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POSITIVELY CURVED COHOMOGENEITY ONE MANIFOLDS AND 3-SASAKIAN GEOMETRY

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Dedicated to Wilhelm Klingenberg on his 80th birthday

Abstract

We provide an exhaustive description of all simply connected positively curved cohomogeneity one manifolds. The description yields a complete classification in all dimensions but 7, where in addition to known examples, our list contains one exceptional space and two infinite families not yet known to carry metrics of positive curvature. The infinite families also carry a 3-Sasakian metric of cohomogeneity one, which is associated to a family of selfdual Einstein orbifold metrics on the 4-sphere constructed by Hitchin.

Since the round sphere of constant positive (sectional) curvature is the simplest and most symmetric topologically non-trivial Riemannian manifold, it is only natural that manifolds with positive curvature always will have a special appeal, and play an important role in Riemannian geometry. Yet, the general knowledge and understanding of these objects is still rather limited. In particular, although only a few obstructions are known, examples are notoriously hard to come by.

The additional structure provided by the presence of a large isometry group has had a significant impact on the subject (for a survey see [**Gr**]). Aside from classification and structure theorems in this context (as in [**HK**], [**GS1**], [**GS2**], [**GK**], [**Wi2**], [**Wi3**] and [**Ro**], [**FR2**], [**FR3**]), such investigations also provide a natural framework for a systematic search for new examples. In retrospect, the classification of simply connected homogeneous manifolds of positive curvature ([**Be**],[**Wa**],[**AW**],[**BB**]) is a prime example. It is noteworthy that in dimensions above 24, only the rank one symmetric spaces, i.e., spheres and projective spaces appear in this classification. The only further known

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examples of positively curved manifolds are all biquotients [**E1**, **E2**, **Ba**], and so far occur only in dimension 13 and below.

A natural measure for the size of a symmetry group is provided by the so-called cohomogeneity, i.e., the dimension of its orbit space. It was recently shown in [**Wi3**] that the lack of positively curved homogeneous manifolds in higher dimensions in the following sense carries over to any cohomogeneity: If a simply connected positively curved manifold with cohomogeneity $k \geq 1$ has dimension at least $18(k+1)^2$, then it is homotopy equivalent to a rank one symmetric space.

This paper deals with manifolds of cohomogeneity one. Recall that in [**GZ**] a wealth of new nonnegatively curved examples were found among such manifolds. Our ultimate goal is to classify positively curved (simply connected) cohomogeneity one manifolds. The spheres and projective spaces admit an abundance of such actions (cf. [**HL**, **St1**, **Iw1**, **Iw2**], and [**Uc1**]). In [**Se**], however, it was shown that in dimensions at most six, these are in fact the only ones. In [**PV2**] it was shown that this is also true in dimension 7, as long as the symmetry group is not locally isomorphic to $\mathsf{S}^3 \times \mathsf{S}^3$. Recently Verdiani completed the classification in even dimensions (see [**PV1**, **V1**, **V2**]):

Theorem (Verdiani)**.** *An even dimensional simply connected cohomogeneity one manifold with an invariant metric of positive sectional curvature is equivariantly diffeomorphic to a compact rank one symmetric space with a linear action.*

The same conclusion is false in odd dimensions. The three exceptional normal homogeneous manifolds of positive curvature admit cohomogeneity one actions: The Berger space $B^7 = \mathsf{SO}(5)/\,\mathsf{SO}(3)$ with a subaction by $\mathsf{SO}(4)$, the Aloff Wallach manifold $W^7 = \mathsf{SU}(3)/\operatorname{diag}(z, z, \bar{z}^2)$ $= \mathsf{SU}(3)\,\mathsf{SO}(3)/\,\mathsf{U}(2)$ with subactions by $\mathsf{SU}(2)\,\mathsf{SO}(3)$, denoted by $W^7_{(1)}$, and by $\mathsf{SO}(3)\,\mathsf{SO}(3)$, denoted by $W^7_{(2)}$, and finally the Berger space $B^{13} = \mathsf{SU}(5)/\,\mathsf{Sp}(2)\,\mathsf{S}^1$ with a subaction by $\mathsf{SU}(4)$. It is perhaps somewhat surprising that none of the remaining homogeneous manifolds of positive curvature admit cohomogeneity one actions. More interestingly, the subfamily $E^7_p = \operatorname{diag}(z, z, z^p)\backslash\,\mathsf{SU}(3)/\operatorname{diag}(1, 1, \bar{z}^{p+2})$, $p \geq 1$ of inhomogeneous positively curved Eschenburg spaces admit cohomogeneity one actions by $\mathsf{SO}(3)\,\mathsf{SU}(2)$ which extend to $\mathsf{SO}(3)\,\mathsf{U}(2)$. Similarly, the subfamily of the inhomogeneous positively curved Bazaikin spaces, $B^{13}_p = \operatorname{diag}(z, z, z, z, z^{2p-1}) \backslash\,\mathsf{SU}(5)/\,\mathsf{Sp}(2)\ \operatorname{diag}(1, 1, 1, 1, \bar{z}^{2p+3}), p \geq 1$ admit cohomogeneity one actions by $\mathsf{SU}(4)$, which extend to $\mathsf{U}(4)$. We point out that $E^7_1 = W^7_{(1)}$ with one of its cohomogeneity one actions, and $B^{13}_1 = B^{13}$.

The goal of this paper is to give an exhaustive description of all simply connected cohomogeneity one manifolds that can possibly support an invariant metric with positive curvature. In addition to the examples already mentioned, it turns out that only one isolated 7-manifold, R and two infinite 7-dimensional families P_k and Q_k potentially admit invariant cohomogeneity one metrics of positive curvature.

We will also exhibit an intriguing connection between these *new candidates for positive curvature* and the cohomogeneity one self dual Einstein *orbifold metrics* on $\mathbb{S}^4$ constructed by Hitchin [**Hi2**]. As a biproduct, the manifolds P_k and Q_k all support 3-Sasakian Riemannian metrics, i.e., their Euclidean cones are Hyper-Kähler (see [**BG**] for a survey), and are in particular Einstein manifolds with positive scalar curvature. In dimension 7 the known examples, due to Boyer, Galicki, Mann and Rees [**BGM**, **BGMR**], are constructed as so-called reductions from the 3-Sasakian metric on a round sphere, and except for $\mathbb{S}^7$, have positive second Betti number. They include the Eschenburg spaces E_p as a special case. The new 3-Sasakian manifolds P_k are particularly interesting since they are, apart from $\mathbb{S}^7 = P_1$, the first 2-connected examples in dimension 7 (see Theorem C). Both P_k and Q_k are also the first seven dimensional non-toric 3-Sasakian manifolds, i.e., do not contain a 3-torus in their isometry group.

To describe the new candidates for positive curvature, recall that any simply connected cohomogeneity one G-manifold admits a decomposition $M = \mathsf{G} \times_{\mathsf{K}^-} \mathbb{D}_- \cup \mathsf{G} \times_{\mathsf{K}^+} \mathbb{D}_+$ where $\mathsf{H} \subset \{\mathsf{K}^-, \mathsf{K}^+\} \subset \mathsf{G}$ are (isotropy) subgroups of G, and $\mathbb{D}_\pm$ are Euclidean discs with $\partial\mathbb{D}_\pm = \mathbb{S}_\pm = \mathsf{K}^\pm/\,\mathsf{H}$. Conversely, any collection of groups $\mathsf{H} \subset \{\mathsf{K}^-, \mathsf{K}^+\} \subset \mathsf{G}$ where $\mathsf{K}^\pm/\,\mathsf{H}$ are spheres, give rise in this fashion to a cohomogeneity one manifold.

Using this notation, we first describe a sequence of 7-dimensional manifolds H_k. They are given by the groups $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \{\mathsf{K}_0^- \cdot \mathsf{H}, \mathsf{K}_0^+ \cdot \mathsf{H}\} \subset_k \mathsf{SO}(3)\,\mathsf{SO}(3)$. Furthermore, the identity components $\mathsf{K}_0^\pm \cong \mathsf{SO}(2)$ depend on integers (p, q) which describe the slope of their embedding into a maximal torus of $\mathsf{SO}(3)\,\mathsf{SO}(3)$. They are $(1, 1)$ for K_0^- embedded into the lower 2×2 block of $\mathsf{SO}(3)$, and $(k, k+2)$ for K_0^+ embedded into the upper 2×2 block.

The universal covers of H_k break up into two families, P_k being the universal cover of H_{2k-1} with $\mathsf{G} = \mathsf{SO}(4)$ and principal isotropy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and Q_k the universal cover of H_{2k} with $\mathsf{G} = \mathsf{SO}(3)\,\mathsf{SO}(3)$ and principal isotropy group $\mathbb{Z}_2$. The additional manifold R is like Q_k but with slopes $(3, 1)$ on the left and $(1, 2)$ on the right.

Our main result can now be formulated as:

Theorem A. *Any odd dimensional simply connected cohomogeneity one manifold M with an invariant metric of positive sectional curvature is equivariantly diffeomorphic to one of the following:*

- *A sphere with a linear action,*
- *One of E_p^7, B_p^{13} or B^7,*

- *One of the 7-manifolds P_k, Q_k, or R,*

with one of the actions described above.

The first in each sequence P_k, Q_k admits an invariant metric with positive curvature since $P_1 = \mathbb{S}^7$ and $Q_1 = W^7_{(2)}$. For more information and further discussion of the non-linear examples we refer to Section 4.

There are numerous 7 dimensional cohomogeneity one manifolds with singular orbits of codimension two, all of which by [**GZ**] have invariant metrics with non-negative curvature. Among these, there are two subfamilies like the above P_k and Q_k, but where the slopes for $\mathsf{K}^{\pm}$ are arbitrary. It is striking that in positive curvature, with one exception, only the above slopes are allowed. The exception is given by the positively curved cohomogeneity one action on B^7, where the isotropy groups are like those for P_k with slopes $(1, 3)$ and $(3, 1)$. In some tantalizing sense then, the exceptional Berger manifold B^7 is associated with the P_k family in an analogous way as the exceptional candidate R is associated with the Q_k family. It is also surprising that all non-linear actions in Theorem A, apart from the Bazaikin spaces B^{13}_p, are cohomogeneity one under a group locally isomorphic to $\mathsf{S}^3 \times \mathsf{S}^3$.

As already indicated, the manifolds H_k have another intriguing characterization. To describe this in more detail, recall that $\mathbb{S}^4$ and $\mathbb{CP}^2$ according to Hitchin are the only smooth self dual Einstein 4-manifolds. However, in the more general context of orbifolds, Hitchin constructed [**Hi1**] a sequence of self dual Einstein orbifolds O_k homeomorphic to $\mathbb{S}^4$, one for each integer $k > 0$, which are invariant under a cohomogeneity one $\mathsf{SO}(3)$ action. It has an orbifold singularity whose angle normal to a smooth $\mathsf{SO}(3)$ orbit $\mathbb{RP}^2$ is equal to $2\pi/k$. Here O_{2k} can also be interpreted as an orbifold metric on $\mathbb{CP}^2$ with normal angle $2\pi/k$, and the cases of $k = 1, 2$ correspond to the smooth standard metrics on $\mathbb{S}^4$ and on $\mathbb{CP}^2$ respectively. In general, any self dual Einstein orbifold gives rise to a 3-Sasakian orbifold metric on the Konishi bundle, which is the $\mathsf{SO}(3)$ orbifold principal bundle of the vector bundle of self dual 2-forms. The action of $\mathsf{SO}(3)$ on the base lifts to form a cohomogeneity one $\mathsf{SO}(3)\,\mathsf{SO}(3)$ action on the total space, and we will prove the following surprising relationship with our positive curvature candidates:

Theorem B. *For each k, the total space of the Konishi bundle corresponding to the selfdual Hitchin orbifold O_k is a smooth 3-Sasakian manifold, which is equivariantly diffeomorphic to H_k with its cohomogeneity one $\mathsf{SO}(3)\,\mathsf{SO}(3)$ action.*

In this context we note that the exceptional manifolds B^7 and R can be described, up to covers, as the $\mathsf{SO}(3)$ orbifold principal bundles of the vector bundle of anti-self dual 2-forms over O_3 and O_4 respectively.

It was shown by O. Dearricott in [**De1**] that Konishi metrics, scaled down in direction of the principal $\mathsf{SO}(3)$ orbits, have positive sectional curvature if and only if the self dual Einstein orbifold base has positive curvature. Unfortunately, the Hitchin orbifold metrics do not have positive curvature for $k > 2$, so this appealing description does not easily yield the desired metrics of positive curvature on P_k and Q_k.

Our candidates also have interesting topological properties:

Theorem C. *The manifolds P_k are two-connected with $\pi_3(P_k) = \mathbb{Z}_k$. For the manifolds Q_k and R we have $H^2(Q_k, \mathbb{Z}) = H^2(R, \mathbb{Z}) = \mathbb{Z}$ and $H^4(Q_k, \mathbb{Z}) = \mathbb{Z}_{2k+1}$, respectively $H^4(R, \mathbb{Z}) = \mathbb{Z}_{35}$.*

We note that the cohomology of Q_k and R occur as the cohomology of one or more of the seven dimensional positively curved Eschenburg biquotients [**E1**],[**E2**]. In fact, surprisingly, Q_k has the same cohomology as E_k. On the other hand the manifolds P_k have the same cohomology as $\mathbb{S}^3$ bundles over $\mathbb{S}^4$, and among such manifolds, so far only $\mathbb{S}^7$ and the Berger space B^7 (see [**GKS**]) are known to admit metrics of positive curvature. It would be interesting to know whether there are other cases where a manifold in the families P_k, Q_k is diffeomorphic to an Eschenburg space or to an $\mathbb{S}^3$ bundle over $\mathbb{S}^4$.

The fact that the manifolds P_k are 2-connected is particularly significant. Recall that by the finiteness theorem of Petrunin-Tuschmann [**PT**] and Fang-Rong [**FR1**], 2-connected manifolds play a special role in positive curvature since there exist only finitely many diffeomorphism types of such manifolds, if one specifies the dimension and the pinching constant, i.e., δ with $\delta \leq sec \leq 1$. Thus, if P_k admit positive curvature metrics, the pinching constants δ_k necessarily go to 0 as $k \to \infty$, and P_k would be the first examples of this type. The existence of such metrics would provide counter examples to a conjecture by Fang and Rong in [**FR2**] (cf. also Fukaya [**Fu**], Problem 15.20).

We conclude the introduction by giving a brief discussion of the proof of our main result and how we have organized it.

The most basic *recognition tool* one has is of course the group diagram itself. However, given just the richness of linear actions on spheres, one would expect that looking primarily for such detailed information might actually hinder classification. It is thus crucial to have other recognitions tools at our disposal, that do not require the full knowledge of a group diagram. In fact, in our proof we often either exclude a potential manifold, or determine what it is before we actually derive a possible group diagram.

For this we first note that Straume [**St1**] has provided a complete classification of all cohomogeneity one actions on homotopy spheres. Aside from linear actions on the standard sphere, there are families of non-linear actions, and also actions on exotic Kervaire spheres. It was observed by Back and Hsiang [**BH**] (Searle [**Se**] in dimension 5) that only the linear ones support invariant metrics of positive curvature (in dimensions other than five they cannot even support invariant metrics of nonnegative curvature [**GVWZ**]). In particular, for our purposes it suffices to recognize the underlying manifold as a homotopy sphere, and we have two specific tools for doing so: One of them is provided by the (equivariant diffeomorphism) classification of positively curved *fixed point homogeneous* manifolds [**GS2**], i.e., manifolds on which a group G acts transitively on the normal sphere to a component of its fixed point set M^{G}. The other is the *Chain Theorem* of [**Wi3**], which classifies up to homotopy 1-connected positively curved manifolds that support an isometric action by one of the classical groups, $\mathsf{SO}(n), \mathsf{SU}(n)$ or $\mathsf{Sp}(n)$ so that its principal isotropy group contains the same type of group as a standard 3×3 block (or 2×2 block in case of $\mathsf{Sp}(n)$).

Our classification of positively curved manifolds with an isometric cohomogeneity one G-action is done by induction on the dimension of the manifold M. Here the induction step is typically done via *reductions*, i.e., by analyzing fixed point sets of subgroups of G and how they sit inside of M. Since such fixed point sets are totally geodesic, they are themselves positively curved manifolds of cohomogeneity at most one and hence in essence known by assumption. In this analysis, the basic *connectivity lemma* of [**Wi2**] which asserts that the inclusion map of a totally geodesic codimension k submanifold in an n dimensional positively curved manifold is $n - 2k + 1$ connected, naturally plays an important role.

Another variable in the proof is $\operatorname{rk}\mathsf{G}$, the rank of G. Here it is a simple but important fact that in positive curvature, the corank of the principal isotropy group H, i.e., $\operatorname{corank}\mathsf{H} = \operatorname{rk}\mathsf{G} - \operatorname{rk}\mathsf{H}$, is 1 in even dimensions, and 0 or 2 in odd dimensions. The equal rank case is fairly simple and induction is not used here (see Section 5).

The following brief description of the content of the sections will hopefully support the overall understanding of the strategy of the proof just outlined.

In Section 1 we recall some essential simple curvature free facts about cohomogeneity one manifolds we will need throughout. This includes a discussion of the *Weyl groups* and *reductions*, i.e., fixed point sets of subgroups, including the *core* of the action.

Sections 2 and 3 form the geometric heart of the paper. It is here we present and derive all our *obstructions* stemming from having an invariant metric of positive curvature. Some of these, which have been derived earlier in more general settings (see [**Wi2**, **Wi3**]), become particularly powerful in the context of cohomogeneity one manifolds. Other than the rank restriction, which enters from the outset, two key obstructions used throughout are primitivity, and restrictions imposed on the isotropy representation of the principal isotropy group. The full strength of primitivity is derived in Section 3 after a classification of all Weyl groups corresponding to non trivial cores. It is also shown here that all Weyl groups are finite and strong bounds on their orders are derived.

In Section 4 we present and discuss some of the properties of the cohomogeneity one actions on the known examples of positive curvature, as well as on the new candidates.

We start the classification in Section 5 with the equal rank case and in Section 6 we deal with the case where G is not semisimple. For semisimple groups G, it turns out to be useful to prove the theorem for groups of rank 2 or 3 first, and this is done in the Section 7 and 8. In a sense these two sections form the core of the classification. It is here that all non spherical examples emerge. The case of semisimple groups G with $\operatorname{rk}\mathsf{G} \geq 4$ is done separately for non-simple groups in Sections 9 and 10 and for simple groups in Section 11.

In Section 12 we exhibit our new infinite families of candidates as 3-Sasakian manifolds (Theorem B), and in Section 13 we prove Theorem C. These sections can be read independently of the rest of the paper.

Since we need the classification in even dimensions, we have added a relatively short proof as a service to the reader in Appendix I. As another service to the reader, we have collected the cohomogeneity one diagrams for the essential actions on rank one symmetric spaces, and other known useful classification results in Appendix II.

It is our pleasure to thank Jost Eschenburg for useful comments.

1. Cohomogeneity one manifolds

We begin by discussing a few useful general facts about closed cohomogeneity one Riemannian G manifolds M and fix notation we will use throughout. Readers with good working knowledge of cohomogeneity one manifolds may want to proceed to Section 2, Section 3 and the classification starting in Section 5 immediately and refer back to this section whenever needed.

Our primary interest is in positively curved, 1- connected G manifolds M with G connected. However, since fixed point sets with induced cohomogeneity one actions play a significant role in our proof, it is important to understand the more general case where G is not connected, and M is connected with possibly non-trivial finite fundamental group.

Since M has finite fundamental group, the orbit space M/G is an interval and not a circle. The end points of the interval correspond to two non-principal orbits, and all interior points to principal orbits. By scaling the metric if necessary we may assume that $M/\mathsf{G} = [-1, 1]$ as a metric space.

Fix a normal geodesic $c : \mathbb{R} \to M$ perpendicular to all orbits (an infinite horizontal lift of M/G). The image $C = c(\mathbb{R})$ is either an embedded circle, or a 1-1 immersed line (cf. [**AA**, Proposition 3.2]). We denote by H the principal isotropy group $\mathsf{G}_{c(0)}$ at $c(0)$, which is equal to the isotropy groups $\mathsf{G}_{c(t)}$ for all $t \neq 1 \mod 2\mathbb{Z}$, and by $\mathsf{K}^{\pm}$ the isotropy groups at $p_{\pm} = c(\pm 1)$. Then M is the union of tubular neighborhoods of the non-principal orbits $B_{\pm} = \mathsf{G}/\mathsf{K}^{\pm}$ glued along their common boundary G/H, i.e., by the slice theorem

$$M = \mathsf{G} \times_{\mathsf{K}^-} \mathbb{D}_- \cup \mathsf{G} \times_{\mathsf{K}^+} \mathbb{D}_+, \tag{1.1}$$

where $\mathbb{D}_{\pm}$ denotes the normal disc to the orbit $\mathsf{G}\,p_{\pm} = B_{\pm}$ at $p_{\pm}$. Furthermore, $\mathsf{K}^{\pm}/\mathsf{H} = \partial\mathbb{D}_{\pm} = \mathbb{S}_{\pm}$ are spheres, whose dimension we denote by $l_{\pm}$. It is important to note that the diagram of groups

$$\begin{tikzcd} & \mathsf{G} & \\ \mathsf{K}^- \arrow[ur, "j_-"] & & \mathsf{K}^+ \arrow[ul, "j_+"'] \\ & \mathsf{H} \arrow[ul, "h_-"] \arrow[ur, "h_+"'] & \end{tikzcd} \tag{1.2}$$

where $j_{\pm}$ and $h_{\pm}$ are the natural inclusions, which we also record as

$$\mathsf{H} \subset \{\mathsf{K}^-, \mathsf{K}^+\} \subset \mathsf{G}, \tag{1.3}$$

determines M. Conversely, such a *group diagram* with $\mathsf{K}^{\pm}/\mathsf{H} = \mathbb{S}^{l_{\pm}}$, defines a cohomogeneity one G-manifold.

In Section 12, we will see that the above construction, as well as the principal bundle construction for cohomogeneity one manifolds in [GZ], naturally carries over to a large class within the more general context of orbifolds.

We point out that the spheres $\mathsf{K}^{\pm}/\mathsf{H}$ are often highly ineffective and we denote by $\mathsf{H}_{\pm}$ their ineffective kernel. It will be convenient to allow the ineffective kernel of G/H to be finite, i.e., to allow the action to be *almost effective.*

A non–principal orbit G/K is called *exceptional* if $\dim \mathsf{G}/\mathsf{K} = \dim \mathsf{G}/\mathsf{H}$ or equivalently $\mathsf{K}/\mathsf{H} = \mathbb{S}^0$. Otherwise G/K is called *singular.* As usual we refer to the collection M_0 of principal orbits, i.e., $M - (B_- \cup B_+)$, as the *regular part* of M.

The Cohomogeneity One Weyl Group.

The *Weyl group*, $\mathsf{W}(\mathsf{G}, M) = \mathsf{W}$ of the action, is by definition the stabilizer of the geodesic C modulo its kernel H. If $\mathsf{N}(\mathsf{H})$ is the normalizer of H in G, it is easy to see (cf. [**AA**]) that W is a dihedral subgroup of $\mathsf{N}(\mathsf{H})/\mathsf{H}$, generated by unique involutions $w_{\pm} \in (\mathsf{N}(\mathsf{H}) \cap \mathsf{K}^{\pm})/\mathsf{H}$, and that $M/\mathsf{G} = C/\mathsf{W}$. Each of these involutions can also be described as the unique element $a \in \mathsf{K}^{\pm}$ mod H such that a^2 but not a lies in H.

Note that W is finite if and only if C is a closed geodesic, and in that case the order $|\mathsf{W}|$ is the number of minimal geodesic segments $C - (B_- \cup B_+)$. Note also that any non principal isotropy group along c is of the form $w\mathsf{K}^{\pm}w^{-1}$ for some $w \in \mathsf{N}(\mathsf{H})$ representing an element of W. The isotropy types $\mathsf{K}^{\pm}$ alternate along C and hence half of them are isomorphic to K^{+} and half to K^{-}, in the case where W is finite.

Group Components.

In this section G is a not necessarily connected Lie group acting with cohomogeneity one on a connected manifold M with finite fundamental group. From the description of M as a double disc bundle (1.1), we see that

$$\text{(1.4)} \qquad \begin{aligned} \mathsf{G}/\mathsf{K}^{\pm} \cong B_{\pm} \to M \quad &\text{is} \quad l_{\mp}\text{-connected.} \\ \mathsf{G}/\mathsf{H} \to M \quad &\text{is} \quad \min\{l_-, l_+\}\text{-connected.} \end{aligned}$$

Recall that by definition a map $f : X \to Y$ is l*-connected* if the induced map $f_i : \pi_i(X) \to \pi_i(Y)$ between homotopy groups is an isomorphism for $i < l$ and surjective for $i = l$.

First observe that it is impossible that both $l_{\pm} = 0$. Indeed, if both normal bundles to $\mathsf{G}/\mathsf{K}^{\pm}$ are trivial M is a bundle over $\mathbb{S}^1$. If one of the orbits, say $\mathsf{G}/\mathsf{K}^{+}$, has non-trivial normal bundle, the two fold cover $\mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{K}^{+}$ gives rise to a two fold cover M' of M on which G acts by cohomogeneity one with diagram $\mathsf{H} \subset \{\mathsf{K}^{-}, w_+\mathsf{K}^{-}w_+\} \subset \mathsf{G}$. We are now either in the first situation, or we can repeat the second argument indefinitely, contradicting that $\pi_1(M)$ is finite.

If both $l_{\pm} > 0$, (1.4) implies that G/H is connected and hence G and G_0 have the same orbits, and in particular the same Weyl group. If one of $l_{\pm}$ say $l_- = 0$ and $l_+ > 0$, (1.4) implies that $\mathsf{G}/\mathsf{K}^{-}$ is connected. Since G/H is a sphere bundle over $\mathsf{G}/\mathsf{K}^{-}$, it follows that G/H has at most two components. This in turn implies that

(1.5) The Weyl group $\mathsf{W}(\mathsf{G}_0, M)$ has index at most 2 in $\mathsf{W}(\mathsf{G}, M)$.

We now assume that M is simply connected and G is connected. The above covering argument then implies that there cannot be any exceptional orbits. If both $l_{\pm} \geq 2$, (1.4) implies that all orbits are simply connected and hence all isotropy groups are connected. If one of $l_{\pm}$ say $l_- = 1$ and $l_+ \geq 2$, then $\mathsf{G}/\mathsf{K}^{-}$ is simply connected and hence K^{-} connected. Since G/H is a circle bundle over $\mathsf{G}/\mathsf{K}^{-}$ it follows that $\pi_1(\mathsf{G}/\mathsf{H})$ and hence $\mathsf{H}/\mathsf{H}_0 \simeq \mathsf{K}^{+}/\mathsf{K}^{+}_0$ are cyclic. In summary,

Lemma 1.6. *Assume that* G *acts on* M *by cohomogeneity one with* M *simply connected and* G *connected. Then:*

(a) *There are no exceptional orbits, i.e.,* $l_\pm \geq 1$.
(b) *If both* $l_\pm \geq 2$, *then* $\mathsf{K}^\pm$ *and* H *are all connected.*
(c) *If one of* $l_\pm$, *say* $l_- = 1$, *and* $l_+ \geq 2$, *then* $\mathsf{K}^- = \mathsf{H} \cdot \mathsf{S}^1 = \mathsf{H}_0 \cdot \mathsf{S}^1$, $\mathsf{H} = \mathsf{H}_0 \cdot \mathbb{Z}_k$ *and* $\mathsf{K}^+ = \mathsf{K}_0^+ \cdot \mathbb{Z}_k$.

The situation where both $l_\pm = 1$ is analyzed in the presence of an invariant positively curved metric is in (3.5). Finally we observe

Lemma 1.7. *Suppose* $\bar{\mathsf{K}}^\pm \subset \mathsf{K}^\pm$ *are subgroups with* $\mathsf{K}^\pm/\bar{\mathsf{K}}^\pm$ *finite,* $\bar{\mathsf{K}}^\pm \not\subset \mathsf{H}$, *and* $\bar{\mathsf{K}}^- \cap \mathsf{H} = \bar{\mathsf{K}}^+ \cap \mathsf{H} =: \bar{\mathsf{H}}$. *Then* $\mathsf{K}^-/\bar{\mathsf{K}}^- \simeq \mathsf{H}/\bar{\mathsf{H}} \simeq \mathsf{K}^+/\bar{\mathsf{K}}^+$ *and the cohomogeneity one manifold* $\bar{M}$ *defined by* $\bar{\mathsf{H}} \subset \{\bar{\mathsf{K}}^-, \bar{\mathsf{K}}^+\} \subset \mathsf{G}$ *is an* $\mathsf{H}/\bar{\mathsf{H}}$ *cover of* M.

In general, a subcover of a compact cohomogeneity one manifold with finite fundamental group and G connected is obtained by a combination of the following three: We can add components to $\mathsf{K}^\pm$ and H as in (1.7), or we can divide G by a central subgroup which does not intersect $\mathsf{K}^\pm$. These two yield orbitspace preserving covering maps. We can also create a subcover where one of the orbits is exceptional, if K^+ is the w conjugate of K^- for an order two element in $\mathsf{N}(\mathsf{H})/\mathsf{H}$ represented by $w \in \mathsf{N}(\mathsf{H})$.

Reductions.

Fixed point sets of subgroups $\mathsf{L} \subset \mathsf{G}$ will play a pivotal role throughout. It is well known that the fixed point set M^{L} of L consists of a disjoint union of totally geodesic submanifolds. If M^{L} is non empty, L is of course a subgroup of an isotropy group, and hence of H or of $\mathsf{K}^\pm$ (up to conjugacy). In general when $\mathsf{L} \subset \mathsf{K} \subset \mathsf{G}$, it is well known that $\mathsf{N}(\mathsf{L})$ acts with finite orbit space on $(\mathsf{G}/\mathsf{K})^{\mathsf{L}}$, and transitively when $\mathsf{L} = \mathsf{K}$, or when L is a maximal torus of K (see e.g., [**Br**], Corollary II.5.7).

Suppose first that $\mathsf{L} \subset \mathsf{K}^-$ is not conjugate to a subgroup of H. Then no component of M^{L} intersects the regular part M_0 of M. In this case, all components of M^{L} are homogeneous, and we usually consider the component in one of $B_\pm$ say B_- containing p_- which equals $\mathsf{N}(\mathsf{L})_0/\mathsf{N}(\mathsf{L})_0 \cap \mathsf{K}^-$. As a particular application of this, we point out that a central involution in G which lies in one of $\mathsf{K}^\pm$ say K^- but not in H has G/K^- as its fixed point set.

If L is conjugate to a subgroup of H, the components of M^{L} which intersect the regular part of M form a cohomogeneity one manifold under the action of $\mathsf{N}(\mathsf{L})$ since $\mathsf{N}(\mathsf{L})$ acts with finite quotient on $(\mathsf{G}/\mathsf{H})^{\mathsf{L}}$. Each component of M^{L} that intersects the regular part is hence a cohomogeneity one manifold under the action of the subgroup of $\mathsf{N}(\mathsf{L})$ stabilizing the component. Unless otherwise stated, the reduction we will consider is the component M_c^{L} of M^{L} containing the geodesic c. We will denote it's stabilizer subgroup of $\mathsf{N}(\mathsf{L})$ by $\mathsf{N}(\mathsf{L})_c$ and refer to $(M_c^{\mathsf{L}}, \mathsf{N}(\mathsf{L})_c)$ as *reductions* (for general actions see [**GS3**]). In general the length of $M_c^{\mathsf{L}}/\mathsf{N}(\mathsf{L})_c$ is an integer multiple of the length of $M/\,\mathsf{G}$. The orbit spaces coincide if both $\mathsf{N}(\mathsf{L})\cap\mathsf{K}^{\pm}$ act nontrivially on the normal spheres of $M_c^{\mathsf{L}}\cap B_{\pm} \subset M_c^{\mathsf{L}}$ at $p_{\pm}$, which are given by $\mathbb{S}_{\pm}^{\mathsf{L}} = \mathsf{N}(\mathsf{L})_c \cap \mathsf{K}^{\pm}/\mathsf{N}(\mathsf{L})_c \cap \mathsf{H}$. If this is the case, $\mathsf{N}(\mathsf{L})_c$ acts (L ineffectively) by cohomogeneity one on M_c^{L} with orbit space $M/\,\mathsf{G}$, and diagram $\mathsf{N}(\mathsf{L})_c\cap\mathsf{H} \subset \{\mathsf{N}(\mathsf{L})_c\cap\mathsf{K}^-, \mathsf{N}(\mathsf{L})_c\cap\mathsf{K}^+\} \subset \mathsf{N}(\mathsf{L})_c$.

In the main part of the induction proof, it is usually sufficient to consider the cohomogeneity one action of the connected component $\mathsf{N}(\mathsf{L})_0$ of $\mathsf{N}(\mathsf{L})_c$ on M_c^{L} keeping in mind that its Weyl group need not be that of M.

If L is a maximal torus of H_0 and $a \in \mathsf{N}(\mathsf{H})$, then $a\mathsf{L}a^{-1} \subset \mathsf{H}_0$ is also conjugate to L by an element in H_0. In particular, one can represent $w_{\pm}$ by elements in the normalizer of L. The same holds by definition of the Weyl group for $\mathsf{L} = \mathsf{H}$, and hence:

Lemma 1.8 (Reduction Lemma)**.** *If* L *is either equal to* H *or given by a maximal torus of* H_0*, then* $\mathsf{N}(\mathsf{L})_c/\mathsf{L}$ *acts by cohomogeneity one on* M_c^{L} *and the corresponding Weyl groups coincide.*

In the most reduced case where $\mathsf{L} = \mathsf{H}$, we refer to M_c^{H} as the *core* of M and $\mathsf{N}(\mathsf{H})_c$ as the *core group.*

Often we consider also the least reduced case, that is we take the fixed point set of an involution or of an element ι whose square, but not ι itself, lies in the center of G, i.e., ι is an involution in some central quotient of G. In this case we can determine $\mathsf{N}(\langle\iota\rangle) = \mathsf{N}(\iota)$ using the well known fact that $\mathsf{G}/\mathsf{N}(\iota)$ is a symmetric space with $\mathrm{rk}(\mathsf{N}(\iota)) = \mathrm{rk}(\mathsf{G})$, and appeal to their classification, see Table G, Appendix II.

In general the codimension of a reduction might be odd. However, if L is a subgroup of a torus in $\mathsf{T} \subset \mathsf{G}$, and M is positively curved and odd dimensional, then all components of M^{L} have even codimension. One can establish this fact by induction on the dimension, where one uses that odd dimensional positively curved manifolds are orientable and that the statement holds for cyclic subgroups $\mathsf{L} \subset \mathsf{T}$.

As a simple consequence of the Rank Lemma 2.1, we also see that in positive curvature, M^{L} has even codimension when $\mathrm{rk}\,\mathsf{N}(\mathsf{L}) = \mathrm{rk}\,\mathsf{G}$ and $\mathrm{rk}\,\mathsf{G} - \mathrm{rk}\,\mathsf{H} = 2$.

Equivalence of diagrams.

Recall that in order to get a group diagram we choose an invariant metric on M. Thus it can happen that different metrics on the manifold give different group diagrams. Of course, one can conjugate all three groups by the same element in G, and one can also switch K^- and K^+.

Let us now fix a point p in the regular part of the manifold and an orientation of the normal bundle $\mathsf{G}\,p$. For each invariant metric g on the manifold we consider the minimal horizontal geodesic $c_g\colon[-\varepsilon_1(g),\varepsilon_2(g)] \to M$ from the left singular orbit to the right with $c_g(0)=p$. We reparametrize these geodesics relative to a fixed parametrization of the orbit space $M/\mathsf{G}=[-1,1]$, where the orbit through p corresponds to 0. The resulting curves $\bar c_g$ are fixed pointwise by H. Using a (smooth) family of such reparametrized geodesics in M^{H} corresponding to convex combinations of two invariant metrics g_1, g_2 and the fact that $\mathsf{N}(\mathsf{H})$ acts transitively on $(\mathsf{G}/\mathsf{H})^{\mathsf{H}}$, we can find a curve $a\colon[-1,1]\to \mathsf{N}(\mathsf{H})_0$ such that the curve $\bar c_{g_2}$ is given by $a(t)\bar c_{g_1}(t)$. This proves that we can find two elements $a_-,a_+\in \mathsf{N}(\mathsf{H})_0$ such that the group diagram for the metric g_2 is obtained from the group diagram for the metric g_1 by conjugating $\mathsf{K}^{\pm}$ with $a_{\pm}$. On the other hand it is easy to see that indeed for any $a_-,a_+\in\mathsf{N}(\mathsf{H})_0$ one can find a metric for which there is a horizontal geodesic from $a_-c(-\varepsilon_1(g))$ to $a_+c(\varepsilon_2(g))$. In fact this can be achieved by changing the metric on the complement of two small tubular neighborhoods of $B_{\pm}$.

All in all we conclude that two group diagrams $\mathsf{H}\subset\{\mathsf{K}^-,\mathsf{K}^+\}\subset\mathsf{G}$ and $\tilde{\mathsf{H}}\subset\{\tilde{\mathsf{K}}^-,\tilde{\mathsf{K}}^+\}\subset\mathsf{G}$ yield the same cohomogeneity one manifold up to equivariant diffeomorphism if and only if after possibly switching the roles of K^- and K^+, the following holds: There is a $b\in\mathsf{G}$ and an $a\in\mathsf{N}(\mathsf{H})_0$ with $\mathsf{K}^-=b\tilde{\mathsf{K}}^-b^{-1}$, $\mathsf{H}=b\tilde{\mathsf{H}}b^{-1}$, and $\mathsf{K}^+=ab\tilde{\mathsf{K}}^+b^{-1}a^{-1}$ (cf. also [**Ne**]).

