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ON DEFORMING CONFOLIATIONS

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Abstract

We present a parabolic deformation of one-forms on compact, orientable, odd-dimensional manifolds. The flow produces contact forms from the class of initial conditions called "conductive confoliations". We give applications of these techniques to new constructions of contact forms on products of contact manifolds with surfaces. In particular, we produce contact forms on the product of any three dimensional manifold with any surface.

1. Introduction

1.1. Results. In this paper, manifolds are assumed to be smooth, compact, orientable and endowed with a Riemannian metric. A one-form η is contact on a $2n+1$ dimensional manifold if at every point $\eta \wedge (d\eta)^n \neq 0$. A global volume form may be chosen to reformulate the (positive) contact condition as

$$\star(\eta \wedge (d\eta)^n) > 0. \tag{1.1}$$

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If η only satisfies the weaker inequality

$$\star\left(\eta \wedge (d\eta)^n\right) \geq 0 \tag{1.2}$$

it is a so-called (positive) "confoliation" [6].

Deformations of confoliations were studied in [1], and later [6], only for the case of 3-dimensional manifolds. As in [1], we present a heat equation tailored to diffuse the "positivity" of the form in the contact region throughout the rest of the manifold. The transport mechanism for this heat equation is related to the two-form

$$\tau = \star\left(\eta \wedge (d\eta)^{n-1}\right). \tag{1.3}$$

and conditions on this form are needed to ensure heat flow throughout the manifold.

Loosely speaking, we call a confoliation "conductive" (see §1.3) if every point on M^{2n+1} can be connected to a point where η is contact by a curve whose tangent vector is in the "range" of τ. The subset of smooth one-forms that are conductive confoliations are denoted by $Con\left(M^{2n+1}\right)$. Our main result is as follows:

Theorem 1.1. *If $\eta \in Con\left(M^{2n+1}\right)$, then η is C^∞ close to a contact form.*

The analytical program to prove this result may be summarized as follows:

1. Define a diffusion equation on one-forms that preserves the confoliation condition for all time;

2. Prove a strong maximum principle so that under suitable conditions confoliations become contact; this is subtle as the transport mechanism for the equation does not produce diffusion in all directions;

3. Characterize the class of so-called "conductive confoliations" that satisfy the conditions necessary in step 2 for the maximum principle.

We use this flow to demonstrate new constructions of contact forms for certain products of contact manifolds with surfaces. The following result resolves a case left open by the work of Lutz [10].[1]

[1] We have been informed by Y. Eliashberg that M. Gromov has an unpublished approach [8] using branched coverings over $M^3 \times S^2$.

Theorem 1.2. *Let Σ^2 be any compact orientable surface, and M^3 be any compact, orientable 3-manifold. Then $M^3 \times \Sigma^2$ is contact.*

We next include an explicit construction of a conductive confoliation on $S^{3+2p} \times \Sigma^2$. The construction of conductivity is more sophisticated than the one needed for the previous result and is included as a case study. It is already established that $S^{3+2p} \times \Sigma^2$ is contact.

At the end of the paper, we indicate a number of additional applications. In particular we extend the techniques to generalizations of contact forms. For example, under analogous conductivity conditions, a 3-form $\eta_{(3)}$ on a 7-manifold may be produced satisfying $\star\left(\eta_{(3)} \wedge d\eta_{(3)}\right) > 0$.

A second generalization is deforming differential forms that have a notion of non-integrability when paired with a background structure. For example, a precondition for a manifold M^{2n+1} to be contact ([2]) is a 1-form η and a globally defined 2-form Φ such that $\star(\eta \wedge \Phi^n) > 0$ everywhere. Again, under suitable conductivity conditions, "intermediate" measurements of non-integrability such as $\star\left(\eta \wedge d\eta^k \wedge \Phi^{n-k}\right) > 0$ are constructed.

1.2. Comments. The existence of a contact form (or a confoliation) on a closed compact manifold is a topological condition. There are known topological obstructions to the existence of contact forms. When contact forms do exist, it is well known by the classical integrability theorem of Darboux that all contact forms of the same dimension are locally equivalent.

In three dimensions, the topological obstructions to finding contact forms vanish. It was shown by Martinet [12] that all three manifolds have contact forms. Work by Gonzolo [9] in fact demonstrated the paralellization of every three manifold by contact forms. In five dimension, existence results were obtained by Lutz [10] for the case of tori bundles over three-manifolds, and in particular for the cross-product of 3-manifolds with two-dimensional tori. However, this approach relied on the special structure of tori and did not extend to general surfaces. Results in this dimension were also obtained by Thomas [15] for simply connected manifolds.

Many properties of contact and confoliation forms in low dimensions have since been studied in depth (see [6]). In contrast, for higher dimensions, a general program for even demonstrating the existence of confoliations and contact forms is lacking. On many spaces where all known obstructions vanish, such as the cross-product of a contact manifold with a surface, contact forms are conjectured to exist. An indication of the remarkable difficulty encountered in this subject is that the existence of contact forms, even on such spaces as the odd-dimensional tori, is an open question.

There are several factors contributing to the difficulty of constructing contact forms on manifolds of dimensions greater than three. Foremost among these is that condition (1.1) places a non-linear condition on the antisymmetrized matrix of first derivatives of η. This makes an approach of adding locally defined contact forms together to form a global contact form difficult. Adding to the formidable computational difficulty of constructing contact forms is the seemingly non-geometric nature of the hyper-plane distribution of the tangent bundle defined by the null space of η. In contrast to integrable distributions that define foliations, contact distributions are "maximally twisted." This makes contact distributions difficult to visualize at scales larger than those given by Darboux.

Since progress in producing high dimensional contact forms has been difficult, new approaches such as those presented in this paper may prove valuable for approaching the general problem of existence. Additionally, the techniques presented in this paper may provide new examples of non-homotopically equivalent contact forms.

1.3. Definitions. In this section we discuss the conditions needed for the class of so-called "conductive confoliations." Recall that these are forms suited for perturbations by our heat equation techniques. Examples illustrating structures in this section will be given in § 1.4. Let $\alpha \in \Lambda^1\left(M^{2n+1}\right)$ and g be a metric on M^{2n+1}. Recall from (1.3) the two-form

$$\tau = \star\left(\alpha \wedge (d\alpha)^{n-1}\right). \tag{1.4}$$

The "square" of τ is defined on two vector fields X and Y by

$$\langle X, Y\rangle_a = \langle i_X\tau, i_Y\tau\rangle_g \tag{1.5}$$

or, in local coordinates,

$$a_{ij} = \tau_{im}\tau_{jn}g^{mn}. \tag{1.6}$$

The vector space of degenerate directions of a is denoted by

$$Null(a) = \{X \,|\, \langle X, Y\rangle_a = 0, \forall Y\} \tag{1.7}$$

$Null(a)^{\perp}$ is its orthogonal complement with respect to g. It may be easily seen that $Null(\tau) = \{X | i_X \tau = 0\} = Null(a)$ and $Null(\tau)^{\perp} = Null(a)^{\perp}$. With this in mind, we make the following definition.

