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A UNIFORMIZATION THEOREM FOR COMPLETE NON-COMPACT KÄHLER SURFACES WITH POSITIVE BISECTIONAL CURVATURE

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Abstract

In this paper, by combining techniques from Ricci flow and algebraic geometry, we prove the following generalization of the classical uniformization theorem of Riemann surfaces. Given a complete non-compact complex two dimensional Kähler manifold M of positive and bounded holomorphic bisectional curvature, suppose its geodesic balls have maximal volume growth, then M is biholomorphic to $\mathbf{C}^{\mathbf{2}}$. This gives a partial affirmative answer to the well-known conjecture of Yau [**41**] on uniformization theorem. During the proof, we also verify an interesting gap phenomenon, predicted by Yau [**42**], which says that a Kähler manifold as above automatically has quadratic curvature decay at infinity in the average sense.

1. Introduction

One of the most beautiful results in complex analysis of one variable is the classical uniformization theorem of Riemann surfaces which states that a simply connected Riemann surface is biholomorphic to either the Riemann sphere, the complex line or the open unit disc. Unfortunately, a direct analog of this beautiful result to higher dimensions does not exist. For example, there is a vast variety of biholomorphically distinct complex structures on $\mathbf{R}^{2n}$ for $n > 1$, a fact which was already known to Poincaré (see [**3**], [**11**] for a modern treatment). Thus, in order to characterize the standard complex structures for higher dimensional complex manifolds, one must impose more restrictions on the manifolds.

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From the point of view of differential geometry, one consequence of the uniformization theorem is that a positively curved compact or non-compact Riemann surface must be biholomorphic to the Riemann sphere or the complex line, respectively. It is thus natural to ask whether there is similar characterization for higher dimensional complete Kähler manifold with positive "curvature". That such a characterization exists in the case of compact Kähler manifold is the famous Frankel conjecture which says that a compact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to a complex projective space. This conjecture was solved by Andreotti–Frankel [**12**] and Mabuchi [**21**] in complex dimensions two and three respectively and the general case was then solved by Mori [**27**], and Siu–Yau [**38**] independently. In this paper, we are thus interested in complete non-compact Kähler manifolds with positive holomorphic bisectional curvature. The following conjecture provides the main impetus.

Yau Conjecture (Yau [**41**])**.** *A complete non-compact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to a complex Euclidean space.*

Greene and Wu [**15**] also proposed a weaker version of this conjecture by assuming that the Kähler manifold has positive Riemannian sectional curvature. In contrast to the compact case, very little is known about this conjecture. The first result in this direction is the following isometrically embedding theorem.

Theorem (Mok–Siu–Yau [**23**], Mok [**24**])**.** *Let M be a complete non-compact Kähler manifold of non-negative holomorphic bisectional curvature of complex dimension $n \geq 2$. Suppose there exist positive constants C_1, C_2 such that for a fixed base point x_0 and some $\varepsilon > 0$,*

$$\text{(i)} \qquad \mathrm{Vol}\,(B(x_0,r)) \geq C_1 r^{2n}, \qquad 0 \leq r < +\infty\ ,$$

$$\text{(ii)} \qquad R(x) \leq \frac{C_2}{1+d^{2+\varepsilon}(x_0,x)} \qquad \textit{on} \quad M\ ,$$

where $\mathrm{Vol}\,(B(x_0,r))$ *denotes the volume of the geodesic ball* $B(x_0,r)$ *centered at* x_0 *with radius* r, $R(x)$ *denotes the scalar curvature and* $d(x_0,x)$ *denotes the geodesic distance between* x_0 *and* x. *Then,* M *is isometrically biholomorphic to* $\mathbf{C}^{\mathbf{n}}$ *with the flat metric.*

Their method is to consider the Poincaré–Lelong equation $\sqrt{-1}\partial\overline{\partial}u =$ Ric. Under the condition (ii) that the curvature has faster than quadratic decay, they proved the existence of a bounded solution u to the Poincaré–Lelong equation. By virtue of Yau's Liouville theorem on complete manifolds with non-negative Ricci curvature, this bounded plurisubharmonic function u must be constant and hence, the Ricci curvature must be identically zero. This implies that the Kähler metric is flat because of the non-negativity of the holomorphic bisectional curvature. However, this argument breaks down if the faster than quadratic decay condition (ii) is weakened to a quadratic decay condition. In this case, although we can still solve the Poincaré–Lelong equation with logarithmic growth, the boundedness of the solution can no longer be guaranteed.

In [**24**], Mok also developed a general scheme for compactifying complete Kähler manifolds of positive holomorphic bisectional curvature. This allowed him to obtain the following improvement of the above theorem.

Theorem (Mok [**24**])**.** *Let M be a complete non-compact Kähler manifold of complex dimension n with positive holomorphic bisectional curvature. Suppose there exist positive constants C_1, C_2 such that for a fixed base point x_0,*

$$\text{(i)} \qquad \operatorname{Vol}(B(x_0,r)) \geq C_1 r^{2n}, \qquad 0 \leq r < +\infty,$$

$$\text{(ii)}' \qquad 0 < R(x) \leq \frac{C_2}{1+d^2(x_0,x)} \qquad \text{on} \quad M,$$

then M is biholomorphic to an affine algebraic variety. Moreover, if in addition the complex dimension $n = 2$ and

(iii) *the Riemannian sectional curvature of M is positive, then M is biholomorphic to $\mathbf{C}^2$.*

To the best of our knowledge, the above result of Mok and its slight improvements by To [**39**], and Chen–Zhu [**8**] are the best results in complex dimension two of the above stated conjecture. Here, we would also like to recall the remark pointed out in [**8**] that there is a gap in the proof of Shi [**34**] or [**35**] (see [**8**] for more explanation) which would otherwise constitute a better result than that of Mok [**24**].

In this paper, we consider only the case of complex dimension two. Our principal result is the following:

Main Theorem. *Let M be a complete non-compact complex two–dimensional Kähler manifold of positive and bounded holomorphic bisectional curvature. Suppose there exists a positive constant C_1 such that for a fixed base point x_0, we have*

$$\text{(i)} \qquad \mathrm{Vol}(B(x_0, r)) \geq C_1 r^4 \qquad 0 \leq r < +\infty,$$

then, M is biholomorphic to $\mathbf{C^2}$.

Before we describe the ideas of the proof of the main theorem, we recall a curvature linear decay estimate which was first established by the first and third authors in [**9**].

Theorem (Chen–Zhu [**9**]). *Let M be a complete non-compact Kähler manifold with positive holomorphic bisectional curvature. Then, for any $x_0 \in M$, there exists a positive constant C such that*

$$\text{(ii)}'' \quad \frac{1}{\mathrm{Vol}\,(B(x_0, r))} \int_{B(x_0,r)} R(x)dx \leq \frac{C}{1+r} \quad \textit{for all } 0 \leq r < +\infty,$$

where $R(x)$ is the scalar curvature of M.

Recently in Theorem 4.2 of [**29**], Ni and Tam had generalized this curvature linear decay estimate to complete Kähler manifolds with non-negative bisectional curvature.