2. Positive Curvature Obstructions

In this section we will discuss a number of severe obstructions on a cohomogeneity one manifold to have an invariant metric with positive curvature. We point out that none of our obstructions are caused by nonnegative curvature only. We also mention that Alexandrov geometry of orbit spaces, which is used extensively to obtain our two geometric recognitions tools (2.11) and (2.8), enters only once directly in our proof, namely the rank two case (7.1).

The simplest obstruction is a direct consequence of the well known fact (see, e.g., [**Gr**]) that an isometric torus action on a positively curved manifold has fixed points in even dimensions and orbits of dimension at most one in odd dimensions. Applying this to a maximal torus in G, and using the fact that a sphere K/H has corank at most one, we get:

Lemma 2.1 (Rank Lemma)**.** *If M is even dimensional, at least one of $\mathsf{K}^{\pm}$ has corank 0 and H has corank 1 in G. If M is odd dimensional, at least one of $\mathsf{K}^{\pm}$ has corank 1 and H has corank 0 or 2.*

A second powerful and much more difficult result expresses in two ways how the *representation of the triple* $\mathsf{H}\subset\{\mathsf{K}^-,\mathsf{K}^+\}$ in G is *maximal.*

The first of these, which we will refer to as *linear primitivity*, follows from [**Wi4**, Corollary 10], and has the Weyl group bound below as an immediate consequence. As we will see in the next section this type of primitivity implies that the Weyl group is finite as well (see (3.1)).

To define the second kind of primitivity, we say that a G - manifold is *non-primitive* if there is a G equivariant map $M \to \mathsf{G}/\mathsf{L}$ for some subgroup $\mathsf{L} \subset \mathsf{G}$ (see [**AA**, p.17]). Otherwise, the action is said to be *primitive*. For cohomogeneity one manifolds, non-primitivity is equivalent to the statement that for some representation we have $\mathsf{H} \subset \{\mathsf{K}^-, \mathsf{K}^+\} \subset \mathsf{L} \subset \mathsf{G}$, i.e., for some invariant metric and some normal geodesic, $\mathsf{K}^\pm$ generate a proper subgroup of G. By the last subsection of section 1 we know exactly all possible groups $\mathsf{K}^\pm$ arising from different metrics. Hence, in terms of the original groups, the action is primitive if K^- and $n\mathsf{K}^+n^{-1}$ generate G, for any fixed $n \in \mathsf{N}(\mathsf{H})_0$.

In the next section we will show that the core with its core action is primitive (3.2). When this is combined with linear primitivity for G, we will show that the G action itself is primitive (see (3.3)):

Lemma 2.2 (Primitivity Lemma). *Let $c\colon \mathbb{R} \to M$ be any horizontal geodesic as above. Then*

(a) (Linear Primitivity) *The Lie algebras of the isotropy groups along c generate $\mathfrak{g}$ as a vectorspace.*
(b) (Lower Weyl Group Bound) *The Weyl group is finite, and $|\mathsf{W}| \geq 2\dim(\mathsf{G}/\mathsf{H})/(l_- + l_+)$.*
(c) (Group Primitivity) *Any of the singular isotropy groups $\mathsf{K}^\pm$, together with any conjugate of the other by an element of the core group, generate G as a group. In particular this is true for conjugation by elements of $\mathsf{N}(\mathsf{H})_0$.*

The following obstructions deal with *isotropy representations*. The first of these is a special case of a more general result in [**Wi3**]. The key observation in the proof is that the direct sum of all irreducible subrepresentations equivalent to the given one form a parallel subbundle along the normal geodesic. Since the norm of the volume form of a parallel bundle is strictly concave in positive curvature, it must vanish at one of the singular orbits. The second part of the lemma follows from the first part and the classification of transitive actions on spheres, see Table C, Appendix II.

Lemma 2.3 (Isotropy Lemma). *Suppose H is non trivial. Then*

(a) *Any irreducible subrepresentation of the isotropy representation of G/H is equivalent to a subrepresentation of the isotropy representation of one of K/H, where K is an isotropy group of some point in $c(\mathbb{R})$*

(b) *The isotropy representation of* G/H_0 *is spherical, i.e.,* H_0 *acts transitively on the unit sphere of any* k *dimensional irreducible subrepresentation if* $k > 1$.

Notice that part a) implies that any subrepresentation of G/H, i.e., the isotropy representation of G/H, is weakly equivalent to a subrepresentation of K^-/H or K^+/H. Recall that two representations of H are weakly equivalent if they are equivalent modulo an automorphism of H. We thus often say that a particular representation has to degenerate in K^+/H or K^-/H.

The fact that the isotropy representations are spherical is a particularly powerful tool. In [**Wi3**] one finds an exhaustive list of such so-called *spherical subgroups* when H and G are simple (apart from the case where H is a rank one group in an exceptional Lie group). We reproduce this list in Table B, since it will be used frequently.

Lemma 2.3 has the following very useful consequence:

Lemma 2.4. *Assume that* G *is simple. Then:*

(a) *If* L_1 *and* L_2 *are two simple normal subgroups of* H*, there exists an irreducible subrepresentation of* H *in* G *on which both* L_1 *and* L_2 *act non-trivially.*

(b) H *can have at most one simple normal subgroup of rank at least two.*

Proof. Assume that G/H has no subrepresentation on which L_1 and L_2 act non-trivially. Decompose $\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{n}$ such that L_1 acts non-trivially on $\mathfrak{m}_1$ and trivially on $\mathfrak{m}_2$, L_2 acts trivially on $\mathfrak{m}_1$ and non-trivially on $\mathfrak{m}_2$ and both act trivially on $\mathfrak{n}$. Note that $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$ since both L_1 and L_2 act non-trivially on any subrepresentation of $\mathfrak{m}_1 \otimes \mathfrak{m}_2$. Similarly $[\mathfrak{m}_1, \mathfrak{n}] \subset \mathfrak{m}_1$ and in summary $[\mathfrak{m}_1, \mathfrak{g}] \subset \mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_1]$. Using the Jacobi identity we see that $[[\mathfrak{m}_1, \mathfrak{m}_1], \mathfrak{n}] \subset \mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_1]$ and $[[\mathfrak{m}_1, \mathfrak{m}_1], \mathfrak{m}_2] = 0$. Thus $\mathfrak{m}_1 + [\mathfrak{m}_1, \mathfrak{m}_1]$ is an ideal of $\mathfrak{g}$, a contradiction.

For part (b), assume that L_1 and L_2 are two simple normal subgroups of H with $\operatorname{rk} \mathsf{L}_i \geq 2$. From the classification of transitive actions on spheres it follows that either L_1 or L_2 must act trivially on the irreducible subrepresentations of H in $\mathsf{K}^{\pm}$. By the Isotropy Lemma the same then holds for each irreducible subrepresentation of H in G and part (a) finishes the proof. q.e.d.

For the singular orbits there are two relevant representations, the *isotropy representation* and the *slice representation*. These are related via equivariance of the *second fundamental form*

$$B^{\pm} : S^2(T_{\pm}) \to T_{\pm}^{\perp} \tag{2.5}$$

where $T_{\pm}$ is the tangent space of $B_{\pm} \cong \mathsf{G}/\mathsf{K}^{\pm}$ at $p_{\pm}$, and $T_{\pm}^{\perp}$ is the normal space.

As an example of an application of this, it sometimes follows that equivariance forces a singular orbit to be totally geodesic. In particular, this singular orbit must then be in the short list of positively curved homogeneous manifolds, see Table C and D in Appendix II.

The next result also follows from equivariance of the second fundamental form applied to a singular orbit.

Lemma 2.6 (Product Lemma)**.** *Suppose* $\mathsf{G} = \mathsf{L}_1 \times \mathsf{L}_2$ *is semisimple and that the identity component of at least one of* $\mathsf{K}^{\pm}$ *is a product subgroup, say* $\mathsf{K}^-_0 = \mathsf{K}_1 \times \mathsf{K}_2$*, and that one of* $\mathsf{N}^{\mathsf{L}_i}(\mathsf{K}_i)/\mathsf{K}_i$ *is finite. Then* M *cannot carry a positively curved* G*- invariant metric if it is odd dimensional.*

Proof. Denote by U_i the subspace tangent to the factor $\mathsf{L}_i/\mathsf{K}_i$. Note that $\dim U_i > 1$ since G is semisimple. A non-trivial subrepresentation of K^-_0 in U_1 cannot be equivalent to one in U_2 and by assumption, one of U_i has no trivial subrepresentations. Thus Schur's Lemma implies that every invariant metric on G/K is (locally) a product metric on $(\mathsf{L}_1/\mathsf{K}_1) \times (\mathsf{L}_2/\mathsf{K}_2)$.

From the classification of transitive actions on spheres, see Table C, we may assume that one of the factors, say K_1, acts transitively on the normal sphere. Since K_1 acts trivially on U_2, no subrepresentation of S^2U_2 is equivalent to the slice representation, and hence $B_{S^2U_2} = 0$. Since any plane generated by one vector in U_1 and one vector in U_2 has intrinsic curvature 0, we see from the Gauss equation that $B(u_1, u_2) \neq 0$ for all nonzero $u_i \in U_i$. Because B is bilinear, this implies that $\dim(U_i) \leq \dim(T^{\perp})$.

If there exists a K_1-invariant subspace $U_1' \subset U_1$ such that the induced representation in U_1' is not equivalent to the slice representation, then the equivariance of B implies that $B_{U_1' \otimes U_2} = 0$, contradicting $B(u_1, u_2) \neq 0$ for all nonzero $u_i \in U_i$. Thus, using in addition the above dimension restriction, the representation of K_1 on all of U_1 is equivalent to the slice representation. In particular, K_1 acts transitively on the unit sphere in U_1, and hence $\mathsf{L}_1/\mathsf{K}_1$ is two point homogeneous. Thus $\mathsf{L}_1/\mathsf{K}_1$ is isometric to a rank one symmetric space. From the classification of rank one symmetric spaces as homogeneous spaces we see that the representation of K_1 is either of real or complex type, but not quaternionic. Recall that a representation π has real type, complex type, or quaternionic type corresponding to $\pi \otimes \mathbb{C}$ being irreducible, $\pi \otimes \mathbb{C} \cong \sigma \oplus \sigma^*$ with $\sigma \not\cong \sigma^*$ or $\pi \otimes \mathbb{C} \cong \sigma \oplus \sigma$. The algebra of endomorphisms that commute with π is then equal to $\mathbb{R}$, $\mathbb{C}$, respectively $\mathbb{H}$.

Since the manifold is odd dimensional and U_1 and the slice have the same dimension, it follows that U_2 is odd dimensional and therefore $\dim(U_2) \geq 3$. Because of $\dim(U_2) \geq 2$ there exists a K_1 invariant irreducible subspace $U' \neq 0$ of $U_1 \otimes U_2$ contained in the kernel of B.

If the representation of K_1 on U_1 is of real type, we claim that U' is necessarily of the form $U_1 \otimes U_2'$, where U_2' is a one dimensional subspace of U_2, which contradicts $B(u_1, u_2) \neq 0$. To see this, choose a basis $e_0, e_1, \ldots, e_k$ of U_2. Any K_1 invariant subspace of $U_1 \otimes U_2$, which we can assume projects onto $U_1 \otimes e_0$, must be of the form $x \otimes e_0 + L_1(x) \otimes e_1 + \cdots + L_k(x) \otimes e_k$, where $x \in U_1$ and L_i endomorphisms of U_1. To be K_1 invariant implies that L_i commute with the representation of K_1 on U_1. Since it is of real type, this means that L_i are scalar multiplication with λ_i, and hence $e_0 + \lambda_1 e_1 + \cdots + \lambda_k e_k$ spans U_2'.

If the representation of K_1 on U_1 is of complex type, we can repeat the previous argument in the complexifications $U_i \otimes \mathbb{C}$. Thus any K_1 invariant irreducible subspace $U' \subset U_1 \otimes U_2$ corresponds to a one dimensional subspace in $U_2 \otimes \mathbb{C}$. Since the kernel of $B_{U_1 \otimes U_2}$ contains $\dim(U_2) - 1$ linearly independent K_1 invariant irreducible subrepresentations, we may view these subrepresentations as a complex hyperplane in $U_2 \otimes \mathbb{C}$. Because of $\dim(U_2) \geq 3$, this hyperplane intersects $U_2 \otimes \mathbb{R}$, and we get a contradiction as before. q.e.d.

We stress that in even dimensions, the statement of the product lemma is no longer valid in general. We will determine the exceptions in (14.2).

We conclude this section with a discussion of the *recognition tools* we will apply in this paper. These tools are indispensable for our proof.

First of all by combining Straume's classification of cohomogeneity one homotopy spheres [**St1**] with the work of Back-Hsiang [**BH**] (and Searle [**Se**] in dimension five) we have

Theorem 2.7. *Any cohomogeneity one homotopy sphere Σ^n with an invariant metric of positive curvature is equivariantly diffeomorphic to the standard sphere $\mathbb{S}^n$ with a linear action.*

The same conclusion is true for all manifolds whose rational cohomology ring is like that of a nonspherical rank one symmetric space (see [**Iw1**, **Iw2**] and [**Uc1**]).

The following very general recognition theorem was proved in [**Wi3**]:

Theorem 2.8 (Chain Theorem)**.** *Suppose $\mathsf{G}_d \in \{\mathsf{SO}(d), \mathsf{SU}(d), \mathsf{Sp}(d)\}$ acts isometrically and nontrivially on a positively curved compact simply connected manifold M. Suppose also that a principal isotropy group of the action contains up to conjugacy a $k \times k$ block with $k \geq 2$ if $\mathsf{G}_d = \mathsf{Sp}(d)$, and $k \geq 3$ otherwise. Then M is homotopy equivalent to a rank one symmetric space.*

In conjunction with the reduction idea, the following basic *connectedness lemma* of [**Wi2**] provides another general topological tool that will aid us in the recognition process.

Theorem 2.9 (Connectedness Lemma)**.** *Let M^n be a compact positively curved Riemannian manifold, and $N^{n-k} \subset M^n$ a compact totally geodesic submanifold. Then*

(a) *The inclusion map $N^{n-k} \to M^n$ is $(n-2k+1)-$connected.*
(b) *If in addition N^{n-k} is fixed pointwise by a compact group L of isometries of M, then the inclusion map is $\big(n-2k+1+\delta(L)\big)$-connected, where $\delta(L)$ is the dimension of a principal orbit of the L action.*
(c) *If also $V^{n-l} \subset M^n$ is a compact totally geodesic submanifold, and $k \leq l$, $k+l \leq n$, then the inclusion map $N^{n-k} \cap V^{n-l} \longrightarrow V^{n-l}$ is $(n-k-l)-$connected.*

As an example of a simple application of this result, combined with Poincare duality, we note (cf. [**Wi2**]):

$$M^{2n+1} \text{ positively curved, } \pi_1(M) = \{1\}, \text{ and } V^{2n-1} \subset M \\ \text{totally geodesic} \implies M \text{ is a homotopy sphere.} \tag{2.10}$$

We finally recall that a G-manifold is fixed point homogeneous if M^{G} is non-empty and G acts transitively on the normal spheres to a component of the fixed point set, equivalently $\dim M/\mathsf{G} - \dim M^{\mathsf{G}} = 1$. The classification of fixed point homogeneous manifolds with positive curvature [**GS2**] will be used frequently.

Theorem 2.11 (Fixed Point Homogeneity)**.** *Let M be a compact simply connected manifold of positive curvature. If M is fixed point homogeneous, then M is equivariantly diffeomorphic to a rank one symmetric space endowed with a linear action.*

Consider the special case, where one of $\mathsf{K}^{\pm}$, say K^-, contains a connected normal subgroup $\mathsf{G}' \lhd \mathsf{G}$. Let $\mathsf{G}'' \lhd \mathsf{G}$ be a normal subgroup with $\mathsf{G}' \cdot \mathsf{G}'' = \mathsf{G}$. Clearly G' acts trivially on G/K^-. Thus if G' acts transitively on the normal sphere $\mathbb{S}^{l_-}$, M is fixed point homogeneous. If not, $\mathsf{G}'' \cap \mathsf{K}^-$ acts transitively on $\mathbb{S}^{l_-}$, and hence G'' has the same orbits as G does. In summary:

Lemma 2.12. *If one of $\mathsf{K}^{\pm}$ contains a normal connected subgroup of G, then either there is a proper normal subgroup of G acting orbit equivalently, or M is fixed point homogeneous.*

This motivates the following:

Definition 2.13. An action is called *essential* if no subaction is fixed point homogeneous, and no normal subaction is orbit equivalent to it.

Note that the above lemma asserts in particular that:

- For an essential G-action, none of $\mathsf{K}^{\pm}$ contain a connected normal subgroup of G.

In the proof of Theorem A we restrict ourselves to essential actions. In the case that the underlying manifold is a sphere this is justified by Theorem 2.7. If the underlying manifold is not a sphere, Theorem 2.11 implies that a non-essential cohomogeneity one action has an orbit equivalent essential normal subaction. By Lemma 4.2 below this subaction already determines the action itself.

In the case of linear actions on spheres, all essential actions but three (and one extension) are irreducible. All non-essential actions without normal essential subactions are reducible and we refer to them as *sum actions* for short (they are in fact characterized by having a fixed point homogeneous subaction). See Appendix II for a description of such sum actions and their isotropy groups. The essential actions on spheres with their isotropy groups, which we use frequently, are collected in Table E (and for the even dimensional rank 1 projective spaces in Table F). We include their normal extensions since, although not essential in the above sense, they will also be used in our induction steps.

3. Weyl Groups

The main objective in this section is to obtain effective upper bounds on the Weyl groups of positively curved cohomogeneity one manifolds, and to prove group primitivity of such manifolds. The main result asserts that except for the cases of $\mathrm{corank}(\mathsf{H}) = 0$, and H finite and non-cyclic, the order of the Weyl group divides $4\,\mathrm{corank}(\mathsf{H}) \leq 8$. We will first analyze the situation in the case of a trivial H and later on reduce the general case to this one.

We begin with the following crucial observation:

Lemma 3.1. *The Weyl group of a positively curved cohomogeneity one manifold is finite.*

Proof. Since the Weyl group is a subgroup of $\mathsf{N}(\mathsf{H})/\,\mathsf{H}$ our claim is obvious when $\mathsf{N}(\mathsf{H})/\,\mathsf{H}$ is finite. When $\dim(\mathsf{N}(\mathsf{H})/\,\mathsf{H}) > 0$ we will use the fact noted earlier, that the Weyl group of M coincides with the Weyl group of its core (1.8). In particular, it suffices to prove our claim for G-actions with trivial principal isotropy group. Now suppose $\mathsf{W} = \langle w_-, w_+\rangle$ is infinite, i.e., the Weyl group elements w_+, w_- are involutions in G and $w_+ \cdot w_-$ generates an infinite cyclic group. Let $\mathsf{T}^h, h \geq 1$ be the identity component of the closure of this cyclic group. Choose a positive integer l with $(w_+w_-)^l \in \mathsf{T}^h$. Clearly $w_-(w_+ \cdot w_-)w_- = w_+(w_+ \cdot w_-)w_+ = (w_+ \cdot w_-)^{-1}$ and similarly

$$w_-(w_+ \cdot w_-)^l w_- = w_+(w_+ \cdot w_-)^l w_+ = (w_+ \cdot w_-)^{-l}.$$

Since the infinite group generated by $(w_+ \cdot w_-)^l$ is dense in T^h, it follows that the maps $\mathsf{T}^h \to \mathsf{T}^h$, $a \mapsto w_\pm a w_\pm$ both coincide with the inverse map $\iota : \mathsf{T} \to \mathsf{T}$ taking t to t^{-1}. Thus $\mathrm{Ad}_{w_+} v = \mathrm{Ad}_{w_-} v = -v$ for all vectors v in the Lie algebra of T^h. On the other hand, since $\mathsf{K}^{\pm}$ can only be $\mathbb{Z}_2$, S^1 or S^3, $w_{\pm}$ is central in $\mathsf{K}^{\pm}$, and hence $\mathrm{Ad}_{w_{\pm}} v_{\pm} = v_{\pm}$ for $v_{\pm}$ in the Lie algebra of $\mathsf{K}^{\pm}$. If we fix a biinvariant metric we deduce that the Lie algebras of $\mathsf{K}^{\pm}$ are perpendicular to the Lie algebra of T^h. Applying the same argument again on any of $w\mathsf{K}^{\pm}w^{-1}$, $w \in \mathsf{W}$, we see that in fact the Lie algebras of $w\mathsf{K}^{\pm}w^{-1}$ for any $w \in \mathsf{W}$ are perpendicular to the Lie algebra of T^h. This contradicts linear primitivity. q.e.d.

It is now possible to classify all cores with their core actions (see also [**Pü**] for the even dimensional case). However, the following suffices for our purposes:

Lemma 3.2 (Core-Weyl Lemma)**.** *Suppose a Lie group* G *acts with cohomogeneity one on a positively curved compact manifold* M *with finite fundamental group and trivial principal isotropy group. Then* G *has at most two components and the action is primitive. Moreover,*

$$|\,\mathsf{W}\,| \quad \textit{divides} \quad 2\,\mathrm{rk}(\mathsf{G}) \cdot |\,\mathsf{G}\,/\,\mathsf{G}_0\,| \leq 8.$$

Furthermore G_0 *is one of the groups* $\mathsf{S}^1, \mathsf{S}^3, \mathsf{T}^2, \mathsf{S}^1 \times \mathsf{S}^3, \mathsf{U}(2), \mathsf{S}^3 \times \mathsf{S}^3$, $\mathsf{SO}(3) \times \mathsf{S}^3$*, or* $\mathsf{SO}(4)$*, and* M *is fixed point homogeneous in all cases but* $\mathsf{G}_0 = \mathsf{SO}(3) \times \mathsf{S}^3$*.*

Proof. First notice that the rank of G is 1 or 2 by the rank lemma. Since the group H is trivial, it follows that $\mathsf{K}^{\pm}$ is isomorphic to one of the groups $\mathbb{Z}_2, \mathsf{S}^1$ or S^3. Moreover, at most one of $\mathsf{K}^{\pm}$ is $\mathbb{Z}_2$ and G has at most two components (cf. (1.4)). Furthermore if G is not connected then the Weyl group of the G_0 action has index 2 in W, and the bound follows from the connected case. It is also easy to see that primitivity follows from primitivity in the connected case. In other words it suffices to consider the connected groups of rank at most two.

We start by excluding the case where G is simple and without central involution, i.e., we suppose G is one of the groups $\mathsf{SO}(3), \mathsf{SU}(3)$, $\mathsf{SU}(3)/\mathbb{Z}_3$, $\mathsf{SO}(5)$, or G_2. The Weyl group is generated by two involutions w_- and w_+ in G and we claim that one can find elements $g \in \mathsf{G}$ arbitrarily close to e such that the group generated by w_- and gw_+g^{-1} is infinite. This in turn implies that there are invariant metrics on M that are C^{∞} close to the given metric for which the normal geodesic goes from p_- to gp_+ and for which the Weyl group is hence infinite. But this contradicts Lemma 3.1. To see the claim we assume, on the contrary, that it is false. Then we could find a small neighborhood U of $e \in \mathsf{G}$ and a map $k : U \to \mathbb{Z}$ with $(w_-gw_+g^{-1})^{k(g)} = e$. Since for each integer k the set of all g satisfying $(w_-gw_+g^{-1})^k = e$ is an algebraic subvariety of G, it follows that all $(w_-gw_+g^{-1})$ have a common order independent of $g \in U$. However, this is false for each of the above groups. In all cases but $\mathsf{SO}(5)$, this follows from the fact that all involutions are unique up to conjugacy, see Table G.

The case $\mathsf{G} \cong \mathsf{SO}(3) \times \mathsf{SO}(3)$, where G is non-simple without central involutions, is ruled out as well: As above, we can find a nearby metric with infinite Weyl group unless $w_- \in 1 \times \mathsf{SO}(3)$ and $w_+ \in \mathsf{SO}(3) \times 1$ (or vice versa) and hence $\mathsf{W} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $\mathsf{SO}(3) \times \mathsf{SO}(3)$ contains no subgroup isomorphic to S^3 it follows that $\dim(\mathsf{K}^{\pm}) \leq 1$, but this contradicts linear primitivity.

Now suppose G has central as well as non-central involutions, i.e., G is one of the groups $\mathsf{U}(2), \mathsf{S}^1 \times \mathsf{SO}(3)$, $\mathsf{SO}(4)$, $\mathsf{S}^3 \times \mathsf{SO}(3)$, or $\mathsf{Sp}(2)$. We can argue as before unless one of the elements, say w_-, is central in G. But then $\mathsf{W} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathsf{W} \cong \mathbb{Z}_2$ and W normalizes the group K^+. From linear primitivity we see that the Lie algebras of the groups K^-, K^+ and $w_+\mathsf{K}^-w_+$ generate the Lie algebra of G as a vector space. Because of $\dim(\mathsf{K}_\pm) \leq 3$ this clearly rules out $\mathsf{Sp}(2)$. For the other groups it follows that either K^- or K^+ is three dimensional and thus isomorphic to S^3, so $\mathsf{S}^1 \times \mathsf{SO}(3)$ is ruled out as well . If $\mathsf{G} = \mathsf{SO}(4)$ or $\mathsf{U}(2)$, every S^3 is normal and hence M is fixed point homogeneous. Note that primitivity in these cases immediately follows from linear primitivity since one of the groups $\mathsf{K}^{\pm}$ is a normal subgroup of G.

If $\mathsf{G} = \mathsf{S}^3 \times \mathsf{SO}(3)$ and one of $\mathsf{K}^{\pm}$ is an S^3 factor, M is fixed point homogeneous as above, and $\mathsf{W} = \mathbb{Z}_2$. If both $\mathsf{K}^{\pm}$ are diagonal 3-spheres, we obtain a contradiction to linear primitivity by observing that they must have at least a one dimensional intersection. If K^- is diagonal and $\mathsf{K}^+ = \mathbb{Z}_2$, the conjugates K^- and $w_+\mathsf{K}^-w_+$ also have a one dimensional intersection. In all other cases, one of $\mathsf{K}^{\pm}$, say K^-, is a diagonal S^3 and K^+ is a circle with slope (p, q) in a maximal torus of G. Notice that linear primitivity also implies that $\mathsf{W} = \mathbb{Z}_2 \times \mathbb{Z}_2$. We will later determine what slopes (p, q) are possible, and the corresponding manifolds are Eschenburg spaces (cf. Section 4). To prove primitivity in this case it is sufficient to show that neither K^- nor a conjugate of K^- can be a subgroup of K^+. But under such an assumption, we would have that $w_+ = w_-$ and thus $W \cong \mathbb{Z}_2$, contradicting the Lower Weyl Group Bound.

It remains to consider the cases where all involutions of G are central, i.e., G is one of the groups S^1, S^3, $\mathsf{S}^1 \times \mathsf{S}^1$, $\mathsf{S}^1 \times \mathsf{S}^3$, $\mathsf{S}^3 \times \mathsf{S}^3$. Clearly the order of the Weyl group is at most $2\,\mathrm{rk}\,\mathsf{G}$. From linear primitivity it follows that the Lie algebras of $\mathsf{K}^{\pm}$ generate the Lie algebra of G as a vector space. This implies that at least one of the groups $\mathsf{K}^{\pm}$ is normal and M is fixed point homogenous and primitive. q.e.d.

We can now use the above lemma and the last paragraph of Section 1 to prove the group primitivity stated in (2.2):

Corollary 3.3 (Group Primitivity)**.** *Suppose that M admits a positively curved cohomogeneity one metric. Consider any other cohomogeneity one metric on M; then the corresponding groups $\mathsf{K}^-, \mathsf{K}^+$ generate G as a Lie group. Equivalently K^- and $n\mathsf{K}^+n^{-1}$ generate G for any $n \in \mathsf{N}(\mathsf{H})_0$.*

Proof. Let $\mathsf{K}^\pm$ denote the isotropy groups with respect to a positively curved metric. By linear primitivity K^- and K^+ generate G as a group. We need to show that for any $a \in \mathsf{N}(\mathsf{H})_0$, the groups K^- and $a\mathsf{K}^+a^{-1}$ generate G as well. But by primitivity of the core, we know that $\mathsf{K}^- \cap \mathsf{N}(\mathsf{H})_c$ and $a(\mathsf{K}^+ \cap \mathsf{N}(\mathsf{H})_c)a^{-1} = a\mathsf{K}^+a^{-1} \cap \mathsf{N}(\mathsf{H})_c$ generate the core group. In particular, the group generated by K^- and $a\mathsf{K}^+a^{-1}$ contains $\mathsf{N}(\mathsf{H})_0$, and hence is equal to the group generated by K^- and K^+. q.e.d.

We have the following useful consequence of primitivity:

Lemma 3.4. *Assume G acts effectively. Then the intersection $\mathsf{H}_- \cap \mathsf{H}_+$ of the ineffective kernels $\mathsf{H}_\pm$ of $\mathsf{K}^\pm/\mathsf{H}$ is trivial.*

Proof. We first observe the following: If for a connected homogeneous space K/H a normal subgroup L of H acts trivially on K/H, then L is normal in K also. Indeed, first observe that $\mathsf{N}(\mathsf{L})$ acts transitively, since it in general acts with finite quotient on the fixed point set of L. Hence $\mathsf{K}/\mathsf{H} = \mathsf{N}(\mathsf{L})/(\mathsf{N}(\mathsf{L}) \cap \mathsf{H}) = \mathsf{N}(\mathsf{L})/\mathsf{H}$ and thus $\mathsf{K} = \mathsf{N}(\mathsf{L})$. In our case, we can apply this to the normal subgroup $\mathsf{H}_- \cap \mathsf{H}_+$ of H which fixes both $\mathbb{S}^{l_\pm}$. Thus $\mathsf{K}^\pm \subset \mathsf{N}(\mathsf{H}_- \cap \mathsf{H}_+)$, and hence by primitivity $\mathsf{N}(\mathsf{H}_- \cap \mathsf{H}_+) = \mathsf{G}$. Since the action is effective, $\mathsf{H}_- \cap \mathsf{H}_+$ is trivial. q.e.d.

When M is simply connected and G is connected, we recall from (1.6) that $\mathsf{K}^\pm$ and H are all connected as long as both $l_\pm \geq 2$. If exactly one of $l_\pm$ is 1, say $l_- = 1$ and $l_+ \geq 2$, K^- is connected, $\mathsf{H}/\mathsf{H}_0 = \mathsf{K}^+/\mathsf{K}^+_0$ is cyclic, and $\mathsf{H} = \mathsf{H}_-$. If in addition G is assumed to act effectively, it follows from the above Lemma 3.4 that K^+ acts effectively on $\mathbb{S}^{l_+}$. In the remaining situation where both $l_\pm = 1$, Lemma 3.4 and $|\,\mathsf{H}/\mathsf{H}_\pm\,| \leq 2$ yield:

Lemma 3.5. *Suppose M is a 1-connected positively curved manifold on which the connected group G acts effectively and isometrically with codimension two singular orbits. Then one of the following holds:*

(a) *$\mathsf{H} = \{1\}$ and both $\mathsf{K}^\pm$ are isomorphic to $\mathsf{SO}(2)$.*
(b) *$\mathsf{H} = \mathsf{H}_- = \mathbb{Z}_2$, $\mathsf{K}^- = \mathsf{SO}(2)$ and $\mathsf{K}^+ = \mathsf{O}(2)$.*
(c) *$\mathsf{H} = \mathsf{H}_- \cdot \mathsf{H}_+ = \mathbb{Z}_2 \times \mathbb{Z}_2$, and both $\mathsf{K}^\pm$ are isomorphic to $\mathsf{O}(2)$.*

Notice that part (a) of (3.5) is not possible when $\operatorname{rk} \mathsf{G} \geq 2$ since the action would then not be group primitive due to the fact that both K^+ and K^- can be conjugated into a common maximal torus.

As a consequence of the Core-Weyl Lemma one obtains an important upper bound for the Weyl group:

Proposition 3.6 (Upper Weyl Group Bound). *Assume that M is simply connected and* G *connected. Then*

(a) *If* H/H_0 *is trivial or cyclic, we have* $|\mathsf{W}| \le 8$ *if the corank of* H *in* G *is two, and* $|\mathsf{W}| \le 4$ *if the corank is one.*
(b) *If* H *is connected and* $l_\pm$ *are both odd,* $|\mathsf{W}| \le 4$ *in the corank two case and* $|\mathsf{W}| \le 2$ *in the corank one case.*
(c) *If none of* $\mathsf{N}(\mathsf{H}) \cap \mathsf{K}^\pm$ *are finite,* $|\mathsf{W}| \le 4$ *in the corank two case and* $|\mathsf{W}| \le 2$ *in the corank one case.*

Proof. We first consider the case where H/H_0 is non-trivial and cyclic. Then (1.6) and (3.5) imply that the codimension of one of the orbits is two and one of the corresponding K groups is connected. Thus $\mathsf{N}(\mathsf{H})/\mathsf{H}$ is not finite since $\mathsf{K} \subset \mathsf{N}(\mathsf{H})$. By passing to the reduction M^{H}, we deduce from the Core-Weyl Lemma 3.2 that $|\mathsf{W}| \le 8$ ($|\mathsf{W}| \le 4$ in the corank one case).

Now assume that H is connected. If $\mathsf{H} = \{e\}$, the claim follows again from the Core-Weyl Lemma. If not, fix a maximal torus $\mathsf{T} \subset \mathsf{H}$. Clearly then M^{T} has positive dimension. By Lemma 1.8, the group $\mathsf{N}(\mathsf{T})_c$ acts on the reduction M_c^{T} with the same Weyl group. By (1.5), the Weyl group of $\mathsf{N}(\mathsf{T})_0/\mathsf{T}$ has index at most two in $\mathsf{W}(\mathsf{G}, M)$.

Next observe that for any torus T of a connected compact Lie group G, $\mathsf{N}(\mathsf{T})_0 \subset \mathsf{C}(\mathsf{T})$, the centralizer of T in G. Because H is a connected Lie group, T is maximal abelian in H and thus $\mathsf{C}(\mathsf{T}) \cap \mathsf{H} = \mathsf{T}$. Hence $\mathsf{N}(\mathsf{T})_0 \cap \mathsf{H} = \mathsf{T}$ and thus $\mathsf{N}(\mathsf{T})_0/\mathsf{T}$ acts with trivial principal isotropy group on the reduction M_c^{T}. It follows that $|\mathsf{W}| \le 8$ ($|\mathsf{W}| \le 4$ in even dimensions) by the Core-Weyl Lemma.

Since the codimension of $\mathbb{S}_\pm^{\mathsf{T}} \subset \mathbb{S}^{l_\pm}$ is always even, $\mathbb{S}_\pm^{\mathsf{T}} \ncong \mathbb{S}^0$ if both $l^\pm$ are odd and hence (1.5) implies that $\mathsf{N}(\mathsf{T})_c/\mathsf{T}$ and $\mathsf{N}(\mathsf{T})_0/\mathsf{T}$ have the same Weyl group, which implies part (b).

For part (c) just note that by assumption both normal spheres of the core M_c^{H} have positive dimension. As we have seen then $\mathsf{N}(\mathsf{H})_c$ and its identity component have the same orbits and Weyl group. Thus from Core-Weyl Lemma $|\mathsf{W}| \le 4$ ($|\mathsf{W}| \le 2$ in even dimensions). q.e.d.

Remark 3.7. The only cases where we have no bound on the Weyl group are hence when H has corank zero, or when H has corank one or two and $\mathsf{H} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

In the equal rank case, $\mathsf{N}(\mathsf{H})/\mathsf{H}$ is always finite, and hence the Core-Weyl Lemma does not apply. However, in this case, information about the Weyl group does not enter in the proof of Theorem A. It will follow as a consequence of the proof that W is one of $\mathsf{D}_1, \mathsf{D}_3, \mathsf{D}_4, \mathsf{D}_6$.

If $\mathsf{H} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, we note that $\mathsf{N}(\mathsf{H})/\mathsf{H}$ is also finite since each of $(\mathsf{N}(\mathsf{H}) \cap \mathsf{K}^\pm)/\mathsf{H}$ is and M^{H} is primitive. In fact this is the case where the Weyl group can become larger. One easily sees that the Weyl groups are D_3 for P_{2k} and D_6 for P_{2k+1}, whereas for Q_k and R it is always

D_4. Hence, as a consequence of our classification, it follows that the Weyl groups for simply connected positively curved cohomogeneity one manifolds are the same as for linear actions on spheres. Notice also that there are many actions among the linear actions on spheres, for example all tensor product actions, where $\mathsf{W} = \mathsf{D}_4$, and some of those with $l^{\pm}$ odd and H not connected (see Table E).

4. Examples and Candidates

To aid the induction step in our proof of Theorem A it is important to know more details about the individual manifolds and actions that occur. The linear actions are of course well known, and the essential ones and their normal extensions are exhibited in Tables E and F in Appendix II. The corresponding details for the remaining spaces and actions, i.e., for the known non-spherical cohomogeneity one manifolds of positive curvature (the second part of Theorem A), and for our new candidates (third part of Theorem A), are provided in the following Table A. In the next seven sections we show that the list is complete. Indeed all the cases in which nonspherical examples occur are covered by Lemma 7.2 and Proposition 8.2.

In this section, we will explain which of these actions correspond to the known cohomogeneity one manifolds of positive curvature. The information in the table is separated into homogeneous examples, biquotients, and candidates (with some overlap). Due to its special significance we have included as a separate entry the linear action of $\mathsf{SO}(4)$ on $\mathbb{S}^7$ and separated the two cohomogeneity one actions on the Aloff Wallach space W^7 by its lower index. All manifolds are assumed to be simply connected.