Definition 1.3. A point $p \in M^{2n+1}$ is *accessible* from $q \in M^{2n+1}$ if there is a smooth path $x : [0,1] \to M^{2n+1}$ from p to q with $x'(s) \in Null(a)^{\perp}$ for all s.

We now define the class of confoliations amenable to perturbation by our heat equation. These are confoliations that are able to conduct heat, i.e., "contactness", to all points of the manifold via paths in the range of a.

Definition 1.4. The space of conductive confoliations, $Con\left(M^{2n+1}\right)$, is defined to be the subset of $\alpha \in \Lambda^1(M^{2n+1})$ such that

1. α is a confoliation: $\star(\alpha \wedge (d\alpha)^n) \geq 0$;

2. every point is accessible from a contact point of α.

Note that at a point where $Rank\left(d\alpha|_{Null(\alpha)}\right) = 2n$, α is contact. A computation in coordinates given by Darboux's theorem yields $Rank(a) = 2n$.

At a point where $Rank\left(d\alpha|_{Null(\alpha)}\right) = 2n-2$ we choose an orthonormal frame $\{Z, X_1, \cdots, X_{2n-2}, Y_1, Y_2\}$ with the properties

$$\alpha(Z) \neq 0, \quad \alpha(X_i) = \alpha(Y_j) = 0, \quad d\alpha(Y_j, \cdot) = 0 \quad \forall i,j. \tag{1.8}$$

One may compute that $\tau(Z,\cdot) = 0, \tau(X_i,\cdot) = 0$ for all i, and $\tau(Y_1, Y_2) \neq 0$. Hence, $Null(a)^{\perp} = Span\{Y_1, Y_2\} = Null(d\alpha)$ and $Rank(a) = 2$.

In contrast, if $Rank\left(d\alpha|_{Null(\alpha)}\right) < 2n-2$, then $\alpha \wedge (d\alpha)^{n-1} = 0$. Hence $\tau = 0$ and $Null(a)^{\perp} = \{0\}$.

1.4. Examples. We now give examples that help illustrate concepts introduced in the previous section.

Example 1. In 3-dimensions, the previous discussion indicates that $Rank(a) = 2$ whether α is contact or not. It is also the case that $Null(a)^{\perp} = Null(\alpha)$.

Example 2. Let $\mathbb{R}^5$ be given with coordinates $\{z, x_1, y_1, x_2, y_2\}$, Euclidean metric, and volume form $dz \wedge dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$. The one-form $\alpha = dz + x_1 dy_1$ is a confoliation. Using $\tau = dx_2 \wedge dy_2$, it follows that $Null(a) = span\{\frac{\partial}{\partial z}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\}$. Hence, $\langle \cdot, \cdot \rangle_a$ is proportional to the Euclidean metric on the $\{x_2, y_2\}$-plane and degenerate otherwise.

If α is contact, say $\alpha = dz + x_1 dy_1 + x_2 dy_2$, then the null space is one-dimensional and given by $Null(a) = span\{\frac{\partial}{\partial z} + x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2}\}$. This is seen from the fact $\tau = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 - x_1 dx_1 \wedge dz - x_2 dx_2 \wedge dz$.

Next, we construct an example of a confoliation that is not conductive. Let $\psi = \psi(x_1, y_1, z)$ be a smooth function such that $\psi(x_1, y_1, z) > 0$ for $|x_1|^2 + |y_1|^2 + |z|^2 < 1$ and $\psi(x_1, y_1, z) = 0$ otherwise. Then for

$$\alpha = dz + x_1 dy_1 + \psi(x_1, y_1, z) x_2 dy_2$$

one may compute

$$\alpha \wedge d\alpha^2 = 2\psi(x_1, y_1, z) dz \wedge dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2.$$

Hence α is contact only in the region where $\psi > 0$. Outside this region, $\alpha = dz + x_1 dy_1$ and it follows from comments at the beginning of this example that non-contact points are not accessible from points in the contact region.

Example 3. We next look at a construction useful for constructing contact forms in lower dimensions and examine why it fails to produce a conductive confoliation in higher dimensions.

Let α be a contact form on M^{2n+1}. Let (r, θ) be polar coordinates on a unit 2-dimensional ball B^2. Then, for functions $a(r)$ and $b(r)$ we consider the family one-form on $M^{2n-1} \times B^2$ defined by

$$\eta = a(r)\alpha + b(r) d\theta.$$

This form is similar to the "propeller" contact one-form (see [16],[1],[6] for geometric interpretations). One may compute

$$\begin{aligned} d\eta &= a(r) d\alpha + a'(r) dr \wedge \alpha + b'(r) dr \wedge d\theta \\ d\eta^{n+1} &= a^{n+1}(r) d\alpha^{n+1} + (n+1) a^n a'(r) d\alpha^n \wedge dr \wedge \alpha \\ &\quad + (n+1) a^n b'(r) d\alpha^n \wedge dr \wedge d\theta \\ \eta \wedge (d\eta)^{n+1} &= (n+1) a^n \begin{vmatrix} a & a' \\ b & b' \end{vmatrix} \alpha \wedge (d\alpha)^n \wedge dr \wedge d\theta \end{aligned} \tag{1.10}$$

To look for conditions on a, b that give a sign to (1.10), we call out the following interpretation of the determinant condition.

Definition 1.5 (Propeller curves). Viewing $X(r) = (a(r), b(r))$ as a curve in $\mathbb{R}^2$ (see Figure(1)), we require

1. $a = -1$ and $b = -r$ near $r = 0$,

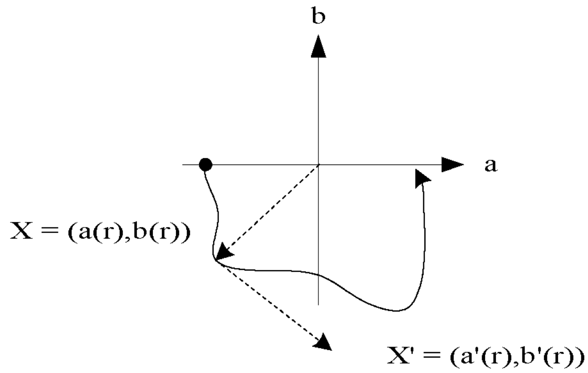

Figure 1: Propeller Curves

2. $a \to 1$ and $a', b, b', \to 0$ as $r \to 1$,

3. X and X' are never collinear and $X(r) \neq (0,0)$,

4. X rotates counter-clockwise, hence $(ab' - ba') > 0$ for $r < 1$.

It is easy to find curves satisfying these conditions. For such a curve, η is well defined. For n even, $\eta \wedge (d\eta)^{n+1} \geq 0$. However, for r_0 such that $a(r_0) = 0$, it is easily seen that $\eta \wedge (d\eta)^{n+1}$ and $\eta \wedge (d\eta)^n$ vanish and hence $\eta \notin Con\left(M^{2n+1}\right)$.