The proof of the Main Theorem will be divided into three parts. In the first part, we will show that M is a Stein manifold homeomorphic to $\mathbf{R^4}$. For this, we evolve the Kähler metric on M by the Ricci flow first studied by Hamilton. Note that the underlying complex structure of M is unchanged under the Ricci flow, thus we can replace the Kähler metric in our main theorem by any one of the evolving metrics. The advantage is that, in our case, properties of the evolving metric are improving during the flow. Moreover, we know that the maximal volume growth condition (i) as well as the positive holomorphic bisectional curvature condition are preserved by the evolving metric. More importantly, by a blow up and blow down argument as in [**7**] and using an observation of Ivey in [**20**], we can prove that the curvature of the evolving metric decays linearly in time. This implies that the injectivity radius of the evolving metric is getting bigger and bigger and any geodesic ball with radius less than half of the injectivity radius is pseudoconvex. We can then construct an increasing one parameter family of exhausting pseudoconvex domains on M. From this, it follows readily that M is a Stein manifold homeomorphic to $\mathbf{R^4}$.

In the second part of the proof, we consider the algebra $P(M)$ of holomorphic functions of polynomial growth on M and we will prove that its quotient field has transcendental degree two over $\mathbf{C}$. For this, we first need to construct two algebraically independent holomorphic functions in the algebra $P(M)$. Using the L^2 estimates of Andreotti–Vesentini [**1**] and Hömander [**19**], it suffices to construct a strictly plurisubharmonic function of logarithmic growth on M. Now, if the scalar curvature decays in space at least quadratically, it was known from [**23**], [**24**] that such a strictly plurisubharmonic function of logarithmic growth can be obtained by solving the Poincaré–Lelong equation, as we mentioned before. However, the decay estimate (ii)$''$ is too weak to apply their result directly. To resolve this difficulty, we make use of the Ricci flow to verify a new gap phenomenon which was already predicted by Yau in [**42**]. More explicitly, by using the time decay estimate of evolving metric in the previous part, we prove that the curvature of the initial metric must decay quadratically in space in certain average sense. Fortunately, this turns out to be enough to insure the existence of a strictly plurisubharmonic function of logarithmic growth. Next, by using the time decay estimate and the injectivity radius estimate of the evolving metric, we prove that the dimension of the space of holomorphic functions in $P(M)$ of degree at most p is bounded by a constant times p^2. Combining this with the existence of two algebraically independent holomorphic functions in $P(M)$ as above, we can prove that the quotient field $R(M)$ of $P(M)$ has transcendental degree two over $\mathbf{C}$ by a classical argument of Poincaré–Siegel. In other words, $R(M)$ is a finite extension field of some $\mathbf{C}(f_1, f_2)$, where f_1, $f_2 \in P(M)$ are algebraically independent over $\mathbf{C}$. Then, from the primitive element theorem, we have $R(M) = \mathbf{C}(f_1, f_2, g/h)$ for some g, $h \in P(M)$. Hence, the mapping $F : M \to \mathbf{C}^4$ given by $F = (f_1, f_2, g, h)$ defines, in an appropriate sense, a birational equivalence between M and some irreducible affine algebraic subvariety Z of $\mathbf{C}^{\mathbf{4}}$.

In the last part of the proof, we will basically follow the approach of Mok in [**24**] and [**26**] to establish a biholomorphic map from M onto a quasi–affine algebraic variety by desingularizing the map F. Our essential contribution in this part is to establish uniform estimates on the multiplicity and the number of irreducible components of the zero divisor of a holomorphic function in $P(M)$. Again, the time decay estimate of the Ricci flow plays a crucial role in the arguments. Based on these estimates, we can show that the mapping $F : M \to Z$ is almost surjective in the sense that it can miss only a finite number of subvarieties in Z, and can be desingularized by adjoining a finite number of holomorphic functions of polynomial growth. This completes the proof that M is a quasi—affine algebraic variety. Finally, by combining with the fact that M is homeomorphic to $\mathbf{R^4}$, we conclude that M is indeed biholomorphic to $\mathbf{C^2}$ by a theorem of Ramanujam [**31**] on algebraic surfaces.

This paper contains eight sections. From Sections 2 to 4, we study the Ricci flow and obtain several geometric estimates for the evolving metric. In Section 5, we show that the two dimensional Kähler manifold is homeomorphic to $\mathbf{R^4}$ and is a Stein manifold. Based on the estimates on the Ricci flow, a space decay estimate on the curvature and the existence of a strictly plurisubharmonic function of logarithmic growth are obtained in Section 6. In Section 7, we establish uniform estimates on the multiplicity and the number of irreducible components of the zero divisor of a holomorphic function of polynomial growth. Finally, in Section 8, we construct a biholomorphic map from the Kähler manifold onto a quasi—affine algebraic variety and complete the proof of the Main Theorem.

Finally, we remark that the main theorem can be slightly generalized to the case of non-negative bisectional curvature. The details had already been carried out in the thesis (Theorem 0.9 of [**6**]) of the first author.

2. Preserving the volume growth

Let $(M, g_{\alpha\overline{\beta}})$ be a complete, non-compact Kähler surface (i.e., a Kähler manifold of complex dimension two) satisfying all the assumptions in the Main Theorem. We evolve the metric $g_{\alpha\overline{\beta}}$ according to the following Ricci flow equation

$$(2.1)\qquad \begin{cases} \dfrac{\partial g_{\alpha\overline{\beta}}(x,t)}{\partial t} = -R_{\alpha\overline{\beta}}(x,t) & x \in M \quad t>0, \\ g_{\alpha\overline{\beta}}(x,0) = g_{\alpha\overline{\beta}}(x) & x \in M, \end{cases}$$

where $R_{\alpha\overline{\beta}}(x,t)$ denotes the Ricci curvature tensor of the metric $g_{\alpha\overline{\beta}}(x,t)$.

Since the curvature of the initial metric is bounded, it is known from [**33**] that there exists some $T_{\max} > 0$ such that (2.1) has a maximal solution on $M \times [0, T_{\max})$ with either $T_{\max} = +\infty$ or the curvature becomes unbounded as $t \to T_{\max}$ when $T_{\max} < +\infty$. By using the maximum principle, one knows that the Kählerity of $g_{\alpha\overline{\beta}}$ and the positivity of holomorphic bisectional curvature (see Mok [**25**], Hamilton [**16**], or Shi [**36**]) are preserved under the evolution of (2.1). In particular, the Ricci curvature remains positive.

Our first result for the solution of the Ricci flow (2.1) is the following proposition.

Proposition 2.1. *Suppose* $(M, g_{\alpha\overline{\beta}})$ *is assumed as above. Then, the maximal volume growth condition* (i) *is preserved under the evolution of* (2.1), *i.e.,*

$$\mathrm{Vol}_t(B_t(x,r)) \geq C_1 r^4 \qquad \text{for all} \quad r > 0\ , \quad x \in M \tag{2.2}$$

with the same constant C_1 *as in condition* (i). *Here,* $B_t(x,r)$ *is the geodesic ball of radius* r *with center at* x *with respect to the metric* $g_{\alpha\overline{\beta}}(\cdot,t)$, *and the volume* Vol_t *is also taken with respect to the metric* $g_{\alpha\overline{\beta}}(\cdot,t)$.

