mathcal{K}_\Lambda^{(k)}$, write $\Psi_r(y) = (a, (\alpha, \beta))$, and then the expression in (3.6) defines an element, $\mathbb{H}$, in the summand $i\Omega^2 \oplus C^\infty(S_-)$ of $i(\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$. By construction, $\mathbb{H}$ is given by $\Sigma_k \chi_{100\delta,k} \mathbb{H}^k$ with both $(1 - \Pi)\mathbb{H}^k = 0$ and also $(1 - Q_\Lambda)\Upsilon_1^{-1}\mathbb{H}^k = 0$. The term $\lambda^k v$ is the differential of $\mathbb{H}^k$ at $y \in \psi_r^{-1}(0)$. In this regard, it proves useful to remember the following: Suppose that $\sigma(\cdot)$ is a differentiable map from the interval $(-1, 1)$ into a vector space, $V$. Then, let $P_{(\cdot)}$ be a family of projections on $V$ which is differentiably parameterized by $(-1, 1)$. Finally, suppose that $P_{(\cdot)}\sigma(\cdot) = 0$. Let $\sigma'$ denote the derivative of $\sigma$ with respect to $t$, and let $P'$ denote the derivative of $P$ with respect to $t$. Then, $P\sigma' = -P'\sigma$, so $P\sigma' \neq 0$ in general. However, $P\sigma'$ automatically vanishes at values of $t$ where $\sigma(t) = 0$.

*Step 2.* Suppose now that $p'$ is annihilated by $L_{\Psi_r(y)}$. Given $v \in T(\times_k \mathcal{K}_\Lambda^{(k)})$, then $p = p' - p_v$ obeys the equation $L_{\Psi_r(y)} p = g' = \Sigma_k \chi_{100\delta,k} \lambda^k v$, and thus is described by Lemma 6.7. More to the point, for a particular choice of $v$, each $u^k$ which appears in (6.29a) can be assumed to be $L^2$-orthogonal to $T(\mathcal{K}_\Lambda^{(k)})$. This is to say that $u^k$ is $L^2$-orthogonal to the kernel of $(1 - Q_\Lambda)\Delta_y$ in the case where $y = y^k$ and $\Lambda = \Lambda^k$.

This last condition on the $u = (u^1, \ldots, u^n)$ will be assumed in the subsequent steps, with the goal to prove that $p = 0$. (This implies that $p'$ is in the image of the differential of $\Psi_r$.) The proof that $p = 0$ is completed in Step 4.

*Step 3.* Because of (6.35), each $u^k$ in (6.29a) obeys

$$(1 - Q_\Lambda)(\Delta_y u^k + \phi_0^k(u)) = f^k, \tag{6.36}$$

where $||f_k||_2 \le \zeta e^{-\sqrt{r}/\zeta}\Sigma_k||x_v^k||_2$. (Both $(1-\chi_{25\delta,k})\chi_{100\delta,k}\lambda^k v$ and $(1-\Pi)\chi_{25\delta,k}\lambda^k v$ are exponentially small because of (6.35).) Meanwhile, since $u^k$ is $L^2$ orthogonal to the kernel of $(1-Q_\Lambda)\Delta_y$, one has $||u^k||_{1,2} \le \zeta||(1-Q_\Lambda)\Delta_y u^k||_2$. This last fact plus Lemma 6.7's estimate for $\phi_0^k$ implies that

$$\Sigma_k||u^k||_{1,2} \le \zeta e^{-\sqrt{r}/\zeta}\Sigma_k||x_v^k||_2, \tag{6.37}$$

when $r > \zeta$ and $\varepsilon < \zeta^{-1}$ for some fixed constant $\zeta$.

*Step 4.* Now project (6.29a) onto $\Lambda = \Lambda_k$. Because $\Upsilon_1^{-1}x(\varsigma^k v) = x_v^k$ by definition, the latter projection and (6.37) imply that for each $k$,

$$||x_v^k||_2 \le \zeta e^{-\sqrt{r}/\zeta}\Sigma_{k'}||x_v^k||_2. \tag{6.38}$$

The latter equation is consistent at large $r$ only if $x_v^k = 0$ for all $k$. That is, only if $\lambda^k v = 0$.