Example 4. Finally, we study two relevant degenerate quadratic forms. Let $\psi(s)$ be a smooth function that is positive for $-1 < s < 1$ and 0 elsewhere. In $\mathbb{R}^2$ with coordinates (x, y) define

$$a_1 = \begin{pmatrix} \psi(x) & 0 \\ 0 & 1 \end{pmatrix} \qquad a_2 = \begin{pmatrix} \psi(y) & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.11}$$

Points outside of $R = (-1,1) \times \mathbb{R}^1$ are not accessible to points inside of R by curves in the range of a_1. This is essentially the situation in example 2 above. In contrast, all points in $\mathbb{R}^2$ are accessible to each other by curves in the range of a_2 (see Figure(2)). This is the model of the quadratic forms a given by conductive confoliations.

2. The flow

Many of the computations in this section may be found in [1] for the special case of three-dimensional manifolds.

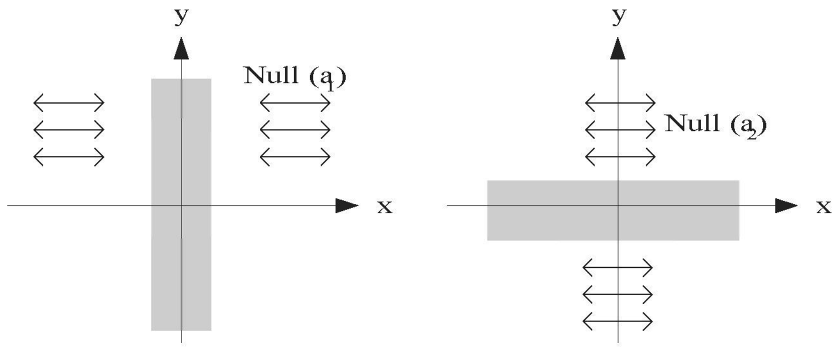

Figure 2: Degenerate Quadratic Forms

2.1. Cross Term Energy. Let $\eta = \alpha + \epsilon\beta$. Then we may expand

$$\begin{aligned}\eta \wedge (d\eta)^n &= \alpha \wedge (d\alpha)^n \\ &\quad + \epsilon \Big(\beta \wedge (d\alpha)^n + n\alpha \wedge (d\alpha)^{n-1} \wedge (d\beta)\Big) + O(\epsilon^2).\end{aligned} \tag{2.1}$$

Definition 2.1. Let the first order cross terms be defined by the function

$$f(\alpha, \beta) = \frac{1}{n} \star \Big(\beta \wedge (d\alpha)^n + n\alpha \wedge (d\alpha)^{n-1} \wedge (d\beta)\Big). \tag{2.2}$$

The normalization of $1/n$ is used for convenience in later equations.

The following is evident, and is justification for studying f more carefully.

Proposition 2.2. *If $\star(\alpha \wedge (d\alpha)^n) \geq 0$ and $f(\alpha, \beta) > 0$ then ϵ may be chosen small enough to make η contact.*

Our problem then, may be neatly summarized by the task of making $f > 0$.

Note that f measures some of the first derivatives of β. Working by analogy with the derivation of the standard heat equation $\frac{\partial}{\partial t}u = \Delta u$ from $E(u) = \int |\nabla u|^2 \, d\mu$ we define the cross term energy of α and β by

$$E(\alpha, \beta) = \int f^2(\alpha, \beta) d\mu \tag{2.3}$$

We take the first variation of the energy with respect to β treating α as a constant. That is,

$$E'(\alpha, \beta) = \frac{d}{du} E(\alpha, \beta + u\beta')|_{u=0}. \tag{2.4}$$

We then integrate by parts.

$$
\begin{aligned}
\frac{n}{2}E'(\alpha,\beta) &= \int f\cdot\Big(\beta'\wedge(d\alpha)^n + n\alpha\wedge\big(d\beta'\big)\wedge(d\alpha)^{n-1}\Big)\\
&= \int \beta'\wedge f(d\alpha)^n + nf\alpha\wedge\big(d\beta'\big)\wedge(d\alpha)^{n-1}\\
&= \int \beta'\wedge f(d\alpha^n) - nd(f\alpha\wedge\beta\wedge(d\alpha)^{n-1})\\
&\quad + nd(f\alpha)\wedge\beta'\wedge(d\alpha)^{n-1}\\
&= \int (n+1)\beta'\wedge f(d\alpha)^n - n\beta'\wedge\alpha\wedge(d\alpha)^{n-1}\wedge df
\end{aligned}
\tag{2.5}
$$

From 2.5 we see that a gradient descent for the energy is given by the variation

$$
\beta' = \star\big(\alpha\wedge(d\alpha)^{n-1}\wedge df\big) - \frac{n+1}{n}f\star(d\alpha)^n \tag{2.6}
$$

2.2. The Evolution Equations. The deformation defined in (2.6) has a number of interesting properties and deserves attention in its own right. However, this article has the primary goal of turning confoliations into contact forms; for this purpose it turns out that the *zero*th-order term in f may be dropped. The reduced evolution has the advantages of simplifying later computations.

The deformation of 2.6 is therefore motivation for the following definition.

Definition 2.3 (The contact flow). Let $\alpha(\cdot),\beta(\cdot,t)\in\Lambda^1\left(M^{2n+1}\right)$ where α is a time-independent and β varies in time. We define the *contact flow* to be

$$
\begin{cases}\frac{\partial}{\partial t}\beta = \star(\alpha\wedge(d\alpha)^{n-1}\wedge df)\\ \beta(\cdot,0)=\alpha(\cdot)\end{cases} \tag{2.7}
$$

where

$$
f = \star\frac{1}{n}\Big(\beta\wedge(d\alpha)^n + n\alpha\wedge(d\alpha)^{n-1}\wedge(d\beta)\Big). \tag{2.8}
$$

Note that one may more generally write $\beta(\cdot,0)=\beta_0(\cdot)$. In our case where $\beta_0(\cdot)=\alpha(\cdot)$ and α is a confoliation, the function $f(\cdot,0) = \frac{n+1}{n}\star(\alpha\wedge(d\alpha)^n)$ and will be non-negative by assumption.