Proof. Define a function $F(x,t)$ on $M \times [0, T_{\max})$ as follows,

$$F(x,t) = \log \frac{\det\Big(g_{\alpha\overline{\beta}}(x,t)\Big)}{\det\Big(g_{\alpha\overline{\beta}}(x,0)\Big)}.$$

By (2.1), we have

$$\begin{aligned} \frac{\partial F(x,t)}{\partial t} &= g^{\alpha\overline{\beta}}(x,t) \cdot \frac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x,t) \\ &= -R(x,t) \leq 0, \end{aligned} \tag{2.3}$$

which implies that $F(\cdot,t)$ is non-increasing in time. Since $R_{\alpha\overline{\beta}}(x,t) \geq 0$, we know from (2.1) that the metric is shrinking in time. In particular,

$$g_{\alpha\overline{\beta}}(x,t) \leq g_{\alpha\overline{\beta}}(x,0) \qquad \text{on} \quad M \times [0, T_{\max}). \tag{2.4}$$

This implies that

$$\begin{aligned} e^{F(x,t)} R(x,t) &= g^{\alpha\overline{\beta}}(x,t) R_{\alpha\overline{\beta}}(x,t) \cdot \frac{\det\Big(g_{\gamma\overline{\delta}}(x,t)\Big)}{\det\Big(g_{\gamma\overline{\delta}}(x,0)\Big)} \\ &\leq g^{\alpha\overline{\beta}}(x,0) R_{\alpha\overline{\beta}}(x,t) \\ &= g^{\alpha\overline{\beta}}(x,0)\Big(R_{\alpha\overline{\beta}}(x,t) - R_{\alpha\overline{\beta}}(x,0)\Big) + R(x,0) \end{aligned} \tag{2.5}$$

$$= - \triangle_0 F(x,t) + R(x,0),$$

where $\triangle_0$ denotes the Laplace operator with respect to the initial metric $g_{\alpha\overline{\beta}}(x,0)$ and $R(x,t)$ denotes the scalar curvature of the metric $g_{\alpha\overline{\beta}}(x,t)$.

Combining (2.3) and (2.5) gives

$$e^{F(x,t)}\frac{\partial F(x,t)}{\partial t} \geq \triangle_0 F(x,t) - R(x,0) \qquad \text{on} \quad M\times[0,T_{\max}). \tag{2.6}$$

Next, we introduce a cutoff function which will be used several times in this paper. Now, as the Ricci curvature of the initial metric is positive, we know from Schoen and Yau (Theorem 1.4.2 in [**32**]) or Shi [**36**] that there exists a positive constant C_3 depending only on the dimension such that for any fixed point $x_0 \in M$ and any number $0 < r < +\infty$, there exists a smooth function $\varphi(x)$ on M satisfying

$$\begin{cases} e^{-C_3\left(1+\frac{d_0(x,x_0)}{r}\right)} \leq \varphi(x) \leq e^{-\left(1+\frac{d_0(x,x_0)}{r}\right)} \ , \\ \left|\nabla\varphi\right|_0(x) \leq \dfrac{C_3}{r}\varphi(x) \ , \\ \left|\triangle_0\varphi\right|(x) \leq \dfrac{C_3}{r^2}\varphi(x) \end{cases} \tag{2.7}$$

for all $x \in M$, where $d_0(x,x_0)$ is the distance between x and x_0 with respect to the initial metric $g_{\alpha\overline{\beta}}(x,0)$ and $|\cdot|_0$ stands for the corresponding C^0 norm of the initial metric $g_{\alpha\overline{\beta}}(x,0)$.

Combining (2.6) and (2.7), we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\int_M \varphi(x)e^{F(x,t)}dV_0 \\ &\quad\geq \int_M \left(\triangle_0 F(x,t) - R(x,0)\right)\varphi(x)dV_0 \\ &\quad\geq \frac{C_3}{r^2}\int_M F(x,t)\varphi(x)dV_0 - \int_M R(x,0)\varphi(x)dV_0, \end{aligned}$$

where dV_0 denotes the volume element of the initial metric $g_{\alpha\overline{\beta}}(x,0)$. The integration by parts in the second inequality is justified because for any $t < T_{\max}$, the boundedness of the curvature implies the boundedness of F and ∇F.

Recall that $F(\cdot,t)$ is non-increasing in time and $F(\cdot,0)\equiv 0$. We integrate the above inequality from 0 to t to get

$$\begin{aligned}\int_M \varphi(x)\left(1-e^{F(x,t)}\right)dV_0 \\ \leq \frac{C_3 t}{r^2}\int_M \left(-F(x,t)\right)\varphi(x)dV_0 + t\int_M R(x,0)\varphi(x)dV_0.\end{aligned} \tag{2.8}$$

Since the metric is shrinking under the Ricci flow, we have

$$B_t(x_0,r)\supset B_0(x_0,r) \qquad \text{for} \quad t\geq 0\ , \quad 0<r<+\infty,$$

and

$$\begin{aligned}&\mathrm{Vol}_t(B_t(x_0,r)) \\ &\geq \mathrm{Vol}_t(B_0(x_0,r)) \\ &= \int_{B_0(x_0,r)} e^{F(x,t)}dV_0 \\ &= \mathrm{Vol}_0(B_0(x_0,r)) + \int_{B_0(x_0,r)}\left(e^{F(x,t)}-1\right)dV_0.\end{aligned} \tag{2.9}$$

Then, by (2.7) and (2.8), the last term in (2.9) satisfies

$$\begin{aligned}&\int_{B_0(x_0,r)}\left(e^{F(x,t)}-1\right)dV_0 \\ &\geq e^{2C_3}\int_M \left(e^{F(x,t)}-1\right)\varphi(x)dV_0 \\ &\geq \frac{C_3 e^{2C_3}t}{r^2}\int_M F(x,t)\varphi(x)dV_0 \\ &\quad - e^{2C_3}t\int_M R(x,0)\varphi(x)dV_0.\end{aligned} \tag{2.10}$$

To estimate the two terms of the right-hand side of (2.10), we consider any fixed $T_0 < T_{\max}$. Since the curvature is uniformly bounded on $M\times[0,T_0]$, it is clear from the equation (2.3) that $F(x,t)$ is also uniformly bounded on $M\times[0,T_0]$.

Set

$$A = \sup\left\{\,|F(x,t)|\,|\, x\in M\ ,\ t\in[0,T_0]\right\}$$

and

$$M(r) = \sup_{a\geq r}\frac{1}{\mathrm{Vol}_0\left(B_0(x_0,a)\right)}\int_{B_0(x_0,a)} R(x,0)dV_0.$$