For subgroups $\mathsf{S}^1 \subset \mathsf{S}^3 \times \mathsf{S}^3$ we have used the notation $\mathsf{C}^i_{(p,q)} = \{(e^{pi\theta}, e^{qi\theta}) \mid \theta \in \mathbb{R}\}$ and $\mathsf{C}^j_{(p,q)} = \{(e^{pj\theta}, e^{qj\theta}) \mid \theta \in \mathbb{R}\}$ and Q denotes the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$.

Some explanations are in order. The embedding of H is not always explicitly given, but can be determined in each case. $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is always embedded as $\{(\pm 1, \pm 1), (\pm i, \pm i)\}$. Otherwise, a $\mathbb{Z}_2$ inside H is always embedded in the circle inside K^+. The embedding of Q depends on the slopes, although it is always embedded diagonally up to conjugacy. e.g., for B^7 it must be of the form $\{\pm(1,1), \pm(i,-i), \pm(j,-j), \pm(k,k)\}$. The embedding of $\mathsf{SU}(2)$ is in a 2×2 block in $\mathsf{SU}(4)$.

Most of these actions are only almost effective, i.e., G and H have a finite normal, hence central subgroup in common. The effective version can easily be determined in each case, and we include in our table the most important part, the effective group $\bar{\mathsf{H}}$. It is also important to notice that the full effective groups for P_k are $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \{\mathsf{O}(2), \mathsf{O}(2)\} \subset_k \mathsf{SO}(4)$ and for Q_k (as well as for R) are $\mathbb{Z}_2 \subset \{\mathsf{SO}(2), \mathsf{O}(2)\} \subset_k \mathsf{SO}(3)\,\mathsf{SO}(3)$.

M^n	G	K^-	K^+	H	$\bar{\mathsf{H}}$	(l_-, l_+)	W
$\mathbb{S}^7$	$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{C}^i_{(1,1)}\mathsf{H}$	$\mathsf{C}^j_{(1,3)}\mathsf{H}$	Q	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(1,1)$	D_6
B^7	$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{C}^i_{(3,1)}\mathsf{H}$	$\mathsf{C}^j_{(1,3)}\mathsf{H}$	Q	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(1,1)$	D_3
$W^7_{(1)}$	$\mathsf{S}^3 \times \mathsf{S}^3$	$\Delta\,\mathsf{S}^3 \cdot \mathsf{H}$	$\mathsf{C}^i_{(1,2)}$	$\mathbb{Z}_2$	1	$(3,1)$	D_2
$W^7_{(2)}$	$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{C}^i_{(1,1)}\mathsf{H}$	$\mathsf{C}^j_{(1,2)}\mathsf{H}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$(1,1)$	D_4
B^{13}	$\mathsf{SU}(4)$	$\mathsf{Sp}(2)\cdot\mathbb{Z}_2$	$\mathsf{SU}(2)\cdot\mathsf{S}^1_{1,2}$	$\mathsf{SU}(2)\cdot\mathbb{Z}_2$	$\mathsf{SU}(2)\cdot\mathbb{Z}_2$	$(7,1)$	D_2
$\mathsf{E}^7_p, p \geq 1$	$\mathsf{S}^3 \times \mathsf{S}^3$	$\Delta\,\mathsf{S}^3 \cdot \mathsf{H}$	$\mathsf{C}^i_{(p,p+1)}$	$\mathbb{Z}_2$	1	$(3,1)$	D_2
$B^{13}_p, p \geq 1$	$\mathsf{SU}(4)$	$\mathsf{Sp}(2)\cdot\mathbb{Z}_2$	$\mathsf{SU}(2)\cdot\mathsf{S}^1_{p,p+1}$	$\mathsf{SU}(2)\cdot\mathbb{Z}_2$	$\mathsf{SU}(2)\cdot\mathbb{Z}_2$	$(7,1)$	D_2
$P_k, k \geq 1$	$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{C}^i_{(1,1)}\mathsf{H}$	$\mathsf{C}^j_{(2k-1,2k+1)}\mathsf{H}$	Q	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(1,1)$	D_3 or D_6
$Q_k, k \geq 1$	$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{C}^i_{(1,1)}\mathsf{H}$	$\mathsf{C}^j_{(k,k+1)}\mathsf{H}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$(1,1)$	D_4
R	$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{C}^i_{(3,1)}\mathsf{H}$	$\mathsf{C}^j_{(1,2)}\mathsf{H}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$(1,1)$	D_4

Table A. Known examples and candidates.

Here the groups K^- and K^+ are embedded in different blocks in each component of $\mathsf{SO}(3)\,\mathsf{SO}(3)$. The isomorphism types of these groups are consistent with, and in fact determined by, Lemma 3.5.

There are obvious and important isomorphisms among some of these cohomogeneity one actions which are apparent from the tables: $P_1 = \mathbb{S}^7$, $Q_1 = W^7_{(2)}$, $E_1 = W^7_{(1)}$ and $B^{13}_1 = B^{13}$.

The Weyl groups can be computed from the given isotropy groups. For example in the case of P_k, one chooses $w_- = (e^{\pi i/4}, e^{\pi i/4})$ and $w_+ = (e^{\pi j/4}, (-1)^k e^{\pi j/4})$ as representatives. One then checks that $(w_- w_+)^3 = 1$ in $\mathsf{N}(\mathsf{H})/\,\mathsf{H}$ for k even, and $(w_- w_+)^6 = 1$ for k odd. Hence $\mathsf{W} = \mathsf{D}_3$ for k even and $\mathsf{W} = \mathsf{D}_6$ for k odd.

The cohomogeneity one actions on the known positively curved manifolds were discovered by the first and last author in 1997, see [**Zi**] and [**GSZ**]. Although one can determine the group diagrams for these actions directly, it will be much simpler for us to use the classification. More precisely we will use Lemma 7.2 and Proposition 8.2 from below, whose proofs are independent of this section.

$\mathbb{S}^7$ with $\mathsf{G} = \mathsf{SO}(4)$

The 7-sphere has a cohomogeneity one action by $\mathsf{SO}(4)$ given by the isotropy representation of the symmetric space $\mathsf{G}_2\,/\,\mathsf{SO}(4)$. A normal subgroup $\mathsf{SU}(2)$ of $\mathsf{SO}(4)$ acts freely on $\mathbb{S}^7$ and hence is given by the Hopf action. If we divide by this action, we obtain an induced action of $\mathsf{SO}(3)$ on $\mathbb{S}^4$, which must be given by the usual action on trace free symmetric 3×3 matrices. The isotropy groups of this action on $\mathbb{S}^4$ are given by $\mathsf{K}^- = \mathsf{O}(2)$, $\mathsf{K}^+ = \mathsf{O}(2)$, and $\mathsf{H} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and hence are the same for the $\mathsf{SO}(4)$ action on $\mathbb{S}^7$. Since $\mathsf{SU}(2)$ acts freely, the slopes for the circles $\mathsf{K}^\pm_0$, viewed as subgroups of $\mathsf{S}^3 \times \mathsf{S}^3$, must have ± 1 in the second component. Using Lemma 7.2, the slopes must be $(1,1)$ and $(3,1)$ and this completely determines the group picture.

$B^7 = \mathsf{SO}(5)/\,\mathsf{SO}(3)$ with $\mathsf{G} = \mathsf{SO}(4)$

In the positively curved homogeneous Berger space $\mathsf{SO}(5)/\,\mathsf{SO}(3)$ the subgroup $\mathsf{SO}(3)$ is embedded via the irreducible representation of $\mathsf{SO}(3)$ on trace free symmetric 3×3 matrices (see [**Be**]). Notice that $\mathsf{SO}(4)\backslash \mathsf{SO}(5)/\,\mathsf{SO}(3) = \mathbb{S}^4/\,\mathsf{SO}(3)$ is one dimensional and thus $\mathsf{SO}(4)$ acts on $\mathsf{SO}(5)/\,\mathsf{SO}(3)$ by cohomogeneity one. Next we observe that the extended $\mathsf{O}(4)$ action is not orbit equivalent to the $\mathsf{SO}(4)$ action since for the $\mathsf{SO}(3)$ action on $\mathbb{S}^4$ the antipodal map takes one singular orbit to the other. This implies that the two singular isotropy groups K^- and K^+ are isomorphic up to an outer automorphism of $\mathsf{SO}(4)$. Combining this property with Lemma 7.2 we see that the action is determined: both singular groups are 1 dimensional and that the slopes of the circles of the corresponding ineffective $\mathsf{S}^3\times\mathsf{S}^3$-action are given by $\{(3,1),(1,3)\}$.

E_p^7 with $\mathsf{G} = \mathsf{SO}(3)\times\mathsf{SU}(2)$

The Eschenburg space $E_p^7 = \operatorname{diag}(z,z,z^p)\backslash\,\mathsf{SU}(3)/\operatorname{diag}(1,1,\bar z^{p+2})$, $p \geq 1$ has positive curvature according to [**E2**]. The group $\mathsf{SU}(2)\times\mathsf{SU}(2)$ acting from left and right in the first two coordinates induces an action on E_p^7 whose orbit through the identity is $\mathsf{SU}(2)\times\mathsf{SU}(2)/(\triangle\,\mathsf{S}^3\cdot\mathsf{H}) = \mathbb{RP}^3$ with $\mathsf{H} = \mathbb{Z}_2 = \langle(1,-1)\rangle$ or $\langle(-1,1)\rangle$. One easily sees that the action of K^- on the slice is nontrivial and hence E_p^7 is cohomogeneity one. The group K^+ is in this case not determined by this information. A computation shows it is a circle with slope $(p+1,p)$ and hence $\mathsf{H} = ((-1)^{p+1},(-1)^p)$, see [**GSZ**]. For p even, the left hand side $\mathsf{SU}(2)$ acts effectively as $\mathsf{SO}(3)$, and for p odd, the right hand side one does. For $p=1$ we obtain the cohomogeneity one picture for $W^7_{(1)}$ and the right hand side $\mathsf{SO}(3)$ acts freely. For $p=2$ the left hand side $\mathsf{SO}(3)$ acts freely, as one sees immediately from the group picture.

$W^7_{(2)}$ with $\mathsf{G} = \mathsf{SO}(3)\times\mathsf{SO}(3)$

For the positively curved Aloff–Wallach space $W^7 = \mathsf{SU}(3)\,/\operatorname{diag}(z,z,\bar z^2)$ [**AW**], we have $\mathsf{N}(\mathsf{H})/\,\mathsf{H} = \mathsf{U}(2)/\,\mathsf{H} = \mathsf{SO}(3)$ which acts freely on the right and hence we can write $B^7 = \mathsf{SU}(3)\,\mathsf{SO}(3)/\,\mathsf{U}(2)$ (see [**Wi1**]). Furthermore the second factor acts freely on W^7, and the action descends to the natural cohomogeneity one action of $\mathsf{SO}(3)$ on $\mathbb{CP}^2 = W^7/\,\mathsf{SO}(3)$. Thus G acts by cohomogeneity 1. From Lemma 7.2 it follows that there is only one cohomogeneity one action of $\mathsf{SO}(3)^2$ on a positively curved simply connected 7-manifold for which one of the factors acts freely. Thus the action is determined, both singular isotropy groups are one dimensional and that the slopes are given by $\{(1,2),(1,1)\}$.

B_p^{13} with $\mathsf{G} = \mathsf{SU}(4)$

The Bazaikin space

$$B_p^{13} = \mathrm{diag}(z,z,z,z,z^{2p-1})\backslash\,\mathsf{SU}(5)/\mathsf{Sp}(2)\,\mathrm{diag}(1,1,1,1,\bar z^{2p+3}),\quad p \geq 1$$

has positive curvature by [**Ba**] (see also [**Zi**] and [**DE**]). The action of $\mathsf{SU}(4) \subset \mathsf{SU}(5)$ on the left induces an action on B_p^{13} whose orbit through the identity is $\mathsf{SU}(4)/(\mathsf{Sp}(2) \cup i\,\mathsf{Sp}(2)) = \mathbb{RP}^5$. The action on the slice is easily seen to be nontrivial and hence B_p^{13} is cohomogeneity one. From the proof of Proposition 8.2 in the case of $\mathsf{G} = \mathsf{SU}(4)$ it follows that $\mathsf{H} = \mathsf{SU}(2)\cdot\mathbb{Z}_2$ and $\mathsf{K}^+ = \mathsf{SU}(2)\cdot\mathsf{S}^1$ where S^1 is allowed to have slopes $(q, q+1)$ inside of a maximal (two dimensional) torus of the centralizer of H. We can now consider the fixed point set of the involution $\mathrm{diag}(-1,-1,1,1,1) \in \mathsf{SU}(5)$ as in [**Ta**] and one shows that its fixed point set is $\mathrm{diag}(z,z,z^{2p-1})\backslash\,\mathsf{SU}(3)/\,\mathrm{diag}(z,z,\bar z^{2p+3}) = \mathrm{diag}(z,z,z^p)\backslash\,\mathsf{SU}(3)/\,\mathrm{diag}(1,1,\bar z^{p+2}) = E_p^7$ (see [**DE**]). Hence the slopes of the $\mathsf{SU}(4)$ action are determined (i.e., $q = p$). Because of $B_1^{13} = B^{13}$, this group picture is determined as well.

We add the following information about these actions, needed in our proof:

Lemma 4.2 (Extensions)**.** *The nonlinear actions in Table* A *have the following normal extensions*

(a) *The manifolds B^7, P_k, Q_k, and R, with their natural cohomogeneity one action, do not admit any connected normal extensions.*
(b) *For the manifolds E_p and B_p^{13}, the natural action has a unique connected normal extension by S^1.*

Proof. For the spaces B^7, P_k, Q_k, and R, which have singular orbits of codimension two, the identity component of the principal isotropy group of the extended action would normalize both singular isotropy groups contradicting primitivity.

For the spaces E_p and B_p^{13}, the natural action has a $\mathsf{U}(1)$ extension, since e.g., $\mathsf{SU}(4) \subset \mathsf{SU}(5)$ lies in $\mathsf{U}(4)$. Since the group diagram of this extension can be derived from that of G, any two extensions are equivariantly diffeomorphic. q.e.d.

One also easily derives the following information from the group diagrams in Table A and Table E.

Lemma 4.3 (Free Actions)**.** *If* G *acts by cohomogeneity one on an odd dimensional simply connected positively curved manifold M and* *there exists a subgroup* $\mathsf{L} \subset \mathsf{G}$ *with* $\mathsf{L} = \mathsf{SU}(2)$ *or* $\mathsf{L} = \mathsf{SO}(3)$ *which acts freely, then*

(a) $M = E_1 = W^7_{(1)}$ *or* $M = E_2$ *with* $\mathsf{L} = \mathsf{SO}(3) \subset \mathsf{SO}(3)\,\mathsf{SU}(2) = \mathsf{G}$.

(b) $M = W^7_{(2)}$ *with* $\mathsf{L} = \mathsf{SO}(3) \subset \mathsf{SO}(3)\,\mathsf{SO}(3) = \mathsf{G}$.

(c) M *is a sphere and the subaction of* $\mathsf{L} \cong \mathsf{S}^3$ *is given by the Hopf action.*

Remark 4.4. The existence of the free $\mathsf{SO}(3)$ actions on E_1 and E_2 was first observed by Shankar in [**Sh**], in connection with his discovery of counter examples to the so-called Chern conjecture. In the case of $E_1 = W^7_{(1)}$ and $W^7_{(2)}$ it is the natural free action of $\mathsf{N}(\mathsf{H})/\,\mathsf{H}$ on W^7.

Also notice that in all three cases the quotient by $\mathsf{SO}(3)$ is equal to $\mathbb{CP}^2$, which one can recognize from the induced cohomogeneity one diagram on the base. In the case of E_1 and E_2 it is the action of $\mathsf{SU}(2)$ on $\mathbb{CP}^2$ which has a fixed point. In the case of $W^7_{(2)}$ it is the cohomogeneity one action by $\mathsf{SO}(3)$ with singular orbits of codimension two.

The proof of Theorem A will occupy the next seven sections. As stated earlier, this is achieved by classifying essential cohomogeneity one actions by compact connected groups on simply connected odd dimensional manifolds with positive (sectional) curvature.

All partial classification results will be formulated in propositions, and

- for simplicity we will abuse language and assume from now on without stating it explicitly, that the manifolds M under consideration are simply connected and positively curved.

When a manifold is recognized via its isotropy groups, we will often say that we have "recovered" a particular action and manifold and leave it up to the reader to find the corresponding entry in Tables E or F and to verify that the groups are indeed recovered up to equivalence of their diagrams.

5. Equal Rank Groups

We are now ready to begin our classification of essential isometric cohomogeneity one G-actions on simply connected positively curved manifolds M. This section is concerned with the simplest situation of the rank lemma, where

- $\mathrm{rk}(\mathsf{H}) = \mathrm{rk}(\mathsf{K}^-) = \mathrm{rk}(\mathsf{K}^+) = \mathrm{rk}(\mathsf{G})$.

In particular, the normal spheres

- $\mathbb{S}^{l_\pm} = \mathsf{K}^\pm/\,\mathsf{H}$ are even dimensional

and hence one of $\mathsf{SO}(2n+1)/\,\mathsf{SO}(2n)$ or $\mathsf{G}_2\,/\,\mathsf{SU}(3)$. Thus

- $\mathsf{H} \subset \{\mathsf{K}^-, \mathsf{K}^+\} \subset \mathsf{G}$ are all connected.

Since an equal rank subgroup of $\mathsf{G}_1 \cdot \mathsf{G}_2$ is of the form $\mathsf{H}_1 \cdot \mathsf{H}_2$ with $\mathsf{H}_i \subset \mathsf{G}_i$, G is clearly semisimple, and hence by the product lemma

- G is simple.

Since the weights of the isotropy representation of an equal rank subgroup are roots, we have

- The irreducible subrepresentations $\mathfrak{m}_i$ of H are pairwise non-equivalent.

We will divide our analysis into the following three cases: (1) H is not semisimple, (2) H is semisimple, but not simple, and (3) H is simple.

Proposition 5.1. *If* G *acts essentially, with non-semisimple* H *of corank zero, then* G *is one of* $\mathsf{SU}(3), \mathsf{Sp}(2)$*, or* G_2 *and the action is the adjoint representation restricted to the sphere.*

Proof. We first show that in fact H is a maximal torus T. If not, let $\mathsf{H}' \lhd \mathsf{H}$ be a simple connected normal subgroup, and $\mathsf{S}^1 \subset Z(\mathsf{H})$. Since $\mathsf{K}^\pm / \mathsf{H}$ are even dimensional spheres, either H' or S^1 must act trivially on the irreducible subrepresentations of H in $\mathsf{K}^\pm$. By the Isotropy Lemma the same then holds for each irreducible subrepresentation of H in G and Lemma 2.4 implies that G is not simple, contradicting our assumption.

Therefore $\mathsf{H} = \mathsf{T}$ and we conclude that $\mathbb{S}^{l_\pm} \cong \mathbb{S}^2$, and $\mathsf{H} / \mathsf{H}_\pm$ both circles. By primitivity we see that $\dim \mathsf{T} = \operatorname{rk} \mathsf{G} \leq 2$. If $\operatorname{rk} \mathsf{G} = 1$ the action is obviously a suspension action which is non essential. It follows that G is one of $\mathsf{SU}(3), \mathsf{Sp}(2)$, or G_2.

To unify the discussion of these three cases we will use the well known fact (see e.g., [**Wo**]) that the Weyl group, $\mathsf{N}(\mathsf{T}) / \mathsf{T}$ of G acts transitively on the set of roots of G of the same length.

The Weyl group of $\mathsf{SU}(3)$ is D_3 acting transitively on its set of three equal length roots. Each root corresponds to a $\mathsf{U}(2) \subset \mathsf{SU}(3)$, and by primitivity the pair $(\mathsf{K}^-, \mathsf{K}^+)$ must be a pair of $\mathsf{U}(2)$ subgroups of $\mathsf{SU}(3)$ corresponding to different roots. We have recovered the diagram for the adjoint action of $\mathsf{SU}(3)$ on $\mathbb{S}^7$.

Both $\mathsf{Sp}(2)$ and G_2 have roots of two lengths. From the Isotropy Lemma it follows that the singular isotropy groups must correspond to roots of different lengths.

The Weyl group of $\mathsf{Sp}(2)$ is D_4 with two long roots $\mathsf{Sp}(1) \times \mathsf{S}^1 \subset \mathsf{Sp}(1) \times \mathsf{Sp}(1) \subset \mathsf{Sp}(2)$ and two short roots $\mathsf{U}(2) \subset \mathsf{Sp}(2)$. All pairs $(\mathsf{K}^-, \mathsf{K}^+)$ corresponding to a long and a short root define the same manifold, namely $\mathbb{S}^9$ with the adjoint action of $\mathsf{Sp}(2)$.

The Weyl group of G_2 is D_6, and has three long roots and three short roots. A short root corresponds to $\mathsf{U}(2) \subset \mathsf{SU}(3)$. There are two $\mathsf{U}(2) \subset \mathsf{SO}(4)$, one a long root and one a short root. Since $\mathsf{K}^\pm$ cannot both be in $\mathsf{SO}(4)$ by primitivity, this leaves, modulo the action of the

Weyl group, only one configuration for the pairs $(\mathsf{K}^-, \mathsf{K}^+)$ and we have recovered the adjoint action of G_2 on $\mathbb{S}^{13}$. q.e.d.

Proposition 5.2. *If* G *acts essentially, with semisimple, nonsimple* H *of corank zero, then* $\mathsf{G} = \mathsf{Sp}(3)$ *and the action is the unique linear action on* $\mathbb{S}^{13}$ *with* $\mathsf{H} = \mathsf{Sp}(1)^3$.

Proof. Suppose H' is a simple normal subgroup of H with $\operatorname{rk}\mathsf{H}' \geq 2$. By Lemma 2.4, we can find a subrepresentation of G/H on which H' and H/H' act non-trivially, which can not degenerate since $\mathsf{K}^\pm/\mathsf{H}$ are even dimensional spheres. Thus H is a semisimple group with rank one factors only. In particular both $\mathbb{S}^{l_\pm}$ are 4-dimensional.

Again by Lemma 2.4, we see that for any two different simple subgroups H_1 and H_2 of H, the isotropy representation of G/H has an irreducible subrepresentation on which both H_i act non trivially. By the isotropy lemma, this representation has to degenerate along the normal geodesic c at some singular orbit, say $\mathsf{K}/\mathsf{H} = \mathsf{Sp}(2)/\mathsf{Sp}(1)\,\mathsf{Sp}(1)$. Note that there is an element $w \in \mathsf{W}$ represented by an element $w \in \mathsf{K}\cap\mathsf{N}(\mathsf{H})$, which acts on H by permuting the two factors H_1 and H_2, and leaving all other factors of H invariant. Thus the action of the Weyl group on the factors of H contains all possible transpositions, and it is hence the full symmetric group. The only symmetric groups which are dihedral are S_2 and S_3. Hence H has at most three factors or equivalently $\operatorname{rk}(\mathsf{G}) \leq 3$. If $\operatorname{rk}(\mathsf{G}) = 2$, G must contain an $\mathsf{Sp}(2)$ or $\mathsf{SO}(5)$, which rules out $\mathsf{G} = \mathsf{SU}(3)$ and G_2, and for $\mathsf{G} = \mathsf{Sp}(2)$ the action must be a suspension action, which is not essential.

If $\operatorname{rk}(\mathsf{G}) = 3$, G contains a semisimple 9-dimensional subgroup H as well as an $\mathsf{Sp}(2)\,\mathsf{Sp}(1)$, which rules out $\mathsf{SU}(4)$ and $\mathsf{SO}(7)$, and in the case of $\mathsf{G} = \mathsf{Sp}(3)$ with $\mathsf{H} = \mathsf{Sp}(1)^3$ leaves, by primitivity, only one configuration for $\mathsf{K}^\pm$ and we have recovered the action of $\mathsf{Sp}(3)$ on $\mathbb{S}^{13}$. q.e.d.

Proposition 5.3. *If* G *acts essentially, with simple* H *of corank zero, then* $\mathsf{G} = \mathsf{F}_4$*, and the action is the unique linear action on* $\mathbb{S}^{25}$ *with* $\mathsf{H} = \mathsf{Spin}(8)$.

Proof. Using that H is a simple equal rank subgroup of G with a spherical isotropy representation, we can deduce from Table B that (G, H) is either $(\mathsf{F}_4, \mathsf{Spin}(8))$ or $(\mathsf{F}_4, \mathsf{Spin}(9))$. The latter case can actually not occur since the 16-dimensional representation of $\mathsf{F}_4/\mathsf{Spin}(9)$ can not possibly degenerate. Recall that the isotropy representation of $\mathsf{F}_4/\mathsf{Spin}(8)$ decomposes into three pairwise nonequivalent 8 dimensional representations of $\mathsf{Spin}(8)$, each contained in a $\mathsf{Spin}(9)$. Clearly the action is determined by primitivity, and we have recovered the unique cohomogeneity one action of $\mathsf{G} = \mathsf{F}_4$ on $\mathbb{S}^{25}$. q.e.d.

We point out that for all actions classified in this section the cohomogeneity one Weyl groups coincide with the core groups $\mathsf{N}(\mathsf{H})/\mathsf{H}$ which are either D_3, D_4 or D_6.

6. Non Semisimple Groups

In this and the following five sections we assume that:

- M is a simply connected cohomogeneity one G-manifold, with an invariant metric of positive curvature,
- G is connected acting essentially with principal isotropy group H of corank two.

Based on the even dimensional classification [**V1**, **V2**], the following is quite simple:

Proposition 6.1. *Suppose* G *is not semisimple and acts essentially with corank* 2*. Then either* $\mathsf{G} = \mathsf{S}^1 \cdot \mathsf{L}$*, where* L *is one of* $\mathsf{SO}(n), \mathsf{Spin}(7)$*, or* G_2*, and the action is a tensor product action on* $\mathbb{S}^{2n-1}, \mathbb{S}^{15}$*, or* $\mathbb{S}^{13}$ *respectively.*

Proof. After passing to a finite covering of G we may assume $\mathsf{G} = \mathsf{S}^1 \times \mathsf{L}$. Since $\mathsf{H} \cap \mathsf{S}^1$ is in the ineffective kernel of the action we can assume it is trivial. Moreover, H does not project surjectively onto S^1, since otherwise the subaction of L would be orbit equivalent to the G-action, which would then not be essential. Assume first that the subaction of the S^1-factor is free. Then $B = M/\mathsf{S}^1$ is an even dimensional simply connected manifold of positive sectional curvature with a cohomogeneity one action of L, and B is not 2-connected. So Verdiani's classification implies that B is a complex projective space. Since M is simply connected, the Euler class of the bundle $\mathsf{S}^1 \to M \to B$ is a generator of $H^2(B, \mathbb{Z})$. Using the Gysin sequence we deduce that M is a homology sphere.

If the subaction of the S^1-factor is not free, we can assume without loss of generality that $\mathsf{K}^- \cap \mathsf{S}^1 \neq 1$. Since $\mathsf{S}^1 \cap \mathsf{H} = 1$, $\mathsf{K}^- \cap \mathsf{S}^1$ acts freely on K^-/H and hence G/K^- is a component of the fixed point set $M^{(\mathsf{K}^- \cap \mathsf{S}^1)}$. By assumption (cf. 2.12) K^- is not normal in G, and $\dim(\mathsf{G}/\mathsf{K}^-) > 1$. Moreover, K^- must project surjectively to S^1, since G/K^- has positive curvature and hence finite fundamental group. On the other hand, since H does not project surjectively to S^1, it follows that G/K^- has codimension 2, and thus M is a homotopy sphere by the connectedness lemma (cf. 2.10).

The actual determination of the action follows from Straume's classification (see Table E). q.e.d.

7. Semisimple Rank 2 Groups

In the next four sections we assume in addition to M being a simply connected cohomogeneity one G-manifold, with an invariant metric of positive curvature, that:

- G is connected, simply connected and semisimple acting essentially with principal isotropy group H of corank two.

In this section we consider the case where $\operatorname{rk}\mathsf{G} = 2$, and hence H is finite. Clearly then $\mathsf{K}_0^{\pm} = \mathsf{S}^1$ or S^3.

We will first deal with the most interesting case, where G is not simple, i.e., $\mathsf{G} = \mathsf{S}^3 \times \mathsf{S}^3$.

Proposition 7.1. *If $\mathsf{G} = \mathsf{S}^3 \times \mathsf{S}^3$ acts essentially with corank 2, M is equivariantly diffeomorphic to one of the following spaces: An Eschenburg space $E_p, p \geq 1$, a $P_k, k \geq 1$, the Berger space B^7, a $Q_k, k \geq 1$, or R with the actions described in Table* A.

Since our actions are not assumed to be effective, we will use the notation $\bar{\mathsf{G}}, \bar{\mathsf{K}}$ and $\bar{\mathsf{H}}$ if the action is made effective. In view of our description provided in Table A in Section 4, the proposition is easily seen to follow from the following:

Lemma 7.2. *Under the condition of the above proposition, there are three possibilities:*

1) $\bar{\mathsf{H}} = 1$, $\bar{\mathsf{K}}^- \cong \mathsf{S}^3$ *and* $\bar{\mathsf{K}}^+ \cong \mathsf{S}^1$. *In* $\mathsf{S}^3 \times \mathsf{S}^3$, $\mathsf{K}^- = \triangle\mathsf{S}^3 \cdot \mathsf{H}$, $\mathsf{K}^+ = \mathsf{C}^i_{(p,p+1)}$ *with* $p \geq 1$, *and* $\mathsf{H} \cong \mathbb{Z}_2$.
2) $\bar{\mathsf{H}} \cong \mathbb{Z}_2$, $\bar{\mathsf{K}}^- \cong \mathsf{SO}(2)$ *and* $\bar{\mathsf{K}}^+ \cong \mathsf{O}(2)$. *In* $\mathsf{S}^3 \times \mathsf{S}^3$, *the groups are* $\mathsf{K}^- = \mathsf{C}^i_{(1,1)} \cdot \mathsf{H}$, $\mathsf{K}^+ = \mathsf{C}^j_{(p,p+1)} \cdot \mathsf{H}$ *with* $p \geq 1$ *and* $\mathsf{H} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, *or the same kind of space with slopes* $\{(3,1),(1,2)\}$.
3) $\bar{\mathsf{H}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *and* $\bar{\mathsf{K}}^- \cong \mathsf{O}(2) \cong \bar{\mathsf{K}}^+$. *In* $\mathsf{S}^3 \times \mathsf{S}^3$, *the groups are* $\mathsf{K}^- = \mathsf{C}^i_{(1,1)} \cdot \mathsf{H}$, $\mathsf{K}^+ = \mathsf{C}^j_{(p,p+2)} \cdot \mathsf{H}$ *with p odd* ≥ 1 *and* $\mathsf{H} \cong \mathsf{Q}$, *or the same kind of space with slopes* $\{(3,1),(1,3)\}$.

Proof. If $l_- = l_+ = 3$, the assumption that the action is essential means that K_0 cannot be one of the S^3 factors. Hence both $\mathsf{K}_0^{\pm} \simeq \mathsf{S}^3$ are embedded diagonally in $\mathsf{S}^3 \times \mathsf{S}^3$, contradicting group primitivity since any two diagonal embeddings are conjugate, and in the effective picture all groups are connected, and in particular $\bar{\mathsf{H}} = \{1\}$.

We now know that at least one of the singular orbits has codimension 2, which for the moment we denote as G/K and where we can assume that, up to conjugacy, $\mathsf{K}_0 = \mathsf{C}^i_{(p,q)}$ for two relatively prime nonnegative integers p, q. Moreover, note that the Product Lemma 2.6 implies that neither p nor q can be 0 since the normalizer of K_0 in one of the S^3 factors is finite.

In the following we will make use of a consequence of the equivariance of the second fundamental form of G/K regarded as a K equivariant linear map $B\colon S^2T \to T^\perp$. The non-trivial irreducible representations of $\mathsf{S}^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ consist of two dimensional representations given by multiplication by $e^{in\theta}$ on $\mathbb{C}$, called a weight n representation. The action of K_0 on $T^\perp = \mathbb{R}^2$ will have weight k if $\mathsf{H} \cap \mathsf{K}_0 = \mathbb{Z}_k$ since $\mathbb{Z}_k$ is the ineffective kernel. As we will show below, only the cases $k = 2, 4$ arise and we claim that $|p-q| = 2$ or $(p,q) = (1,1)$ in the case of $k = 4$, and $|p-q| = 1$ in the case $k = 2$.

To see this, we first observe that the action of K_0 on T has weights 0 on W_0 spanned by $(-qi, pi)$, weight $2p$ on the two plane W_1 spanned by $(j,0)$ and $(k,0)$ and weight $2q$ on the two plane W_2 spanned by $(0,j)$ and $(0,k)$. The action on $S^2(W_1 \oplus W_2)$ therefore has weights 0 and $4p$ on S^2W_1, 0 and $4q$ on S^2W_2 and $2p+2q$ and $2p-2q$ on $W_1 \otimes W_2$.

Next, we claim that for any homogeneous metric on G/K_0 there exists a vector $w_1 \in W_1$ and $w_2 \in W_2$ such that the 2-plane spanned by w_1 and w_2 tangent to G/K has curvature 0 intrinsically. Indeed, if $(p,q) \neq (1,1)$ or equivalently $p \neq q$, $\mathrm{Ad}(\mathsf{K}_0)$ invariance of the metric on G/K_0 implies that the two planes $\mathrm{span}\{(j,0),(0,j)\}$ and $\mathrm{span}\{(k,0),(0,k)\}$ and the line W_0 are orthogonal to each other. Hence $\mathrm{Ad}((j,j))$ induces an isometry on G/K_0, which implies that the two plane spanned by the commuting vectors $w_1 = (j,0) \in W_1$ and $w_2 = (0,j) \in W_2$ is the tangent space of the fixed point set of $\mathrm{Ad}((j,j))$ and thus has curvature 0. If $(p,q) = (1,1)$, $\mathrm{Ad}(\mathsf{K}_0)$ invariance implies that the inner products between W_1 and W_2 are given by $\langle (X,0),(0,Y)\rangle = \langle \phi(X), Y\rangle$ where $\phi\colon W_1 \to W_2$ is an $\mathrm{Ad}(\mathsf{K}_0)$ equivariant map. Hence, if we choose $j' = \phi(j)$ and $k' = \phi(k)$, the two planes $\mathrm{span}\{(j,0),(0,j')\}$ and $\mathrm{span}\{(k,0),(0,k')\}$ are orthogonal to each other, so that by the same argument $w_1 = (j,0) \in W_1$ and $w_2 = (0,j') \in W_2$ span a 2-plane with curvature 0.

If we now assume that $(p,q) \neq (1,1)$ at least one of the numbers $4p$ or $4q$ is not equal to the normal weight $k > 0$. The equivariance of the second fundamental form then implies that $B_{S^2W_i}$ vanishes for at least one i and hence by the Gauss equations $B(w_1, w_2) \neq 0$ for the above vectors w_1 and w_2. If $(p,q) = (1,1)$ the same holds if $k = 2$. Using the equivariance of the second fundamental form once more we see that $W_1 \otimes W_2$ contains a subrepresentation whose weight is equal to the normal weight k. Hence, $|2p+2q| = k$ or $|2p-2q| = k$, which proves our claim.

In addition we observe that H cannot contain an element h of the form $(a, \pm 1)$ or $(\pm 1, a)$ with a being a noncentral element. Indeed, this would imply that $\mathsf{N}(h)_0 = \mathsf{S}^1 \times \mathsf{S}^3$ or $\mathsf{S}^3 \times \mathsf{S}^1$ and hence M^h would be a totally geodesic submanifold of codimension 2 in M. By (2.10) M would be $\mathbb{S}^7$ with a linear action. But there is only one action on $\mathbb{S}^7$ with $\mathsf{K}_0^\pm = \mathsf{S}^1$, see Table E, and for that action H does indeed not contain such elements (cf. Table A).

Now let us consider the case where say $(l_-, l_+) = (3, 1)$. Since the action is assumed essential we have $\mathsf{K}_0^- = \triangle\,\mathsf{S}^3$ and $\mathsf{K}_0^+ = \mathsf{S}^1$. From the fact that $\triangle\,\mathsf{S}^3$ can be extended only by the central element $(1, -1)$, we see that $\bar{\mathsf{K}}^-$ is connected and $\bar{\mathsf{H}} = 1$. Thus $\mathsf{H} = \mathbb{Z}_2$ since $\mathsf{H} = 1$, and hence $k = 1$, contradicting the above equivariance argument. Thus $\mathsf{H} = \{(1, \pm 1)\}$ or $\{(\pm 1, 1)\}$, and $\mathsf{K}^+ \supset \mathsf{H}$ is connected since M is simply connected (cf. (1.6)). We can assume that, up to conjugacy and switching the two factors in $\mathsf{S}^3 \times \mathsf{S}^3$, $\mathsf{K}^+ = \mathsf{K}_0^+ = (e^{ip\theta}, e^{iq\theta})$ for two relatively prime positive integers p, q such that $q \geq p$. Using $k = 2$, the above equivariance argument implies that $q - p = 1$ and hence $(p, q) = (p, p+1)$ with $p > 0$.

It remains to consider the cases where $(l_-, l_+) = (1, 1)$, i.e., $\mathsf{K}_0^\pm = \mathsf{S}^1$. By Lemma 3.5 $\bar{\mathsf{H}}$ contains only elements of order two, which implies that H can only contain elements of order two or four. This in turn implies that the normal weights of the two singular orbits are 2 or 4.

We now have slopes p_-, q_- on the left and p_+, q_+ on the right. We next proceed to derive the following strong restrictions:

$$1 = \min\{|q_+|, |q_-|\} = \min\{|p_+|, |p_-|\}.$$

The first step utilizes the Alexandrov geometry of the quotients $M/\,\mathsf{S}^3\times 1$ and $M/1 \times \mathsf{S}^3$.