It is important to study the induced evolution of f since, by Proposition 2.2, f is the key to determining whether $\eta = \epsilon\alpha + \beta$ can be made contact. The quadratic form a_{ij} as defined in §1.3 is used to define a Laplacian-like second order operator

$$\Delta_a f = a^{pq} \nabla_p \nabla_q f \tag{2.9}$$

where $a^{pq} = g^{pi} g^{qj} a_{ij}$.

Proposition 2.4. *The evolution of f may be written as*

$$\frac{\partial}{\partial t} f = \Delta_a f + \nabla_X f \tag{2.10}$$

where $X = X(\alpha, \nabla\alpha)$ is a vector field depending only on α and its first derivatives.

Proof. From the definition of f and equation (2.7), one has

$$\begin{aligned}
\frac{\partial}{\partial t} f &= \star\frac{1}{n}\left(\frac{\partial}{\partial t}\beta \wedge (d\alpha)^n + n\alpha \wedge (d\alpha)^{n-1} \wedge \left(d\frac{\partial}{\partial t}\beta\right)\right) \\
&= \star\left(\alpha \wedge (d\alpha)^{n-1} \wedge \left(d\star\left(\alpha \wedge (d\alpha)^{n-1} \wedge df\right)\right)\right) & (2.11) \\
&\quad + \star\frac{1}{n}\left(\star\left(\alpha \wedge (d\alpha)^{n-1} \wedge df\right) \wedge (d\alpha)^n\right). & (2.12)
\end{aligned}$$

Term (2.12) contributes only first derivatives of f while term (2.11) contributes first and second derivatives. We leave it as an exercise to rewrite (2.11) in local coordinates as

$$g^{ij} g^{pq} g^{rs} \tau_{ip} \nabla_q \left(\tau_{jr} \nabla_s f\right) = \Delta_a f + g^{rs}\left(g^{ij} g^{pq} \tau_{ip} \nabla_q \tau_{jr}\right) \nabla_s f \tag{2.13}$$

q.e.d.

The proof of existence of solutions follows the approach developed in §3 of [1].

Theorem 2.5. *The contact flow defined by equations 2.7 has a unique, smooth solution on $M^{2n+1} \times [0, \infty)$.*

Proof. We start with the observation that the contact flow may be viewed as a coupled system for the pair (β, f) defined jointly by equations (2.7) and (2.10) with initial data $(\alpha, f(\cdot, 0))$ Now, the evolution of f given in equation (2.10) above is decoupled from β and may be studied separately.

The second order operator on f is not strictly elliptic as a will not have full rank. However, one can regularize the equation for f by artificially adding in a positive multiple ϵ of the full Laplacian

$$\frac{\partial}{\partial t} f_\epsilon = \epsilon \Delta f_\epsilon + \Delta_a f_\epsilon + \nabla_X f_\epsilon \tag{2.14}$$

By standard theory of parabolic equations, equation(2.14) has short time existence on a positive time interval.

As in Theorem 3.4 [1], the modified equation is used to obtain estimates *independent of* ϵ on all derivatives of f_ϵ. We show below estimates on f_ϵ and its first derivatives.

Using

$$\frac{\partial}{\partial t} f_\epsilon = \epsilon \Delta f_\epsilon + a^{ij} \nabla_i \nabla_j f_\epsilon + X^i \nabla_i f_\epsilon$$

one may compute

$$\frac{\partial}{\partial t} f_\epsilon^2 = \epsilon \left(\Delta f_\epsilon^2 - 2|\nabla f_\epsilon|^2 \right) + \Delta_a f_\epsilon^2 - 2a^{ij} \nabla_i f_\epsilon \nabla_j f_\epsilon + \nabla_X f_\epsilon^2 \tag{2.16}$$

and the weak maximum principle implies $\max f_\epsilon^2(t) \leq \max f_\epsilon^2(0)$ for all time.

We then compute

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla f_\epsilon|^2 = {} & \epsilon \left(\Delta |\nabla f_\epsilon|^2 - 2|\nabla^2 f_\epsilon|^2 - 2g^{pq} g^{rs} R_{pr} \nabla_q f_\epsilon \nabla_s f_\epsilon \right) \\ & + \Delta_a |\nabla f_\epsilon|^2 - 2a^{ij} g^{pq} \nabla_i \nabla_p f_\epsilon \nabla_j \nabla_q f_\epsilon \\ & + 2g^{pq} \nabla_p a^{ij} \nabla_i \nabla_j f_\epsilon \nabla_q f_\epsilon \\ & + 2a^{ij} g^{pq} g^{rs} R_{pijr} \nabla_q f_\epsilon \nabla_s f_\epsilon + X^i \nabla_i |\nabla f_\epsilon|^2 \\ & + 2g^{pq} \nabla_p X^i \nabla_i f_\epsilon \nabla_q f_\epsilon \end{aligned} \tag{2.17}$$

The evolution of $|\nabla f_\epsilon|^2$ may be estimated as follows

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla f_\epsilon|^2 \leq {} & \epsilon \left(\Delta |\nabla f_\epsilon|^2 + 2C |\nabla f_\epsilon|^2 \right) \\ & + \Delta_a |\nabla f_\epsilon|^2 + \nabla_X |\nabla f_\epsilon|^2 + 2C |\nabla f_\epsilon|^2 \end{aligned} \tag{2.18}$$

for some constant C. To obtain this inequality, it is necessary to show that there exists a constant C_1 such that

$$g^{pq} \nabla_p a^{ij} \nabla_i \nabla_j f_\epsilon \nabla_q f_\epsilon \leq g^{pq} a^{ij} \nabla_i \nabla_p f_\epsilon \nabla_j \nabla_q f_\epsilon + C_1 |\nabla f_\epsilon|^2. \tag{2.19}$$

Using $a^{ij} = g^{ir}g^{js}g^{kl}\tau_{rk}\tau_{sl}$ and

$$\nabla_p a^{ij} = g^{ir}g^{js}g^{kl}\left(\nabla_p \tau_{rk}\right)\tau_{sl} + g^{ir}g^{js}g^{kl}\tau_{rk}\left(\nabla_p \tau_{sl}\right)$$

equation(2.19) results from an application of Cauchy's inequality.

Now, letting $W_\epsilon = |\nabla f_\epsilon|^2 + Cf_\epsilon^2$,

$$\frac{\partial}{\partial t} W_\epsilon \leq \epsilon \Delta W_\epsilon + \Delta_a W_\epsilon + \nabla_X W_\epsilon + CW_\epsilon \tag{2.20}$$

and the weak maximum principle implies

$$\max |\nabla f_\epsilon|^2(t) \leq \max W_\epsilon(t) \leq \max W_\epsilon(0)e^{Ct}$$

for all time. Hence the first derivatives of f_ϵ are bounded independent of ϵ.

The interested reader can consult [1] for similar computations of higher derivative estimates.