Then, the decay estimate (ii)$''$ implies that $M(r) \to 0$ as $r \to +\infty$. By using the standard volume comparison theorem and (2.7), we have

$$
\begin{aligned}
(2.11) \qquad & \int_M R(x,0)\varphi(x)dV_0 \\
&\le \int_M R(x,0)e^{-\left(1+\frac{d_0(x,x_0)}{r}\right)}dV_0 \\
&= \int_{B_0(x_0,r)} R(x,0)e^{-\left(1+\frac{d_0(x,x_0)}{r}\right)}dV_0 \\
&\quad + \sum_{k=0}^{\infty}\int_{B_0(x_0,2^{k+1}r)\setminus B_0(x_0,2^k r)} R(x,0)e^{-\left(1+\frac{d_0(x,x_0)}{r}\right)}dV_0 \\
&\le \int_{B_0(x_0,r)} R(x,0)dV_0 + \sum_{k=0}^{\infty} e^{-2^k}\left(2^{k+1}\right)^4 \\
&\qquad \cdot \frac{\mathrm{Vol}_0\left(B_0(x_0,r)\right)}{\mathrm{Vol}_0\left(B_0(x_0,2^{k+1}r)\right)}\int_{B_0(x_0,2^{k+1}r)} R(x,0)dV_0 \\
&\le C_4 \cdot M(r)\cdot \mathrm{Vol}_0\left(B_0(x_0,r)\right),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
(2.12) \qquad & \int_M \varphi(x)dV_0 \\
&\le \int_{B_0(x_0,r)} e^{-\left(1+\frac{d_0(x,x_0)}{r}\right)}dV_0 \\
&\quad + \sum_{k=0}^{\infty}\int_{B_0(x_0,2^{k+1}r)\setminus B_0(x_0,2^k r)} e^{-\left(1+\frac{d_0(x,x_0)}{r}\right)}dV_0 \\
&\le \mathrm{Vol}_0\left(B_0(x_0,r)\right) + \sum_{k=0}^{\infty} e^{-2^k}\left(2^{k+1}\right)^4 \cdot \mathrm{Vol}_0\left(B_0(x_0,r)\right) \\
&\le C_4 \mathrm{Vol}_0\left(B_0(x_0,r)\right),
\end{aligned}
$$

where C_4 is some positive constant independent of r.

Substituting (2.10)–(2.12) into (2.9) and dividing by r^4, we obtain

$$\begin{aligned}\frac{\mathrm{Vol}_t(B_t(x_0,r))}{r^4} \geq \frac{\mathrm{Vol}_0(B_0(x_0,r))}{r^4} - \frac{C_3 e^{2C_3} A T_0}{r^2}\left(C_4 \frac{\mathrm{Vol}_0(B_0(x_0,r))}{r^4}\right) \\ - e^{2C_3} T_0 \left(C_4 M(r) \cdot \frac{\mathrm{Vol}_0(B_0(x_0,r))}{r^4}\right) \\ \geq C_1 - \frac{C_3 e^{2C_3} A T_0 \cdot C_4 C_1}{r^2} - e^{2C_3} T_0 C_4 C_1 \cdot M(r)\end{aligned}$$

by condition (i). Then, letting $r \to +\infty$, we deduce that

$$\lim_{r\to+\infty} \frac{\mathrm{Vol}_t\left(B_t(x_0,r)\right)}{r^4} \geq C_1.$$

Hence, by using the standard volume comparison theorem, we have

$$\mathrm{Vol}_t\left(B_t(x,r)\right) \geq C_1 r^4 \quad \text{for all } x \in M\ ,\ 0 \leq r < +\infty,\ t \in [0, T_0].$$

Finally, since $T_0 < T_{\max}$ is arbitrary, this completes the proof of the proposition. q.e.d.

3. Singularity models

In Sections 3 and 4, we will continue our study of the Ricci flow (2.1). We will use rescaling arguments to analyse the behavior of the solution of (2.1) near the maximal time $T_{\max}$.

First of all, let us recall some basic terminologies. According to Hamilton (see for example, definition 16.3 in [**18**]), a solution to the Ricci flow, where either the manifold is compact or at each time t the evolving metric is complete and has bounded curvature, is called a singularity model if it is not flat and is of one of the following three types. Here, we have used Rm and $|Rm|$ to denote the Riemannian curvature tensor and its corresponding norm with respect to the evolving metric.

Type I: The solution exists for $-\infty < t < \Omega$ for some $0 < \Omega < +\infty$ and

$$|Rm| \leq \frac{\Omega}{\Omega - t}$$

everywhere with equality somewhere at $t = 0$;

Type II: The solution exists for $-\infty < t < +\infty$ and

$$|Rm| \leq 1$$

everywhere with equality somewhere at $t = 0$;

Type III: The solution exists for $-A < t < +\infty$ for some $0 < A < +\infty$ and

$$|Rm| \leq \frac{A}{A+t}$$

everywhere with equality somewhere at $t = 0$.

The singularity models of Type I and II are called ancient solutions in the sense that the existence time interval of the solution contains $(-\infty, 0]$.

Next, we recall the local injectivity radius estimate of Cheeger, Gromov and Taylor [**5**]. Let N be a complete Riemannian manifold of dimension m with $\lambda \leq$ sectional curvature of $N \leq \Lambda$ and let r be a positive constant satisfying $r \leq \frac{\pi}{4\sqrt{\Lambda}}$ if $\Lambda > 0$, then the injectivity radius of N at a point x is bounded from below as follows,

$$\mathrm{inj}_N(x) \geq r \cdot \frac{\mathrm{Vol}(B(x,r))}{\mathrm{Vol}(B(x,r)) + V_\lambda^m(2r)},$$

where $V_\lambda^m(2r)$ denotes the volume of a ball with radius $2r$ in the m dimensional model space V_λ^m with constant sectional curvature λ. In particular, it implies that for a complete Riemannian manifold N of dimension 4 with sectional curvature bounded between -1 and 1, the injectivity radius at a point x can be estimated as

$$\mathrm{inj}_N(x) \geq \frac{1}{2} \cdot \frac{\mathrm{Vol}(B(x,\frac{1}{2}))}{\mathrm{Vol}(B(x,\frac{1}{2})) + V} \tag{3.1}$$

for some absolute positive constant V. Furthermore, if in addition, N satisfies the maximal volume growth condition

$$\mathrm{Vol}\,(B(x,r)) \geq C_1 r^4\ , \quad 0 \leq r < +\infty,$$

then, (3.1) gives

$$\mathrm{inj}_N(x) \geq \beta > 0 \tag{3.2}$$

for some positive constant β depending only on C_1 and V.

Now, return to our setting. Let $(M, g_{\alpha\overline{\beta}})$ be a complete, non-compact Kähler surface satisfying the same assumptions as in the Main Theorem and let $g_{\alpha\overline{\beta}}(x,t)$ be the solution of the Ricci flow (2.1) on $M \times [0, T_{\max})$. Denote

$$R_{\max}(t) = \sup_{x\in M} R(x,t).$$

We have shown in Proposition 2.1 that the solution $g_{\alpha\overline{\beta}}(\cdot,t)$ satisfies the same maximal volume growth condition (i) as the initial metric.

Since condition (i) is invariant under rescaling of metrics, by a simple rescaling argument, we get the following injectivity radius estimate for the solution $g_{\alpha\overline{\beta}}(\cdot, t)$,

$$\text{inj}(M, g_{\alpha\overline{\beta}}(\cdot, t)) \geq \frac{\beta}{\sqrt{R_{\max}(t)}} \qquad \text{for } t \in [0, T_{\max}). \tag{3.3}$$

Then, by applying a result of Hamilton (see Theorems 16.4 and 16.5 in [**18**]), we know that there exists a sequence of dilations of the solution which converges to one of the singularity models of Type I, II or III. We will analyse this limit in Section 4.

We conclude this section with a lemma which will be very useful in our analysis of the Type I and Type II limits. We remark that in the case of compact shrinking Kähler–Ricci solitons with non-negative isotropic curvature condition, the lemma is proved by Ivey [**20**]. The following argument adapts that in [**20**].