In general, for an isometric G action on M, it is a consequence of the slice theorem that the *strata*, i.e., components in $M/\,\mathsf{G}$ of orbits of the same type are (locally) totally geodesic (cf. [**Gr**]). In the case of $M/\,\mathsf{S}^3 \times 1$, the isotropy groups are effectively trivial on the regular part since $(a, 1)$ cannot lie in H unless it lies in the center. Along $B_\pm$ the isotropy groups are $\mathbb{Z}_{q_-}$ and $\mathbb{Z}_{q_+}$. This implies that the image of both $B_\pm$ in $M/\,\mathsf{S}^3 \times 1$ are totally geodesic if $\min\{|q_+|, |q_-|\} > 2$. Since these strata are two dimensional and $M/\,\mathsf{S}^3$ is four dimensional, both strata cannot be totally geodesic according to Petrunin's analogue [**Pe**] of Frankel's theorem for Alexandrov spaces. Hence we have $\min\{|q_+|, |q_-|\} \leq 2$ and $\min\{|p_+|, |p_-|\} \leq 2$. Furthermore, if equality holds in one of these inequalities, then G acts effectively as $\mathsf{SO}(3) \times \mathsf{S}^3$.

According to Lemma 3.5, two cases remain corresponding to $\bar{\mathsf{H}} = \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ since $\bar{\mathsf{H}} = 1$ and $l_\pm = 1$ contradicts group primitivity. In either case H contains an element h of order four. Combined with the above restrictions on h, we have $h^2 = (-1, -1)$. Thus $\bar{\mathsf{G}} \neq \mathsf{SO}(3) \times \mathsf{S}^3$ and $1 = \min\{|q_+|, |q_-|\} = \min\{|p_+|, |p_-|\}$ as claimed above.

If $\bar{\mathsf{H}} = \mathbb{Z}_2$, we can assume that $\bar{\mathsf{K}}^- = \mathsf{SO}(2)$, $\bar{\mathsf{K}}^+ = \mathsf{O}(2)$ and the non-trivial element $\bar{h} \in \bar{\mathsf{H}}$ is in the second component of $\bar{\mathsf{K}}^+$. Clearly, H contains an element h, whose image in $\bar{\mathsf{H}}$ is $\bar{h}$, and by the above each component in h is an unit imaginary quaternion. Since $\bar{h}$ acts trivially on $\mathbb{S}^{l_-}$ and by reflection on $\mathbb{S}^{l_+}$, so does h. In particular, h commutes with K_0^- and we can arrange w.l.o.g. that $\mathsf{K}_0^- = \mathsf{C}^i_{(p_-, q_-)}$ for two relatively prime positive integers p_-, q_- with $q_- \geq p_-$. Then h is one of $(i, \pm i)$, and hence p_-, q_- are both odd. Also, since conjugation by h must preserve K_0^+ and induce a reflection on it, we can assume, after possibly conjugating with an element in $\mathsf{N}(h)$, that $\mathsf{K}_0^+ = \mathsf{C}^j_{(p_+, q_+)}$ with positive integers p_+ and q_+ which are relatively prime.

For the precise group picture in $\mathsf{S}^3 \times \mathsf{S}^3$, there are two possible subcases. Either $\mathsf{H} = \mathbb{Z}_4 = \langle h \rangle = \{\pm(1,1), \pm h\}$ or $\mathsf{H} = \mathbb{Z}_4 \oplus \mathbb{Z}_2 = \langle h, (1,-1)\rangle = \{(\pm 1, \pm 1), (\pm i, \pm i)\}$. To rule out $\mathsf{H} = \mathbb{Z}_4$, assume first that p_+ and q_+ are both odd. In this case $\mathsf{H} \cap \mathsf{K}_0^+ = \mathbb{Z}_2$. Thus the normal weight is 2 and equivariance implies that $|p_+ \pm q_+| = 1$, a contradiction. If one is even and the other odd, $\mathsf{H} \cap \mathsf{K}_0^+ = 1$, which contradicts again the above equivariance argument. Now assume that $\mathsf{H} = \mathbb{Z}_4 \oplus \mathbb{Z}_2 = \{(\pm 1, \pm 1), (\pm i, \pm i)$, which implies that $\mathsf{H} \cap \mathsf{K}_0^+ = \mathbb{Z}_2$ and hence $q_+ - p_+ = \pm 1$. On the left, we have that $\mathsf{K}_0^- \cap \mathsf{H} = \langle h \rangle = \mathbb{Z}_4$ and hence the normal weight is 4, which implies that $q_- - p_- = 2$, or $(p_-, q_-) = (1,1)$. Together with the above Frankel argument, this implies that we have the possibility $(p_-, q_-) = (1,1)$ and $q_+ - p_+ = \pm 1$ or $(p_-, q_-) = (1,3)$ and $(p_+, q_+) = (2,1)$. In the first case we can also assume that $q_+ > p_+$ by interchanging the two factors if necessary, and hence $(p_+, q_+) = (p, p+1)$ with $p \geq 1$.

Finally, we assume that $\bar{\mathsf{H}} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In this case there are up to sign two noncentral order 4 elements h_- and h_+ in H, whose images $\bar{h}_-$ and $\bar{h}_+$ in $\bar{\mathsf{H}}$ are in the second components of $\bar{\mathsf{K}}^+$ and of $\bar{\mathsf{K}}^-$ respectively, as well as in the identity components K_0^- and K_0^+ respectively. Notice that h_- and h_+ must anticommute in G since both components of h_- and h_+ as well as $h_- h_+$ are unit imaginary quaternions. Since $\bar{h}_\pm$ act on $\mathbb{S}^{l_\pm}$ as expected from the previous case, we can arrange that $\mathsf{K}_0^\pm$ are of the form $\mathsf{K}_0^- = \mathsf{C}^i_{(p_-, q_-)}$ and $\mathsf{K}_0^+ = \mathsf{C}^j_{(p_+, q_+)}$ respectively, and correspondingly $h_- = (\pm i, \pm i)$ and $h_+ = (\pm j, \pm j)$ and thus all p_i, q_i are odd. We can also arrange, as above, that $q_- \geq p_- > 0$ and $p_+, q_+ > 0$.

There are now two possibilities for H. Either $\mathsf{H} = \triangle \mathsf{Q}$ (up to signs of the components) or $\mathsf{H} = \triangle \mathsf{Q} \oplus \langle (1,-1) \rangle$. In the latter case, since $(1,-1)$ generates another component for K^- and for K^+, M is not simply connected by Lemma 1.7. Thus $\mathsf{H} = \triangle \mathsf{Q}$, the weights on both normal spaces are 4 and hence $q_\pm - p_\pm = \pm 2$ or $(p_\pm, q_\pm) = (1,1)$. Combining all of the above now yields only two possibilities. Either $\{(p_-, q_-), (p_+, q_+)\} = \{(1,3), (3,1)\}$ or $\{(1,1), (p_+, q_+)\}$ with $q_+ - p_+ = 2$, where we used the fact that $\{(p_-, q_-), (p_+, q_+)\} = \{(1,1), (1,1)\}$ would not be group primitive. q.e.d.

We now turn to the simple rank two groups:

Proposition 7.3. *There are no actions of corank two of any of the groups* $\mathsf{SU}(3)$, $\mathsf{Sp}(2)$ *or* G_2.

Proof. From the Core-Weyl Lemma, we see that for the effective versions $\bar{\mathsf{H}} \neq 1$. In particular, (1.6) implies that $l_\pm$ cannot both be 3.

Now suppose one of $l_\pm$ is 3, and w.l.o.g. then $\bar{\mathsf{K}}^- = \mathsf{S}^1$, and $\bar{\mathsf{K}}^+ = \mathsf{S}^3 \cdot \bar{\mathsf{H}}$ and hence $\bar{\mathsf{H}}$ is cyclic by(1.6). It follows that $\mathsf{N}(\mathsf{H}) \cap \mathsf{K}^\pm$ are both at least 1-dimensional and by part c) the Upper Weyl Group Bound $|\mathsf{W}| \leq 4$. But this yields a contradiction to the Lower Weyl Group Bound if $\mathsf{G} = \mathsf{Sp}(2)$, or G_2. If $\mathsf{G} = \mathsf{SU}(3)$, then $\mathsf{N}(\mathsf{H})_0 = \mathsf{U}(2)$ or T^2. In either case it follows that w_+ may be represented by a central element in $\mathsf{N}(\mathsf{H})_0$. Using $\mathsf{S}^1 = \bar{\mathsf{K}}^- \subset \mathsf{N}(\mathsf{H})_0$ it follows that the Weyl group normalizes K^-. But then linear primitivity implies that equality can not hold in the lower Weyl group bound – a contradiction.

It remains to consider the situation where both $l_\pm = 1$, and thus, by Lemma (3.5), either $\bar{\mathsf{H}} = \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In the latter case we know that $\mathsf{N}(\mathsf{H})/\,\mathsf{H}$ must be finite since each of $(\mathsf{N}(\mathsf{H}) \cap \mathsf{K}^\pm)/\,\mathsf{H}$ are and M^{H} is primitive. However, for $\bar{\mathsf{G}} = \mathsf{SO}(5)$ we can diagonalize both involutions simultaneously. In one case, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is contained in an $\mathsf{SO}(3)$ block and the normalizer contains a circle. In the other case, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is contained in an $\mathsf{SO}(4)$ block, and the normalizer contains a torus. Similar arguments can be applied to all the other groups individually as well. These, however, are also all covered by the a general result due to Borel [**Bo**], which asserts in particular that any $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \bar{\mathsf{G}}$ is contained in a torus unless $\pi_1(\bar{\mathsf{G}})$ has 2-torsion.

If $\bar{\mathsf{H}} = \mathbb{Z}_2$ and hence $\bar{\mathsf{K}}^- = \mathsf{S}^1, \bar{\mathsf{K}}^+ = \mathsf{O}(2)$ the Lower Weyl Group Bound implies that $|W| \geq \dim \mathsf{G}/\,\mathsf{H} = \dim \mathsf{G}$ and $|W| \leq 8$ by the Upper Weyl Group Bound. Hence $\mathsf{G} = \mathsf{SU}(3)$, and it follows that $\mathsf{N}(\mathsf{H}) = \mathsf{U}(2)$ since this is the only equal rank symmetric subgroup of $\mathsf{SU}(3)$. In particular $N(\mathsf{H})$ is connected and the Core-Weyl Lemma gives the contradiction $|W| \leq 4$. q.e.d.

8. Semisimple Rank 3 Groups

If G has rank 3 and H has corank 2, one has the two subcases $\mathsf{H}_0 = \mathsf{S}^1$, or H_0 is one of S^3 or $\mathsf{SO}(3)$. Also recall that H/H_0 is cyclic. By the Isotropy Lemma $\max\{l_-, l_+\} \geq 2$, and by the rank Lemma $l_\pm$ cannot both be even.

In the case of $\mathsf{H}_0 = \mathsf{S}^1$, one has the possibilities $(l_-, l_+) = (1, 2)$, $(1, 3)$, $(2, 3)$, $(3, 3)$ (up to order) and in the latter two cases all groups are connected. Furthermore, $\mathsf{K}_0 = \mathsf{T}^2$ if $l_\pm = 1$, $\mathsf{K}_0 = \mathsf{SO}(3)$, or S^3 if $l_\pm = 2$ and $\mathsf{K}_0 = \mathsf{U}(2)$, or $\mathsf{S}^1 \times \mathsf{S}^3$ if $l_\pm = 3$.

If H_0 is 3-dimensional, one has the possibilities $l_\pm = 1, 3, 5, 7$ and $\mathsf{K}_0 = \mathsf{U}(2)$, $\mathsf{S}^3 \times \mathsf{S}^1$, or $\mathsf{SO}(3) \times \mathsf{S}^1$ if $l_\pm = 1$, $\mathsf{K}_0 = \mathsf{SO}(4)$, or $\mathsf{S}^3 \times \mathsf{S}^3$ if $l_\pm = 3$, $\mathsf{K}_0 = \mathsf{SU}(3)$ if $l_\pm = 5$ and $\mathsf{Sp}(2)$ if $l_\pm = 7$. If $\mathsf{H}_0 = \mathsf{SU}(2)$ (in every effective version), the lowest dimension of a representation is 4, which must degenerate somewhere and hence one of $\mathsf{K}_0^\pm = \mathsf{SU}(3)$ or $\mathsf{Sp}(2)$.

We will first deal with the case where G has a normal subgroup of rank one, i.e., almost effectively $\mathsf{G} = \mathsf{S}^3 \times \mathsf{L}$, where $\operatorname{rk} \mathsf{L} = 2$.

Proposition 8.1. *If* $\operatorname{rk} \mathsf{G} = 3$ *and* G *has a normal subgroup of rank one, an essential action of* G *with corank 2 is the tensor product action of* $\mathsf{SU}(2)\,\mathsf{SU}(3)$.

Proof. Before we start with the four possible subcases, let us notice that a three dimensional subgroup H_0 of $\mathsf{S}^3 \times \mathsf{L}$ must be contained in L since the action is almost effective and essential.

Case 1. $\mathsf{G} = \mathsf{S}^3 \times \mathsf{S}^3 \times \mathsf{S}^3$

If H_0 were three dimensional, the projection onto one of the factors would be onto and hence the action would be inessential. Thus $\mathsf{H}_0 = \mathsf{S}^1$, and one of $l_\pm$ is 2 or 3. First suppose, e.g., $l_- = 3$. Then the semisimple part of K^- is S^3 whose involution is a Weyl group element. Being central in G, it has G/K^- as a fixed point component, contradicting the fact that it cannot have positive curvature. Hence we are left with $l_- = 2$ and $l_+ = 1$. In particular $\mathsf{K}^-_0 = \mathsf{S}^3$ and $\mathsf{K}^+ = \mathsf{T}^2$. By the Product Lemma it follows that we can assume that $\mathsf{K}^-_0 = \{(q,q,q) | q \in \mathsf{S}^3\}$ and hence $\mathsf{H}_0 = \{(z,z,z) | z \in \mathsf{S}^1\}$. Clearly then the cyclic group $\mathsf{K}^-/\mathsf{K}^-_0 = \mathsf{H}/\mathsf{H}_0$ has at most two elements. Since $\mathsf{K}^+ \cong \mathsf{T}^2 \subset \mathsf{N}(\mathsf{H}_0)_0 \cong \mathsf{T}^3$ we can represent the Weyl group element w_+ by an element of the form $\iota = (\iota_1, \iota_2, \iota_3)$ of order 2 if $\mathsf{H} = \mathsf{H}_0$, and order 4 otherwise. Since we can also replace ι by $\iota(i,i,i)$ we can arrange that $\iota_p^2 = 1$ holds for at least two indices p. But then a component of M^{w_+} is a totally geodesic submanifold of G/K^+ of the form $\mathsf{S}^3 \times \mathsf{S}^3 \times \mathsf{S}^3 / \mathsf{T}^2$ or $\mathsf{S}^3 \times \mathsf{S}^3 \times \mathsf{S}^1 / \mathsf{T}^2$, neither one of which can have positive curvature.

Case 2. $\mathsf{G} = \mathsf{S}^3 \times \mathsf{SU}(3)$

We first settle the case that H is 3-dimensional. The only three dimensional spherical subgroup of $\mathsf{SU}(3)$ is $\mathsf{SU}(2)$ (cf. Table B in Appendix II). Since its normalizer is $\mathsf{S}^3 \times \mathsf{U}(2)$, the action by H_0 is fixed point homogeneous, M is a sphere, and the action is inessential.

Now suppose $\mathsf{H}_0 = \mathsf{S}^1$. We can then assume that H_0 is not contained in the S^3 factor since otherwise M would again be fixed point homogeneous. We distinguish between two subcases:

a) The involution $\iota \in \mathsf{H}_0$ is not in the center of G, i.e., $\iota = (\pm 1, b)$, and we can assume $b = \operatorname{diag}(-1,-1,1)$.

b) The involution of H_0 is central in G.

Subcase a). Then $\mathsf{N}(\iota)_0 = \mathsf{S}^3 \times \mathsf{U}(2)$ acts on M^ι_c by cohomogeneity one with one dimensional principal isotropy group. Thus M^ι_c has dimension 7 and M dimension 11 and hence M^ι_c is simply connected by the Connectedness Lemma.

Let us first assume that M_c^ι is a sphere. The Connectedness Lemma implies that M is 4-connected. We may assume that the action of $\mathsf{S}^3 \times \mathsf{U}(2)$ on M_c^ι has finite kernel, since otherwise we can deduce from part (b) of the Connectedness Lemma that M is 5−connected and hence a sphere. By assumption $\mathsf{N}(\iota)_0$ acts linearly on M_c^ι. There are two types of linear actions by $\mathsf{S}^3 \times \mathsf{U}(2)$ on the 7-sphere: one is a sum action and the other the tensor product action. If it were a sum action, the S^3 factor would have a fixed point and hence would be contained in some $\mathsf{K}^\pm$, contradicting the assumption that the action on M is essential.

Hence it is the tensor product action and thus S^3 acts freely on M_c^ι. This implies that the action of S^3 on M is also free since all G orbits meet M_c^ι and S^3 is normal in G. Since M is 4-connected, the quotient $M/\mathsf{SU}(2)$ is two connected but not 4-connected and by Verdiani's classification in even dimensions $M/\mathsf{SU}(2) = \mathbb{HP}^2$. From the Gysin sequence it follows first that the Euler class of the bundle $\mathsf{S}^3 \to M \to \mathbb{HP}^2$ is a generator of $H^4(\mathbb{HP}^2, \mathbb{Z})$ (again since M is 4-connected), and then that M is a homology sphere. From Table E we then see that it must be the tensor product action of $\mathsf{SU}(2)\,\mathsf{SU}(3)$.

Next we exclude the case that M_c^ι is not a sphere. Since any two involutions in $\mathsf{SU}(3)$ are conjugate, we can choose an element $g \in \mathsf{SU}(3)$ such that ι and $g\iota g^{-1}$ span a dihedral group $\mathsf{D}_2 = \mathbb{Z}_2^2$. By Frankel, $M_c^\iota \cap gM_c^\iota$ is non-empty and by transversality at least 3-dimensional. Since D_2 is contained in a torus the codimension is even. From the assumption that M_c^ι is not a sphere, we conclude that it cannot have dimension 5 by (2.10), and hence it is 3-dimensional. Since $M_c^\iota \cap gM_c^\iota \to M_c^\iota$ is 3-connected by part (c) of the Connectedness Lemma, $M_c^\iota \cap gM_c^\iota$ is simply connected and hence must be $\mathbb{S}^3$. In particular M_c^ι is 2-connected. The only 2-connected positively curved 7-manifolds in our classification theorem are B^7 and P_k. However, as we have seen in Lemma 4.2, for these manifolds the group does not have a connected normal extension. It follows that the $\mathsf{S}^3 \times \mathsf{U}(2)$ action has a one dimensional kernel, which must be the center of $\mathsf{U}(2)$, and hence this is actually an action by $\mathsf{SU}(2)\,\mathsf{SO}(3)$. But this group does not act on B^7 or P_k or any of its subcovers; see Table A.

Subcase b). In this case H_0 has only one involution, namely $(-1, \mathrm{diag}(1, 1, 1))$.

Consider the cyclic subgroup C_4 of order four in H_0. We may assume $\mathsf{C}_4 \not\subset \mathsf{S}^3$ and thus $\mathsf{N}(\mathsf{C}_4) = \mathsf{Pin}(2) \times \mathsf{U}(2) \supset \mathsf{N}(\mathsf{H})$. Let M' be a component of $\mathrm{Fix}(\mathsf{C}_4)$ on which $\mathsf{N}(\mathsf{C}_4)_0$ acts with cohomogeneity one. By induction assumption M' is up to covering a 5-sphere endowed with a linear action. This shows that K^- (or K^+) is a 4-dimensional subgroup of $\mathsf{N}(\mathsf{C}_4)_0$.

Clearly the semisimple part $\mathsf{SU}(2)$ of K^- is normal in $\mathsf{N}(\mathsf{C}_4) \supset \mathsf{N}(\mathsf{H})$ and $\mathsf{SU}(2) \cdot \mathsf{H} = \mathsf{K}^-$. Hence $\mathsf{N}(\mathsf{H})$ and thereby the Weyl group normalizes K^-. Because of $\mathrm{rk}(\mathsf{K}^-) = 2$ it is clear that $\mathsf{N}(\mathsf{H})_0 \not\subset \mathsf{K}^-$. Combining this with linear primitivity we see that K^+/H contains a trivial subrepresentation. Therefore $\mathsf{K}^+/\mathsf{H} \cong \mathbb{S}^3$, or $\mathbb{S}^1$. The latter case would imply $\mathsf{K}^\pm \subset \mathsf{N}(\mathsf{C}_4)$, which contradicts primitivity. In the former case the Weyl group has order at most 4, by the upper Weyl group bound, Proposition 3.6. Since K^- is normalized by the Weyl group, linear primitivity says that the Lie algebras of K^-, K^+ and $w_-\mathsf{K}^+w_-$ span the Lie algebra of G. But this is clearly impossible as these groups have H in common.

Case 3. $\mathsf{G} = \mathsf{S}^3 \times \mathsf{Sp}(2)$

Again we first settle the case that H is 3-dimensional. There are two spherical 3 dimensional subgroups of $\mathsf{Sp}(2)$: $\mathsf{Sp}(1) \times 1$ and $\triangle \mathsf{Sp}(1)$ (cf. Table B). In the first case H_0 acts transitively in the unit sphere orthogonal to M^{H_0} since $\mathsf{N}(\mathsf{H}_0) = \mathsf{S}^3 \times \mathsf{Sp}(1) \times \mathsf{Sp}(1)$ and is hence fixed point homogeneous. In the second case G/H effectively becomes $\mathsf{S}^3 \times \mathsf{SO}(5)/\mathsf{SO}(3)$ and the Chain Theorem applies.

We can now assume $\mathsf{H}_0 = \mathsf{S}^1$ and one, say K^- has rank 2. If K^-_0 contains one of the involutions $\iota = (\pm 1, \pm\,\mathrm{diag}(1, -1))$, up to conjugation, we obtain a contradiction as follows. If ι lies in H, M^ι_c is cohomogeneity one under $\mathsf{N}(\iota)_0 = (\mathsf{S}^3)^3$ with one dimensional principal isotropy group. As we saw in Case 1, such an action does not exist. If ι does not lie in H, it has $(\mathsf{S}^3)^3/(\mathsf{K}^- \cap \mathsf{N}(\iota))$ as a fixed point component, which cannot have positive curvature.

We may assume that K^- contains the center of $\mathsf{S}^3 \times \mathsf{Sp}(2)$. Since G/K^- cannot be totally geodesic, it follows that the center of $\mathsf{S}^3 \times \mathsf{Sp}(2)$ is contained in H. Therefore H is not connected and we may assume that $\mathsf{K}^- \cong \mathsf{T}^2$ (see Lemma 1.6). By the product lemma K^- projects to a maximal torus of $\mathsf{Sp}(2)$. Since H contains no involution as above, it follows that the Weyl group element w_- can be represented by an element $\iota := (*, \mathrm{diag}(\pm 1, \pm 1)) \in \mathsf{K}^-$. Clearly the fixed point set of ι would be a homogeneous space which does not have positive sectional curvature.

Case 4. $\mathsf{G} = \mathsf{S}^3 \times \mathsf{G}_2$

We first rule out the case that H is 3-dimensional. The only 3-dimensional spherical subgroup of G_2 is $\mathsf{SU}(2) \subset \mathsf{SU}(3) \subset \mathsf{G}_2$. Although this does not immediately follow from Table B, it is easily verified by considering the four three dimensional subgroups of G_2. Since a four dimensional representation of $\mathsf{H}_0 = \mathsf{SU}(2)$ must degenerate, one of $\mathsf{K}^\pm_0 = \mathsf{SU}(3) \subset \mathsf{G}_2$ (no $\mathsf{Sp}(2)$ exists in G_2), which contradicts the Product Lemma.

Hence $\mathsf{H}_0 = \mathsf{S}^1$ and we can assume that $\operatorname{rk} \mathsf{K}^- = 2$. Among the involutions in K_0^- there is one of the form $\iota = (\pm 1, b)$ with b an non-trivial involution, which has normalizer $\mathsf{SO}(4)$ (see Table G). Thus $\mathsf{N}(\iota)_0 = \mathsf{S}^3 \times \mathsf{SO}(4)$. If ι lies in a principal isotropy group, the reduction M_c^ι has $\mathsf{S}^3 \times \mathsf{SO}(4)$ acting by cohomogeneity one with a one dimensional principal isotropy group, but such an action does not exist as we saw in the first case. Otherwise ι has a homogeneous fixed point component $\mathsf{S}^3 \times \mathsf{SO}(4)/(\mathsf{K}^- \cap \mathsf{N}(\iota)_0)$ which cannot have positive curvature. q.e.d.

It remains to deal with the cases where G is simple.

Proposition 8.2. *If* G *is simple with* $\operatorname{rk} \mathsf{G} = 3$ *acting essentially and with corank* 2, *then it either the linear reducible representation of* $\mathsf{SU}(4)$ *on* $\mathbb{S}^{13}$ *or the cohomogeneity one action of* $\mathsf{SU}(4)$ *on one of the Bazaikin spaces* $B_p^{13}, p \geq 1$ (*see Table* A).

Proof. There are three cases to consider, corresponding to $\mathsf{G} = \mathsf{SU}(4)$, $\mathsf{Sp}(3)$ or $\mathsf{Spin}(7)$. We first consider the most interesting case where $\mathsf{G} = \mathsf{SU}(4)$.

Case 1. $\mathsf{G} = \mathsf{SU}(4)$

We will first rule out the case that $\mathsf{H}_0 = \mathsf{S}^1$. We can assume that $\mathsf{H}_0 = \operatorname{diag}(z^{p_1}, z^{p_2}, z^{p_3}, z^{p_4}) \subset \mathsf{SU}(4)$ and hence the isotropy representation of G/H has weights $p_i - p_j$. By the Isotropy Lemma there can be at most two distinct non-zero weights and one easily sees that this leaves only four possibilities $(p_1, p_2, p_3, p_4) = (1, -1, 0, 0), (1, 1, -1, -1), (1, 1, 1, -3)$, and $(3, 3, -1, -5)$. In the last two cases $\mathsf{N}(a) = \mathsf{U}(3)$ for some element $a \in \mathsf{H}_0$ corresponding to z with $z^8 = 1$. But then the reduction M_c^a is a cohomogeneity one manifold manifold under $\mathsf{U}(3)$ with one dimensional principal isotropy group, which does not exist by induction.

If $(p_1, p_2, p_3, p_4) = (1, -1, 0, 0)$ we choose the involution $\iota = \operatorname{diag}(-1, -1, 1, 1) \in \mathsf{H}_0$. Then $\mathsf{N}(\iota)_0/\iota = \mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(2))/\iota = \mathsf{SO}(3)\,\mathsf{U}(2)$ acts by cohomogeneity one on the seven dimensional reduction M_c^ι with one dimensional principal isotropy group. By induction, up to covers, such a 7-dimensional cohomogeneity one manifold could be only a sphere with a sum action or the Eschenburg space E_p. But in both cases, the isotropy group is not contained in the $\mathsf{SO}(3)$ factor as it is for M_c^ι.

If $(p_1,\, p_2,\, p_3,\, p_4) = (1,\, 1,\, -1,\, -1)$, we observe that $\mathsf{N}(\mathsf{H}_0)_0/\mathsf{H}_0 = \mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(2))/\operatorname{diag}(z, z, \bar{z}, \bar{z})$ is equal to $\mathsf{SO}(4)$ since $\mathsf{SU}(2)\,\mathsf{SU}(2))$ acts transitively with isotropy $\operatorname{diag}(-1, -1, -1, -1)$. In the full normalizer $\mathsf{N}(\mathsf{H}_0)/\mathsf{H}_0$ we have a second component corresponding to the element that interchanges the two normal $\mathsf{SU}(2)$ subgroups of $\mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(2))$. Hence $\mathsf{N}(\mathsf{H}_0)/\mathsf{H}_0 = \mathsf{O}(4)$. Furthermore, M^{H_0} has only one seven dimensional component since the inclusion $M^{\mathsf{H}_0} \subset M$ is 1-connected by part (b) of the Connectedness Lemma. Hence $\mathsf{O}(4)$ acts by cohomogeneity one on M^{H_0} with cyclic principal isotropy group. Such a manifold is either Q_k or a space form. But in Q_k the slopes of $\mathsf{K}^+ = \mathsf{S}^1$ are $(k, k+1)$ and hence its (ineffective) $\mathsf{SO}(4)$ action does not extend to $\mathsf{O}(4)$. It also cannot be a space form, since the action on its cover would be a sum or modified sum action and hence $|\mathsf{W}| \leq 4$, which gives a contradiction to the lower Weyl group bound $l_- + l_+ \geq 7$.

We can now assume that H_0 is three dimensional. But the only spherical 3-dimensional subgroups of $\mathsf{SU}(4)$ are $\mathsf{SU}(2) \subset \mathsf{SU}(3) \subset \mathsf{SU}(4)$ or $\Delta\,\mathsf{SU}(2) \subset \mathsf{SU}(2)\,\mathsf{SU}(2) \subset \mathsf{SU}(4)$, (cf. Table B). In the latter case $\mathsf{G}/\mathsf{H}_0 = \mathsf{SO}(6)/\,\mathsf{SO}(3)$ and the Chain Theorem applies.

Hence we can assume $\mathsf{H}_0 = \mathsf{SU}(2)$ embedded as the lower 2×2 block. By the isotropy lemma one of $\mathsf{K}_0^{\pm}$ is equal to $\mathsf{SU}(3)$ or $\mathsf{Sp}(2)$. It is important to observe that $\mathsf{N}(\mathsf{H}_0)/\mathsf{H}_0 = \mathsf{U}(2)$ acts transitively on all possible embeddings of $\mathsf{SU}(3)$ or $\mathsf{Sp}(2)$ in $\mathsf{SU}(4)$ containing the same H_0 (in the case of $\mathsf{Sp}(2)$ this is best seen in $\mathsf{SO}(6)$).

Assume first that $\mathsf{K}_0^- = \mathsf{SU}(3)$. If $l_+ = 1$, $\mathsf{K}^+ = \mathsf{SU}(2)\cdot\mathsf{S}^1$ is connected and, modulo $\mathsf{N}(\mathsf{H}_0)/\mathsf{H}_0$, both $\mathsf{K}^{\pm}$ are contained in $\mathsf{U}(3)$, which contradicts primitivity. If $l_+ = 3$ and hence $\mathsf{K}^+ = \mathsf{SU}(2)\,\mathsf{SU}(2)$, the element $-\operatorname{Id} \in \mathsf{SU}(4)$ is in K^+ and represents a Weyl group element. Since it is central, B_+ is totally geodesic, but it cannot have positive curvature. If $l_+ = 5$ and hence $\mathsf{K}^+ = \mathsf{SU}(3)$, the action is not primitive. If $l_+ = 7$ we have $\mathsf{K}^+ = \mathsf{Sp}(2)$. All embeddings are determined, modulo $\mathsf{N}(\mathsf{H}_0)/\mathsf{H}_0$, and we have the linear action of $\mathsf{SU}(4)$ on $\mathbb{S}^{13}$.

This leaves $\mathsf{K}_0^- = \mathsf{Sp}(2)$. If also $\mathsf{K}^+ = \mathsf{Sp}(2)$, the action is not primitive. The case of $\mathsf{K}^+ = \mathsf{SU}(2)\,\mathsf{SU}(2)$ is dealt with as above, and $\mathsf{K}^+ = \mathsf{SU}(3)$ was already considered. It only remains to consider the case where $l_+ = 1$. Since $\mathsf{K}^+ = \mathsf{K}_0^+ \subset \mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(2))$ we can assume up to conjugacy that $\mathsf{K}^+ = \mathsf{H}_0 \operatorname{diag}(z^k, z^l, \bar{z}^{(k+l)/2}, \bar{z}^{(k+l)/2})$. Notice that $-\operatorname{Id} \in \mathsf{SU}(4)$ cannot be in H since it is in $\mathsf{K}_0^- = \mathsf{Sp}(2)$ and $\mathsf{K}^-/\,\mathsf{H} = \mathbb{S}^7$. But $-\operatorname{Id}$ can also not be in K^+, since then it would represent w_+ in contradiction to the fact that B_+ has zero curvatures and hence cannot be totally geodesic. This implies that $(k, l) = (2p, 2q)$ with $(p, q) = 1$, p and q not both odd. Choosing $z = i$ and multiplying by $\operatorname{diag}(1, 1, \pm(i, -i))$ we see that $\iota = \operatorname{diag}(1, -1, -1, 1)$ or $\iota = \operatorname{diag}(-1, 1, -1, 1)$ is in K^+. If it does not lie in H, it has $\mathsf{U}(2)\,\mathsf{SU}(2)/\,\mathsf{T}^2 \subset B^+$ as a fixed point component, which does not have positive curvature. Hence H is not connected. Since $\mathsf{Sp}(2) \subset \mathsf{SU}(4)$ can only be extended by $\mathbb{Z}_2$, $\mathsf{H}/\mathsf{H}_0 = \mathbb{Z}_2$, with ι representing a second component. Thus M_c^{ι} is cohomogeneity one under the action of $\mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(2))/\langle\iota\rangle$ with $\mathsf{N}^{\mathsf{H}}(\iota)/\langle\iota\rangle = \operatorname{diag}(1, 1, z, \bar{z})$ as its principal isotropy group. Moreover, the subaction by $\mathsf{SU}(2)\,\mathsf{SU}(2)/\langle\iota\rangle = \mathsf{SO}(3)\,\mathsf{SU}(2)$ is again cohomogeneity one with trivial principal isotropy group. In this reduction $\mathsf{K}^+ = (z^{2p}, z^{2q})$ which effectively becomes (z^p, z^q). This reduction must be an Eschenburg space, and hence $(p, q) = (q+1, q)$ with $q \geq 1$. Hence our original manifold must be a Bazaikin space B^{13}_p (cf. Table A).

Case 2. $\mathsf{G} = \mathsf{Sp}(3)$

The symmetric subgroups of $\mathsf{Sp}(3)$ are $\mathsf{Sp}(2)\,\mathsf{Sp}(1)$ and $\mathsf{U}(3)$ where the latter is only a normalizer under an order 4 element (e.g., $i\,\mathrm{Id}$). If $\mathsf{H}_0 = \mathsf{S}^1$ we have, for appropriate a, $\mathsf{N}(a) = \mathsf{Sp}(2)\,\mathsf{Sp}(1)$ or $\mathsf{U}(3)$ with one dimensional principal isotropy group which does not exist by induction.

Now assume that H_0 is three dimensional. The 3 dimensional spherical subgroups of $\mathsf{Sp}(3)$ are, according to Table B, $\mathrm{diag}(q,\ q,\ q)$, $\mathrm{diag}(q, q, 1)$ or $\mathrm{diag}(q, 1, 1)$ with $q \in \mathsf{Sp}(1)$. In the first case, we can choose $\iota = i\mathrm{Id} \in \mathsf{H}_0$ and hence $\mathsf{N}(\iota) = \mathsf{U}(3)$ acts by cohomogeneity one on M^ι_c with one dimensional principal isotropy group, which does not exist by induction. In the second and third case we can choose an involution $\iota \in \mathsf{H}_0$ with $\mathsf{N}(\iota) = \mathsf{Sp}(2)\,\mathsf{Sp}(1)$ which acts by cohomogeneity one on the reduction M^ι_c with three dimensional principle isotropy group. By induction it must be a linear sum or modified sum action which contains a standard $\mathsf{Sp}(1) \subset \mathsf{Sp}(2)$ in its principal isotropy group. Thus $\mathsf{H}_0 = \mathrm{diag}(1, 1, q)$ and hence $\mathsf{Sp}(2)$ acts with finite principal isotropy group on the reduction $M^{\mathsf{H}_0}_c$, which, as we saw in Section 6, is not possible.

Case 3. $\mathsf{G} = \mathsf{Spin}(7)$

The symmetric subgroups of $\mathsf{SO}(7)$ are $\mathsf{SO}(6)$, $\mathsf{SO}(5)\,\mathsf{SO}(2)$ and $\mathsf{SO}(4)\,\mathsf{SO}(3)$, and correspondingly for $\mathsf{Spin}(7)$. If $\mathsf{H}_0 = \mathsf{S}^1$, we can choose $\iota \in \mathsf{H}_0$ with ι^2 but not ι in the center of $\mathsf{Spin}(7)$, and $\mathsf{N}(\iota)_0$ is one of the groups $\mathsf{Spin}(6), \mathsf{Spin}(5)\,\mathsf{Spin}(2)$ or $\mathsf{Spin}(4)\,\mathsf{Spin}(3)$. Hence they act by cohomogeneity one on the reduction M^ι_c with one dimensional principle isotropy group. But such a manifold does not exist by induction.

Now suppose H is 3-dimensional. If H_0 is a 3×3 block in $\mathsf{Spin}(7)$ we are done by the Chain Theorem. Thus by Table B we can assume that $\mathsf{H}_0 = \mathsf{SU}(2)$ is embedded as a normal subgroup of a 4×4 bock. By the Isotropy Lemma a four dimensional representation of H_0 must degenerate, which means that one of $\mathsf{K}^\pm_0$ must be $\mathsf{SU}(3)$ or $\mathsf{Sp}(2)$. There is only one embedding of $\mathsf{Sp}(2)$ and, since it corresponds to $\mathsf{SO}(5) \subset \mathsf{SO}(7)$, its central element is central in $\mathsf{Spin}(7)$. It then has $\mathsf{G}/\mathsf{K} = \mathsf{Spin}(7)/\,\mathsf{Sp}(2)$ as its fixed point set which does not admit positive curvature.