A smooth solution to equation (2.10) for f may now be obtained as a limit of solutions for f_ϵ as $\epsilon \to 0$. The solution for f yields a solution β to evolution (2.7). The estimates above give a solution (β, f) existing for all time. The weak maximum principle implies the solution for f, hence β, is unique. Thus condition (2.8) holds true for all time. q.e.d.

The following is a straightforward application of the weak maximum principle to the evolution equation for f (2.10). As desired, a flow that preserves the confoliation condition has been produced.

Proposition 2.6. *If $f(\cdot, 0) \geq 0$, then $f(\cdot, t) \geq 0$ for all $t \geq 0$.*

The strong maximum principle was developed in [3] for the case of constant rank degenerate elliptic operators and later by [14] for a general class of degenerate parabolic operators. For convenience, we present a special case of the result in §4 of [14] as needed for our perturbation of conductive confoliations to contact forms.

Let L be an operator on $M^{2n+1} \times [0, \infty)$ of the form

$$L = \nabla_i \left(a^{ij}\nabla_j\right) + b^i\nabla_i - \frac{\partial}{\partial t} \tag{2.21}$$

such that a and b are smooth and $a^{ij} \geq 0$. For a point $x_0 \in M^{2n+1}$ define $A(x_0)$ to be the closure of points $(x(t), t)$ in $M^{2n+1} \times [0, \infty)$ where $x(t) : [0, t] \to M^{2n+1}$ is a smooth path satisfying $x(0) = x_0$ and, for all $t \geq 0$, $x'(t) + b(t) = w(t)$ where $w^i = a^{ij}v_j$ for some v. Essentially, $A(x_0)$ contains points in $M^{2n+1} \times [0, \infty)$ reachable from $(x_0, 0)$ by graphs of curves whose spatial tangent vectors are in the range of a plus the drift term b.

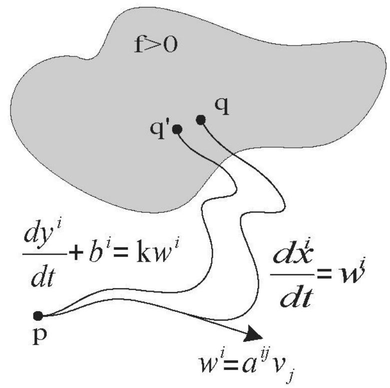

Figure 3: Accessible Points

Theorem 2.7. *[14] Let u be a solution on $M^{2n+1}\times[0,T)$ to $L(u)\leq 0$ such that $u\geq 0$ at $t=0$. If $u(x_0,0)>0$ then $u(x,t)>0$ for all $(x,t)\in A(x_0)$.*

We encourage the reader to think about the diffusion properties L in the case where a is the degenerate form given by either a_1 or a_2 in §1.4, example 4.

The operator for f given in equation (2.10) may be written in the form of equation (2.21). We are now ready to show that conductive confoliations may perturbed to contact forms by the contact flow.

Theorem 2.8. *If $\alpha\in Con\left(M^{2n+1}\right)$ then α is C^∞ close to a contact form.*

Proof. The definition of conductive confoliations (1.3) requires that every point be accessible by a curve whose tangent is in the range of a. To apply the strong maximum principle stated above, we need consider curves that additionally incorporate the first order drift term b.

For a confoliation, if p is accessible from some point q where $f(q,0)>0$, then there exists a smooth path from q to p satisfying $x'=w$ where $w^i=a^{ij}v_j$ for a covector field v on M^{2n+1}. However, by choosing k large enough, one may find a path $y(t)$ satisfying $y'(s)+b(s)=k\cdot w(s)$ connecting q' to p where q' is some point near q and $f(q',0)>0$ (see Figure (3)).

Since every point on a conductive confoliation is accessible from a point where the intial conditions are positive, it follows from the strong maximum principle that conductive confoliations instantly become contact under the contact flow. q.e.d.

3. New constructions of contact forms

In this section, we give applications of the contact flow to new constructions of contact forms. An interesting question is whether a contact manifold crossed with a surface is contact. A general answer to this is unknown. As mentioned in the introduction, the question of existence even in the case of such standard spaces as the odd-dimensional tori of dimension > 5 is open. Case studies of product manifolds can provide insight for existence of contact forms on more general manifolds.

In [10], Lutz gave a construction for constructing contact forms on torus bundles over manifolds that can be described as fibered knots. The contact forms produced are invariant under the group action of the tori on the total space. For dimension 5 his techniques yielded contact forms on every T^2 fiber bundle over a 3-manifold. In particular, Lutz produced forms on $M^3 \times T^2$ and gave the first known contact form on the torus T^5. The constructions in [10] make specific use the fact that the fibers are tori and the techniques do not carry over to more general types of product manifolds.

Below, we extend Lutz's results to all surface cross-products in dimension 5. We do this by producing a very simple conductive confoliation and then deforming it to a contact form.

Theorem 3.1. *Let Σ^2 be any compact, orientable surface, and M^3 be any compact, orientable 3-manifold. Then $M^3 \times \Sigma^2$ is contact.*

Proof. Let $\{\psi_1, \psi_2, \psi_3\}$ be a frame of one-forms on M^3 such that ψ_1 is contact. In fact, by work of Gonzalo [9], the frame may be chosen so that ψ_2 and ψ_3 are also contact, but these extra conditions are unnecessary for our construction.

Let (x_1, x_2) represent coordinates on a ball $B^2 \subset \Sigma^2$. Define

$$\alpha = a(x_1, x_2)\psi_1 + b(x_1, x_2)\psi_2 + c(x_1, x_2)\psi_3. \tag{3.1}$$

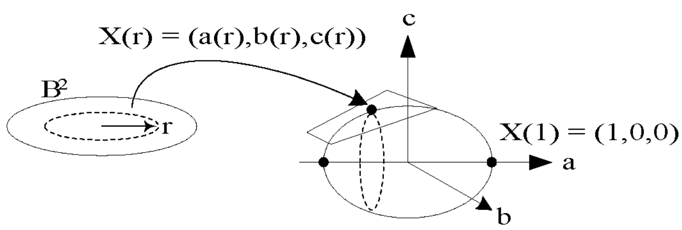

Figure 4: Propeller Surface

Then

$$(3.2)\qquad \begin{aligned} d\alpha = {} & a(x_1,x_2)d\psi_1 + da(x_1,x_2)\wedge\psi_1 \\ & + b(x_1,x_2)d\psi_2 + db(x_1,x_2)\wedge\psi_2 \\ & + c(x_1,x_2)d\psi_3 + dc(x_1,x_2)\wedge\psi_3. \end{aligned}$$

Since $\alpha \wedge d\alpha \wedge d\alpha$ needs to contain $dx \wedge dy$ to be a non-zero multiple of the volume form, no terms arising from $d\alpha$ in the collection $\{d\psi_1, d\psi_2, d\psi_3\}$ can contribute to the final wedge product.