Lemma 3.1. *Suppose $(\widetilde{M}, \widetilde{g}_{\alpha\overline{\beta}}(\cdot, t))$ is a complete ancient solution to the Ricci flow on a non-compact Kähler surface with non-negative and bounded holomorphic bisectional curvature for all time. Then, the curvature operator of the metric $\widetilde{g}_{\alpha\overline{\beta}}(\cdot, t)$ is non-negative definite everywhere on $\widetilde{M} \times (-\infty, 0]$.*

Proof. Choose a local orthonormal coframe $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ on an open set $U \subset \widetilde{M}$ so that $\omega_1 + \sqrt{-1}\omega_2$ and $\omega_3 + \sqrt{-1}\omega_4$ are $(1, 0)$ forms over U. Then, the self–dual forms

$$\varphi_1 = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4, \qquad \varphi_2 = \omega_2 \wedge \omega_3 + \omega_1 \wedge \omega_4,$$
$$\varphi_3 = \omega_3 \wedge \omega_1 + \omega_2 \wedge \omega_4$$

and the anti-self-dual forms

$$\psi_1 = \omega_1 \wedge \omega_2 - \omega_3 \wedge \omega_4, \qquad \psi_2 = \omega_2 \wedge \omega_3 - \omega_1 \wedge \omega_4,$$
$$\psi_3 = \omega_3 \wedge \omega_1 - \omega_2 \wedge \omega_4$$

form a basis of the space of 2 forms over U. In particular, $\varphi_1, \psi_1, \psi_2$ and ψ_3 give a basis for the space of $(1, 1)$ forms over U.

On a Kähler surface, it is well known that its curvature operator has image in the holonomy algebra $u(2)\,(\subset so(4))$ spanned by $(1, 1)$ forms. Thus, the curvature operator $\mathbf{M}$ in the basis $\{\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3\}$ has the following form,

$$\mathbf{M} = \begin{pmatrix} a & 0 & 0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & & & \\ b_2 & 0 & 0 & & A & \\ b_3 & 0 & 0 & & & \end{pmatrix},$$

where A is a 3×3 symmetric matrix.

Let V be a real tangent vector of the Kähler surface $\widetilde{M}$. Denote by J the complex structure of the Kähler surface $\widetilde{M}$. It is clear that the complex 2-plane $V \wedge JV$ is dual to $(1,1)$ form $u\varphi_1 + v_1\psi_1 + v_2\psi_2 + v_3\psi_3$ satisfying the decomposability condition $u^2 = v_1^2 + v_2^2 + v_3^2$. Then, after normalizing u to 1 by scaling, we see that the holomorphic bisectional curvature is non-negative if and only if

$$a + b \cdot v + b \cdot w + {}^t vAw \geq 0, \tag{3.4}$$

for any unit vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ in $\mathbf{R^3}$, where b is the vector (b_1, b_2, b_3) in $\mathbf{M}$.

Denote by $a_1 \leq a_2 \leq a_3$ the eigenvalues of A. Recall that $\operatorname{tr} A = a$ by the Bianchi identity, so if we choose v to be the eigenvector of A with eigenvalue a_3 and choose $w = -v$, (3.4) gives

$$a_1 + a_2 \geq 0. \tag{3.5}$$

In particular, we have $a_2 \geq 0$.

To proceed further, we need to adapt the maximum principle for parabolic equations on compact manifold in Hamilton [**16**] to $\widetilde{M}$. Let

$$(a_i)_{\min}(t) = \inf_{x \in \widetilde{M}} a_i(x,t)\ , \qquad i = 1,2,3$$

and

$$K = \sup_{(x,t) \in \widetilde{M} \times (-\infty, 0]} |Rm(x,t)|\,.$$

By assumption, the ancient solution $\widetilde{g}_{\alpha\overline{\beta}}(\cdot, t)$ has bounded holomorphic bisectional curvature, hence K is finite. Thus, by the derivative estimate of Shi [**33**] (see also Theorem 7.1 in [**18**]), the higher order derivatives of the curvature are also uniformly bounded. In particular, we can use the maximum principle of Cheng–Yau (see Proposition 1.6 in [**10**]) and then, as observed in [**18**], this implies that the maximum principle of Hamilton in [**16**] actually works for the evolution equations of the curvature of $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$ on the complete non-compact manifold $\widetilde{M}$. Thus, from [16], we obtain

$$\begin{aligned}\frac{d\,(a_1)_{\min}}{dt} &\geq ((a_1)_{\min})^2 + 2\,(a_2)_{\min}\cdot(a_3)_{\min}\\ &\geq 3\,((a_1)_{\min})^2\end{aligned}$$

by (3.5). Then, for fixed $t_0 \in (-\infty, 0)$ and $t > t_0$,

$$\begin{aligned}(a_1)_{\min}(t) &\geq \frac{1}{(a_1)_{\min}^{-1}(t_0) - 3\,(t-t_0)}\\ &\geq \frac{1}{-K^{-1} - 3\,(t-t_0)}.\end{aligned}$$

Letting $t_0 \to -\infty$, and observing that $(a_1)_{\min}(t)$ is non-decreasing, we get

$$a_1 \geq 0 \qquad \text{for all} \quad (x,t) \in \widetilde{M} \times (-\infty, 0] \tag{3.6}$$

i.e., $A \geq 0$.

Finally, to prove the non-negativity of the curvature operator $\mathbf{M}$, we recall its corresponding ODE from [16],

$$\frac{d\mathbf{M}}{dt} = \mathbf{M}^2 + \begin{pmatrix} 0 & 0 \\ 0 & A^{\#} \end{pmatrix},$$

where $A^{\#} \geq 0$ is the adjoint matrix of A.

Let m_1 be the smallest eigenvalue of the curvature operator $\mathbf{M}$. By using the maximum principle of Hamilton ([16] or [18]) again, we have

$$\frac{d\,(m_1)_{\min}}{dt} \geq (m_1)_{\min}^2\,,$$

where $(m_1)_{\min}(t) = \inf\limits_{x\in\widetilde{M}} m_1(x,t)$. Therefore, by the same reasoning in the derivation of (3.6), we have

$$m_1 \geq 0 \qquad \text{for all} \;\; (x,t) \in \widetilde{M} \times (-\infty, 0]. \tag{3.7}$$

So, $\mathbf{M} \geq 0$ and the proof of the lemma is completed. q.e.d.

4. Time decay estimate on curvature

Let $(M, g_{\alpha\overline{\beta}}(x))$ be a complete non-compact Kähler surface satisfying all the assumptions in the Main Theorem and $(M, g_{\alpha\overline{\beta}}(\cdot,t))$, $t \in [0, T_{\max})$ be the maximal solution of the Ricci flow (2.1) with $g_{\alpha\overline{\beta}}(\cdot)$ as the initial metric. Clearly, the maximal solution is of either one of the following types.

Type I: $T_{\max} < +\infty$ and $\sup\,(T_{\max} - t)\,R_{\max}(t) < +\infty$;
Type II(a): $T_{\max} < +\infty$ and $\sup\,(T_{\max} - t)\,R_{\max}(t) = +\infty$;
Type II(b): $T_{\max} = +\infty$ and $\sup\,tR_{\max}(t) = +\infty$;
Type III: $T_{\max} = +\infty$ and $\sup\,tR_{\max}(t) < +\infty$.