We can therefore assume that $\mathsf{K}^-_0 = \mathsf{SU}(3)$. Observe now that $\mathsf{N}(\mathsf{H}_0)_0/\mathsf{H}_0 = (\mathsf{Spin}(4) \times \mathsf{Spin}(3)/\triangle\mathbb{Z}_2)/\,\mathsf{SU}(2) = \mathsf{S}^3 \times \mathsf{S}^3/(-1, -1) = \mathsf{SO}(4)$ acts by cohomogeneity one on the reduction $M^{\mathsf{H}_0}_c$ with cyclic principal isotropy group H/H_0. All non-spherical examples and candidates in dimension 7, as well as their subcovers, either do not admit a cohomogeneity one action of $\mathsf{SO}(4)$, or only allow for actions with a noncyclic principal isotropy group. Thus M_c^{H} is a space form. Using once more that the principal isotropy group is cyclic, we see that the action is inessential and thus both singular orbit have codimension 4, a contradiction as the left singular orbit has codimension 2. q.e.d.

9. Semisimple Groups with a Rank 1 Normal Subgroup

In this section we will complete the analysis of simply connected, positively curved cohomogeneity one G-manifolds, where G has a normal subgroup of rank one:

Proposition 9.1. *Suppose a semi-simple* G *of rank at least four has a normal subgroup of rank one, and acts essentially with corank* 2*. Then* $\mathsf{G} = \mathsf{SU}(2)\,\mathsf{SU}(n)$*,* $M = \mathbb{S}^{4n-1}$ *and the action is the tensor product action.*

Proof. Let $\mathsf{G} = \mathsf{S}^3 \times \mathsf{L}$, where L is a simply connected semisimple group with $\operatorname{rk} \mathsf{L} \geq 3$ and hence $\operatorname{rk} \mathsf{H} \geq 2$ and $\operatorname{rk} \mathsf{H} \cap \mathsf{L} \geq 1$.

First observe that if $\mathsf{H} \cap \mathsf{S}^3 \lhd \mathsf{H}$ is not contained in the center of S^3, then the reduction $M_c^{\mathsf{H}\cap\mathsf{S}^3}$ has codimension 2 in M, and hence M is a sphere, and we are done by the classification of essential actions on spheres. Thus, if we set $\mathsf{S} = \mathsf{S}^3$ if $\mathsf{H} \cap \mathsf{S}^3$ is trivial, and $\mathsf{S} = \mathsf{SO}(3)$ if $\mathsf{H} \cap \mathsf{S}^3$ is non-trivial, we can assume that $\mathsf{G} = \mathsf{S} \times \mathsf{L}$ and $\mathsf{H} \cap \mathsf{S}$ is trivial.

In the proof we will use the following useful notation for the groups $\mathsf{K}^{\pm}$ and H: $\mathsf{K}_{\mathsf{S}} = \mathsf{K} \cap \mathsf{S}$ and $\mathsf{K}_{\mathsf{L}} = \mathsf{K} \cap \mathsf{L}$. Furthermore, there exists a connected normal subgroup K_{Δ} of K_0 embedded diagonally in $\mathsf{S} \times \mathsf{L}$ such that $\mathsf{K}_0 = (\mathsf{K}_{\mathsf{S}} \cdot \mathsf{K}_{\Delta} \cdot \mathsf{K}_{\mathsf{L}})_0$. It follows that K_{Δ} is a rank one group and, by the Product Lemma, K_{S} is finite, if non-empty.

We divide the proof into three subcases: (1) $\mathsf{S} = \mathsf{S}^3$ acts freely, (2) $\mathsf{S} = \mathsf{SO}(3)$ acts freely, and (3) S does not act freely. As it turns out, only the first case can occur.

Case 1. S^3 acts freely

In this case $B := M/\mathsf{S}^3$ is an even dimensional simply connected cohomogeneity one L - manifold of positive curvature. By Verdiani's classification, B is a rank one symmetric space and the action of L on B is linear.

Fix a maximal torus $\mathsf{T} = \mathsf{T}^h$ of $\mathsf{H}_{\mathsf{L}} = \mathsf{H} \cap \mathsf{L} \subset \mathsf{L}$, which has positive dimension by assumption, and consider the reduction $M' = M_c^{\mathsf{T}}$. Since $\mathsf{N}(\mathsf{T})/\mathsf{T} = \mathsf{S}^3 \times \mathsf{N}^{\mathsf{L}}(\mathsf{T})/\mathsf{T}$, the reduction M' supports a cohomogeneity one action by a group $\mathsf{S}^3 \times \mathsf{L}'$, where L' has rank 1 if $\mathsf{H}_0 \subset \mathsf{L}$, or rank 2 if H_{Δ} is non-trivial. The group L' also acts on the reduction B_c^{T} as well as on $M'/\mathsf{S}^3 \subset B_c^{\mathsf{T}}$ and in both cases with principal isotropy group $\mathsf{N}^{\mathsf{H}}(\mathsf{T})$. Hence $B' := M'/\mathsf{S}^3 = B_c^{\mathsf{T}}$.

The totally geodesic fixed point set B' is again a rank one symmetric space and must be simply connected since it is orientable. This in turn implies that M' is simply connected.

Since T is a maximal torus in H_L, the principal isotropy group of the $\mathsf{S}^3 \times \mathsf{L}'$ action on M' has at most finite intersection with the L' factor. As the subaction of the S^3-factor is free, our results in the previous two sections combined with Lemma 4.3 imply that $M' = \mathbb{S}^{4k+3}$ and $B' = \mathbb{HP}^k$.

The Euler class of the S^3 bundle $M \to B$ pulls back to the Euler class of $M' \to B'$ which is a generator in $H^4(B', \mathbb{Z}) = \mathbb{Z}$. This is only possible if $B \cong \mathbb{HP}^l$. The Euler class of $M \to B = \mathbb{HP}^l$ is therefore also a generator of $H^4(\mathbb{HP}^l, \mathbb{Z})$, and the Gysin sequence implies that M is a homology sphere. Table E now shows that it is the tensor product action of $\mathsf{SU}(2)\,\mathsf{SU}(n)$.

Case 2. $\mathsf{SO}(3)$ acts freely

In this case $B = M/\mathsf{SO}(3)$ is an even dimensional positively curved cohomogeneity one L-manifold. Since $M \to B$ is a principal $\mathsf{SO}(3)$ bundle and M is simply connected we see that B is simply connected, but not 2-connected. By Verdiani's classification B is a complex projective space. In the long homotopy sequence $\pi_2(M) \to \pi_2(B) \to \pi_1(\mathsf{SO}(3)) = \mathbb{Z}_2 \to \pi_1(M)$ the map in the middle can be regarded as representing the second Stiefel Whitney class in $H^2(B, \mathbb{Z}_2)$. Hence it is non-trivial for the bundle $M \to B$.

Consider as above a maximal torus $\mathsf{T} = \mathsf{T}^h$ of H_L and the corresponding reductions $M' \subset M$ and $B' \subset B$. Since the L action on B is linear, it follows that B' is a complex projective space as well, and by naturality, the principle $\mathsf{SO}(3)$ bundle $M' \to B'$ has a non vanishing second Stiefel Whitney class also. This in turn implies that M' is simply connected.

Also as above, we note that M' comes with a cohomogeneity one action of $\mathsf{SO}(3) \times \mathsf{L}'$ where $\mathrm{rk}(\mathsf{L}') \in \{1, 2\}$. Since $\mathsf{SO}(3)$ acts freely, it follows from our previous sections and Lemma 4.3 that $M' = E_1, E_2$ with $\mathsf{L}' = \mathsf{SU}(2)$ or $M' = W^7_{(2)}$ with $\mathsf{L}' = \mathsf{SO}(3)$. In all three cases $B' \cong \mathbb{CP}^2$ (see Remark 4.4) and the action of L' on $\mathbb{CP}^2$ is the action of $\mathsf{SU}(2)$ with a fixed point in the first two cases and in the third case the action of $\mathsf{SO}(3)$ on $\mathbb{CP}^2$ induced by the tensor product action of $\mathsf{SO}(2) \times \mathsf{SO}(3)$ on $\mathbb{S}^5$.

Consider first the case that B' is endowed with the standard $\mathsf{SU}(2)$ cohomogeneity one action which has a fixed point. Clearly only another "sum" action on a higher dimensional complex projective space can have this as a reduction. Because of $\mathrm{rk}(\mathsf{L}) \geq 3$, it follows that a normal simple subgroup $\mathsf{L}' \subset \mathsf{L}$ of rank at least 2 has a non-empty fixed point set in B, and in fact acts fixed point homogeneously. Since the action of $\mathsf{SO}(3)$ on the fibers only extends to an action of $\mathsf{SO}(4)$ and the action of L' fixes one $\mathsf{SO}(3)$ orbit in M, it follows that M is fixed point homogeneous. Clearly this is not possible since spheres do not support free actions of $\mathsf{SO}(3)$.

Assume now that $B' \cong \mathbb{CP}^2$ is equipped with the cohomogeneity one action of $\mathsf{SO}(3)$ with both singular orbits of codimension two. The only way this is a reduction of an L-action on a higher dimensional complex projective space is that up to orbit equivalence the L action is given by an $\mathsf{SO}(h+1)$-action on $\mathbb{CP}^h$ for some $h \geq 5$. Indeed, one sees that for all other actions in Table F, one of the normal spheres has odd codimension, which is preserved under a reduction by a torus.

The codimension of the singular orbits of the $\mathsf{SO}(h+1)$-action are 2 and $h-1$. The singular isotropy group for the orbit of codimension $h-1$ has a simple identity component of rk ≥ 2 and $\mathsf{K}^- = \mathsf{SO}(2) \cdot \mathsf{H}$ (see Table F). For the lifted picture upstairs in M, i.e., in the diagram $\mathsf{H} \subset \{\mathsf{K}^-, \mathsf{K}^+\} \subset \mathsf{SO}(3) \times \mathsf{L}$, we see that the projections of K^+ and H to the $\mathsf{SO}(3)$ factor are trivial and the projection of K^- is one dimensional. But this contradicts group primitivity.

Case 3. S^3 or $\mathsf{SO}(3)$ does not act freely.

In this subsection S is one of S^3 or $\mathsf{SO}(3)$, and we assume that $\mathsf{H}_{\mathsf{S}} = \mathsf{H} \cap \mathsf{S} = 1$, but S does not act freely on M. In particular one of $\mathsf{K}^{\pm}_{\mathsf{S}}$, say $\mathsf{K}^-_{\mathsf{S}}$, is non-trivial.

Choose an element $\iota \in \mathsf{K}^-_{\mathsf{S}}$. Since ι is not in H, the component V of M^ι containing $c(-1)$ is an odd dimensional positively curved homogeneous space $\mathsf{N}(\iota)_0/\mathsf{K}^- \cap \mathsf{N}(\iota)_0$. From the classification of positively curved homogeneous spaces we deduce that

- $V = \mathsf{L}/\mathsf{K}^-_{\mathsf{L}}$.

Since $\mathsf{K}^- \cap \mathsf{N}(\iota)_0$ has corank one in $\mathsf{N}(\iota)_0$ and $\mathrm{rk}\,\mathsf{N}(\iota)_0 = \mathrm{rk}\,\mathsf{S} \times \mathsf{L}$, it follows that K^- has corank one in $\mathsf{G} = \mathsf{S} \times \mathsf{L}$. The Product Lemma hence implies that $(\mathsf{K}^-_{\Delta})_0$ is non-empty. Indeed, since $\mathsf{S} \times \mathsf{L}$ and K^- do not have a normal subgroup in common, we have either $(\mathsf{K}^-_{\mathsf{S}})_0 = \mathsf{S}^1$, which has finite normalizer in S, or $\mathsf{K}^-_0 = (\mathsf{K}^-_{\mathsf{L}})_0$ is of equal rank in L which has finite normalizer in L. Thus it also follows that the projection of K^- into $\mathsf{L} \subset \mathsf{S} \times \mathsf{L}$, which is isomorphic to $\mathsf{K}^-_{\Delta} \cdot \mathsf{K}^-_{\mathsf{L}}$, has equal rank in L and hence $\mathsf{N}^{\mathsf{L}}((\mathsf{K}^-_{\mathsf{L}})_0)$ has equal rank also, i.e., $(\mathsf{K}^-_{\mathsf{L}})_0$ is a regular subgroup of L.

The cover $\tilde{V} = \mathsf{L}/(\mathsf{K}^-_{\mathsf{L}})_0$ of V is hence an odd dimensional homogeneous space of positive curvature with L semisimple of rank ≥ 3 and $(\mathsf{K}^-_{\mathsf{L}})_0$ regular. From the classification of 1-connected, positively curved homogeneous spaces (Table C and Table D), we see that

- The pair $(\mathsf{L}, (\mathsf{K}^-_{\mathsf{L}})_0)$ is one of $(\mathsf{Sp}(d), \mathsf{Sp}(d-1))$ or $(\mathsf{SU}(d+1), \mathsf{SU}(d))$ with $d \geq 3$.

Note that since $(\mathsf{K}^-_{\mathsf{L}})_0$ is simple, $\mathsf{K}^-_{\mathsf{S}}$ is finite and K^-_{Δ} of rank one, it follows that $\mathsf{K}^-_{\mathsf{L}}$ acts transitively on $\mathbb{S}^{l_-}$, unless $\mathsf{K}^-_{\mathsf{L}} = \mathsf{H}_{\mathsf{L}}$. In the latter case we can apply the Chain Theorem, and hence we can assume that $\mathsf{K}^-_{\mathsf{L}}$ indeed acts transitively on $\mathbb{S}^{l_-}$.

Consider the case $(\mathsf{L}, (\mathsf{K}^-_{\mathsf{L}})_0) = (\mathsf{Sp}(d), \mathsf{Sp}(d-1))$. Clearly, the odd dimensional sphere $\mathbb{S}^{l_-} = \mathsf{Sp}(d-1)/(\mathsf{H}_{\mathsf{L}})_0$ is equal to $\mathsf{Sp}(d-1)/\mathsf{Sp}(d-2)$. If $d \geq 4$, we can again apply the Chain Theorem. In the remaining case consider the reduction $M_c^{\mathsf{Sp}(1)}$ corresponding to a standard $\mathsf{Sp}(1) \subset \mathsf{H}_{\mathsf{L}} \subset \mathsf{Sp}(3) = \mathsf{L}$ which has a cohomogeneity one action by $\mathsf{S} \times \mathsf{Sp}(2)$. From our classification in the previous section it follows that it must be a sum action or a modified sum action. But in that case both $\mathsf{K}^{\pm} \cap \mathsf{S}$ are either trivial or all of S. This is a contradiction since $\mathsf{K}^-_{\mathsf{S}}$ is nontrivial and finite.

In the case of $(\mathsf{L}, (\mathsf{K}^-_{\mathsf{L}})_0) = (\mathsf{SU}(d+1), \mathsf{SU}(d))$, $d \geq 3$, we see as above that $\mathbb{S}^{l_-} = \mathsf{SU}(d)/(\mathsf{H}_{\mathsf{L}})_0$ is one of $\mathsf{SU}(d)/\mathsf{SU}(d-1)$, or $\mathsf{SU}(4)/\mathsf{Sp}(2)$. In particular, we can appeal to the Chain Theorem when $d \geq 4$.

If $\mathbb{S}^{l_-} = \mathsf{SU}(4)/\mathsf{Sp}(2)$, we obtain a contradiction to the Isotropy Lemma since the 8-dimensional representation of $\mathsf{SU}(5)/\mathsf{Sp}(2)$ on the orthogonal complement of $\mathsf{U}(4)$ can only degenerate in $\mathsf{Sp}(3)/\mathsf{Sp}(2)$, but $\mathsf{Sp}(3) \not\subset \mathsf{SU}(5)$.

It remains to consider the case $\mathbb{S}^{l_-} = \mathsf{SU}(3)/\mathsf{SU}(2)$. Since $\mathsf{K}^-_0 \supset \mathsf{SU}(3)$, the group $(\mathsf{K}^-_{\Delta})_0$ must be S^1 and hence $\mathsf{K}^-_0 = \Delta\,\mathsf{S}^1 \cdot \mathsf{SU}(3)$ and $\mathsf{H}_0 = \mathsf{S}^1 \cdot \mathsf{SU}(2)$, although the precise embedding of $\mathsf{S}^1 \subset \mathsf{H}_0$ is still to be determined. In any case, the projection of H_0 onto the first factor S is also given by a circle and hence H_0 has a two dimensional representation (inside S) which necessarily degenerates in K^+. Hence $\mathbb{S}^{l_+}$ is either $\mathbb{S}^2 = \mathsf{S}^3/\mathsf{S}^1$ or $\mathbb{S}^3 = \mathsf{S}^3 \cdot \mathsf{S}^1/\mathsf{S}^1$ and all groups are connected. In both cases primitivity implies that K^+ projects onto S and hence in both cases $\Delta\,\mathsf{SU}(2) \cdot \mathsf{SU}(2)$ must be contained in K^+.

If $\mathsf{K}^+/\mathsf{H} = \mathbb{S}^2$, we have $\mathsf{K}^+ = \Delta\,\mathsf{SU}(2) \cdot \mathsf{SU}(2)$ which determines the embedding of H, and hence the whole group diagram is determined. The action is the tensor product action of $\mathsf{SU}(2) \times \mathsf{SU}(4)$ on $\mathbb{S}^{15}$, but this contradicts the fact that the action of $\mathsf{S} = \mathsf{SU}(2)$ was assumed to be non free on the left singular orbit.

If $\mathsf{K}^+/\mathsf{H} = \mathbb{S}^3$, we have $\mathsf{K}^+ = \Delta\,\mathsf{SU}(2) \cdot \mathsf{SU}(2) \cdot \mathsf{S}^1$ and hence w_+ can be represented by a central element in G. But then G/K^+ is totally geodesic, which is not possible. q.e.d.

10. Non Simple Groups without Rank 1 Normal Subgroups

It remains to consider semisimple groups G without normal subgroups of rank one. In this section we deal with the non simple case, and prove the following

Proposition 10.1. *Let G be a non simple semisimple group without normal subgroups of rank one. If G acts essentially with corank 2, it is the tensor product action of $\mathsf{Sp}(2)\,\mathsf{Sp}(k)$ on $\mathbb{S}^{8n-1}$.*

Proof. Allowing a finite kernel $\mathsf{F} \subset \mathsf{H}$ for the action, we can assume that $\mathsf{G} = \mathsf{L}_1 \times \mathsf{L}_2$ with $\mathrm{rk}(\mathsf{L}_i) \geq 2$, and none of the L_i have normal subgroups of rank one. We let $\mathrm{pr}_i\colon \mathsf{G} \to \mathsf{L}_i$ denote the projections, and set $\mathsf{K}_i^{\pm} = \mathsf{K}^{\pm} \cap \mathsf{L}_i$, and $\mathsf{H}_i = \mathsf{H} \cap \mathsf{L}_i$. There are connected normal subgroups K_Δ of K_0 and H_Δ of H_0 embedded diagonally in $\mathsf{L}_1 \times \mathsf{L}_2$ such that $\mathsf{K}_0^{\pm} = (\mathsf{K}_1^{\pm} \cdot \mathsf{K}_\Delta \cdot \mathsf{K}_2^{\pm})_0$ and $\mathsf{H}_0 = (\mathsf{H}_1 \cdot \mathsf{H}_\Delta \cdot \mathsf{H}_2)_0$.

We first claim that at least one of the four groups $\mathsf{K}_i^{\pm}$ acts transitively on $\mathbb{S}^{l_\pm}$:

If one of H_i, say H_1, is non trivial when the action is made effective, then one of $\mathsf{K}_1^{\pm}$ acts transitively, since otherwise they both act freely or trivially, which implies that H_1 would be a subset of $\mathsf{H}_- \cap \mathsf{H}_+ = \mathsf{F}$, contradicting primitivity (3.4).

If both $\mathsf{H}_i \subset \mathsf{F}$, we see that $\mathsf{H}_0 = \mathsf{H}_\Delta$ embeds diagonally in $\mathsf{L}_1 \times \mathsf{L}_2$, and as a consequence $\mathrm{rk}\,\mathsf{H} = \mathrm{rk}\,\mathsf{L}_i = 2$. Now assume w.l.o.g. that K^- has corank one in G. From the Product Lemma it follows as before that K_Δ^- is not trivial of rank one and hence each of K_i^- has rank one. Thus all simple subgroups of K^-, and hence of H as well, have rank one. In particular $\mathbb{S}^{l_-}$ is one of $\mathbb{S}^1 = \mathsf{T}^2/\mathsf{S}^1$, $\mathbb{S}^3 = \mathsf{S}^3 \cdot \mathsf{S}^1/\Delta\,\mathsf{S}^1$, or $\mathbb{S}^3 = \mathsf{S}^3 \cdot \mathsf{S}^3/\Delta\,\mathsf{S}^3$. If one of K_i^- is three dimensional, it clearly must act transitively on K^-/H and the same is true if K^- and hence H are abelian. Hence we need to rule out the case $\mathsf{K}_\Delta^- = \mathsf{S}^3$ and $(\mathsf{K}_1^-)_0 \cong (\mathsf{K}_2^-)_0 \cong \mathsf{S}^1$, with $\mathsf{H}_0 = \mathsf{T}^2$ embedded into the maximal torus of K^-, such that it is onto $\mathsf{K}_1^- \cdot \mathsf{K}_2^-$. Since $\mathrm{rk}(\mathrm{pr}_i\,\mathsf{K}^-) = \mathrm{rk}(\mathsf{L}_i) = 2$, we see that the isotropy representation of $\mathsf{L}_1 \times \mathsf{L}_2/\mathsf{K}_0^-$ consists of a 3-dimensional representation and all other irreducible subrepresentations are even dimensional and pairwise inequivalent. It follows that there is an induced Riemannian submersion

$$\pi : \mathsf{L}_1 \times \mathsf{L}_2/\mathsf{K}_0^- \to \mathsf{L}_1/\,\mathrm{pr}_1(\mathsf{K}_0^-) \times \mathsf{L}_2/\,\mathrm{pr}_2(\mathsf{K}_0^-)$$

where the latter is equipped with a product metric. Let $\iota = (\iota_1, \iota_2)$ denote the central element in $\mathsf{K}_\Delta^- \cong \mathsf{S}^3$. Since ι acts by the antipodal map on the slice, the fixed point component V of M^ι containing $c(-1)$ is the positively curved homogeneous manifold $(\mathsf{N}(\iota_1) \times \mathsf{N}(\iota_2))/\mathsf{K}^- \subset \mathsf{L}_1 \times \mathsf{L}_2/\mathsf{K}^-$. Since $\mathsf{K}^- \cong \mathsf{S}^3 \times \mathsf{T}^2$, the classification of positively curved homogeneous spaces (cf. Table C and D) implies that $V = \mathsf{S}^3 \times \mathsf{S}^3/\Delta\,\mathsf{S}^3$ effectively. Hence neither ι_i can be central in L_i and we let $U_i \subset T(\mathsf{L}_i/\mathsf{K}_i^-) \subset T(\mathsf{L}_1 \times \mathsf{L}_2/\mathsf{K}^-)$ be the proper subspaces on which ι acts by $-id$. Then $U_1 \oplus U_2$ is horizontal with respect to the submersion π. But in the base, any plane spanned by $u_i \in U_i, i = 1, 2$ has curvature zero, so in the total space it has nonpositive curvature intrinsically.

This, however, yields the desired contradiction since by equivariance of the second fundamental form, $U_1 \oplus U_2$ is totally geodesic.

All in all it is no loss of generality to assume that say

- K_1^- acts transitively $\mathbb{S}^{l_-}$.

Since in this case $\mathsf{K}^- = \mathsf{K}_1^- \cdot \mathsf{H}$, the Weyl group element w_- may be represented by an element in L_1. Thus $\mathrm{pr}_2(w_- \mathsf{K}^+ w_-) = \mathrm{pr}_2(\mathsf{K}^+)$ and since $\mathrm{pr}_2(\mathsf{K}^-) = \mathrm{pr}_2(\mathsf{H}) \subset \mathrm{pr}_2(\mathsf{K}^+)$, we can employ Linear Primitivity to see that $\mathrm{pr}_2(\mathsf{K}^+) = \mathsf{L}_2$. In particular $\mathsf{K}_2^+ \lhd \mathsf{G}$, and hence $\mathsf{K}_2^+ = \{1\}$ since the action is essential. It follows that $\mathsf{K}_\Delta^+ \cong \mathsf{L}_2$ has rank two and thus:

- K^+ has corank two in G, and $\mathrm{rk}\, \mathsf{L}_2 = 2$.

Since K^+ and H have the same rank, either $\mathsf{K}_1^+ = \mathsf{H}_1$, or $\mathsf{K}_\Delta^+ = \mathsf{H}_\Delta$. The latter would imply that the subaction by L_1 is cohomogeneity one. Hence we can assume that $\mathsf{K}_1^+ = \mathsf{H}_1$, and K_Δ^+ acts transitively on $\mathbb{S}^{l_+}$. Since l_+ is even, L_2 is either $\mathsf{Sp}(2)$ or G_2, corresponding to $\mathbb{S}^{l_+}$ either $\mathsf{Sp}(2)/\mathsf{Sp}(1)\,\mathsf{Sp}(1)$ or $\mathsf{G}_2/\mathsf{SU}(3)$. The latter, however, is impossible since then H would contain $\mathsf{SU}(3)$ embedded diagonally in $\mathsf{L}_1 \times \mathsf{G}_2$ in contradiction to the Isotropy Lemma. In summary, using in addition the fact that K^- must be of corank one and $\mathsf{H}_2 = \{1\}$, we have:

- $\mathsf{L}_1 \times \mathsf{L}_2 = \mathsf{L}_1 \times \mathsf{Sp}(2)$
- $\mathsf{K}^+ = \mathsf{H}_1 \,\Delta\, \mathsf{Sp}(2)$ and $\mathsf{H} = \mathsf{H}_1 \,\Delta\, \mathsf{Sp}(1)^2$
- $\mathsf{K}_1^- = \mathsf{K}_1^- \mathsf{K}_\Delta^- \mathsf{K}_2^-$ with K_Δ^- of rank one and K_2^- acting freely.

Since $\mathsf{Sp}(1)^2$ in H is embedded diagonally, one $\mathsf{Sp}(1)$ must agree with K_Δ^- and the other must be embedded diagonally in $\mathsf{K}_1^- \mathsf{K}_2^-$. From the classification of transitive actions on spheres, it follows that $\mathsf{K}_2^- = \mathsf{Sp}(1)$ and $\mathsf{K}_1^- = \mathsf{Sp}(k)$ with $k \geq 1$ and hence $\mathsf{H}_1 = \mathsf{Sp}(k-1)$. It remains to determine L_1. From the group diagram we have so far, it follows that $\mathrm{pr}_1(\mathsf{K}^-) = \mathsf{Sp}(k)\,\mathsf{Sp}(1)$ and $\mathrm{pr}_1(\mathsf{K}^+) = \mathsf{Sp}(k-1)\,\mathsf{Sp}(2)$ are equal rank subgroups of L_1. This implies that $\mathsf{L}_1 = \mathsf{Sp}(k+1)$. The group diagram is now determined and the action is the tensor product action of $\mathsf{Sp}(k+1)\,\mathsf{Sp}(2)$ on $\mathbb{S}^{4k+11}$. q.e.d.

11. Simple Groups

In this section we will show that a simple group of rank at least four either does not act isometrically on an odd dimensional positively curved 1-connected manifold, or that it acts linearly on a sphere.

Proposition 11.1. *There are no actions of corank two for* $\mathsf{G} = \mathsf{Sp}(k), k \geq 4$.

Proof. Recall that we already saw that $\mathsf{G} = \mathsf{Sp}(2)$ and $\mathsf{G} = \mathsf{Sp}(3)$ do not act with corank two on a positively curved cohomogeneity one manifold .

If H contains $\mathsf{Sp}(1)$ embedded as a standard 1×1 block, then the reduction $M_c^{\mathsf{Sp}(1)}$ is odd dimensional, and $\mathsf{Sp}(k-1)$ acts by cohomogeneity one on it. By induction, such an action does not exist. Thus we may assume that H does not contain a 1×1 block.

Since $\mathrm{rk}(\mathsf{H}) \geq 2$, we can find an involution $\iota_1 \in \mathsf{T} \subset \mathsf{H}_0$ that is not central in $\mathsf{Sp}(k)$. The reduction $M_c^{\iota_1}$ is odd dimensional and supports a cohomogeneity one action of $\mathsf{Sp}(k-l) \cdot \mathsf{Sp}(l)$. From our induction hypothesis, this action is a tensor product or a sum action and hence H contains a 1×1 block unless $(k, l) = (4, 2)$. It remains to consider the tensor product action of $\mathsf{Sp}(2) \times \mathsf{Sp}(2)$, whose principal isotropy group and hence also H contains $\Delta(\mathsf{Sp}(1) \times \mathsf{Sp}(1)) \subset \Delta\,\mathsf{Sp}(2) \subset \mathsf{Sp}(2) \times \mathsf{Sp}(2) \subset \mathsf{Sp}(4)$. Now pick $\iota_2 = \mathrm{diag}(-1, 1, -1, 1) \in \mathsf{H} \subset \mathsf{Sp}(4)$, and note that the reduction $M_c^{\iota_2}$ supports a cohomogeneity one action of $\mathsf{Sp}(2) \times \mathsf{Sp}(2)$ corresponding to the $(1, 3)$ and $(2, 4)$ blocks, but with principal isotropy containing the above $\Delta(\mathsf{Sp}(1) \times \mathsf{Sp}(1))$ since ι_2 is central in it. In particular, the principal isotropy group of this action has a three dimensional intersection with either of the two $\mathsf{Sp}(2)$ factors. But such a linear action does not exist. q.e.d.

The case of $\mathsf{G} = \mathsf{SU}(k)$ with $k \geq 5$ is harder since there is an exceptional cohomogeneity one action of $\mathsf{SU}(5)$ on $\mathbb{S}^{19}$, and because $\mathsf{SU}(4)$ acts essentially on both $\mathbb{S}^{13}$ and on the Bazaikin spaces B_p, which can hence occur in a reduction.

Proposition 11.2. *The linear action of* $\mathsf{SU}(5)$ *on* $\mathbb{S}^{19}$ *is the only essential cohomogeneity one action by* $\mathsf{SU}(k), k \geq 5$ *of corank two.*

Proof. We first claim that H contains $\mathsf{SU}(2)$ embedded as a standard 2×2–block. To see this, choose an element $\iota \in \mathsf{H}_0$ of order 2 that is not central in $\mathsf{SU}(k)$. Then $\mathsf{S}(\mathsf{U}(k-2l)\,\mathsf{U}(2l))$ acts by cohomogeneity one on the reduction M_c^ι. For $\max\{k-2l, 2l\} \geq 4$ we see that either the kernel of the action and in particular H contains a 2×2 block, or else the action must be a tensor product action, a sum action, or the action of $\mathsf{U}(5)$ on $\mathbb{S}^{19}$ or $\mathsf{U}(4)$ on $\mathbb{S}^{13}$. In either case we again obtain a 2×2 block in H . Thus we may assume $(k, l) = (5, 1)$ and the universal cover of M_c^ι is $\mathbb{S}^{11}$ endowed with the tensor product action of $\mathsf{SU}(3)\,\mathsf{U}(2)$ with principal isotropy group T^2. Since in this case $\iota = \mathrm{diag}(1, 1, 1, -1, -1)$, it follows that M_c^ι admits an action of $\mathsf{SU}(3) \cdot \mathsf{SO}(3) \cdot \mathsf{S}^1$, and is therefore $\mathbb{RP}^{11}$. From the connectedness lemma we deduce that the codimension of M_c^ι is strictly larger than 10. Thus $\dim(M) = 23$ and $\mathsf{H}_0 \cong \mathsf{T}^2$. The singular orbits in M_c^ι have codimensions 3 and 4. Since $\mathsf{H}_0 \cong \mathsf{T}^2$, these codimensions necessarily coincide with the codimensions in M, all groups are connected, and we see that $\mathsf{K}^-, \mathsf{K}^+ \subset N(\iota)$ – a contradiction to primitivity.

From the fact that $\mathsf{H} \subset \mathsf{SU}(k)$ contains $\mathsf{SU}(2)$ embedded as a standard 2×2–block we proceed as follows: The reduction $M_c^{\mathsf{SU}(2)}$ supports a cohomogeneity one action by $\mathsf{SU}(k-2) \cdot \mathsf{S}^1$. By induction, this corank two action satisfies one of the following:

- The action is a sum action and $\mathsf{SU}(k-3) \subset \mathsf{SU}(k-2)$ is contained in the principal isotropy group, or $k = 6$ and the action is a sum action of $\mathsf{Spin}(6) \cdot \mathsf{S}^1$ which contains $\mathsf{Sp}(2)$ in its principal isotropy group.
- The action is orbit equivalent to the subaction of the $\mathsf{SU}(k-2)$-factor. This can only occur for $k = 6$ for the exceptional actions on $\mathbb{S}^{13}$ or B_p, and for $k = 7$ for the exceptional action on $\mathbb{S}^{19}$. In all cases, the isotropy group contains an $\mathsf{SU}(2)$ embedded as a 2×2 block, and in the last case $\mathsf{SU}(2)^2$ embedded as two 2×2–blocks.
- $k = 6$ and the action is given as the tensor product action of $\mathsf{S}^1 \cdot \mathsf{Spin}(6)$ on $\mathbb{S}^{11}$ and the principal isotropy group contains $\mathsf{SU}(2)^2 \subset \mathsf{SU}(4)$ embedded as two 2×2–blocks.

Clearly then for $k \geq 8$, we see that H contains $\mathsf{SU}(k-3)$ embedded as a standard $(k-3) \times (k-3)$ block and we are done by the Chain Theorem. It remains to deal with the cases $k = 5, 6, 7$.

$$\mathsf{G} = \mathsf{SU}(5)$$

By the above reduction argument we see that H contains another $\mathsf{SU}(2)$ block. If $\dim(\mathsf{H}) > 6$, then H_0 is an equal rank extension of $\mathsf{SU}(2)^2 \subset \mathsf{SU}(5)$ and hence $\mathsf{H}_0 = \mathsf{Sp}(2) \subset \mathsf{SU}(4) \subset \mathsf{SU}(5)$. But the irreducible 8-dimensional representation of $\mathsf{SU}(4) \subset \mathsf{SU}(5)$ restricted to $\mathsf{H}_0 = \mathsf{Sp}(2)$ cannot degenerate since $\mathsf{Sp}(3)$ is not contained in $\mathsf{SU}(5)$. Thus $\mathsf{H}_0 = \mathsf{SU}(2)^2$.

Note that the 8-dimensional representation of $\mathsf{SU}(4) \subset \mathsf{SU}(5)$ restricted to $\mathsf{H}_0 = \mathsf{SU}(2) \times \mathsf{SU}(2)$ splits as a sum of two four dimensional representations, each of which is acted on non trivially by exactly one of the $\mathsf{SU}(2)$ factors. We may assume that such a representation degenerates in K^-, and hence $\mathsf{K}_0^- = \mathsf{SU}(3) \cdot \mathsf{SU}(2) \subset \mathsf{SU}(5)$. There is also a 4-dimensional irreducible subrepresentation of $\mathsf{H}_0 = \mathsf{SU}(2) \times \mathsf{SU}(2) \subset \mathsf{Sp}(2)$ and the Isotropy Lemma implies that $\mathsf{K}_0^+ = \mathsf{Sp}(2)$. All groups are connected and we have recovered the picture of $\mathbb{S}^{19}$.

$$\mathsf{G} = \mathsf{SU}(6)$$

First suppose that the rank three group H contains $\mathsf{Sp}(2) \subset \mathsf{SU}(4)$. We can assume that $\mathsf{Sp}(2)$ is a normal subgroup of H, since otherwise H is $\mathsf{SU}(4)$ and the chain theorem applies, or H is $\mathsf{Sp}(3)$, which is maximal and thus G has a fixed point. Since the isotropy representation of $\mathsf{SU}(6)/\mathsf{Sp}(2)$ has an irreducible 8-dimensional subrepresentation coming from $\mathsf{Sp}(2) \subset \mathsf{SU}(4) \subset \mathsf{SU}(5)$, we can employ the Isotropy Lemma to see that one of the isotropy groups, say K^-, contains $\mathsf{Sp}(3)$ as a normal subgroup. But this is impossible since we also have $\mathrm{rk}(\mathsf{K}^-) = 4$ and $\mathsf{Sp}(3) \subset \mathsf{SU}(6)$ is a maximal connected subgroup.

Now we can assume that H_0 contains another $\mathsf{SU}(2)$ block. Let ι be the product of the central elements of the 2 blocks, i.e., up to conjugacy $\iota = \mathrm{diag}(1, 1, -1, -1, -1, -1) \in \mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(4))$ lies in H_0. The reduction M_c^ι is an odd dimensional manifold which supports a cohomogeneity one action by $\mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(4))/\iota = \mathsf{SU}(2) \cdot \mathsf{S}^1 \cdot \mathsf{SO}(6)$, whose principal isotropy group contains the lower 4×4-block $\mathsf{SO}(4) = \mathsf{SU}(2)\,\mathsf{SU}(2)/\iota$ of $\mathsf{SO}(6)$. If the action is a sum action H contains $\mathsf{Sp}(2)$, which we already dealt with.