Hence,

$$\alpha \wedge d\alpha \wedge d\alpha = 2\Delta\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge dx_1 \wedge dx_2$$

where the subscripts on a, b, c indicate derivatives with respect to $\frac{\partial}{\partial x_i}$ and

$$(3.3)\qquad \Delta = \begin{vmatrix} a & a_1 & a_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix}$$

The vector $X = (a, b, c)$ defines a map $X : B^2 \to \mathbb{R}^3$ that extends to all of Σ^2. Similar to the case of propeller curves (1.5), a geometric interpretation of $\Delta \neq 0$ is that X is never contained in the tangent plane to its image. "Propeller surfaces" are maps X that have "spherical" images, take the origin $O \in B^2$ to $(-1, 0, 0)$, and are constant $(a, b, c) = (1, 0, 0)$ outside of $B^2 \subset \Sigma^2$. An illustration of a propeller surface is given in figure (4)

More explicitly, we see this by choosing a rotationally symmetric map. If (x_1, x_2) are polar coordinates (r, θ), then for functions $(u(r), v(r))$ we may rewrite the conditions

$$\begin{pmatrix} a(r,\theta) \\ b(r,\theta) \\ c(r,\theta) \end{pmatrix} = \begin{pmatrix} u(r) \\ v(r)\sin\theta \\ v(r)\cos\theta \end{pmatrix}$$

and

$$\alpha \wedge d\alpha \wedge d\alpha = -v \begin{vmatrix} u & u' \\ v & v' \end{vmatrix} \cdot \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge dr \wedge d\theta. \tag{3.4}$$

Choose a product metric on $M^3 \times \Sigma^2$. For convenience we choose

$$\mu = -\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge dx_1 \wedge dx_2$$

to be the volume form on $M^3 \times \Sigma^2$. A propeller curve $(u(r), v(r))$ in $\mathbb{R}^2$, defined by conditions (1.5), makes α well defined at the origin, a non-negative multiple of the volume form, and a positive multiple of the volume form on the interior of $M^3 \times B^2$. Therefore, α satisfies the confoliation condition in (1.4).

Outside $M^3 \times B^2$ one has $\alpha = \psi_1$ and $Null(a)^\perp = T\Sigma^2$. Thus diffusion for the evolution equation will occur along the surface Σ and the conductivity condition of (1.4) is satisfied. q.e.d.

As mentioned in the introduction, the next result, namely that $S^{3+2p} \times \Sigma^2$ is contact, is already known. The interest in the case study below is that the construction of the confoliation is explicit, high dimensional, and uses more than one propeller to make the form conductive.

Theorem 3.2. *Let Σ^2 be any compact, orientable surface then $S^{3+2p} \times \Sigma^2$ is contact.*

Proof. Let $m = 2 + p$ and

$$h : S^{2m-1} \times \Sigma^2 \hookrightarrow \mathbb{R}^{2m} \times \Sigma^2$$

be the standard embedding of the sphere into Euclidean space and the identity on the other components. Let $(\xi_1, \cdots, \xi_{2m})$ be Cartesian coordinates on $\mathbb{R}^{2m}$.

Let $\{B_i^2\}_{i=1}^m$ be disjoint balls in Σ^2. Below, we will construct non-intersecting propellers $\{\mathfrak{P}_i\}_{i=1}^m$ over each space $S^{2m-1} \times B_i^2$. Each $\mathfrak{P}_i$ is constructed by a different identification of $\mathbb{R}^{2m}$ with $\mathbb{R}^4 \times \mathbb{R}^{2p}$ given by the $2m$ coordinate re-labelings $(\xi_{2i-1}, \xi_{2i}, \cdots, \xi_{2m-1}, \xi_{2m}, \xi_1, \xi_2 \cdots, \xi_{2i-2})$.

Other than the relabeling of the coordinates, the $\mathfrak{P}_i$ are constructed identically. So, below we will give a generic construction given any such coordinate identification.

We now begin the construction of a propeller on

$$S^{3+2p} \times B^2 \subset \mathbb{R}^4 \times \mathbb{R}^{2p} \times B^2$$

and for convenience we further distinguish the coordinates on the Euclidean factor by

$$(y, z) = (y_1, \cdots, y_4, z_1, \cdots, z_{2p})$$

As in the previous theorem, denote the coordinates on a ball $B^2 \subset \Sigma^2$ by (x_1, x_2).

We also introduce the notation

$$\begin{aligned} V_y &= dy_1 \wedge \cdots \wedge dy_4 \\ r_y &= \sqrt{y_1^2 + \cdots + y_4^2} \\ \nu_y &= r_y dr_y = \sum_{i=1}^{4} y_i dy_i \end{aligned} \tag{3.5}$$

and similarly define V_z and ν_z. Let

$$\begin{aligned} \nu &= \nu_y + \nu_z \\ r &= \sqrt{r_y^2 + r_z^2} \end{aligned}$$

Finally, let V_Σ be the volume form on Σ^2.

Define the following forms

$$\begin{aligned} \psi_1 &= y_1 dy_2 - y_2 dy_1 + y_3 dy_4 - y_4 dy_3 \\ \psi_2 &= y_1 dy_3 - y_3 dy_1 + y_4 dy_2 - y_2 dy_4 \\ \psi_3 &= y_1 dy_4 - y_4 dy_1 + y_2 dy_3 - y_3 dy_2 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \alpha_1 &= a(x_1, x_2)\psi_1 + b(x_1, x_2)\psi_2 + c(x_1, x_2)\psi_3 \\ \alpha_2 &= \sum_{i=1}^{p} (z_{2i-1} dz_{2i} - z_{2i} dz_{2i-1}) \\ \alpha &= \alpha_1 + \alpha_2. \end{aligned} \tag{3.7}$$

Though $h^\star\alpha$ is properly the one-form of interest, it is computationally most convienent to check the conductivity conditions in $\mathbb{R}^{4+2p} \times \Sigma^2$. The user may verify that for any form λ on $\mathbb{R}^{4+2p} \times \Sigma^2$

$$h^\star\lambda = i_{\partial_r} (\nu \wedge \lambda) \,|_{r=1}$$

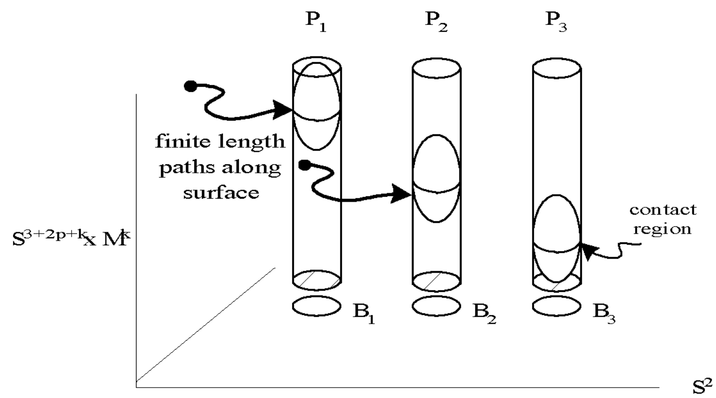

Figure 5: Multiple propellers create a conductive confoliation

where i_{∂_r} denotes interior product with respect to the radial vector in $\mathbb{R}^{4+2p}$.