In Section 3, we have proved that the maximal solution satisfies the following injectivity radius estimate

$$\mathrm{inj}(M, g_{\alpha\overline{\beta}}(\cdot,t)) \geq \frac{\beta}{\sqrt{R_{\max}(t)}} \qquad \text{on} \quad [0, T_{\max})$$

for some $\beta > 0$. By applying a result of Hamilton (Theorems 16.4 and 16.5 in [**18**]), we know that there exists a sequence of dilations of the solution converging to a singularity model of the corresponding type. Note that since the maximal solution is complete and non-compact, the limit must also be complete and non-compact. The following is the main result of this section which says that this limit must be of Type III or equivalently, the maximal solution must be of Type III.

Theorem 4.1. *Let $(M, g_{\alpha\overline{\beta}}(x))$ be a complete non-compact Kähler surface as above. Then, the Ricci flow (2.1) with $g_{\alpha\overline{\beta}}(x)$ as the initial metric has a solution $g_{\alpha\overline{\beta}}(x,t)$ for all $t \in [0,+\infty)$ and $x \in M$. Moreover, the scalar curvature $R(x,t)$ of the solution satisfies*

$$0 \leq R(x,t) \leq \frac{C}{1+t} \qquad \text{on} \quad M \times [0,+\infty) \tag{4.1}$$

for some positive constant C.

Proof. We prove by contradiction. Thus, suppose the maximal solution is of Type I or Type II and let $(\widetilde{M}, \widetilde{g}_{\alpha\overline{\beta}}(x,t))$ be the limit of a sequence of dilations of the maximal solution which is then a singularity model of Type I or Type II respectively. After a study of its properties, we can blow down the singularity model and apply a dimension reduction argument to obtain the desired contradiction.

Now, recall that the maximal solution satisfies the maximal volume growth condition (i) by Proposition 2.1. Since condition (i) is also invariant under rescaling, we see that the singularity model $(\widetilde{M}, \widetilde{g}_{\alpha\overline{\beta}}(x,t))$ also satisfies the maximal volume growth condition, i.e.,

$$\mathrm{Vol}_t\left(\widetilde{B}_t(x,r)\right) \geq C_1 r^4 \qquad \text{for all} \quad 0 \leq r < +\infty \quad \text{and} \quad x \in \widetilde{M}, \tag{4.2}$$

where $\mathrm{Vol}_t\left(\widetilde{B}_t(x,r)\right)$ denotes the volume of the geodesic ball $\widetilde{B}_t(x,r)$ of radius r with center at x with respect to the metric $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$.

It is clear that the limit $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$ has non-negative holomorphic bisectional curvature. Thus, from Lemma 3.1, the curvature operator of the metric $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$ is non-negative definite everywhere.

Denote by $\widetilde{R}(\cdot,t)$ the scalar curvature of $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$ and $\widetilde{d}_t(x,x_0)$ the geodesic distance between two points $x,x_0\in\widetilde{M}$ with respect to the metric $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$. We claim that at time $t=0$, we have

$$\limsup_{\widetilde{d}_0(x,x_0)\to+\infty}\widetilde{R}(x,0)\widetilde{d}_0^2(x,x_0)=+\infty \tag{4.3}$$

for any fixed $x_0\in\widetilde{M}$.

Suppose not, that is the curvature of the metric $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,0)$ has quadratic decay. Now, by applying a result of Shi (see Theorem 8.2 in [**36**]), we know that the solution $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$ of the Ricci flow exists for all $t\in(-\infty,+\infty)$ and satisfies

$$\lim_{t\to+\infty}\sup\left\{\widetilde{R}(x,t)\,\middle|\; x\in\widetilde{M}\right\}=0. \tag{4.4}$$

On the other hand, by the Li–Yau–Hamilton inequality of Cao [**4**], we have

$$\frac{\partial\widetilde{R}}{\partial t}\geq 0\qquad\text{on}\quad \widetilde{M}\times(-\infty,+\infty)\ . \tag{4.5}$$

Thus, combining (4.4) and (4.5), we deduce that

$$\widetilde{R}\equiv 0\qquad\text{for all}\quad (x,t)\in\widetilde{M}\times(-\infty,+\infty)$$

and hence $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$ is flat for all $t\in(-\infty,+\infty)$. But, by definition, a singularity model cannot be flat. This proves our claim (4.3).

With the estimate (4.3), we can then apply a lemma of Hamilton (Lemma 22.2 in [**18**]) to find a sequence of points $x_j\in\widetilde{M}$, a sequence of radii $r_j>0$ and a sequence of positive numbers δ_j, $j=1,2,\ldots$, with $\delta_j\to 0$ such that

(a) $\widetilde{R}(x,0)\leq(1+\delta_j)\widetilde{R}(x_j,0)$ for all x in the ball $\widetilde{B}_0(x_j,r_j)$ of radius r_j centered at x_j with respect to the metric $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,0)$;

(b) $r_j^2\widetilde{R}(x_j,0)\to+\infty$;

(c) if $s_j=\widetilde{d}_0(x_j,x_0)$, then $\lambda_j=s_j/r_j\to+\infty$;

(d) the balls $\widetilde{B}_0(x_j, r_j)$ are disjoint.

Denote the minimum of the holomorphic sectional curvature of the metric $\widetilde{g}_{\alpha\overline{\beta}}(\cdot, 0)$ at x_j by h_j. We claim that the following holds

$$\varepsilon_j = \frac{h_j}{\widetilde{R}(x_j, 0)} \to 0 \qquad \text{as} \quad j \to +\infty. \tag{4.6}$$

Suppose not, there exists a subsequence $j_k \to +\infty$ and some positive number $\varepsilon > 0$ such that

$$\varepsilon_{j_k} = \frac{h_{j_k}}{\widetilde{R}(x_{j_k}, 0)} \geq \varepsilon \qquad \text{for all} \quad k = 1, 2, \dots \tag{4.7}$$

Since the solution $\widetilde{g}_{\alpha\overline{\beta}}(\cdot, t)$ is ancient, it follows from the Li-Yau-Hamilton inequality of Cao [4] that the scalar curvature $\widetilde{R}(x, t)$ is pointwisely non-decreasing in time. Then, by using the local derivative estimate of Shi [33] (or see Theorem 13.1 in [18]) and (a), (b), we have

$$\begin{aligned} \sup_{x \in \widetilde{B}_0(x_{j_k}, r_{j_k})} \left| \nabla \widetilde{R}m(x, 0) \right|^2 &\leq C_5 \widetilde{R}^2(x_j, 0) \left(\frac{1}{r_{j_k}^2} + \widetilde{R}(x_j, 0) \right) \\ &\leq 2C_5 \widetilde{R}^3(x_j, 0), \end{aligned} \tag{4.8}$$

where $\widetilde{R}m$ is the curvature tensor of $\widetilde{g}_{\alpha\overline{\beta}}$ and C_5 is a positive constant depending only on the dimension.