If the action is the tensor product action, it is $\mathsf{SU}(2)$ ineffective and H contains the third 2×2-block. Then $\mathsf{H}_0 = \mathsf{SU}(2)^3$, since otherwise $\mathsf{H}_0 = \mathsf{Sp}(1)\,\mathsf{Sp}(2)$, which we already dealt with. At one singular orbit, say $\mathsf{K}^-/\,\mathsf{H}$, the trivial representation of H_0 has to degenerate, which can only happen in a codimension 2 orbit. Thus H_0 is normal in K^-. Also, at least one of the three $\mathsf{SU}(2)$ factors of H is also normal in K^+, contradicting primitivity.

$$\mathsf{G} = \mathsf{SU}(7)$$

From the reduction argument above, it follows that H contains $\mathsf{SU}(2)^3$ embedded as three 2×2-blocks. Hence the element $\iota = \mathrm{diag}(1, -1, -1, -1, -1, -1, -1)$ lies in H up to conjugacy. The reduction M_c^ι admits a cohomogeneity one action of $\mathsf{SU}(6) \times \mathsf{S}^1$ which must be a sum action. Hence H contains $\mathsf{SU}(5)$ and the chain theorem applies. q.e.d.

For $\mathsf{G} = \mathsf{Spin}(k), k \geq 8$ we have:

Proposition 11.3. *There are no essential cohomogeneity one actions of corank two by* $\mathsf{Spin}(k), k \geq 8$*, other than the exceptional linear actions of* $\mathsf{Spin}(8)$ *on* $\mathbb{S}^{15}$ *and* $\mathsf{Spin}(10)$ *on* $\mathbb{S}^{31}$*.*

Proof. We will separately treat the cases $k = 8, 9, 10$, and $k \geq 11$.

$$\mathsf{G} = \mathsf{Spin}(8)$$

In the case of $\mathsf{Spin}(8)$ we can assume, by the Chain Theorem, that H even up to an outer automorphism of $\mathsf{Spin}(8)$ does not contain a 3×3 block. This is particularly useful since there exists an outer automorphism which takes the standard $\mathsf{SU}(4) \subset \mathsf{Spin}(8)$ into the standard $\mathsf{Spin}(6) \subset \mathsf{Spin}(8)$ and $\mathsf{Sp}(2)$ into $\mathsf{Spin}(5)$.

Since $\mathrm{rk}(\mathsf{H}_0) = 2$, H_0 is one of G_2, $\mathsf{Sp}(2)$, $\mathsf{SU}(3)$, $\mathsf{S}^3 \cdot \mathsf{S}^3$, $\mathsf{S}^1 \cdot \mathsf{SU}(2)$ or T^2. We deal with each case separately, and we apply Table B to determine the embeddings.

If $\mathsf{H}_0 = \mathsf{G}_2$, the groups $\mathsf{K}^\pm$ must be $\mathsf{Spin}(7)$. There are 3 such $\mathsf{Spin}(7)$ in $\mathsf{Spin}(8)$ which are taken into each other by the outer automorphisms of $\mathsf{Spin}(8)$. Primitivity then determines the group diagram and M is $\mathbb{S}^{15}$.

If $\mathsf{H}_0 = \mathsf{Sp}(2) \subset \mathsf{SU}(4) \subset \mathsf{Spin}(8)$, an outer automorphism takes $\mathsf{Sp}(2)$ into a 5×5 block, and the Chain Theorem applies.

If $\mathsf{H}_0 = \mathsf{SU}(3) \subset \mathsf{SU}(4) = \mathsf{Spin}(6) \subset \mathsf{Spin}(8)$ the subgroup $\mathsf{L} = \mathsf{SU}(2)$ in H_0 is normal in $\mathsf{SU}(2)\,\mathsf{SU}(2) \subset \mathsf{SU}(4)$ which also, via an outer automorphism, is a 4×4 block in $\mathsf{Spin}(8)$. The normalizer of this $\mathsf{SU}(2)$ is therefore $(\mathsf{S}^3)^4$, and hence $(\mathsf{S}^3)^3$ acts by cohomogeneity one on the reduction M_c^{L} with a one dimensional principal isotropy group. As we know such an action does not exist.

If $\mathsf{H}_0 = \mathsf{S}^3 \cdot \mathsf{S}^3$ we see from Table B that the S^3 factors either sit as a 3×3 block, as a Hopf action on $\mathbb{R}^8$, or as a normal subgroup of a 4×4 block. In the second case, up to an outer automorphism, the embedding is also given by a 3×3 block. By the Chain Theorem it suffices to consider the case that both S^3 factors are given as normal subgroups of a 4×4 block. But then up to an automorphism H_0 is a 4×4 block.

If $\mathsf{H}_0 = \mathsf{S}^3 \cdot \mathsf{S}^1$, we can assume as before that S^3 is given by a normal subgroup of a 4×4 block. Then $M_c^{\mathsf{S}^3}$ admits a cohomogeneity one action of $\mathsf{Spin}(4) \times \mathsf{S}^3$ with one dimensional principal isotropy group. But such an action does not exist.

If $\mathsf{H}_0 = \mathsf{T}^2$ is abelian, choose an element $\iota \in \mathsf{H}_0$ for which ι^2 but not ι is in the center of $\mathsf{Spin}(8)$. Then the reduction M_c^ι admits a cohomogeneity one action of $\mathsf{Spin}(4) \cdot \mathsf{Spin}(4)$ or $\mathsf{Spin}(6) \cdot \mathsf{Spin}(2) = \mathsf{SU}(4) \cdot \mathsf{S}^1$ with a 2-dimensional principal isotropy group. By our induction assumption such an action does not exist.

$$\mathsf{G} = \mathsf{Spin}(9)$$

We can think of the maximal torus T^2 in H_0 as a subtorus in $\mathsf{S}^1 \cdot \mathsf{SU}(4) \subset \mathsf{Spin}(8)$. Choose an involution $\iota \in \mathsf{T}^2 \cap \mathsf{SU}(4)$. The normalizer $\mathsf{N}(\iota)_0$ is then either $\mathsf{Spin}(8)$ or $\mathsf{Spin}(5) \cdot \mathsf{Spin}(4)$, and the reduction M_c^ι supports a cohomogeneity one action by $\mathsf{N}(\iota)_0 / \langle \iota \rangle$ with principal isotropy group of corank 2.

It is easy to rule out the possibility $\mathsf{N}(\iota)_0 = \mathsf{Spin}(8)$. Indeed, the reduction M_c^ι clearly has codimension ≤ 8 and $\dim M \geq 22$ since $\dim \mathsf{H} \leq 14$. Thus M_c^ι is simply connected by the Connectedness Lemma. Hence the action of $\mathsf{Spin}(8)$ would have to be the exceptional action on $\mathbb{S}^{15}$, which contradicts the fact that the action is by $\mathsf{Spin}(8)/\langle \iota \rangle \cong \mathsf{SO}(8)$.

Thus we may assume that $\mathsf{N}(\iota)_0 = \mathsf{Spin}(4) \cdot \mathsf{Spin}(5)$. If the action on M_c^ι were almost effective or $\mathsf{Spin}(4)$ or $\mathsf{Spin}(5)$ its ineffective kernel, H would contain a 3×3-block. Hence we can assume that a normal subgroup of $\mathsf{Spin}(4)$ is contained in H and that the action is a sum action of $\mathsf{Spin}(3) \cdot \mathsf{Spin}(5)$. If the second factor acts as $\mathsf{SO}(5)$, H again contains a 3×3-block. If on the other hand the second factor acts as $\mathsf{Sp}(2)$, H contains $\mathsf{Sp}(1) \subset \mathsf{Sp}(1) \times \mathsf{Sp}(1) \subset \mathsf{Sp}(2)$ which is a normal subgroup in $\mathsf{Spin}(4) \subset \mathsf{Spin}(5)$. In this case, the involution $(-1, -1) \in \mathsf{Sp}(1) \times \mathsf{Sp}(1) \in \mathsf{H}$ has $\mathsf{Spin}(8)$ as its normalizer. As seen above, this is impossible.

$$\mathsf{G} = \mathsf{Spin}(10)$$

We choose an involution $\iota \in \mathsf{H}$ that is not central in $\mathsf{Spin}(10)$. Then $\mathsf{N}(\iota)_0$ is given by $\mathsf{Spin}(2) \cdot \mathsf{Spin}(8)$ or by $\mathsf{Spin}(4) \cdot \mathsf{Spin}(6)$, and it acts on the reduction M_c^ι with cohomogeneity one and with principal isotropy group of corank 2.

If $\mathsf{N}(\iota)_0 = \mathsf{Spin}(4) \cdot \mathsf{Spin}(6)$, then we argue as in the case of $\mathsf{Spin}(4) \cdot \mathsf{Spin}(5) \subset \mathsf{Spin}(9)$ that H contains an $\mathsf{SU}(2)$ normal in $\mathsf{Spin}(4)$ and an $\mathsf{SU}(2) \subset \mathsf{SU}(2)\,\mathsf{SU}(2) \subset \mathsf{SU}(4)$ from the sum or tensor product action of $\mathsf{SU}(2)\,\mathsf{SU}(4)$. This $\mathsf{SU}(2)$ is a normal subgroup of $\mathsf{Spin}(4) \subset \mathsf{Spin}(6)$ and we can find a different ι with $N(\iota)_0 = \mathsf{Spin}(2) \cdot \mathsf{Spin}(8)$.

Assume now that $N(\iota)_0 = \mathsf{Spin}(8) \cdot \mathsf{Spin}(2)$. If H contains the $\mathsf{Spin}(2)$-factor, then by induction it must also contain $\mathsf{G}_2 \subset \mathsf{Spin}(8)$. It follows that the isotropy representation of G/H_0 contains a nontrivial tensor product of $\mathsf{Spin}(2)$ and G_2 coming from the tensor product representation of $\mathsf{Spin}(8) \cdot \mathsf{Spin}(2)$ in $\mathsf{Spin}(10)$. But then G/H_0 is not spherical.

The only other possibility for the action of $\mathsf{Spin}(8) \cdot \mathsf{Spin}(2)$ on the reduction M_c^ι is that up to an outer automorphism and possibly a covering it is a tensor product or sum action. By the Chain Theorem we can also assume that H contains no 6×6-block. Hence, if it is a tensor product action, we can assume that H contains $\mathsf{SU}(4)$, and since $\mathsf{SU}(4)$ is not of equal rank in any group, it follows that $\mathsf{H}_0 = \mathsf{SU}(4)$. Similarly, if the reduction comes from a sum action, $\mathsf{H}_0 = \mathsf{Spin}(7) \subset \mathsf{Spin}(8)$ via the spin representation.

If $\mathsf{H}_0 = \mathsf{SU}(4)$, then H_0 has a six dimensional representation from $\mathsf{H}_0 = \mathsf{Spin}(6) \subset \mathsf{Spin}(8)$ and an eight dimensional representation orthogonal to $\mathsf{Spin}(8)$. They necessarily have to degenerate in different orbits and hence H is connected, $\mathsf{K}^- = \mathsf{Spin}(7)$, $\mathsf{K}^+ = \mathsf{SU}(5)$ and we have recovered the action of $\mathsf{Spin}(10)$ on $\mathbb{S}^{31}$.

If $\mathsf{H}_0 = \mathsf{Spin}(7)$, then H_0 has a 7-dimensional, two 8-dimensional and a trivial representation. The $8-$dimensional representation can only degenerate in $\mathsf{K}_0^- = \mathsf{Spin}(9)$ and the trivial representation in $\mathsf{K}^+ = \mathsf{Spin}(2) \cdot \mathsf{Spin}(7)$. The order two element in the center of $\mathsf{Spin}(10)$ is contained in $\mathsf{Spin}(9)$ and hence not in H. Since $\mathsf{H}_0 = \mathsf{Spin}(7)$ has a one dimensional centralizer in $\mathsf{Spin}(10)$, $\mathsf{K}^+ = \mathsf{Spin}(2)\,\mathsf{Spin}(7) \subset \mathsf{Spin}(2)\,\mathsf{Spin}(8) \subset \mathsf{Spin}(10)$. It follows that the central element of $\mathsf{Spin}(10)$ must also be contained in $\mathsf{Spin}(2) \subset \mathsf{K}^+$ and hence G/K^+ is totally geodesic – a contradiction.

$\mathsf{G} = \mathsf{Spin}(k)$ with $k \geq 11$

We let C denote the center of $\mathsf{Spin}(k)$. We first consider the special case that the subaction of C on M has more than one orbit type. Then we may assume $\mathsf{K}^- \cap \mathsf{C} \neq \mathsf{H} \cap \mathsf{C}$. Clearly $\mathsf{K}^- \cap \mathsf{C}$ acts freely on the normal sphere and hence G/K^- is totally geodesic. This implies K^- contains $\mathsf{Spin}(k-1)$ and H contains $\mathsf{Spin}(k-2)$ – a contradiction.

Thus C acts with one orbit type and M/C is a manifold. We now drop the assumption that M is simply connected and replace M by M/C. We also replace $\mathsf{Spin}(k)$ by $\mathsf{SO}(k)$ and C by the center of $\mathsf{SO}(k)$.

Choose an involution $\iota \in \mathsf{H} \subset \mathsf{SO}(k)$ which is not contained in C. Then $N(\iota) = \mathsf{SO}(2h) \cdot \mathsf{SO}(k-2h)$. Given that $\mathrm{rk}(\mathsf{H}) \geq 3$ for $k \geq 11$ and $\mathrm{rk}(\mathsf{H}) \geq 4$ for $k \geq 12$ we can arrange for $h \geq 2$ and $k - 2h \geq 3$.

Notice that $\mathrm{Fix}(\iota)$ has a component M' with a cohomogeneity one action of $\mathsf{SO}(2h) \cdot \mathsf{SO}(k-2h)$. The kernel of the action contains ι as well as C. Thus the center of $\mathsf{SO}(2h) \cdot \mathsf{SO}(k-2h)$ is contained in kernel of the action. We can assume that up to a covering this action is induced by a representation of $\mathsf{Spin}(2h) \times \mathsf{Spin}(k-2h)$ on a sphere (with principal isotropy group of corank 2). Furthermore the center of $\mathsf{Spin}(2h) \times \mathsf{Spin}(k-2h)$ acts on the sphere with one orbit type. It is easy to see that such a representation does not exist. q.e.d.

Proposition 11.4. *There are no cohomogeneity one actions with corank two of any of* $\mathsf{G} = \mathsf{F}_4, \mathsf{E}_6, \mathsf{E}_7,$ *or* E_8.

Proof. If $\mathsf{G} = \mathsf{F}_4$, choose an involution $\iota_1 \in \mathsf{H}_0$. Then $\mathsf{N}(\iota)_0 = \mathsf{Spin}(9)$ or $\mathsf{Sp}(1) \cdot \mathsf{Sp}(3)$ acts by cohomogeneity one on the reduction M_c^ι with corank two. As we have seen, this rules out $\mathsf{N}(\iota)_0 = \mathsf{Spin}(9)$. If $\mathsf{N}(\iota)_0 = \mathsf{Sp}(1) \cdot \mathsf{Sp}(3)$ then H contains $\mathsf{Sp}(2) \subset \mathsf{Sp}(1) \cdot \mathsf{Sp}(3) \subset \mathsf{F}_4$, and there is a different involution $\iota_2 = \mathrm{diag}(-1,-1) \in \mathsf{Sp}(2)$. Its normalizer cannot be another $\mathsf{Sp}(1) \cdot \mathsf{Sp}(3)$ since ι_2 is central in $\mathsf{Sp}(2)$ and hence cannot be central in the new $\mathsf{Sp}(3)$. Therefore we again have $\mathsf{N}(\iota)_0 = \mathsf{Spin}(9)$ and we obtain a contradiction.

If $\mathsf{G} = \mathsf{E}_6$, choose an involution $\iota \in \mathsf{H}_0$. Then $\mathsf{N}(\iota)_0 = \mathsf{SU}(6) \cdot \mathsf{SU}(2)$ or $\mathsf{Spin}(10) \cdot \mathsf{S}^1$ and by induction we see that H for any of the possible actions of these groups on the reduction $M_c^{\iota_1}$ must contain $\mathsf{SU}(4)$.

Choose next $\iota_2 = \mathrm{diag}(1,1,-1,-1) \in \mathsf{SU}(4)$. Since $\mathsf{N}(\iota_2)_0 \cap \mathsf{SU}(4) = \mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(2))$ it follows that H contains another $\mathsf{SU}(4)$ whose intersection with the first $\mathsf{SU}(4)$ is at most seven dimensional. Thus $\dim(\mathsf{H}) \geq 23$. Using Table B, it follows that $\mathsf{H}_0 = \mathsf{Spin}(8) \subset \mathsf{Spin}(9) \subset \mathsf{F}_4 \subset \mathsf{E}_6$, where we have used the fact that $\mathsf{H}_0 = \mathsf{Spin}(9)$ is not allowed since the 16 dimensional spin representation cannot degenerate. The centralizer of H_0 in E_6 is at least two dimensional since the dimension of $\mathsf{E}_6 / \mathsf{Spin}(8)$ equals $50 \equiv 2 \bmod 8$ and $\mathsf{E}_6 / \mathsf{Spin}(8)$ has a spherical isotropy representation.

At one of the singular orbits the trivial representation has to degenerate. This can only occur in a codimension 2 orbit. At the other singular orbit one of the 8-dimensional representation has to degenerate. But $l_- = 1$ and $l_+ = 8$ is a contradiction to the Lower and Upper Weyl Group Bound.

If $\mathsf{G} = \mathsf{E}_7$ or $\mathsf{G} = \mathsf{E}_8$, choose a noncentral involution $\iota \in \mathsf{H}_0$. Then $\mathsf{N}(\iota)_0 = \mathsf{SU}(8)$, $\mathsf{Spin}(12)/\mathbb{Z}_2 \cdot \mathsf{S}^3$ or $\mathsf{E}_6 \cdot \mathsf{S}^1$ in the case of E_7 and $\mathsf{N}(\iota)_0 = \mathsf{Spin}(16)/\mathbb{Z}_2$ or $\mathsf{E}_7 \cdot \mathsf{S}^3$ in the case of E_8. But by induction we know that none of these groups can act isometrically by cohomogeneity one on a positively curved manifold with corank two. q.e.d.

12. 3-Sasakian Structure of the Exceptional Families

In this section we establish the relationship (Theorem B) between the manifolds P_k and Q_k and the interesting orbifold examples due to Hitchin [**Hi1**]:

Theorem 12.1 (Hitchin)**.** *There exists a unique self dual Einstein orbifold metric O_k on $\mathbb{S}^4$ with the following properties:*

a) *It is invariant under the cohomogeneity one action by $\mathsf{G} = \mathsf{SO}(3)$ with singular orbits of codimension two.*
b) *It is smooth on $M \setminus B_+$.*
c) *Along the right hand side singular orbit $B_+ = \mathbb{RP}^2$ it is smooth in the orbit direction and has angle equal to $2\pi/k$ perpendicular to it.*

For the cohomogeneity one action of $\mathsf{SO}(3)$ on $\mathbb{S}^4$ the isotropy groups are given by $\mathsf{K}^\pm = \mathsf{O}(2)$ embedded in two different blocks and $\mathsf{H} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. There exists a similar action by $\mathsf{SO}(3)$ on $\mathbb{CP}^2$ given by multiplication with real matrices on homogenous coordinates in $\mathbb{CP}^2$. One easily shows that in this case $\mathsf{K}^- = \mathsf{SO}(2)$, $\mathsf{K}^+ = \mathsf{O}(2)$, again in two different blocks, and $\mathsf{H} = \mathbb{Z}_2$ generating the second component in K^+. Conjugation in $\mathbb{CP}^2$ then gives rise to an $\mathsf{SO}(3)$ equivariant two fold branched cover $\mathbb{CP}^2 \to \mathbb{S}^4$ with branching locus the real points $\mathsf{G}/\mathsf{K}^+ = \mathbb{RP}^2$ and a two fold cover along the left hand side singular orbits. When $k = 2\ell$ is even, one can thus pull back the metric $O_{2\ell}$ to become an orbifold metric on $\mathbb{CP}^2$ with normal angle $2\pi/\ell$.

We now describe the relationship with 3-Sasakian geometry, see [**BG**] for a general reference. Among the equivalent definitions, we will use the following: A metric is called 3-Sasakian if $\mathsf{SU}(2)$ acts isometrically and almost freely with totally geodesic orbits of curvature 1. Moreover, for U tangent to the $\mathsf{SU}(2)$ orbits and X perpendicular to them, $X \wedge U$ is required to be an eigenvector of the curvature operator $\hat{R}$ with eigenvalue 1, in particular the sectional curvature $\sec(X, U)$ is equal to 1. The dimension of the base is a multiple of 4, and its induced metric is quaternionic Kähler with positive scalar curvature, although it is in general only an orbifold metric. Conversely, given a quaternionic

Kähler orbifold metric on M with positive scalar curvature, one constructs the so-called Konishi bundle whose total space has a 3-Sasakian orbifold metric, such that the quotient gives back the original metric on M. In this fashion one obtains a one-to-one correspondence between 3-Sasakian orbifold metrics and quaternionic Kähler orbifold metrics with positive scalar curvature. If the base has dimension 4, quaternionic Kähler is equivalent to being self-dual Einstein and the Konishi bundle is the $\mathsf{SO}(3)$ principle orbifold bundle of self dual 2-forms on the base with the metric given by the naturally defined connection metric. Hence the Hitchin metrics give rise to 3-Sasakian orbifold metrics on a seven dimensional orbifold H_k^7. The cohomogeneity one action by $\mathsf{SO}(3)$ on the base admits a lift to the total space H_k^7 which commutes with the almost free principal orbifold $\mathsf{SO}(3)$ action. The joint action by $\mathsf{SO}(3)\,\mathsf{SO}(3)$ on H_k^7 is hence an isometric cohomogeneity one action. In general, one would expect the metric on H_k^7 to have orbifold singularities since the base does. However, we first observe that this is not the case. Although the claim also follows from the proof of Theorem 12.3, we give a simple and more geometric proof.

Theorem 12.2. *For each k, the total space H_k^7 of the Konishi bundle corresponding to the selfdual Hitchin orbifold O_k is a smooth 3-Sasakian manifold.*

Proof. Notice that the singular orbit B_+ in $O_k^4, k > 2$ must be totally geodesic. Indeed, being an orbifold singularity, one can locally lift the metric on a normal slice $\mathbb{D}^2$ to $\mathbb{RP}^2$ to its k-fold branched cover $\hat{\mathbb{D}} \to \mathbb{D}$ with an isometric action by $\mathbb{Z}_k$ such that $\hat{\mathbb{D}}/Z_k = \mathbb{D}$. Hence the singular orbit is a fixed point set of a locally defined group action and thus totally geodesic.

The $\mathsf{SO}(3)$ principle bundle H_k^7 is smooth over all smooth orbits in H_k^4. If it has orbifold singularities, they must consist of an $\mathsf{SO}(3)\,\mathsf{SO}(3)$ orbit which projects to B_+, and is again totally geodesic by the same argument as above. This five dimensional orbit is now 3-Sasakian with respect to the natural semi-free $\mathsf{SO}(3)$ action on H_k^7, since it is totally geodesic and contains all $\mathsf{SO}(3)$ orbits. But the quotient is 2-dimensional which contradicts the fact that the base of such a manifold has dimension divisible by 4. q.e.d.

As mentioned in the Introduction, except for $\mathbb{S}^7 = P_1$, the manifolds P_k are the first 2-connected seven dimensional 3-Sasakian manifolds. The cohomology rings of the manifolds Q_k happen to coincide with the cohomology rings of all the previously known 3-Sasakian 7-manifolds with second Betti number one. These are exactly the Eschenburg spaces $\mathrm{diag}(z^a, z^b, z^c)\backslash\ \mathsf{SU}(3)/\,\mathrm{diag}(1, 1, \bar{z}^{a+b+c})$ with a, b, c positive pairwise relatively prime integers [**BGM**]. They contain the 3-Sasakian manifolds E_k as a special case. All of these, as well as those with second

Betti number at least two [**BGMR**], are constructed from the constant curvature 3-Sasakian metric on $\mathbb{S}^{4n+3}$, equipped with the Hopf action, as 3-Sasakian reductions with respect to an isometric abelian group action commuting with the Hopf action. As a consequence all of them are toric, i.e., admit an isometric action by a 2-torus commuting with the $\mathsf{SU}(2)$ action. In contrast, the examples P_k and Q_k, for $k \geq 2$, are not toric, since the orbifolds O_k, for $k \geq 3$, have $\mathsf{SO}(3)$ as the identity component of their isometry group.

Before verifying that the above manifolds H^7_k coincide with the ones described in the introduction, we first discuss a general framework for cohomogeneity one orbifolds.

Observe that a group diagram as in (1.2), where we assume that $h_\pm$ are embeddings, but $j_\pm$ are only homomorphisms with finite kernel and $j_- \circ h_- = j_+ \circ h_+ = j_0$ with $\mathsf{K}^\pm / \mathsf{H} = \mathbb{S}^{l_\pm}$, defines a cohomogeneity one orbifold O: The regular orbits, being hypersurfaces, have no orbifold singularities, and we can therefore assume that j_0 is an embedding, although we still allow the action of G to be ineffective otherwise. A neighborhood of a singular orbit is given by $D(B_\pm) = \mathsf{G} \times_{\mathsf{K}^\pm} \mathbb{D}^{l_\pm+1}$ where $\mathsf{K}^\pm$ acts on G via right multiplication: $g \cdot k = g j_\pm(k)$ and on $\mathbb{D}^{l_\pm+1}$ via the natural linear extension of the action of $\mathsf{K}^\pm$ on $\mathbb{S}^{l_\pm}$. This then can be written as $D(B_\pm) = \mathsf{G} \times_{(\mathsf{K}^\pm / \ker j_\pm)} (\mathbb{D}^{l_\pm+1} / \ker j_\pm)$ and the singularity normal to the smooth singular orbit $\mathsf{G} / j_\pm(\mathsf{K}^\pm)$ is $\mathbb{S}^{l_\pm} / \ker j_\pm$. It is easy to see that any cohomogeneity one orbifold can be described in this fashion. In fact this follows since the frame bundle of a cohomogeneity one orbifold is a cohomogeneity one manifold, and thus orbifolds inherit cohomogeneity one diagrams as described. In all the cases of interest here, we note that both $l_\pm = 1$, and the orbifolds are therefore (topologically) manifolds.

We are now ready to prove:

Theorem 12.3. *Our manifolds P_k and Q_k are equivariantly diffeomorphic to the universal covers of the* 3*-Sasakian manifolds H_{2k-1} and H_{2k} respectively.*

Proof. Since the metrics in the Hitchin examples are smooth near B_-, it follows that $\mathsf{K}^- \cong \mathsf{O}(2)$ and hence $\mathsf{H} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence we can assume that j_- is an embedding of $\mathsf{K}^- \cong \mathsf{O}(2)$ into the lower block in $\mathsf{SO}(3)$, h_- the diagonal embedding $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \mathsf{O}(2)$, and via $j_- \circ h_-$ the group H is embedded as the set of diagonal matrices in $\mathsf{SO}(3)$. As in the case of smooth $\mathsf{SO}(3)$ invariant metrics on $\mathbb{S}^4$, the Hitchin orbifold metrics collapse in different directions corresponding to $\mathsf{K}^\pm_0$, and the normal angle along B_+ is $2\pi/k$. If we define the homomorphism $\phi_k \colon \mathsf{SO}(2) \to \mathsf{SO}(2)$ by $A \to A^k$, we see that $j_+(A) \in \mathsf{SO}(3)$ is $\phi_k(A)$ for $A \in \mathsf{K}^+_0$ followed by an embedding into $\mathsf{SO}(3)$, which we can assume is in the upper block in order to be consistent with the H-irreducible 1-dimensional subspaces of $\mathfrak{so}(3)$.

On the right hand side a neighborhood of the singular orbit is given by $D(B_+) = \mathsf{SO}(3) \times_{\mathsf{K}^+} \mathbb{D}^2_+$ where K_0^+ acts on $\mathsf{SO}(3)$ via ϕ_k and on $\mathbb{D}^2_+$ via ϕ_2 since $\mathsf{K}_0^+ \cap \mathsf{H} = \mathbb{Z}_2$. The description of the disc bundle $D(B_+)$ gives rise to a description of the corresponding (smooth) $\mathsf{SO}(4)$ principle orbifold bundle $\mathsf{SO}(3) \times_{\mathsf{K}^+} \mathsf{SO}(4)$ where the action of K_0^+ on $\mathsf{SO}(3)$ is given by ϕ_k as above, and the action on $\mathsf{SO}(4)$ is given via $\mathsf{SO}(2) \subset \mathsf{SO}(4)\colon A \in \mathsf{SO}(2) \to (\phi_k(A), \phi_2(A))$ acting on the splitting $T \oplus T^\perp$ into tangent space and normal space of the singular orbit. Similarly for the left hand side where $k = 1$. In order to take orientations into account and their consistent match for the gluing in the middle, we start with an oriented basis $\dot c(t), i, j, k$ for the regular orbits, where we have used for simplicity the isomorphism $\mathfrak{so}(3) \cong \mathfrak{su}(2)$. On the left hand side the i direction collapses, T is oriented by j, k and $T^\perp$ by $\dot c(-1), i$. Here i corresponds to the derivative of the Jacobi field along c induced by i. On the right hand side the j direction collapses, T is oriented by k, i and $T^\perp$ by $\dot c(1), j$. Here j corresponds to the negative of the derivative of the Jacobi field along c induced by j. Furthermore, one easily checks that $\mathsf{SO}(2) \subset \mathsf{O}(2)$ has a positive weight on T where we have endowed the isotropy groups on the left and on the right with orientations induced by i and j respectively. Hence $\mathsf{K}_0^\pm \subset \mathsf{SO}(3)\,\mathsf{SO}(4)$ sits inside the natural maximal torus in $\mathsf{SO}(3)\,\mathsf{SO}(4)$ with slopes $(1, 1, 2)$ on the left, and $(k, k, -2)$ on the right.

We can now determine the group picture for the $\mathsf{SO}(3)\,\mathsf{SO}(3)$ action on the principle bundle of the vector bundle of self dual two forms. This vector bundle can also be viewed as follows: If P is the $\mathsf{SO}(4)$ principle bundle of the orbifold tangent bundle of $\mathbb{S}^4$, then the quotient $P/\,\mathsf{SU}(2)$ under a normal $\mathsf{SU}(2)$ in $\mathsf{SO}(4)$ is an $\mathsf{SO}(3)$ principle bundle and by dividing by the two normal subgroups, one obtains the principle bundles for the vector bundle of self dual and the vector bundle of anti self dual 2 forms. This is due to the fact that the splitting $\Lambda^2 V \cong \Lambda^2_+ V \oplus \Lambda^2_- V$ for an oriented four dimensional vector space corresponds to the splitting of Lie algebra ideals $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ under the isomorphism $\Lambda^2 V \cong \mathfrak{so}(4)$. Alternatively we can first project under the two fold cover $\mathsf{SO}(4) \to \mathsf{SO}(3)\,\mathsf{SO}(3)$ and then divide by one of the $\mathsf{SO}(3)$ factors. Under the homomorphism $\mathsf{SO}(4) \to \mathsf{SO}(3)\,\mathsf{SO}(3)$ and the natural maximal tori in $\mathsf{SO}(4)$ and in $\mathsf{SO}(3)\,\mathsf{SO}(3)$, a slope (p, q) circle goes into one with slope $(p + q, p - q)$. Hence the slopes of $\mathsf{K}_0^\pm$ in $\mathsf{SO}(3)\,\mathsf{SO}(3)\,\mathsf{SO}(3)$ are $(1, 3, -1)$ on the left, and $(k, k-2, k+2)$ on the right. This also implies that both $\mathsf{SO}(3)$ factors act freely on P. If we divide by one of the $\mathsf{SO}(3)$ factors to obtain the two $\mathsf{SO}(3)$ (orbifold) principal bundles, the slopes of the circles $\mathsf{K}_0^\pm$ viewed inside $\mathsf{SO}(3)\,\mathsf{SO}(3)$

become $(1,3)$ on the left and $(k,k-2)$ on the right for one principal bundle, and $(1,-1)$ on the left and $(k,k+2)$ on the right for the other.

To see which principal bundle is the correct one for the Hitchin metric, recall that in [**Hi1**] one chooses an orientation on the regular orbits in order to derive the correct differential equation for Eintein metrics which are self dual with respect to the given orientation. For this fixed orientation Hitchin constructs the solution for the self dual Einstein metrics and checks smoothness at the singular orbits. Hence either the family of principal bundles with slopes $\{(1,3),(k,k-2)\}$ or the one with slopes $\{(1,-1),(k,k+2)\}$ are the desired $\mathsf{SO}(3)$ principle bundle for all k. But we know that for $k=1$ the principle bundle $H_1=\mathbb{RP}^7$ has slopes $\{(1,-1),(1,3)\}$ and for $k=2$ the bundle $H_2=W^7/\mathbb{Z}_2$ has slopes $\{(1,-1),(1,2)=(2,4)\}$ (see Section 4). Hence the slopes for the principle bundles defined by the Hitchin metric are $\{(1,-1),(k,k+2)\}$, which is, up to covers, the P family for k odd and the Q family for k even. q.e.d.

The second family of $\mathsf{SO}(3)$ principle bundles in the above proof are the principal bundles of the vector bundle of anti-self dual two forms, which the proof shows are also smooth. We note that in the case of $k=3$ one obtains the slopes for the exceptional manifold B^7 and in the case of $k=4$ the ones for R.

We note that in order for the cohomogeneity group diagram on the frame bundle or the principle bundle H_k to be consistent, $\mathsf{K}^+\cong\mathsf{O}(2)$ for k odd, and $\mathsf{K}^+\cong\mathsf{O}(2)\times\mathbb{Z}_2$ for k even. This also determines the embedding of K^+ into $\mathsf{SO}(3)\,\mathsf{SO}(3)$ and hence the orbifold group diagram for the Hitchin metrics. The manifolds H_k are two-fold subcovers of P_k and Q_k. In the case of P_k we divide by the full center and in the case of Q_k we add another component to all three isotropy groups (see Lemma 1.7). We also point out that for $k=2\ell$ the total space of the Konishi bundle associated to the lifted orbifold metric on $\mathbb{CP}^2$ is equal to Q_ℓ.

There is another interesting connection between self-dual Einstein metrics and positive curvature. O. Dearricott [**De1**] proved that if one allows to scale the 3-Sasakian metric on a 7-manifold with arbitrarily small scale in the direction of the $\mathsf{SU}(2)$ orbits, then the metric on the total space has positive sectional curvature if and only if the base self dual Einstein metric does. One can apply this to the Boyer-Galicki-Mann 3-Sasakian metrics [**BGM**] on the Eschenburg spaces $E_{a,b,c}=\operatorname{diag}(z^a,z^b,z^c)\backslash\ \mathsf{SU}(3)/\operatorname{diag}(1,1,\bar z^{a+b+c})$ whose self dual Einstein orbifold quotient is a weighted projective space $\mathbb{CP}^2[a+b,a+c,b+c]$. O. Dearricott showed in [**De2**] that many (but not all) of the weighted projective spaces have positive sectional curvature. The total space also admits an Eschenburg metric with positive curvature, but the Dearricott metrics are different in that the projection is a Riemannian submersion with totally geodesic fibers, whereas in the Eschenburg metric the fibers are not totally geodesic. It is hence natural to ask if the Hitchin metrics have positive sectional curvature for some k besides the values $k = 1, 2$ where this is true by construction. Hitchin gave an explicit formula for the functions describing his metrics for $k = 4, 6$ in [**Hi1**] and for $k = 3$ in [**Hi2**], which are simply rational functions of a parameter t along the geodesic in the first two cases and algebraic functions in the third case. One can now compute the sectional curvatures of the self dual Einstein metrics in these special cases and one shows, surprisingly, that the curvatures near the non-smooth singular orbit are all positive, but some become negative near the smooth singular orbit. On the other hand, it is not hard to construct 4-dimensional positively curved orbifold metrics with these prescribed orbifold singularities. Nevertheless, it is natural to suggest that there could be some significance in the existence of the 3-Sasakian metrics on P_k and Q_k and the question whether these spaces have a metric where all sectional curvatures are positive.

13. Topology of the Exceptional Examples

In order to prove Theorem C, we study the corresponding larger classes of cohomogeneity one manifolds with arbitrary slopes.

The class containing P_k consists of the cohomogeneity one manifolds $M = M_{(p_-,q_-),(p_+,q_+)}$ where $\mathsf{H} \subset \{\mathsf{K}^-, \mathsf{K}^+\} \subset \mathsf{G}$ is given by $\mathsf{G} = \mathsf{S}^3 \times \mathsf{S}^3, \{\mathsf{K}^-, \mathsf{K}^+\} = \{\mathsf{C}^i_{(p_-,q_-)} \cdot \mathsf{H}, \mathsf{C}^j_{(p_+,q_+)} \cdot \mathsf{H}\}$ and $\mathsf{H} = \mathsf{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$, where (p_-, q_-) as well as (p_+, q_+) are relatively prime odd integers, and $\mathsf{C}^k_{(p,q)} \subset \mathsf{S}^3 \times \mathsf{S}^3$ is the subgroup of elements $\{(e^{kp\theta}, e^{kq\theta})\}$ as in Section 7. It follows that $\mathsf{K}^\pm/\mathsf{K}^\pm_0 = \mathbb{Z}_2$, where the second component is generated by (j, j) on the left and (i, i) on the right, up to signs (of both coordinates). The embedding of Q is determined by the slopes and is $\Delta\mathsf{Q}$, up to sign changes in both coordinates. All cohomology groups, unless otherwise stated, are understood to be with $\mathbb{Z}$ coefficients.