The formulas

$$\begin{aligned}
&(d\alpha_2)^p = 2^p\,(p!)\cdot V_z \\
&\alpha_2 \wedge \nu_z \wedge (d\alpha_2)^{p-1} = -2^{p-1}(p-1)!r_z^2 \cdot V_z \\
&\alpha_1 \wedge \nu_y \wedge (da \wedge \psi_1 + db \wedge \psi_2 + dc \wedge \psi_3)^2 = 2r_y^4\Delta \cdot V_\Sigma \wedge V_y \\
&(ad\psi_1 + bd\psi_2 + cd\psi_3) \wedge (da \wedge \psi_1 + db \wedge \psi_2 + dc \wedge \psi_3)^2 \\
&\qquad\qquad = -4r_y^2\Delta \cdot V_\Sigma \wedge V_y
\end{aligned} \tag{3.8}$$

are useful for deriving

$$\frac{2^{-(p+1)}}{p!} \star \left(\nu \wedge \alpha \wedge (d\alpha)^{2+p}\right) = -r_y^2 r^2 \Delta \tag{3.9}$$

where Δ is as defined in equation (3.3).

We choose the propeller surface $X = (a, b, c)$ as in the previous theorem so that $\Delta > 0$ in B^2 and $(a, b, c) = (1, 0, 0)$ outside the ball. In fact, we can also ask that $a^2 + b^2 + c^2 > 0$. Hence, α is contact over B^2 as long as $r_y \neq 0$.

We now check for conductivity at points where α is not contact. Either the point is contained in a propeller region or it is not. Two such points are shown in Figure (5). We first show that any point in a propeller for which α is not contact has the property that $Null(a)^\perp = T\Sigma^2$ and hence diffusion will take place along the surface direction.

Next, we show that points outside the propellers also diffuse along the surface directions. Finally, we demonstrate that α is actually conductive in that every point is finite distance from a positive heat source, i.e., a contact region. First, in the region where $r_y = 0$, one has $r_z^2 = 1$, $\nu = \nu_z$, $\alpha = \alpha_2$ but $d\alpha = d\alpha_1 + d\alpha_2$, and $(d\alpha_1)^2 = 8\left(a^2+b^2+c^2\right)\cdot V_y$. One may verify that in this region

$$\nu \wedge \alpha \wedge (d\alpha)^{p+1} = \binom{p+1}{2} (\alpha_2 \wedge \nu_z) \wedge (d\alpha_1)^2 \wedge (d\alpha_2)^{p-1} \tag{3.10}$$

hence from identities (3.8)

$$\frac{2^{-(p+1)}}{(p+1)!} \star \left(\nu \wedge \alpha \wedge (d\alpha)^{p+1}\right) = -\left(a^2+b^2+c^2\right) \cdot V_{\Sigma^2}. \tag{3.11}$$

Equation (3.11) implies, for $r_y = 0$ inside the propeller, that $Null(a)^{\perp} = T\Sigma^2$. Hence diffusion occurs along the surface direction.

Next, outside the propellers $\alpha = \psi_1 + \alpha_2$ and one may compute

$$\star\left(h^{\star}\left(\alpha \wedge (d\alpha)^{p+1}\right)\right) = 2^{p+1}(p+1)! \cdot V_{\Sigma^2} \tag{3.12}$$

That is, α is a contact form for S^{3+2p} (see also [10]). Hence $Null(a)^{\perp} = T\Sigma^2$ and diffusion occurs along the surface direction.

Finally, now that we have seen that heat diffuses along the surface directions, we must demonstrate that there is actually some heat source that can be reached. This is readily seen as follows. For any point $(p,x) \in S^{2m-1} \times \Sigma^2$ where $p = (p_1, \cdots, p_{2m}) \in S^{2m-1} \subset \mathbb{R}^{2m}$ there is at least one coordinate function p_{2i-1} or p_{2i} that is nonzero. For this i, by our construction of $\mathfrak{P}_i$, $\alpha(p,x')$ is contact for all $x' \in B_i^2$. Therefore, α is conductive. q.e.d.

There are a number of different initial conditions that offer varying degrees of symmetry. As an example, for $S^{4p-1} \times \Sigma^2$ only one propeller of the following type is necessary to provide a conductive confoliation. For coordinates $(y_1, \cdots, y_{4p}) \in S^{4p-1} \subset \mathbb{R}^{4p}$ let

$$\psi_1 = \sum_{i=0}^{p-1} \left(y_{4i+1}dy_{4i+2} - y_{4i+2}dy_{4i+1} + y_{4i+3}dy_{4i+4} - y_{4i+4}dy_{4i+3}\right)$$

$$\psi_2 = \sum_{i=0}^{p-1} \left(y_{4i+1}dy_{4i+3} - y_{4i+3}dy_{4i+1} + y_{4i+4}dy_{4i+2} - y_{4i+2}dy_{4i+4}\right)$$

$$\psi_3 = \sum_{i=0}^{p-1} \left(y_{4i+1}dy_{4i+4} - y_{4i+4}dy_{4i+1} + y_{4i+2}dy_{4i+3} - y_{4i+3}dy_{4i+2}\right)$$

and let $X(x_1, x_2) = (a(x_1, x_2), b(x_1, x_2), c(x_1, x_2))$ be a propeller surface.

For $h : S^{4p-1} \times \Sigma^2 \hookrightarrow \mathbb{R}^{4p} \times \Sigma^2$,

$$\alpha = h^\star \left(a(x_1, x_2)\psi_1 + b(x_1, x_2)\psi_2 + c(x_1, x_2)\psi_3\right). \tag{3.13}$$

One may compute inside the propeller that

$$\star\left(\alpha \wedge (d\alpha)^{2p}\right) = (2p-2)!\left(4(a^2+b^2+c^2)\right)^{p-1}\Delta > 0 \tag{3.14}$$

where Δ is as defined earlier.

Since ψ_1 is contact on S^{4p-1}, α is a conducting confoliation on $S^{4p-1} \times \Sigma^2$.

4. Other applications

In this section, we describe a number of new applications of the program outlined above. First, we introduce a notion of integrability for higher degree forms.