For any $x \in \widetilde{B}_0(x_{j_k}, r_{j_k})$, we obtain from (4.7) and (4.8) that the minimum of the holomorphic sectional curvature $h_{\min}(x)$ at x, satisfies

$$\begin{aligned} h_{\min}(x) &\geq h_{j_k} - \sqrt{2C_5} \widetilde{R}^{3/2}(x_{j_k}, 0) \widetilde{d}_0(x, x_{j_k}) \\ &\geq \widetilde{R}(x_{j_k}, 0) \left(\varepsilon - \sqrt{2C_5} \cdot \sqrt{\widetilde{R}(x_{j_k}, 0)} \cdot \widetilde{d}_0(x, x_{j_k}) \right) \\ &\geq \frac{\varepsilon}{2} \widetilde{R}(x_{j_k}, 0) \end{aligned} \tag{4.9}$$

if

$$\widetilde{d}_0(x, x_{j_k}) \leq \frac{\varepsilon}{2\sqrt{2C_5} \cdot \sqrt{\widetilde{R}(x_{j_k}, 0)}}.$$

Thus, from (a) and (4.9), there exists $k_0 > 0$ such that for any $k \geq k_0$ and

$$x \in \widetilde{B}_0 \left(x_{j_k}, \frac{\varepsilon}{2\sqrt{2C_5} \cdot \sqrt{\widetilde{R}(x_{j_k}, 0)}} \right),$$

we have

(4.10)
$$\frac{\varepsilon}{2}\widetilde{R}(x_{j_k},0) \le \text{holomorphic sectional curvature at } x \le 2\widetilde{R}(x_{j_k},0).$$

We have proved that the metric $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,0)$ has non-negative definite curvature operator. In particular, the sectional curvature is non-negative. Then, by the generalized Cohn–Vossen inequality in real dimension 4 [**14**], we have

$$\int_{\widetilde{M}} \Theta \le \chi\left(\widetilde{M}\right) < +\infty \tag{4.11}$$

where Θ is the Gauss–Bonnet–Chern integrand for the metric $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,0)$ and $\chi\left(\widetilde{M}\right)$ is the Euler number of the manifold $\widetilde{M}$ which has finite topology type by the soul theorem of Cheeger–Gromoll.

On the other hand, from the proof of Theorem 1.3 of Bishop–Goldberg [**2**] (see p. 523 of [**2**]), the inequality (4.10) implies that

$$\Theta(x) \ge C(\varepsilon)\widetilde{R}^2(x_{j_k},0) \quad \text{for all } x \in \widetilde{B}_0\left(x_{j_k}, \frac{\varepsilon}{2\sqrt{2C_5}\cdot\sqrt{\widetilde{R}(x_{j_k},0)}}\right), \tag{4.12}$$

where $C(\varepsilon)$ is some positive constant depending only on ε. Now, by combining (4.2), (b), (d), (4.11) and (4.12), we get

$$\begin{aligned}
+\infty > \chi\left(\widetilde{M}\right) &\ge \sum_{k=k_0}^{\infty} \int_{\widetilde{B}_0\left(x_{j_k}, \frac{\varepsilon}{2\sqrt{2C_5}\cdot\sqrt{\widetilde{R}(x_{j_k},0)}}\right)} \Theta \\
&\ge C(\varepsilon)\sum_{k=k_0}^{\infty} \widetilde{R}^2(x_{j_k},0)\cdot C_1\left(\frac{\varepsilon}{2\sqrt{2C_5}\cdot\sqrt{\widetilde{R}(x_{j_k},0)}}\right)^4 \\
&= C(\varepsilon)\sum_{k=k_0}^{\infty} \frac{C_1\varepsilon^4}{64C_5^2} \\
&= +\infty,
\end{aligned}$$

which is a contradiction. Hence, our claim (4.6) is proved.

Now, we are going to blow down the singularity model $(\widetilde{M}, \widetilde{g}_{\alpha\overline{\beta}}(x,t))$. For the above chosen x_j, r_j and δ_j, let x_j be the new origin O, dilate the space by a factor λ_j so that $\widetilde{R}(x_j,0)$ become 1 at the origin at $t=0$, and dilate in time by λ_j^2 so that it is still a solution to the Ricci flow. The balls $\widetilde{B}_0(x_j,r_j)$ are dilated to the balls centered at the origin of radii $\widetilde{r}_j = r_j^2\widetilde{R}(x_j,0)\to+\infty$ (by (b)). Since the scalar curvature of $\widetilde{g}_{\alpha\overline{\beta}}(x,t)$ is pointwise non-decreasing in time by the Li–Yau–Hamilton inequality, the curvature bounds on $\widetilde{B}_0(x_j,r_j)$ also give bounds for previous times in these balls. And the maximal volume growth estimate (4.2) and the local injectivity radius estimate of Cheeger, Gromov and Taylor [**5**] imply that

$$\mathrm{inj}_{\widetilde{M}}\left(x_j,\widetilde{g}_{\alpha\overline{\beta}}(\cdot,0)\right)\geq\frac{\beta}{\sqrt{\widetilde{R}(x_j,0)}},$$

for some positive constant β independent of j.

So, we have everything to take a limit for the dilated solutions. By applying the compactness theorem in [**17**] and combining (4.2), (4.6), (a) and (b), we obtain a complete non-compact solution, still denoted by $(\widetilde{M},\widetilde{g}_{\alpha\overline{\beta}}(x,t))$, for $t\in(-\infty,0]$ such that

(e) the curvature operator is still non-negative;
(f) $\widetilde{R}(x,t)\leq 1$, for all $x\in\widetilde{M}$, $t\in(-\infty,0]$, and $\widetilde{R}(0,0)=1$;
(g) $\mathrm{Vol}_t\left(\widetilde{B}_t(x,r)\right)\geq C_1r^4$ for all $x\in\widetilde{M}$, $0\leq r\leq+\infty$;
(h) there exists a complex 2–plane $V\wedge JV$ at the origin O so that at $t=0$, the corresponding holomorphic sectional curvature vanishes.

If we consider the universal covering of $\widetilde{M}$, the induced metric of $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$ on the universal covering is clearly still a solution to the Ricci flow and satisfies all of above (e)–(h). Thus, without loss of generality, we may assume that $\widetilde{M}$ is simply connected.

Next, by using the strong maximum principle on the evolution equation of the curvature operator of $\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t)$ as in [**16**] (see Theorem 8.3 of [**16**]), we know that there exists a constant $K>0$ such that on the time interval $-\infty<t<-K$, the image of the curvature operator of $(\widetilde{M},\widetilde{g}_{\alpha\overline{\beta}}(\cdot,t))$ is a fixed Lie subalgebra of $so(4)$ of constant rank on $\widetilde{M}$. Because $\widetilde{M}$ is Kähler, the possibilities are limited to $u(2)$, $so(2)\times so(2)$ or $so(2)$.

In the case $u(2)$, the sectional curvature is strictly positive. Thus, this case is ruled out by (h). In the cases $so(2)\times so(2)$ or $so(2)$, according to [**16**], the simply connected manifold $\widetilde{M}$ splits as a product $\widetilde{M}=\Sigma_1\times\Sigma_2$, where Σ_1 and Σ_2 are two Riemann surfaces with non-negative curvature (by (e)), and at least one of them, say Σ_1, has positive curvature (by (f)).