Theorem 13.1. *The manifolds $M = M_{(p_-,q_-),(p_+,q_+)}$ are 2-connected. If $\frac{p_-}{q_-} \neq \pm\frac{p_+}{q_+}$ their cohomology is determined by $\pi_3(M) = \mathbb{Z}_k$ with $k = (p_-^2 q_+^2 - p_+^2 q_-^2)/8$. Otherwise $H^3(M) = H^4(M) = \mathbb{Z}$.*

Proof. We will use the same method as in [**GZ**, Poposition 3.3] although the details will be significantly more difficult. In order to show that M is simply connected, one uses Van Kampen on the cover $U_\pm = D(B_\pm) = \mathsf{G} \times_{\mathsf{K}^\pm} \mathbb{D}^{\ell_\pm+1}$, which deformation retract to $B_\pm = \mathsf{G}/\mathsf{K}^\pm$, and $U_- \cap U_+ = \mathsf{G}/\mathsf{H}$. We denote the projections of the sphere bundles by $\pi_\pm\colon \mathsf{G}/\mathsf{H} = \mathsf{G} \times_{\mathsf{K}^\pm} \mathbb{S}^{\ell_\pm} = \partial D(B_\pm) \to B_\pm = \mathsf{G}/\mathsf{K}^\pm$. For a homogeneous space G/L with G simply connected, the fundamental group is given by the group of components L/L_0. This determines the homomorphisms

$\pi_\pm\colon \pi_1(\mathsf{G}\,/\,\mathsf{H}) \to \pi_1(\mathsf{G}\,/\mathsf{K}^\pm)$ and it follows that $\pi_1(M) = 0$. For the cohomology groups, we use the Mayer-Vietoris sequence on the same decomposition, which gives a long exact sequence

$$(13.2) \quad \to H^{i-1}(B_-) \oplus H^{i-1}(B_+) \xrightarrow{\pi_-^* - \pi_+^*} H^{i-1}(\mathsf{G}\,/\,\mathsf{H}) \to H^i(M) \to H^i(B_-) \oplus H^i(B_+) \to$$

We first determine the cohomology groups of the singular and regular orbits. Denote by $\mu_\pm\colon B^0_\pm = \mathsf{G}\,/\mathsf{K}^\pm_0 \to B_\pm$ the natural projections, which are two fold covers. One knows that $B^0_\pm$ are diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^2$, independent of the slopes, see e.g., [**WZ**, Proposition 2.3]. For $B_\pm$ we will show that it has the same cohomology as that of $\mathbb{S}^3 \times \mathbb{RP}^2$, although we do not know if they are diffeomorphic.

Lemma 13.3. *The cohomology of the* G *orbits are given by*

(a) $B_\pm$ *is non-orientable with* $\pi_1(B_\pm) = \mathbb{Z}_2$, $H^0(B_\pm) = H^3(B_\pm) = \mathbb{Z}$, $H^1(B_\pm) = H^4(B_\pm) = 0$ *and* $H^2(B_\pm) = H^5(B_\pm) = \mathbb{Z}_2$. *Furthermore,* $\mu^*_\pm\colon H^3(B_\pm) \to H^3(B^0_\pm)$ *are isomorphisms.*

(b) *The principal orbit is orientable with* $\pi_1(\mathsf{G}\,/\,\mathsf{H}) = \mathsf{Q}$, $H^0(\mathsf{G}\,/\,\mathsf{H}) = H^6(\mathsf{G}\,/\,\mathsf{H}) = \mathbb{Z}$, $H^1(\mathsf{G}\,/\,\mathsf{H}) = H^4(\mathsf{G}\,/\,\mathsf{H}) = 0$, $H^2(\mathsf{G}\,/\,\mathsf{H}) = H^5(\mathsf{G}\,/\,\mathsf{H}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *and* $H^3(\mathsf{G}\,/\,\mathsf{H}) = \mathbb{Z} \oplus \mathbb{Z}$.

Proof. For the principal orbits, we observe that a normal subgroup $\mathsf{S}^3 \subset \mathsf{G}$ acts freely and hence give rise to a principal S^3 bundle $\mathsf{G}\,/\,\mathsf{H} \to \mathsf{S}^3\,/Q$. This bundle must be trivial since the classifying space $\mathbb{HP}^\infty$ is 4-connected. Hence $\mathsf{G}\,/\,\mathsf{H} \cong \mathbb{S}^3 \times (\mathbb{S}^3/\mathsf{Q})$ and the cohomology groups of $\mathsf{G}\,/\,\mathsf{H}$ easily follow.

We first note that a singular orbit $B = \mathsf{G}\,/\mathsf{K}$ with $\mathsf{K} = \mathsf{C}^i_{(p,q)}\,\mathsf{H}$ is non-orientable. This follows since the action of K/K_0 on the tangent space of $\mathsf{G}\,/\mathsf{K}$ does not preserve orientation. Considering the projection onto the second factor in $\mathsf{S}^3 \times \mathsf{S}^3$, we obtain fibrations $L_q \to B \xrightarrow{\sigma} \mathbb{RP}^2$ and $L_q \to B^0 \xrightarrow{\sigma_0} \mathbb{S}^2$ where the fibers of these homogeneous fibrations are lens spaces, since they are of the form $((\mathsf{S}^3 \times \mathsf{S}^1)\,\mathsf{H})/\mathsf{K} = \mathsf{S}^3\,/\{z^p\}$ with $z^q = 1$. It is well known that $H^*(B, \mathbb{Z}_{p'}) = H^*(B^0, \mathbb{Z}_{p'})^\rho$ for p' a prime different from 2, where ρ is the deck transformation of the two fold cover $\mu\colon B^0 \to B$. The spectral sequence for σ_0 implies that $\sigma_0^*\colon H^2(\mathbb{S}^2, \mathbb{Z}) \to H^2(B^0, \mathbb{Z})$ is an isomorphism. Since the deck groups of $B_0 \to B$ and $\mathbb{S}^2 \to \mathbb{RP}^2$ are compatible with the fibrations σ and σ_0, it follows that $\rho^* = -Id$ on $H^2(B^0, \mathbb{Z})$. Since ρ reverses orientation, $\rho^* = +Id$ on $H^3(B^0, \mathbb{Z}) = \mathbb{Z}$. Thus $H^i(B, \mathbb{Z}_p) = \mathbb{Z}_p$ for $i = 0, 3$ and 0 otherwise. Since q is odd, $H^*(L_q, \mathbb{Z}_2) = H^*(\mathbb{S}^3, \mathbb{Z}_2)$ and hence in the spectral sequence for σ with $\mathbb{Z}_2$ coefficients all differentials necessarily vanish. Thus $H^i(B, \mathbb{Z}_2) = \mathbb{Z}_2$ for every i. This, together with the universal coefficient theorem, easily determines the cohomology of $B_\pm$.

We finally show that $\mu^*\colon H^3(B) = \mathbb{Z} \to H^3(B^0) = \mathbb{Z}$ is an isomorphism. By the universal coefficient theorem, it suffices to show that $\mu^*\colon H^3(B, \mathbb{Z}_{p'}) \to H^3(B^0, \mathbb{Z}_{p'})$ is an isomorphism for every prime p'. If p' is odd, this is clearly the case by what we proved above. For $p' = 2$ we use the observation that all differentials in the spectral sequence for σ with $\mathbb{Z}_2$ coefficients vanish. This implies that the edge homomorphism $H^3(B, \mathbb{Z}_2) = \mathbb{Z}_2 \to H^3(L_q, \mathbb{Z}_2) = \mathbb{Z}_2$ is onto and hence an isomorphism. The same argument applies to the fibration σ_0 and hence $\mu^*\colon H^3(B, \mathbb{Z}_2) \to H^3(B^0, \mathbb{Z}_2)$ is an isomorphism as well. q.e.d.

The homomorphisms $\pi_\pm^*\colon \pi_1(\mathsf{G}/\mathsf{H}) \to \pi_1(B_\pm)$ determine $\pi_\pm^*\colon H^2(B_\pm) = \mathbb{Z}_2 \to H^2(\mathsf{G}/\mathsf{H}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ via the universal coefficient theorem, and show that $H^2(B_-) \oplus H^2(B_+) \to H^2(\mathsf{G}/\mathsf{H})$ is an isomorphism. Hence $H^2(M) = 0$ and M is 2-connected. Since we also have $H^4(B_\pm) = 0$, the Mayer Vietoris sequence implies that $H^3(M)$ is the kernel and $H^4(M)$ the cokernel of $\pi_-^* - \pi_+^*\colon H^3(B_-) \oplus H^3(B_+) = \mathbb{Z} \oplus \mathbb{Z} \to H^3(\mathsf{G}/\mathsf{H}) = \mathbb{Z} \oplus \mathbb{Z}$.

To determine $\pi_\pm^*$, we consider the commutative diagram, dropping the signs for the moment:

$$\begin{tikzcd} \mathsf{S}^3 \times \mathsf{S}^3 \arrow[r, "\tau"] \arrow[d, "\eta"] & \mathsf{S}^3 \times \mathsf{S}^3 / \mathsf{K}_0 \arrow[d, "\mu"] \\ \mathsf{S}^3 \times \mathsf{S}^3 / \mathsf{H} \arrow[r, "\pi"] & \mathsf{S}^3 \times \mathsf{S}^3 / \mathsf{K} \end{tikzcd} \tag{13.4}$$

where all arrows are given by their natural projections. In [**GZ**, (3.6)] it was shown that the image of a generator in $H^3(\mathsf{G}/\mathsf{K}_0) = \mathbb{Z}$ is equal to $(-q^2, p^2)$, using the natural basis in $H^3(\mathsf{G}) = \mathbb{Z} \oplus \mathbb{Z}$. Since μ^* is an isomorphism in degree 3, π^* is determined as soon as we know the integral lattice $\mathrm{Im}(\eta)^* \subset H^3(\mathsf{S}^3 \times \mathsf{S}^3)$. Since $\mathsf{S}^3 \times \mathsf{S}^3 / \mathsf{H} \cong S^3 \times (\mathsf{S}^3/\mathsf{Q})$ and $\mathsf{S}^3 \to \mathsf{S}^3/\mathsf{Q}$ is an 8-fold cover, this sublattice must have index 8. Using (13.4) for the slopes $(1,1)$ and $(1,3)$, we see that $(-1,1)$ and $(-9,1)$ lie in the lattice and must be a basis, since the element $(1,0)$ has order 8 in the quotient group. Using the basis $(-1,1)$ and $(4,4)$ the matrix of $\pi_-^* - \pi_+^*$ becomes:

$$\begin{pmatrix} \frac{1}{2}(p_-^2 + q_-^2) & -\frac{1}{2}(p_+^2 + q_+^2) \\ \frac{1}{8}(p_-^2 - q_-^2) & -\frac{1}{8}(p_+^2 - q_+^2) \end{pmatrix}.$$

Since (p_-, q_-) are relatively prime, one easily sees that $(\frac{1}{2}(p_\pm^2 + q_\pm^2), \frac{1}{8}(p_\pm^2 - q_\pm^2))$ are relatively prime as well, which implies that the cokernel of $\pi_-^* - \pi_+^*$ is a cyclic group. If we assume that $\frac{p_-}{q_-} \neq \pm\frac{p_+}{q_+}$, the kernel is 0 and the cokernel is cyclic with order $\det(\pi_-^* - \pi_+^*) = ((p_-^2 q_+^2 - p_+^2 q_-^2)/8$. Otherwise kernel and cokernel are equal to $\mathbb{Z}$. q.e.d.

Next we consider the extension $N = N_{(p_-,q_-),(p_+,q_+)}$ of the Q family, given by $\mathsf{H} = \{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{\mathsf{K}^-, \mathsf{K}^+\} = \{\mathsf{C}^i_{(p_-,q_-)}\,\mathsf{H}, \mathsf{C}^j_{(p_+,q_+)}\,\mathsf{H}\} \subset \mathsf{G} = \mathsf{S}^3 \times \mathsf{S}^3$ with (p_-, q_-) as well as (p_+, q_+) relatively prime, p_+ even and p_-, q_-, q_+ odd. Notice that the component groups $\mathsf{K}^{\pm}/\mathsf{K}^{\pm}_0$ are determined by the fact that $(i, i) \in \mathsf{K}^-_0$ and $(1, -1) \in \mathsf{K}^+_0$.

Theorem 13.5. *The manifolds $N = N_{(p_-,q_-),(p_+,q_+)}$ are simply connected with $H^2(N) = \mathbb{Z}$, $H^3(N) = 0$ and $H^4(N) = \mathbb{Z}_k$ with $k = p_-^2 q_+^2 - p_+^2 q_-^2$.*

Proof. We indicate the changes in the proof which are necessary, and start with the cohomology of the orbits. They contain torsion groups S, T and integers c, d which are to be determined later.

Lemma 13.6. *The cohomology of the G orbits are given by*

(a) *B_- is orientable with $\pi_1(B_-) = \mathbb{Z}_2$, $H^0(B_-) = H^3(B_-) = H^5(B_-) = \mathbb{Z}$, $H^1(B_-) = 0$, $H^2(B_-) = \mathbb{Z} \oplus \mathbb{Z}_2$ and $H^4(B_-) = \mathbb{Z}_2$. Furthermore, $\mu_-^*\colon H^3(B_-) \to H^3(B_-^0)$ is multiplication by c, a power of 2.*

(b) *B_+ is non-orientable with $\pi_1(B_+) = \mathbb{Z}_4$, $H^0(B_+) = \mathbb{Z}$, $H^1(B_+) = H^4(B_+) = 0$, $H^2(B_+) = \mathbb{Z}_4$, $H^3(B_+) = \mathbb{Z} \oplus S$ and $H^5(B_+) = \mathbb{Z}_2$. Furthermore, $\mu_+^*\colon H^3(B_+) \to H^3(B_+^0)$ is multiplication by d, a power of 2, on the free part.*

(c) *The principal orbit is orientable with $\pi_1(\mathsf{G}/\mathsf{H}) = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, $H^0(\mathsf{G}/\mathsf{H}) = H^6(\mathsf{G}/\mathsf{H}) = \mathbb{Z}$, $H^1(\mathsf{G}/\mathsf{H}) = 0$, $H^2(\mathsf{G}/\mathsf{H}) = H^5(\mathsf{G}/\mathsf{H}) = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, $H^3(\mathsf{G}/\mathsf{H}) = \mathbb{Z} \oplus \mathbb{Z} \oplus T$ and $H^4(\mathsf{G}/\mathsf{H}) = T$,*

where S and T are torsion groups of the form $(\mathbb{Z}_2)^m$.

Proof. For $B = B_- = \mathsf{S}^3 \times \mathsf{S}^3 / \mathsf{C}^i_{(p,q)}\,\mathsf{H}$ one has $(i, i) \in \mathsf{K}^-_0$ and $(1, -1)$ generates the second component. Hence B_- is orientable with $\pi_1 = \mathbb{Z}_2$. Projection onto the second coordinate in $\mathsf{S}^3 \times \mathsf{S}^3$ gives rise to fibrations $L_{2q} \to B \xrightarrow{\sigma} \mathbb{S}^2$ and $L_q \to B^0 \xrightarrow{\sigma_0} \mathbb{S}^2$. Notice that the fiber for the first fibration is $((\mathsf{S}^3 \times \mathsf{S}^1)\,\mathsf{H})/\mathsf{K} = \mathsf{S}^3/\{z^p, -1\}$ with $z^q = 1$, which is $\mathsf{S}^3/\mathbb{Z}_{2q}$ since p and q are odd. As before, one now shows that for any prime p' different from 2, $H^i(B, \mathbb{Z}_{p'}) = \mathbb{Z}_{p'}$ for $i = 0, 2, 3, 5$ and 0 otherwise. Since $H^*(L_{2q}, \mathbb{Z}_2) = H^*(\mathbb{RP}^3, \mathbb{Z}_2)$ and $H^1(B, \mathbb{Z}_2) = \mathbb{Z}_2$ it follows that all differentials vanish in the spectral sequence for σ with $\mathbb{Z}_2$ coefficients. This determines $H^*(B, \mathbb{Z}_2)$ and the cohomology groups of B easily follow. Since $\mu^*\colon H^3(B, \mathbb{Z}_{p'}) = \mathbb{Z}_{p'} \to H^3(B^0, \mathbb{Z}_{p'}) = \mathbb{Z}_{p'}$ is an isomorphism for every prime $p' \neq 2$, it must be multiplication by a power of two over the integers.

For $B = B_+ = \mathsf{S}^3 \times \mathsf{S}^3 / \mathsf{C}^i_{(p,q)}\,\mathsf{H}$ with p even q odd, the element (i, i) generates the 4 components of K. Hence B is non-orientable with $\pi_1(B) = \mathbb{Z}_4$. We now also consider the fibrations $L_{2q} \to B \xrightarrow{\sigma} \mathbb{RP}^2$

and $L_q \to B^0 \xrightarrow{\sigma_0} \mathbb{S}^2$. Using the 4-fold cover $\mu\colon B^0 \to B$, it follows as before that for any prime $p' \neq 2$ we have $H^i(B, \mathbb{Z}_{p'}) = \mathbb{Z}_{p'}$ for $i = 0, 3$ and 0 otherwise.

We now consider the spectral sequence of the fibration σ with $\mathbb{Z}_2$ coefficients. Since $H^1(B, \mathbb{Z}_2) = \mathbb{Z}_2$, it follows that $d_2\colon E_2^{0,1} = \mathbb{Z}_2 \to E_2^{2,0} = \mathbb{Z}_2$ is an isomorphism and hence $d_2\colon E_2^{0,2} = \mathbb{Z}_2 \to E_2^{2,1} = \mathbb{Z}_2$ vanishes and $d_2\colon E_2^{0,3} = \mathbb{Z}_2 \to E_2^{2,2} = \mathbb{Z}_2$ is an isomorphism as well. This determines $H^*(B, \mathbb{Z}_2)$ and one easily derives the cohomology of B, up to a non-zero torsion group $S = (\mathbb{Z}_2)^k$ in dimension three. It also follows that $\mu^*\colon H^3(B, \mathbb{Z}_{p'}) = \mathbb{Z}_{p'} \to H^3(B^0, \mathbb{Z}_{p'}) = \mathbb{Z}_{p'}$ is an isomorphism for $p' \neq 2$, and hence $\mu^*\colon H^3(B, \mathbb{Z}) = \mathbb{Z} \oplus S \to H^3(B^0, \mathbb{Z}) = \mathbb{Z}$ is multiplication by a power of two on the free part.

G/H is clearly orientable with $\pi_1(\mathsf{G}/\mathsf{H}) = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Using the 8-fold cover $\eta\colon \mathsf{S}^3 \times \mathsf{S}^3 \to \mathsf{G}/\mathsf{H}$ and the fact that the deck transformations are homotopic to the identity, it follows that the non-zero groups in $H^i(B, \mathbb{Z}_p)$ for $p \neq 2$ are $\mathbb{Z}_p$ for $i = 0, 6$ and $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for $i = 3$. In the spectral sequence for the fibration $\mathsf{S}^1 \times \mathsf{S}^1 \to \mathsf{G}/\mathsf{H} \to \mathbb{S}^2 \times \mathbb{S}^2$ with $\mathbb{Z}_2$ coefficients, all differentials must be 0 since $H^1(\mathsf{G}/\mathsf{H}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This determines $H^*(\mathsf{G}/\mathsf{H}, \mathbb{Z}_2)$ and hence $H^*(\mathsf{G}/\mathsf{H}, \mathbb{Z})$, up to a non-zero torsion group $T = (\mathbb{Z}_2)^k$ in dimensions three and four. q.e.d.

The homomorphisms on the group of components again show that N is simply connected and that the homomorphism $H^2(B_-) \oplus H^2(B_+) = \mathbb{Z}_4 \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \to H^2(\mathsf{G}/\mathsf{H}) = \mathbb{Z}_4 \oplus \mathbb{Z}_2$ is an isomorphism on the torsion part, since it is determined by the corresponding homomorphisms on the fundamental groups. Therefore Mayer Vietoris implies that $H^2(N) = \mathbb{Z}$. By Poincare duality $H^5(N) = \mathbb{Z} \to H^5(B_-) = \mathbb{Z}$ is an isomorphism, which means that the homomorphism $H^4(B_-) \oplus H^4(B_+) = \mathbb{Z}_2 \to H^4(\mathsf{G}/\mathsf{H}) = T$ is onto and hence an isomorphism with $T = \mathbb{Z}_2$. Since the universal coefficient theorem for N implies that $H^3(N)$ cannot have any torsion, $\pi_-^* - \pi_+^*\colon H^3(B_-) \oplus H^3(B_+) = \mathbb{Z} \oplus \mathbb{Z} \oplus S \to H^3(\mathsf{G}/\mathsf{H}) = \mathbb{Z} \oplus \mathbb{Z} \oplus T$ is injective on its torsion part, and hence an isomorphism with $S = T = \mathbb{Z}_2$. Thus $H^3(N)$ is the kernel on the free part and $H^4(N)$ its cokernel.

We next determine the lattice generated by the image of η^* in $H^3(S^3 \times S^3)$. In the spectral sequence for the fibration $\mathsf{S}^3 \times \mathsf{S}^3 \to \mathsf{S}^3 \times \mathsf{S}^3 / \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to B_{\mathbb{Z}_2} \times B_{\mathbb{Z}_4}$ the fundamental group $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ of the base acts trivially in homology and hence the local coefficients become ordinary $\mathbb{Z}$ coefficients. The only non-zero differential is $d_2\colon E_2^{0,3} = \mathbb{Z} \oplus \mathbb{Z} \to E_2^{4,0} = H^4(B_{\mathbb{Z}_2} \times B_{\mathbb{Z}_4}, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$, whose kernel is equal to the image of the edge homomorphism, which can be viewed as η^*. Clearly, $(-2, 2)$ and $(2, 2)$ lie in this kernel and must be a basis of the lattice since the spectral sequence also implies that its quotient group has order 8. In this basis, the matrix of $\pi_-^* - \pi_+^*$ on the free part is given by

$$\begin{pmatrix} \frac{c}{4}(p_-^2 + q_-^2) & -\frac{d}{4}(p_+^2 + q_+^2) \\ \frac{c}{4}(p_-^2 - q_-^2) & -\frac{d}{4}(p_+^2 - q_+^2) \end{pmatrix}$$

where c, d are the integers from Lemma 13.6, which we showed are powers of two. We now claim that $c = 2$ and $d = 4$, which then implies (13.5) as before.

If this were not the case, the order of $H^4(N, Z)$ would be even since $(p_-^2 + q_-^2, p_-^2 - q_-^2) = 2$ and $(p_+^2 + q_+^2, p_+^2 - q_+^2) = 1$, and we now show that it must in fact be odd. To see this, we repeat the above Mayer Vietoris argument for cohomology with $\mathbb{Z}_2$ coefficients. First observe that in the Gysin sequence of $\mathsf{S}^1 \to \mathsf{G}/\mathsf{H} \to B_\pm$ one has

$$\cdots \to H^3(B_\pm) = \mathbb{Z}_2^2 \xrightarrow{\pi_\pm^*} H^3(\mathsf{G}/\mathsf{H}) = \mathbb{Z}_2^4 \to H^2(B_\pm) = \mathbb{Z}_2^2 \to \cdots$$

and hence $\pi_\pm^*$ are injective. Thus from the Mayer Vietoris sequence

$$0 \to H^3(N) \to H^3(B_-) \oplus H^3(B_+) = \mathbb{Z}_2^4 \xrightarrow{\pi_-^* - \pi_+^*} H^3(\mathsf{G}/\mathsf{H}) = \mathbb{Z}_2^4 \to H^4(N) \to 0$$

it follows that $H^3(N, \mathbb{Z}_2) = H^4(N, \mathbb{Z}_2) = 0$ which completes the proof. q.e.d.

14. Appendix I: Classification in Even Dimensions

In this appendix we give a relatively short proof of Verdiani's theorem based on the obstructions, ideas and strategy presented here to handle the odd dimensional case.

Theorem 14.1 (Verdiani)**.** *Suppose* G *acts on an even dimensional positively curved simply connected compact manifold* M *with cohomogeneity one. Then* M *is equivariantly diffeomorphic to a rank* 1 *symmetric space.*

One of the reasons that make the even dimensional case less involved is that one of the groups K^- or K^+ has the same rank as G, and thus $\mathrm{rk}(\mathsf{G}) - \mathrm{rk}(\mathsf{H}) = 1$ as we saw in the Rank Lemma. Moreover, the Upper Weyl Group Bound now says that $|W| = 2$ or 4, and $|W| = 2$ if H is connected and l_- and l_+ are both odd. This becomes especially powerful if combined with the Lower Weyl Group Bound. Another noteworthy difference is that the main body of work is confined to the case of simple groups, and that induction is only used occasionally.

In [**Iw1**, **Iw2**, **Uc1**] it was shown that any cohomogeneity one manifold whose rational cohomology ring is like that of $\mathbb{CP}^n$, $\mathbb{HP}^n$ or $\mathrm{Ca}\mathbb{P}^2$ is equivariantly diffeomorphic to one of the known linear cohomogeneity one actions. Hence it is again sufficient to recover M up to homotopy type.

We treat the cases G simple, or not separately. For G simple we distinguish among the subcases: H contains a normal simple subgroup of $\mathrm{rk} \geq 2$, or not.

G is not simple.

Proposition 14.2. *If G is not simple and acts essentially with corank one, it is the tensor product action of $\mathsf{SU}(2)\,\mathsf{SU}(k)$ on $\mathbb{CP}^{2k-1}$.*

Proof. We can assume that $\mathsf{G} = \mathsf{L}_1 \times \mathsf{L}_2$, and say $\mathrm{rk}(\mathsf{K}^-) = \mathrm{rk}(\mathsf{G})$, and hence $\mathsf{K}_0^- = \mathsf{K}_1 \cdot \mathsf{K}_2$. Since by assumption K^- does not contain a normal subgroup of G, we see that G is semisimple and $\mathsf{G}/\mathsf{K}_0^-$ is necessarily isometric to a product. If each of the factors has dimension > 2, then we can derive a contradiction as in the proof of the Product Lemma in odd dimensions. Using the conventions therein, we may assume that $\mathsf{L}_2/\mathsf{K}_2 = \mathbb{S}^2$, and that K_1 acts transitively on the normal sphere. It follows again that $\mathsf{L}_1/\mathsf{K}_1$ is isometric to a rank one projective space, and as before, we derive a contradiction if the isotropy representation of $\mathsf{L}_1/\mathsf{K}_1$ is of real type. Hence the isotropy representation is of complex type and $\mathsf{L}_1/\mathsf{K}_1 = \mathsf{SU}(k)/\mathsf{U}(k-1)$, $k \geq 2$ or $\mathsf{L}_1/\mathsf{K}_1 = \mathsf{G}_2/\mathsf{SU}(3)$.

Because of primitivity K^+ necessarily projects surjectively onto $\mathsf{L}_2 = \mathsf{S}^3$. The projection of H_0 onto L_2 cannot be 3 dimensional since then the subaction of L_1 would be orbit equivalent to the G action. If the projection is trivial, $\mathsf{K}^-/\mathsf{H} = \mathbb{S}^1$ and $\mathsf{K}^+/\mathsf{H} = \mathbb{S}^3$. Furthermore, H is connected since K_1 and K_2 are both connected. But then the Upper Weyl Group Bound implies that $|W| = 2$, which combined with the Lower Weyl Group Bound gives $\dim \mathsf{G}/\mathsf{H} \leq 4$, a contradiction. Hence the projection is one dimensional and $\mathsf{K}^+/\mathsf{H} = \mathbb{S}^2$.

This completely determines the group picture. Indeed, if $\mathsf{L}_1/\mathsf{K}_1 = \mathsf{SU}(k)/\mathsf{U}(k-1)$, we have $\mathsf{K}^+ = \mathsf{L} \times \mathsf{S}^3$ with the second factor embedded diagonally in $\mathsf{L}_1 \cdot \mathsf{L}_2$. Hence $\mathsf{L} \subset \mathsf{U}(k-2)$ and $\mathsf{H} = \mathsf{L} \times \mathsf{S}^1$. In order for K^-/H to be a sphere, we need $\mathsf{L} = \mathsf{U}(k-2)$ and we recover the tensor product action of $\mathsf{SU}(k) \times \mathsf{SU}(2)$ on $\mathbb{CP}^{2k-1}$.

In the case of $\mathsf{L}_1 \cdot \mathsf{L}_2 = \mathsf{G}_2 \times \mathsf{S}^3$ we have $\mathsf{K}^+ = \mathsf{S}^3 \times \mathsf{S}^3$. K^+ projects onto $\mathsf{SO}(4) \subset \mathsf{G}_2$, and the second factor of K^+ projects onto L_2. The tangentspace T^+ of the orbit $B_+ = \mathsf{G}_2\, x\, \mathsf{S}^3/\mathsf{K}^+$ decomposes as an 8 dimensional and a 3 dimensional invariant subspace. The natural representation in S^2T^+ splits into two trivial two 5-dimensional and into subrepresentations on which $\mathsf{G}_2 \cap \mathsf{K}^+ \cong \mathsf{S}^3 \subset \mathsf{H}$ acts nontrivially.

This in turn implies that B_+ is totally geodesic, a contradiction. q.e.d.

G is simple.

Proposition 14.3. *Assume G is simple with corank 1 and all simple factors in H have rank one. If the action is essential, then (M, G) is* *one of the following pairs:* $(\mathbb{CP}^6,\ \mathsf{G}_2)$, $(\mathbb{CP}^9,\ \mathsf{SU}(5))$, $(\mathbb{HP}^2,\ \mathsf{SU}(3))$, $(\mathbb{HP}^3, \mathsf{SU}(4))$, *or* $(\mathrm{Ca}\mathbb{P}^2, \mathsf{Sp}(3))$, *and the actions are given by Table* F.

Proof. Since all normal factors in H_0 are one and three dimensional, we have $l_\pm = 1, 2, 3, 4, 5$, or 7 and at least one of them is odd. The Weyl group has order at most 4, and the order is 2 if l_+ and l_- are both one of $3, 5$ or 7.

We first treat the case where $\operatorname{rk}\mathsf{G} \leq 2$. For $\mathsf{G} = \mathsf{S}^1$ clearly $M = \mathbb{S}^2$. If $\mathsf{G} = \mathsf{S}^3$, $\mathsf{K}^\pm \cong \mathsf{S}^3$ corresponds to $M = \mathbb{S}^4$, $\mathsf{K}^- \cong \mathsf{S}^3, \mathsf{K}^+ \cong \mathsf{S}^1$ corresponds to $M = \mathbb{CP}^2$, and $\mathsf{K}^\pm \cong \mathsf{S}^1$ is not primitive. In all three cases, the action is not essential.

If $\mathsf{G} = \mathsf{SU}(3)$, then H cannot be three dimensional since otherwise G is not primitive or has a fixed point. Therefore H_0 is a circle. The Lower Weyl Group Bound implies that one of $l_\pm$, say l_-, is 3 and hence $\mathsf{K}_0^- = \mathsf{U}(2)$. Since $\mathsf{U}(2)$ is maximal, K^+ and hence H are connected. From the Upper and Lower Weyl Group Bound it now follows that $l_+ = 1$ or 3 is not possible. Thus $\mathsf{K}^+ = \mathsf{SO}(3)$ or $\mathsf{SU}(2)$. If $\mathsf{K}^+ = \mathsf{SO}(3)$ and hence $\mathbb{S}_+ = \mathbb{S}^2$, we note there is only one embedding into $\mathsf{SU}(3)$ and its isotropy representation is irreducible. One then easily shows (Clebsch Gordon formula) that the representation of $\mathsf{SO}(3)$ in S^2T_+ has no three dimensional subrepresentation, which implies that $\mathsf{SU}(3)/\,\mathsf{SO}(3)$ is totally geodesic, a contradiction. If $\mathsf{K}^+ = \mathsf{SU}(2)$, we have recovered the group picture for $\mathbb{HP}^2$.

If $\mathsf{G} = \mathsf{Sp}(2)$ and H is three dimensional, then Table B implies that $\mathsf{G}/\mathsf{H}_0 \cong \mathbb{S}^7$ or $\mathsf{SO}(5)/\,\mathsf{SO}(3)$. In the former case G either has a fixed point or is not primitive. In the latter case the Chain Theorem applies. So we may assume $\dim(\mathsf{H}) = 1$. The Lower Weyl Group Bound implies that $l_- = 2\,, l_+ = 3$ and hence all groups are connected. If $\mathsf{H} = \{\operatorname{diag}(z^k, z^l) \mid z \in \mathsf{S}^1\}$, the isotropy representation has weights $2k, 2l$, and $2k \pm 2l$. But since $\mathsf{Sp}(2)/\,\mathsf{H}$ by the Isotropy Lemma can have at most two nonequivalent nontrivial subrepresentations, it follows that $\mathsf{H} = \{\operatorname{diag}(z, z) \mid z \in \mathsf{S}^1\}$ or $\{\operatorname{diag}(1, z) \mid z \in \mathsf{S}^1\}$. In the former case M_c^{H} admits a cohomogeneity one action of $\mathsf{N}(\mathsf{H})/\,\mathsf{H} \cong \mathsf{SO}(3)$ with trivial principal isotropy group, which contradicts the Core-Weyl Lemma. If $\mathsf{H} = \{\operatorname{diag}(1, z)\}$, we may assume that $\mathsf{K}^- = \{\operatorname{diag}(1, g) \mid g \in \mathsf{S}^3\}$. Hence K^- is normalized by the normalizer of H and in particular by the Weyl group. By Linear Primitivity, the Lie algebras of $\mathsf{K}^-, \mathsf{K}^+, w_+\mathsf{K}^+w_+$ span the Lie algebra of G, which is not possible since $\dim \mathsf{K}^+ = 4$.

Finally, if $\mathsf{G} = \mathsf{G}_2$ and H is one dimensional, we obtain a contradiction to the Lower Weyl Group Bound since $l_\pm \leq 3$. The only three dimensional spherical subgroup is $\mathsf{H} = \mathsf{SU}(2) \subset \mathsf{SU}(3) \subset \mathsf{G}_2$, as one easily verifies. The subgroups of G_2 only allow $l_\pm = 1$ or 5, and using the Lower Weyl Group Bound, we see that $\mathsf{K}_0^- = \mathsf{SU}(3)$, $\mathsf{K}_0^+ = \mathsf{SU}(2)\,\mathsf{S}^1$ and H is not connected by the Upper Weyl Group Bound. Because of

$\mathsf{G}_2 / \mathsf{SU}(3) \cong \mathbb{S}^6$, K^- and thereby H has at most two, and hence two components. Now all groups $\mathsf{K}^{\pm}$, H are determined, and we have recovered the picture of $\mathbb{CP}^6$ endowed with the linear action of G_2.

If the rank of G is 3 or larger, we first observe that $\dim \mathsf{H} \leq 3 \operatorname{rk} \mathsf{H} = 3(\operatorname{rk} \mathsf{G} - 1)$ since all simple factors of H have rank one. Hence:

$$\dim(\mathsf{G}) - 3 \operatorname{rk} \mathsf{G} \leq \dim(\mathsf{G} / \mathsf{H}) - 3.$$

By the Lower Weyl Group Bound

$$\dim(\mathsf{G} / \mathsf{H}) \leq 2(7+4) = 22$$

and hence $\dim \mathsf{G} - 3 \operatorname{rk}(\mathsf{G}) \leq 19$.

First we consider the case that there is an orbit, say $\mathsf{G} / \mathsf{K}^-$, of codimension 8. Then $\mathsf{Sp}(2)$ is a normal subgroup of K^- and $\operatorname{rk}(\mathsf{K}^-) = \operatorname{rk}(\mathsf{G})$ since $\operatorname{rk} \mathsf{K}^- - \operatorname{rk} \mathsf{H} = 1$. The only simple Lie groups satisfying the above dimension estimate and containing $\mathsf{Sp}(2)$ as a regular subgroup, besides $\mathsf{Sp}(2)$ itself, are $\mathsf{Spin}(7)$ and $\mathsf{Sp}(3)$. In the case of $\mathsf{G} = \mathsf{Spin}(7)$ the central element in $\mathsf{Sp}(2)$ is necessarily central in $\mathsf{Spin}(7)$, but does not lie in H. But then $\mathsf{Spin}(7)/\mathsf{K}^-$ is totally geodesic, which is not possible. In the case of $\mathsf{G} = \mathsf{Sp}(3)$, H contains an $\mathsf{Sp}(1)$-block and $M_c^{\mathsf{Sp}(1)}$ admits a cohomogeneity one action by $\mathsf{Sp}(2)$, whose principal isotropy group has rank one. As we saw earlier, this isotropy group must be three dimensional and hence H is 6-dimensional, which implies $\mathsf{K}_0^- = \mathsf{Sp}(1) \cdot \mathsf{Sp}(2)$. Since this group is maximal in $\mathsf{Sp}(3)$, it follows that K^- and hence H are connected. The Lower Weyl Group Bound implies that $|W| = 4$ and hence, by the Upper Weyl Group Bound, one of the codimensions must be odd. Hence $\mathsf{K}^+ = \mathsf{Sp}(2)$, $\mathsf{H} = \mathsf{Sp}(1)^2 \subset \mathsf{Sp}(2) \subset \mathsf{Sp}(3)$ and we have recovered the cohomogeneity one action of $\mathsf{Sp}(3)$ on $\mathrm{Ca}\mathbb{P}^2$.

If there are no orbits of codimension 8,

$$\dim(\mathsf{G} / \mathsf{H}) \leq 2(4+5) = 18$$

and hence $\dim(\mathsf{G}) - 3 \operatorname{rk}(\mathsf{G}) \leq 15$. We now assume that there is an orbit, say $\mathsf{G} / \mathsf{K}^-$ of codimension 5. Then $\mathsf{Sp}(2) \subset \mathsf{K}^-$. The only groups, other than $\mathsf{Sp}(2)$, satisfying the above dimension estimate and containing $\mathsf{Sp}(2)$ are $\mathsf{Sp}(3)$, $\mathsf{SU}(5)$, $\mathsf{Spin}(7)$ and $\mathsf{Spin}(6)$.