The condition $x_v^k = 0$ for all $k$ implies via (6.40) that $u^k = 0$ for all $k$. This last conclusion yields (via (6.29b) and (6.29c)) that $p = p' - p_v = 0$.

*Step 5.* It has now been established that the kernel of $L$ at points in the $\Psi_r$ image of $\psi_r^{-1}(0)$ are tangent to the image of $\Psi_r$. This step demonstrates that the kernel of $L$ at such points lies in the $\Psi_r$ image of the kernel of the differential of $\psi_r$. For this purpose, consider a point $y \in \psi_r^{-1}(0)$ and an element $v$ in the kernel of $L$ at $\Psi_r(y)$. Let $\gamma : (-1,1) \to \times_k\mathcal{K}_\Lambda^{(k)}$ be a smooth path which obeys $\gamma(0) = y$ and $(\Psi_r)_*(\gamma'|_0) = v$. Here, $\gamma'$ is the derivative of $\gamma$.

Because $v$ is in the kernel of $L$, the norm of $\mathbb{H}$ (that is, (3.6)) at $\Psi_r(\gamma(t))$ is bounded by a uniform multiple of $t^2$. This implies that the norm of $\psi_r(\gamma(t))$ is also bounded by a uniform multiple of $t^2$. (The norm of $\underline{\Upsilon_1\psi_r}$ at any $y$ is bounded from above by a multiple of that of $\mathbb{H}(\Psi_r(y))$ since the former is obtained by projecting the latter.)

It follows from the preceding remark that the differential of $\psi_r$ annihilates the derivative of $\gamma$ at $t = 0$. Thus, as claimed, $v$ lies in the image by the differential of $\Psi_r$ of an element in the kernel of the differential of $\psi_r$.

*Step 6.* The purpose of this last step is to prove that the differential of $\Psi_r$ maps kernel($\psi_{r*}$) into the image of kernel($L$) at points in $\psi_r^{-1}(0)$. To begin the proof, consider a point $\Xi = (a,(\alpha,\beta)) \in (Conn(E) \times C^\infty(S_+))/C^\infty(X;S^1)$ and, as described in the introduction to this section, identify a neighborhood of $\Xi$ with a ball $\mathcal{B} \subset \mathcal{T}_\Xi$

about the origin. The Seiberg-Witten equations in (1.13) restrict to $\mathcal{B}$ to define a map $\mathcal{F} : \mathcal{B} \to i\Omega^{2+} \oplus C^\infty(S_-)$. Together, the differential of $\mathcal{F}$ at $0 \in \mathcal{B}$ and the slice condition in (6.1) define the operator $L$ for $\Xi$.

Let $y \in \psi_r^{-1}(0)$ and let $y' \in \times_k \mathcal{K}_\Lambda^{(k)}$ be near to $y$. Then, $\Psi_r(y')$ satisfies an equation of the form

$$\mathcal{F}(\Psi_r(y')) = \mathcal{G}(y')\psi_r(y'), \tag{6.39}$$

where $\mathcal{G}$ is a smooth map from a neighborhood of $y$ in $\times_k \mathcal{K}_\Lambda^{(k)}$ into the space of injective homomorphisms from $\times_k \Lambda_k$ to $i\Omega^{2+} \oplus C^\infty(S_-)$.

With the preceding understood, fix a non-zero vector $v$ in the kernel of the differential of $\psi_r$ at $y$. Let $\gamma : (-1, 1) \to \times_k \mathcal{K}_\Lambda^{(k)}$ be a smooth path with the property that $\gamma(0) = y$ and $\gamma'(0) = v$. Use (6.47) to compute the derivative of $\mathcal{F}(\gamma(\cdot))$ at $t = 0$. The resulting expression proves that the differential of $\Psi_r$ maps kernel$(\psi_{r*})$ into kernel$(L)$.

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