On M^{2n+1}, let j, k, m be non-negative integers such that $n = (k+1)(j+m)+k$, $\alpha \in \Lambda^{2k+1}(M^{2n+1})$ and $\Phi \in \Lambda^{2k+2}(M^{2n+1})$. Define the function

$$\star\left(\alpha \wedge d\alpha^j \wedge \Phi^m\right) \in \Lambda^0(M^{2n+1}). \tag{4.1}$$

Note that 4.1 agrees with the notion of non-integrability 1.1 for contact forms when $k = 0$ and $m = 0$. Note also that we ask for α to be odd degree. This is because an even degree form α_{2k} has $d(\alpha_{2k} \wedge \alpha_{2k}) = 2\alpha_{2k} \wedge d\alpha_{2k}$ and Stokes theorem implies $\int \alpha_{2k} \wedge d\alpha_{2k} = 0$. So there can be no contact-like notion of positivity.

As with before, if $\star\left(\alpha \wedge d\alpha^j \wedge \Phi^m\right) \geq 0$, one can attempt to construct a new $(2k+1)$-form $\eta = \alpha + \epsilon\beta$ such that equation (4.1) is strictly positive. As in §2.1, we expand the cross terms

$$\begin{aligned} \eta \wedge d\eta^j \wedge \Phi^m = \alpha &\wedge d\alpha^j \wedge \Phi^m \\ &+ \epsilon\left(\beta \wedge d\alpha^j \wedge \Phi^m + j\alpha \wedge d\alpha^{j-1} \wedge d\beta \wedge \Phi^m\right) \\ &+ O(\epsilon^2). \end{aligned} \tag{4.2}$$

Again, defining

$$f = \frac{1}{j}\star\left(\beta \wedge d\alpha^j \wedge \Phi^m + j\alpha \wedge d\alpha^{j-1} \wedge d\beta \wedge \Phi^m\right) \tag{4.3}$$

it is clear that if $\star\left(\alpha \wedge d\alpha^j \wedge \Phi^m\right) \geq 0$ and $f > 0$, then for ϵ small enough one can obtain $\star\left(\alpha \wedge d\alpha^j \wedge \Phi^m\right) > 0$.

So the more general "non-integrability" flow is as follows.

Definition 4.1. Let $\alpha(\cdot), \beta(\cdot, t) \in \Lambda^{2k+1}\left(M^{2n+1}\right)$ and $\Phi(\cdot) \in \Lambda^{2k+2}$ where α and Φ are time-independent and β varies in time. The general *non-integrability flow* is

$$\begin{cases} \frac{\partial}{\partial t}\beta = \star(\alpha \wedge d\alpha^{j-1} \wedge \Phi^m \wedge df) \\ \beta(\cdot, 0) = \beta_0(\cdot) \end{cases} \tag{4.4}$$

Most importantly, this produces a nice evolution for f. As before, we define the k-form

$$\tau = \star(\alpha \wedge d\alpha^{j-1} \wedge \Phi^m) \tag{4.5}$$

and define the "metric"

$$\langle X, Y \rangle_a = \langle i_X \tau, i_Y \tau \rangle \tag{4.6}$$

where the inner product $< \cdot, \cdot >$ on the right hand side is the one induced by g on $k-1$-forms. As before, we say that a point p is accessible from another point q if there is a curve $x(s)$ connecting the two with $x'(s)$ in the range of a.

Then, it may be computed that the evolution of f is given by

$$\frac{\partial}{\partial t} f = \Delta_a f + \nabla_X f \tag{4.7}$$

where Δ_a is defined as in the previous section and $X = X(\alpha, \nabla\alpha, \Phi, \nabla\Phi)$ is a vector field depending only on α, Φ and their first covariant derivatives.

As with the confoliation case, a maximum principle may be applied to the evolution of f yielding the general result.

Theorem 4.2. *Let j, k, m be non-negative integers such that $n = (k+1)(j+m)+k$. Note that for $\alpha \in \Lambda^{2k+1}\left(M^{2n+1}\right)$ and $\Phi \in \Lambda^{2k+2}\left(M^{2n+1}\right)$ the wedge-product $\star\left(\alpha \wedge d\alpha^j \wedge \Phi^m\right) \in \Lambda^0\left(M^{2n+1}\right)$. If α and Ω satisfy*

1. $\star\left(\alpha \wedge d\alpha^j \wedge \Phi^m\right) \geq 0$;
2. *every point p is accessible from a point q satisfying $\star\left(\alpha \wedge d\alpha^j \wedge \Phi^m\right)(q) > 0$;*

then α is C^∞ close to a $(2k+1)$-form η such that $\star\left(\eta \wedge d\eta^j \wedge \Phi^m\right) > 0$.

We now give some examples designed to be representative and illustrative of some of the phenomena discussed above. All manifolds will be assumed to be compact and orientable.

Example 1. If M^{2k+1} has a conductive confoliation one-form λ and N^{2k} has symplectic form ω then define the 3-form $\alpha = \lambda \wedge \omega$. Then α satisfies the two conditions of theorem 4.2 and is C^∞ close to a 3-form η satisfying $\eta \wedge (d\eta)^k > 0$.

Example 2. let M^{2n+1} be an almost contact manifold. That is, M^{2n+1} has a 1-form α and a 2-form Φ such that $\star(\alpha \wedge \Phi^n) > 0$. If, for some k, α additionally satisfies $\star\left(\eta \wedge d\eta^k \wedge \Phi^{n-k}\right) \geq 0$ and condition 2 of theorem 4.2, then α may be perturbed to one with a strict inequality. This is interesting because it offers a range of non-integrability between foliation and contact forms.

Example 3. For the product manifold $M^{2k-1} \times N^{2k}$ there is $(2k-1)$-form η such that $\star(\eta \wedge d\eta) > 0$. This is an example of a generalization of contact forms on 3-manifolds to higher dimensions.

To see this, one may construct initial conditions similar to those in example (5) of §1 and in the proof of theorem 3.1. Let μ be a volume form for M. In a ball B centered around a point in N, make the volume form $r^{2k-1} dr \wedge \nu$ where r is the radial coordinate and ν is a volume form on the sphere S^{2n-1}. For $\alpha = a(r)\mu_M + b(r)\nu$ one has

$$\alpha \wedge d\alpha = \det \begin{vmatrix} a & a_r \\ b & b_r \end{vmatrix} \mu \wedge dr \wedge \nu$$

As with the previous propeller constructions, choose a propeller curve $X(r) = (a(r), b(r))$ such that $b(r)$ behaves as r^{2k+1} near $r = 0$. Therefore, $\alpha \wedge d\alpha > 0$ for $r < 1$ and $\alpha \wedge d\alpha = 0$ elsewhere. It is easy to check that α satisfies the condition for heat conduction outside $M \times B$ and the result follows.

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