In the case of $\mathsf{G} = \mathsf{Spin}(6)$ or $\mathsf{Spin}(7)$, there is a unique embedding of $\mathsf{Sp}(2) = \mathsf{Spin}(5)$ in G and hence H contains a 4×4-block and the Chain Theorem applies. In the case of $\mathsf{G} = \mathsf{Sp}(3)$, $\mathsf{H} = \mathsf{Sp}(1)^2 \subset \mathsf{Sp}(2)$ and the isotropy representation of G / H contains a four dimensional representation, which can only degenerate in an orbit of codimension 8, which we already dealt with. Thus we may assume $\mathsf{G} = \mathsf{SU}(5)$. Because of $\operatorname{rk}(\mathsf{K}^-) = 3$, we have $\mathsf{K}_0^- = \mathsf{Sp}(2) \cdot \mathsf{S}^1$ and hence $\mathsf{H}_0 = \mathsf{SU}(2)^2 \cdot \mathsf{S}^1$. There is a four dimensional subrepresentation of the isotropy representation of $\mathsf{G} / \mathsf{H}_0$ which is not equivalent to a subrepresentation of $\mathsf{K}^-/\mathsf{H}_0$. From the Isotropy Lemma we deduce $\mathsf{K}^+ = \mathsf{SU}(3) \cdot \mathsf{SU}(2) \cdot \mathsf{S}^1$ and all groups are connected. We have recovered the linear action of $\mathsf{SU}(5)$ on $\mathbb{CP}^9$.

We are left with the case that $l_\pm = 1, 2, 3$ or 5. Hence

$$\dim(\mathsf{G}\,/\,\mathsf{H}) \leq 2(2+5) = 14$$

and $\dim(\mathsf{G}) - 3\,\mathrm{rk}(\mathsf{G}) \leq 11$. The only simply connected compact simple Lie groups satisfying this dimension estimate are S^3, $\mathsf{SU}(3)$, $\mathsf{Sp}(2)$, G_2, and $\mathsf{SU}(4)$.

This only leaves us to consider the case $\mathsf{G} = \mathsf{SU}(4)$. If H_0 is abelian, we obtain a contradiction to the Lower Weyl Group Bound. There are two six dimensional subgroups of $\mathsf{SU}(4) = \mathsf{Spin}(6)$, one from $\mathsf{SO}(4) \subset \mathsf{SO}(6)$, where the Chain Theorem applies, and the other from $\mathsf{SO}(3)\,\mathsf{SO}(3) \subset \mathsf{SO}(6)$ which contradicts the Isotropy Lemma.

In the case of $\dim(\mathsf{H}_0) = 4$, Table B implies that H contains an $\mathsf{SU}(2)$-block. Since the four dimensional representation of $\mathsf{SU}(2)$ has to degenerate, $\mathsf{K}^- = \mathsf{U}(3)$. Since $\mathsf{U}(3)$ is maximal in $\mathsf{SU}(4)$, K^- and hence also H are connected. If $|W| = 2$, the Lower Weyl Group Bound implies that $l_+ = 7$, which is not possible since $\mathsf{Sp}(2)$ is maximal in $\mathsf{SU}(4)$. Now the Upper Weyl Group Bound implies that l_+ is even. Thus $l_+ = 2$, $\mathsf{K}^+ = \mathsf{SU}(2)^2, \mathsf{H} = \mathsf{S}^1\,\mathsf{SU}(2)$, and we have recovered the action of $\mathsf{SU}(4)$ on $\mathbb{HP}^3$. q.e.d.

For G simple it remains to consider the case where H has a higher rank normal subgroup.

Proposition 14.4. *Assume* G *is simple with corank* 1 *and* H *contains a simple subgroup of rank* ≥ 2. *If the action is essential, the pair* (M, G) *is one of the following:* $(\mathbb{CP}^{n-1}, \mathsf{SO}(n))$, $(\mathbb{HP}^{n-1}, \mathsf{SU}(n))$, $(\mathbb{CP}^{15}, \mathsf{Spin}(10))$, $(\mathbb{S}^{14}, \mathsf{Spin}(7))$, *or* $(\mathbb{CP}^{7}, \mathsf{Spin}(7))$ *with the actions given by Table* F.

Proof. By Lemma 2.4, there can be only one connected normal subgroup of H which has rank larger than one, which we denote by H'.

$$\mathsf{G} = \mathsf{Sp}(k) \text{ or } \mathsf{SU}(k)$$

If $\mathsf{G} = \mathsf{Sp}(k)$, Table B implies that H' is given by an $h \times h$ block $h \geq 2$, and the Chain Theorem applies.

If $\mathsf{G} = \mathsf{SU}(k)$, Table B implies that either H' is given by an $h \times h$ block and the Chain Theorem applies, or $\mathsf{H}' = \mathsf{Sp}(2) \subset \mathsf{SU}(4) \subset \mathsf{SU}(k)$. The latter case can be ruled out as follows. If $k = 4$, then clearly G has a fixed point. If $k \geq 5$, there is an eight dimensional irreducible representation of H which can only degenerate in an isotropy group K^- containing $\mathsf{Sp}(3)$, and furthermore $\mathrm{rk}(\mathsf{K}^-) = \mathrm{rk}(\mathsf{G})$. But this is impossible since $\mathsf{Sp}(3)$ is not a regular subgroup of $\mathsf{SU}(k)$.

$\mathsf{G} = \mathsf{Spin}(k)$

If $\mathsf{G} = \mathsf{Spin}(k)$, then by Table B, either H' is given by a block and the Chain Theorem applies, or $\mathsf{H}' \cong \mathsf{G}_2, \mathsf{Spin}(7), \mathsf{SU}(4), \mathsf{Sp}(2)$, or $\mathsf{SU}(3)$.

If $\mathsf{H}' = \mathsf{G}_2$ or $\mathsf{Spin}(7)$, we first claim that $\mathsf{H}_0 = \mathsf{H}'$. Indeed, if $\mathsf{H}_0 \neq \mathsf{H}'$, it follows from Table C that only one of H' or $\mathsf{H}_0/\,\mathsf{H}'$ can act non-trivially on each irreducible subrepresentation of $\mathsf{G}\,/\,\mathsf{H}$. But by Lemma 2.4, this contradicts our assumption that G is simple. If $\mathsf{H}_0 = \mathsf{G}_2$, the Rank Lemma implies $\mathsf{G} = \mathsf{Spin}(7)$ and G has a fixed point. If $\mathsf{H}_0 = \mathsf{Spin}(7)$, then $\mathsf{G} = \mathsf{Spin}(8)$ or $\mathsf{G} = \mathsf{Spin}(9)$ and $\mathsf{G}\,/\mathsf{H}_0$ is a sphere. Then G either has a fixed point or the action is not primitive.

If $\mathsf{H}' = \mathsf{SU}(4)$ or $\mathsf{Sp}(2)$, then $k \geq 8$. If $k = 8$, H' is, up to an outer automorphism of $\mathsf{Spin}(8)$, given as a 6×6 or a 5×5 block and the Chain Theorem applies. If $k \geq 9$, let ι be the order 2 central element in H' so that $\mathsf{N}(\iota)_0 = \mathsf{Spin}(k-8) \cdot \mathsf{Spin}(8)$ acts with cohomogeneity one on the reduction M_c^ι and the principal isotropy group contains, up to outer automorphisms, a 5×5 or a 6×6 block. By induction it must be induced by a tensor product action, $\mathsf{H}' = \mathsf{SU}(4)$, $k = 10$ and $\mathsf{H}_0 = \mathsf{SU}(4) \cdot \mathsf{S}^1$ by the Rank Lemma. Hence $\mathsf{K}_0^- = \mathsf{SU}(5) \cdot \mathsf{S}^1$ since the 8 dimensional representation has to degenerate. The isotropy representation of $\mathsf{SO}(10)/\,\mathsf{U}(5)$, when restricted to $\mathsf{U}(4)$, contains a six dimensional representation, which has to degenerate in $\mathsf{K}^+/\,\mathsf{H}$ and hence $\mathsf{K}^+ = \mathsf{Spin}(7) \cdot \mathsf{S}^1$. We have recovered the action of $\mathsf{Spin}(10)$ on $\mathbb{CP}^{15}$.

Finally, we consider the case $\mathsf{H}' = \mathsf{SU}(3) \subset \mathsf{Spin}(6) \subset \mathsf{Spin}(k)$. We first rule out $k \geq 8$. In this case $\mathrm{rk}(\mathsf{H}) \geq 3$, and by Lemma 2.4 there exists an irreducible summand in the isotropy representation of $\mathsf{G}\,/\,\mathsf{H}$ on which both H' and $\mathsf{H}\,/\,\mathsf{H}'$ act nontrivially. Thus not all the six dimensional representations of $\mathsf{SU}(3)$ can degenerate in G_2. Hence an isotropy group, say K^-, contains $\mathsf{SU}(4)$ as a normal subgroup, and we consider the fixed point set of the central involution $\iota \in \mathsf{SU}(4)$. Since it is central in a $\mathsf{Spin}(8)$-block and acts as $-\,\mathrm{id}$ on the slice, it has a homogeneous fixed point component $\mathsf{Spin}(k-8) \cdot \mathsf{Spin}(8)/(\mathsf{K}^- \cap \mathsf{Spin}(k-8) \cdot \mathsf{Spin}(8))$ which cannot have positive curvature.

In the case of $k = 6$, either G has a fixed point, or the action is not primitive. Thus we may assume $k = 7$ and hence $\mathsf{H}_0 = \mathsf{SU}(3)$. The isotropy representation of $\mathsf{Spin}(7)/\,\mathsf{SU}(3)$ consists of the sum of a trivial representation, a 6 dimensional representation corresponding to the isotropy representation in $\mathsf{SU}(4) = \mathsf{Spin}(6)$, and a second 6 dimensional representation orthogonal to it. Thus the only connected subgroups in between $\mathsf{SU}(3)$ and $\mathsf{Spin}(7)$ are $\mathsf{U}(3)$, $\mathsf{Spin}(6)$ and G_2, and the normalizer $\mathsf{N}(\mathsf{H}_0)/\mathsf{H}_0 \cong \mathsf{S}^1$ acts transitively on the possible embeddings of $\mathsf{Spin}(6)$ and G_2.

If $\mathsf{K}_0^- = \mathsf{SU}(4) \cong \mathsf{Spin}(6)$ occurs as isotropy group, then $\mathsf{K}_0^+ = \mathsf{S}^1 \cdot \mathsf{SU}(3)$ or $\mathsf{K}_0^+ = \mathsf{Spin}(6)$ and the action is not primitive. Thus H is connected, $\mathsf{K}^+ \cong \mathsf{G}_2$, and we have recovered the action on $\mathbb{S}^{14}$.

If $\mathsf{SU}(4)$ does not occur as isotropy group, primitivity implies that $\mathsf{K}_0^- = \mathsf{G}_2$, and $\mathsf{K}_0^+ = \mathsf{S}^1 \cdot \mathsf{SU}(3)$. As the center of $\mathsf{Spin}(7)$ is contained in K^+, it must be contained in H also, since otherwise $\mathsf{Spin}(7)/\mathsf{K}^+$ would be totally geodesic. This also shows that $\mathsf{Spin}(7)/\mathsf{K}^- \cong \mathbb{RP}^6$, and K^- and H have precisely two components. We have recovered the linear action of $\mathsf{Spin}(7)$ on $\mathbb{CP}^7$.

$$\mathsf{G} = \mathsf{F}_4\,,\ \mathsf{E}_6\,,\ \mathsf{E}_7\,,\ \mathsf{E}_8$$

If G is one of F_4, E_6, E_7, or E_8, Table B implies that H' is one of the groups $\mathsf{Spin}(k)$, $k \le 8$, G_2 or $\mathsf{SU}(3)$, where we again used the fact that $\mathsf{H}_0 = \mathsf{Spin}(9)$ is not possible. If $\mathsf{H}' \ne \mathsf{Spin}(7)$, we have $\dim(\mathsf{K}^\pm/\,\mathsf{H}) \le 8$ and hence $\dim(\mathsf{G}\,/\,\mathsf{H}) \le 32$ by the Lower Weyl Group Bound. This implies that $\dim \mathsf{G} - 3\,\mathrm{rk}\,\mathsf{G} \le 29$, which is clearly not possible.

For $\mathsf{H}' = \mathsf{Spin}(7)$, it follows as before that $\mathsf{H}' = \mathsf{H}_0$ and thereby $\mathsf{G} = \mathsf{F}_4$. In one singular orbit the 8-dimensional representation of $\mathsf{Spin}(7)$ has to degenerate. This implies $\mathsf{K}_0^- = \mathsf{Spin}(9) \subset \mathsf{F}_4$ and since $\mathsf{Spin}(9)$ is maximal in F_4, K^- and H are connected. Since $l_\pm$ are one of $1, 7, 15$, the Upper Weyl Group Bound implies that $|\mathsf{W}| = 2$, which contradicts the Lower Weyl Group Bound. q.e.d.

15. Appendix II: Group Diagrams for Compact Rank One Symmetric Spaces

In this appendix we will collect various known information that will be used throughout the proof of Theorem A.

To describe the representations, we use the notation ρ_n, μ_n, and ν_n for the defining representations of $\mathsf{SO}(n)$, $\mathsf{U}(n)$, and $\mathsf{Sp}(n)$ respectively. Δ_n denotes the spin representation for $\mathsf{SO}(n)$ and $\Delta_{2n}^\pm$ the half spin representation. Also ϕ denotes a 2 dimensional representation of S^1 and for all others we use ψ_N for an N-dimensional irreducible representation.

In Table B we reproduce the list of spherical simple subgroups of the simple Lie groups from [**Wi3**, Proposition 7.2-7.4] since it will be an important tool in our classification. All embeddings are standard embeddings among classical groups, and $\mathsf{Spin}(7) \subset \mathsf{SO}(8)$ is the embedding via the spin representation. We point out that the case of a rank one group in the exceptional Lie groups was not included in [**Wi3**]. But in our proof, this case will only be needed for a rank one group in G_2, where one easily shows that $\mathsf{SU}(2) \subset \mathsf{SU}(3)$ is the only spherical subgroup.

In Table C we list the transitive actions on spheres and their isotropy representation. Notice that $\nu_n \hat{\otimes} \nu_1$ is the representation on $\mathbb{H}^n = \mathbb{R}^{4n}$ given by left multiplication of $\mathsf{Sp}(n)$ and right multiplication of $\mathsf{Sp}(1)$ on

G	H	Inclusions
$\mathsf{SU}(n)$	$\mathsf{SU}(2)$	$\mathsf{SU}(2) \subset \mathsf{SU}(n)$ given by $p\,(\mu_2) \oplus q\, id$
$\mathsf{SU}(n)$	$\mathsf{Sp}(2)$	$\mathsf{Sp}(2) \subset \mathsf{SU}(4) \subset \mathsf{SU}(n)$
$\mathsf{SU}(n)$	$\mathsf{SU}(k)$	$k \times k$ block
$\mathsf{SO}(n)$	$\mathsf{Sp}(1)$	$\mathsf{Sp}(1) \subset \mathsf{SO}(n)$ given by $p\,\nu_1 \oplus q\, id$
$\mathsf{SO}(n)$	$\mathsf{SU}(3)$	$\mathsf{SU}(3) \subset \mathsf{SO}(6) \subset \mathsf{SO}(n)$
$\mathsf{SO}(n)$	$\mathsf{Sp}(2)$	$\mathsf{Sp}(2) \subset \mathsf{SU}(4) \subset \mathsf{SO}(8) \subset \mathsf{SO}(n)$
$\mathsf{SO}(n)$	G_2	$\mathsf{G}_2 \subset \mathsf{SO}(7) \subset \mathsf{SO}(n)$
$\mathsf{SO}(n)$	$\mathsf{SU}(4)$	$\mathsf{SU}(4) \subset \mathsf{SO}(8) \subset \mathsf{SO}(n)$
$\mathsf{SO}(n)$	$\mathsf{Spin}(7)$	$\mathsf{Spin}(7) \subset \mathsf{SO}(8) \subset \mathsf{SO}(n)$
$\mathsf{SO}(n)$	$\mathsf{SO}(k)$	$k \times k$ block
$\mathsf{Sp}(n)$	$\mathsf{Sp}(1)$	$\mathsf{Sp}(1) = \{\mathrm{diag}(q, q, \ldots, q, 1, \ldots, 1) \mid q \in \mathsf{S}^3\}$
$\mathsf{Sp}(n)$	$\mathsf{Sp}(k)$	$k \times k$ block
G_2	$\mathsf{SU}(3)$	maximal subgroup
$\mathsf{F}_4, \mathsf{E}_6$ $\mathsf{E}_7, \mathsf{E}_8$	$\mathsf{Spin}(k)$	$\mathsf{H} \subset \mathsf{Spin}(9) \subset \mathsf{F}_4 \subset \mathsf{E}_6 \subset \mathsf{E}_7 \subset \mathsf{E}_8$ $\mathsf{H} = \mathsf{Spin}(k), k = 5, \ldots, 9$ standard embedding
$\mathsf{F}_4, \mathsf{E}_6 \cdots \mathsf{E}_8$	$\mathsf{SU}(3)$	$\mathsf{SU}(3) \subset \mathsf{SU}(4) \subset \mathsf{Spin}(8) \subset \mathsf{F}_4 \subset \mathsf{E}_6 \subset \mathsf{E}_7 \subset \mathsf{E}_8$
$\mathsf{F}_4, \mathsf{E}_6 \cdots \mathsf{E}_8$	G_2	$\mathsf{G}_2 \subset \mathsf{Spin}(7) \subset \mathsf{Spin}(8) \subset \mathsf{F}_4 \subset \mathsf{E}_6 \subset \mathsf{E}_7 \subset \mathsf{E}_8$

Table B. G/H with spherical isotropy representations.

quaternionic vectors and $\nu_n \hat{\otimes} \phi$ is the subrepresentation under $\mathsf{U}(1) \subset \mathsf{Sp}(1)$. Notice also that for each irreducible subrepresentation $\mathfrak{m}$ in the isotropy representation of K/H the group H acts transitively on the unit sphere in $\mathfrak{m}$, as long as $\dim \mathfrak{m} > 1$. This elementary but important property is used in the Isotropy Lemma 2.3.

In Table D we list the remaining simply connected homogeneous spaces with positive curvature which will be used when one needs to check whether a singular orbit can be totally geodesic.

Information about cohomogeneity one actions on spheres is taken from [**HL**] and [**St1**] where the group G and the principal isotropy group H are given. In the literature one finds only partial information about the singular isotropy groups $\mathsf{K}^{\pm}$, see e.g., [**Br**], [**HL**], [**MS**], [**St1**], [**St2**], [**Uc2**]. In our proof, the groups $\mathsf{K}^{\pm}$ and their embeddings amusingly are obtained along the way. In other words, once an essential action

n	K	H	Isotropy representation
n	$\mathsf{SO}(n+1)$	$\mathsf{SO}(n)$	ρ_n
$2n+1$	$\mathsf{SU}(n+1)$	$\mathsf{SU}(n)$	$\mu_n \oplus id$
$2n+1$	$\mathsf{U}(n+1)$	$\mathsf{U}(n)$	$\mu_n \oplus id$
$4n+3$	$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)$	$\nu_n \oplus 3\,id$
$4n+3$	$\mathsf{Sp}(n+1)\,\mathsf{Sp}(1)$	$\mathsf{Sp}(n)\Delta\,\mathsf{Sp}(1)$	$\nu_n\hat{\otimes}\nu_1 \oplus id\,\hat{\otimes}\rho_3$
$4n+3$	$\mathsf{Sp}(n+1)\,\mathsf{U}(1)$	$\mathsf{Sp}(n)\Delta\,\mathsf{U}(1)$	$\nu_n\hat{\otimes}\phi \oplus id\,\hat{\otimes}\phi \oplus id$
15	$\mathsf{Spin}(9)$	$\mathsf{Spin}(7)$	$\rho_7 \oplus \Delta_7$
7	$\mathsf{Spin}(7)$	G_2	ϕ_7
6	G_2	$\mathsf{SU}(3)$	μ_3

Table C. Transitive actions on $\mathbb{S}^n$.

n	G	K
2n	$\mathsf{SU}(n+1)$	$\mathsf{U}(n)$
4n	$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)\,\mathsf{Sp}(1)$
4n	$\mathsf{Sp}(n+1)$	$\mathsf{Sp}(n)\,\mathsf{U}(1)$
16	F_4	$\mathsf{Spin}(9)$
6	$\mathsf{SU}(3)$	T^2
12	$\mathsf{Sp}(3)$	$\mathsf{Sp}(1)^3$
24	F_4	$\mathsf{Spin}(8)$
7	$\mathsf{SU}(3)$	$\mathsf{S}^1 = \mathrm{diag}(z^p, z^q, \bar{z}^{p+q})$ $(p,q)=1,\ pq(p+q)\neq 0$
7	$\mathsf{U}(3)$	T^2
13	$\mathsf{SU}(5)$	$\mathsf{Sp}(2)\cdot\mathsf{S}^1$

Table D. Homogeneous spaces $M^n = \mathsf{G}/\mathsf{K}$ with positive curvature, which are not spheres.

arises in the induction proof with G and H from Straume's list, our obstructions leave only one possibility for $\mathsf{K}^{\pm}$. Moreover, all essential actions indeed arise in the proof along the way. In Table E we describe the essential group actions on odd dimensional spheres, where π is the representation of G on $\mathbb{R}^n$, and in Table F the ones on even dimensional rank one symmetric spaces. We also include the normal extensions since these extensions will be used in the induction proof. The cohomogeneity one actions on $\mathbb{CP}^n$ and $\mathbb{HP}^n$ are obtained from an action on an odd dimensional sphere when $\mathsf{U}(1)$ or $\mathsf{Sp}(1)$ is a normal subgroup in G with induced action given by a Hopf action. Conversely, an action on $\mathbb{CP}^n$ or $\mathbb{HP}^n$ lifts to such an action on a sphere.

n	G	π	K^-	K^+	H	(l_-, l_+)	W
$8k+7$	$\mathsf{Sp}(2)\,\mathsf{Sp}(k+1)$	$\nu_2\hat{\otimes}\nu_{k+1}$	$\triangle\,\mathsf{Sp}(2)\,\mathsf{Sp}(k-1)$	$\mathsf{Sp}(1)^2\,\mathsf{Sp}(k)$	$\mathsf{Sp}(1)^2\,\mathsf{Sp}(k-1)$	$(4, 4k+1)$	D_4
$4k+7$	$\mathsf{SU}(2)\,\mathsf{SU}(k+2)$	$\mu_2\hat{\otimes}\mu_{k+2}$	$\triangle\,\mathsf{SU}(2)\,\mathsf{SU}(k)$	$\mathsf{S}^1\cdot\mathsf{SU}(k+1)$	$\mathsf{S}^1\cdot\mathsf{SU}(k)$	$(2, 2k+1)$	D_4
	$\mathsf{U}(2)\,\mathsf{SU}(k+2)$	$\mu_2\hat{\otimes}\mu_{k+2}$	$\triangle\,\mathsf{U}(2)\,\mathsf{SU}(k)$	$\mathsf{T}^2\cdot\mathsf{SU}(k+1)$	$\mathsf{T}^2\cdot\mathsf{SU}(k)$		
$2k+3$	$\mathsf{SO}(2)\,\mathsf{SO}(k+2)$	$\rho_2\hat{\otimes}\rho_{k+2}$	$\triangle\,\mathsf{SO}(2)\,\mathsf{SO}(k)$	$\mathbb{Z}_2\cdot\mathsf{SO}(k+1)$	$\mathbb{Z}_2\cdot\mathsf{SO}(k)$	$(1, k)$	D_4
15	$\mathsf{SO}(2)\,\mathsf{Spin}(7)$	$\rho_2\hat{\otimes}\Delta_7$	$\triangle\,\mathsf{SO}(2)\,\mathsf{SU}(3)$	$\mathbb{Z}_2\cdot\mathsf{Spin}(6)$	$\mathbb{Z}_2\cdot\mathsf{SU}(3)$	$(1, 7)$	D_4
13	$\mathsf{SO}(2)\,\mathsf{G}_2$	$\rho_2\hat{\otimes}\phi_7$	$\triangle\,\mathsf{SO}(2)\,\mathsf{SU}(2)$	$\mathbb{Z}_2\cdot\mathsf{SU}(3)$	$\mathbb{Z}_2\cdot\mathsf{SU}(2)$	$(1, 5)$	D_4
7	$\mathsf{SO}(4)$	$\nu_1\hat{\otimes}\nu_3$	$\mathsf{S}(\mathsf{O}(2)\,\mathsf{O}(1))$	$\mathsf{S}(\mathsf{O}(1)\,\mathsf{O}(2))$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$(1, 1)$	D_6
15	$\mathsf{Spin}(8)$	$\rho_8\oplus\Delta_8^{\pm}$	$\mathsf{Spin}(7)$	$\mathsf{Spin}(7)$	G_2	$(7, 7)$	D_2
13	$\mathsf{SU}(4)$	$\mu_4\oplus\rho_6$	$\mathsf{SU}(3)$	$\mathsf{Sp}(2)$	$\mathsf{SU}(2)$	$(5, 7)$	D_2
	$\mathsf{U}(4)$	$\mu_4\oplus\rho_6$	$\mathsf{S}^1\cdot\mathsf{SU}(3)$	$\mathsf{S}^1\cdot\mathsf{Sp}(2)$	$\mathsf{S}^1\cdot\mathsf{SU}(2)$		
19	$\mathsf{SU}(5)$	$\Lambda^2\mu_5$	$\mathsf{Sp}(2)$	$\mathsf{SU}(2)\,\mathsf{SU}(3)$	$\mathsf{SU}(2)^2$	$(4, 5)$	D_4
	$\mathsf{U}(5)$	$\Lambda^2\mu_5$	$\mathsf{S}^1\cdot\mathsf{Sp}(2)$	$\mathsf{S}^1\cdot\mathsf{SU}(2)\,\mathsf{SU}(3)$	$\mathsf{S}^1\cdot\mathsf{SU}(2)^2$		
31	$\mathsf{Spin}(10)$	$\Delta_{10}^{\pm}$	$\mathsf{SU}(5)$	$\mathsf{Spin}(7)$	$\mathsf{SU}(4)$	$(9, 6)$	D_4
	$\mathsf{S}^1\cdot\mathsf{Spin}(10)$	$\Delta_{10}^{\pm}$	$\Delta\,\mathsf{S}^1\cdot\mathsf{SU}(5)$	$\mathsf{S}^1\cdot\mathsf{Spin}(7)$	$\mathsf{S}^1\cdot\mathsf{SU}(4)$		
7	$\mathsf{SU}(3)$	ad	$\mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(1))$	$\mathsf{S}(\mathsf{U}(1)\,\mathsf{U}(2))$	T^2	$(2, 2)$	D_3
9	$\mathsf{SO}(5)$	ad	$\mathsf{U}(2)$	$\mathsf{SO}(3)\,\mathsf{SO}(2)$	T^2	$(2, 2)$	D_4
13	G_2	ad	$\mathsf{U}(2)$	$\mathsf{U}(2)$	T^2	$(2, 2)$	D_6
13	$\mathsf{Sp}(3)$	$\Lambda^2\nu_3 - 1$	$\mathsf{Sp}(2)\,\mathsf{Sp}(1)$	$\mathsf{Sp}(1)\,\mathsf{Sp}(2)$	$\mathsf{Sp}(1)^3$	$(4, 4)$	D_3
25	F_4	ψ_{26}	$\mathsf{Spin}(9)$	$\mathsf{Spin}(9)$	$\mathsf{Spin}(8)$	$(8, 8)$	D_3

Table E. Essential cohomogeneity one actions and extensions on odd dimensional spheres, $k \geq 1$.

We will also use some knowledge about non-essential actions on spheres, apart from the extensions of essential ones. If a subaction by $\mathsf{L} \subset \mathsf{G}$ on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is fixed point homogeneous, an analysis of the submetry $M/\mathsf{L} \to M/G$ shows that G is either a suspension action or L is a normal subgroup of G contained in one of the singular isotropy groups. Thus we can write $\mathsf{G} = \mathsf{L}_1\mathsf{L}_2\mathsf{L}'$ and $\mathbb{R}^n = V_1 \oplus V_2$ such that G preserves V_i and $\mathsf{L}_i = \{g \in \mathsf{G} \mid g_{|V_j} = \mathrm{Id}, j \neq i\}$, where at least one of L_i is non-trivial. We will call such actions *sum actions* for short. But notice that this is not equivalent to the representation of G on $\mathbb{R}^n$ being reducible, as the actions of $\mathsf{SU}(4)$ (and $\mathsf{U}(4)$) on $\mathbb{S}^{13}$, of $\mathsf{Spin}(8)$ on $\mathbb{S}^{15}$, and of $\mathsf{Spin}(7)$ on $\mathbb{S}^{14}$ illustrate (corresponding to both L_i being trivial). Clearly $\mathsf{L}_i\mathsf{L}'$ acts transitively on the unit sphere in V_i, which easily implies that L_i does so as well. The most elementary sum actions are given by $\mathsf{G} = \mathsf{G}(n)\,\mathsf{G}'(m)$ operating on $V^n \oplus V^m$ as $f_n \hat{\otimes} id \oplus id \hat{\otimes} f_m$ where $\mathsf{G}(n)$ is any of the classical Lie groups with their defining representations f_n. The property we sometimes use is that the principal isotropy group is given by $\mathsf{G}(n-1)\,\mathsf{G}'(m-1)$. This also includes the case where $m = 1$, i.e., $\mathsf{G}(m) = \{e\}$, which corresponds to actions with a fixed point. Such sum actions can be further modified by replacing the action of $\mathsf{G}(n)$ on V^n and $\mathsf{G}(m)$ on V^m by any one of the other transitive actions on spheres. The corresponding isotropy groups are given in Table C. If L' is non-trivial, it is necessarily of rank one and Table C easily implies that there are only three such actions:

1) (G, H) is given by $(\mathsf{U}(1)\,\mathsf{G}(n)\,\mathsf{G}(m)\,,\, \Delta\,\mathsf{U}(1)\,\mathsf{G}(n-1)\,\mathsf{G}'(m-1))$ acting via $\phi^k \hat{\otimes} f_n \hat{\otimes} id \oplus \phi^l \hat{\otimes} id \hat{\otimes} f_m$ with $(k,l) = 1$ and $\mathsf{G}(n)$ is one of $\mathsf{SU}(n)$ or $\mathsf{Sp}(n)$. If $\mathsf{G}(m) = \{e\}$, and hence $V^m = \mathbb{C}$, the principal isotropy group is $\Delta\mathbb{Z}_l\,\mathsf{G}(n-1)$.
2) (G, H) is given by $(\mathsf{Sp}(1)\,\mathsf{Sp}(n)\,\mathsf{Sp}(m), \Delta\,\mathsf{Sp}(1)\,\mathsf{Sp}(n-1)\,\mathsf{Sp}(m-1))$ acting via $\nu_1 \hat{\otimes} \nu_n \hat{\otimes} id \oplus \nu_1 \hat{\otimes} id \hat{\otimes} \nu_m$. If $\mathsf{Sp}(m) = \{e\}$, and $V^m = \mathbb{H}$, we have $\mathsf{H} = \mathsf{Sp}(n-1)$.
3) (G, H) is given by $(\mathsf{Sp}(1)\,\mathsf{Sp}(n), \Delta\,\mathsf{U}(1)\,\mathsf{Sp}(n-1))$ acting via $\nu_1 \hat{\otimes} \nu_n \oplus \rho_3 \hat{\otimes} id$, which is an action on an even dimensional sphere.

	G	K^-	K^+	H	(l_-, l_+)	W
$\mathbb{S}^4$	$\mathsf{SO}(3)$, $\pi = S^2\rho_3 - 1$	$\mathsf{S}(\mathsf{O}(2)\,\mathsf{O}(1))$	$\mathsf{S}(\mathsf{O}(1)\,\mathsf{O}(2))$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(1,1)$	D_3
$\mathbb{S}^{14}$	$\mathsf{Spin}(7)$, $\pi = \rho_7 \oplus \Delta_7$	$\mathsf{Spin}(6)$	G_2	$\mathsf{SU}(3)$	$(1,6)$	D_2
$\mathbb{CP}^{k+1}$	$\mathsf{SO}(k+2)$	$\mathsf{SO}(2)\,\mathsf{SO}(k)$	$\mathsf{O}(k+1)$	$\mathbb{Z}_2 \cdot \mathsf{SO}(k)$	$(1,k)$	D_2
$\mathbb{CP}^{2k+3}$	$\mathsf{SU}(2)\,\mathsf{SU}(k+2)$	$\Delta\,\mathsf{U}(2)\,\mathsf{SU}(k)$	$\mathsf{T}^2\,\mathsf{SU}(k+1)$	$\mathsf{T}^2\,\mathsf{SU}(k)$	$(2, 2k+1)$	D_2
$\mathbb{CP}^6$	G_2	$\mathsf{U}(2)$	$\mathbb{Z}_2 \cdot \mathsf{SU}(3)$	$\mathbb{Z}_2 \cdot \mathsf{SU}(2)$	$(1,5)$	D_2
$\mathbb{CP}^7$	$\mathsf{Spin}(7)$	$\mathsf{S}^1\,\mathsf{SU}(3)$	$\mathbb{Z}_2 \cdot \mathsf{Spin}(6)$	$\mathbb{Z}_2 \cdot \mathsf{SU}(3)$	$(1,7)$	D_2
$\mathbb{CP}^9$	$\mathsf{SU}(5)$	$\mathsf{S}^1 \cdot \mathsf{Sp}(2)$	$\mathsf{S}(\mathsf{U}(2)\,\mathsf{U}(3))$	$\mathsf{S}^1 \cdot \mathsf{SU}(2)^2$	$(4,5)$	D_2
$\mathbb{CP}^{15}$	$\mathsf{Spin}(10)$	$\mathsf{S}^1\,\mathsf{SU}(5)$	$\mathsf{S}^1\,\mathsf{Spin}(7)$	$\mathsf{S}^1\,\mathsf{SU}(4)$	$(9,6)$	D_2
$\mathbb{HP}^{k+1}$	$\mathsf{SU}(k+2)$	$\mathsf{SU}(2)\,\mathsf{SU}(k)$	$\mathsf{U}(k+1)$	$\mathsf{S}^1 \cdot \mathsf{SU}(k)$	$(2, 2k+1)$	D_2
	$\mathsf{S}^1\,\mathsf{SU}(k+2)$	$\Delta\,\mathsf{S}^1\,\mathsf{SU}(2)\,\mathsf{SU}(k)$	$\mathsf{S}^1\,\mathsf{U}(k+1)$	$\mathsf{T}^2 \cdot \mathsf{SU}(k)$		
$\mathrm{Ca}\mathbb{P}^2$	$\mathsf{Sp}(3)$	$\mathsf{Sp}(2)$	$\mathsf{Sp}(1)\,\mathsf{Sp}(2)$	$\mathsf{Sp}(1)^2$	$(11,8)$	D_2
	$\mathsf{S}^1\,\mathsf{Sp}(3)$	$\Delta\,\mathsf{S}^1\,\mathsf{Sp}(2)$	$\mathsf{S}^1\,\mathsf{Sp}(1)\,\mathsf{Sp}(2)$	$\mathsf{S}^1\,\mathsf{Sp}(1)^2$		
	$\mathsf{Sp}(1)\,\mathsf{Sp}(3)$	$\Delta\,\mathsf{Sp}(1)\,\mathsf{Sp}(2)$	$\mathsf{Sp}(1)^2\,\mathsf{Sp}(2)$	$\mathsf{Sp}(1)^3$		

Table F. Essential cohomogeneity one actions and extensions in even dimensions, $k \geq 1$.

For the even dimensional rank one symmetric spaces one also has sum actions by $\mathsf{Sp}(n)\ \mathsf{Sp}(m)$ on $\mathbb{HP}^{n+m-1}$ and by $\mathsf{SU}(n)\ \mathsf{SU}(m)$ or $\mathsf{S}(\mathsf{U}(n)\,\mathsf{U}(m))$ on $\mathbb{CP}^{n+m-1}$, where one or both unitary groups can also be replaced by symplectic groups. If one of the groups is absent, they are the actions with a fixed point. Finally, in Table G we list the symmetric spaces G/K where K and G have the same rank. They occur as normalizers of elements ι whose square, but not ι itself, lies in the center of G. We point out that in this table the group $\mathsf{Spin}(4k)/\mathbb{Z}_2$ is the quotient of $\mathsf{Spin}(4k)$ which is not isomorphic to $\mathsf{SO}(4k)$.

G	K
$\mathsf{SO}(2n)$	$\mathsf{SO}(2k)\,\mathsf{SO}(2n-2k)$, $\mathsf{U}(n)$
$\mathsf{SO}(2n+1)$	$\mathsf{SO}(2k+1)\,\mathsf{SO}(2n-2k)$
$\mathsf{SU}(n)$	$\mathsf{S}(\mathsf{U}(k)\,\mathsf{U}(n-k))$
$\mathsf{Sp}(n)$	$\mathsf{Sp}(k)\,\mathsf{Sp}(n-k)$, $\mathsf{U}(n)$
G_2	$\mathsf{SO}(4)$
F_4	$\mathsf{Spin}(9)$, $\mathsf{Sp}(3)\,\mathsf{Sp}(1)$
E_6	$\mathsf{SU}(6)\,\mathsf{SU}(2)$, $\mathsf{Spin}(10)\cdot\mathsf{S}^1$
E_7	$\mathsf{SU}(8)$, $\mathsf{Spin}(12)/\mathbb{Z}_2\cdot\mathsf{S}^3$, $\mathsf{E}_6\cdot\mathsf{S}^1$
E_8	$\mathsf{Spin}(16)/\mathbb{Z}_2$, $\mathsf{E}_7\cdot\mathsf{S}^3$

Table G. Equal rank symmetric subgroups.

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