Denote by $\widetilde{g}^{(1)}_{\alpha\overline{\beta}}(\cdot,t)$ the corresponding metric on Σ_1. Clearly, it follows from (g) and standard volume comparison that for any $x \in \Sigma_1$, $t \in (-\infty,-K)$, we have

$$\mathrm{Vol}B_{\Sigma_1}(x,r) \geq C_6 r^2 \qquad \text{for } 0 \leq r < +\infty, \tag{4.13}$$

where both the geodesic ball $B_{\Sigma_1}(x,r)$ and the volume are taken with respect to the metric $\widetilde{g}^{(1)}_{\alpha\overline{\beta}}(\cdot,t)$ on Σ_1, C_6 is a positive constant depending only on C_1. Also, as the curvature of $\widetilde{g}^{(1)}_{\alpha\overline{\beta}}(x,t)$ is positive, it follows from Cohn–Vossen inequality that

$$\int_{\Sigma_1} \widetilde{R}^{(1)}(x,t)d\sigma_t \leq 8\pi, \tag{4.14}$$

where $\widetilde{R}^{(1)}(x,t)$ is the scalar curvature of $(\Sigma_1, \widetilde{g}^{(1)}_{\alpha\overline{\beta}}(x,t))$ and $d\sigma_t$ is the volume element of the metric $\widetilde{g}^{(1)}_{\alpha\overline{\beta}}(x,t)$.

Now, the metric $\widetilde{g}^{(1)}_{\alpha\overline{\beta}}(x,t)$ is a solution to the Ricci flow on the Riemann surface Σ_1 over the ancient time interval $(-\infty,-K)$. Thus, (4.13) and (4.14) imply that for each $t \in (-\infty,-K)$, the curvature of $\widetilde{g}^{(1)}_{\alpha\overline{\beta}}(x,t)$ has quadratic decay in the average sense of Shi [**36**] and then the *a priori* estimate of Shi (see Theorem 8.2 in [**36**]) implies that the solution $\widetilde{g}^{(1)}_{\alpha\overline{\beta}}(x,t)$ exists for all $t \in (-\infty,+\infty)$ and satisfies

$$\lim_{t\to+\infty} \sup\left\{ \widetilde{R}^{(1)}(x,t) \,\middle|\, x \in \Sigma_1 \right\} = 0. \tag{4.15}$$

Again, by the Li–Yau–Hamilton inequality of Cao [**4**], we know that $\widetilde{R}^{(1)}(x,t)$ is pointwisely non-decreasing in time. Therefore, we conclude that

$$\widetilde{R}^{(1)}(x,t) \equiv 0 \qquad \text{on} \quad \Sigma_1 \times (-\infty,+\infty).$$

This contradicts with the fact that $(\Sigma_1, \widetilde{g}_{\alpha\overline{\beta}}(\cdot,t))$ has positive curvature for $t < -K$. Hence, we have sought the desired contradiction and have completed the proof of Theorem 4.1. q.e.d.

5. Topology and steinness

In this section, we use the estimates obtained in the previous sections to study the topology and the complex structure of the Kähler surface in our Main Theorem. Our result is shown in the following theorem.

Theorem 5.1. *Suppose* $(M, g_{\alpha\overline{\beta}})$ *is a complete non-compact Kähler surface satisfying the assumptions in the Main Theorem. Then, M is homeomorphic to* $\mathbf{R}^4$ *and is a Stein manifold.*

The proof of Theorem 5.1 is exactly the same as in [**8**]. For the convenience of the readers, we give a sketch of the arguments and refer to the cited reference for details.

Sketch of proof. We evolve the metric $g_{\alpha\overline{\beta}}(x)$ by the Ricci flow (2.1). From Theorem 4.1, the solution $g_{\alpha\overline{\beta}}(x,t)$ exists for all $t \in [0, +\infty)$ and satisfies

$$R(x,t) \le \frac{C}{1+t} \qquad \text{on} \quad M \times [0, +\infty) \tag{5.1}$$

for some positive constant C. Also, Proposition 2.1 tells us that the volume growth condition (i) is preserved under the Ricci flow. By using the local injectivity radius estimate of Cheeger–Gromov–Taylor, this implies that

$$\operatorname{inj}\Big(M, g_{\alpha\overline{\beta}}(\cdot,t)\Big) \ge C_7(1+t)^{\frac{1}{2}} \qquad \text{for} \quad t \in [0, +\infty) \tag{5.2}$$

with some positive constant C_7.

Since the Ricci curvature of $g_{\alpha\overline{\beta}}(x,t)$ is positive for all $x \in M$ and $t \ge 0$, the Ricci flow equation (2.1) implies that the ball $B_t(x_0, \frac{C_7}{2}(1+t)^{\frac{1}{2}})$ of radius $\frac{C_7}{2}(1+t)^{\frac{1}{2}}$ with respect to the metric $g_{\alpha\overline{\beta}}(\cdot,t)$ contains the ball $B_0(x_0, \frac{C_7}{2}(1+t)^{\frac{1}{2}})$ of the same radius with respect to the initial metric $g_{\alpha\overline{\beta}}(\cdot,0)$. Combining this with (5.2), we deduce that

$$\pi_p(M, x_0) = 0 \qquad \text{for any} \quad p \ge 1$$

and

$$\pi_q(M, \infty) = 0 \qquad \text{for} \quad 1 \le q \le 2\ ,$$

where $\pi_q(M,\infty)$ is the qth homotopy group of M at infinity.

Thus, by the resolution of the generalized Poincaré conjecture on four manifolds by Freedman [**13**], we know that M is homeomorphic to $\mathbf{R}^4$.

Next, the injectivity radius estimate (5.2) also tells us that, for t large enough, the exponential maps provide diffeomorphisms between big geodesic balls $B_t(x_0, \frac{C_7}{2}(1+t)^{\frac{1}{2}})$ of M with big Euclidean balls on $\mathbf{C}^2$. The curvature estimate (5.1) and the standard comparison theorem implies that the distance function is plurisubharmonic in $B_t(x_0, c_7'(1+t)^{\frac{1}{2}})$ for some constant c_7' independent of t (see (7.3)). Since the metric is shrinking , the balls $B_t(x_0, c_7'(1+t)^{\frac{1}{2}})$ form an increasing one parameter family of exhausting pseudoconvex domains in M. Then, by the theorem of Docquier–Grauert (see Theorem(5.2) in [**37**]), M is Stein .

6. Space decay estimate on curvature and the Poincaré–Lelong equation

Let $(M, g_{\alpha\overline{\beta}})$ be a complete non-compact Kähler surface satisfying all the assumptions in the Main Theorem. The main purpose of this section is to establish the existence of a strictly plurisubharmonic function of logarithmic growth on M. To this end, we first prove a curvature decay estimate at infinity of the metric $g_{\alpha\overline{\beta}}$.

Theorem 6.1. *Let $(M, g_{\alpha\overline{\beta}})$ be a complete non-compact Kähler surface as above. Then, there exists a constant $C > 0$ such that for all $x \in M,\ r > 0$, we have*

$$\int_{B(x,r)} R(y)\,\frac{1}{d^2(x,y)}\,dy \le C\log(2+r). \tag{6.1}$$

Proof. Let $g_{\alpha\overline{\beta}}(x,t)$ be the solution of the Ricci flow (2.1) with $g_{\alpha\overline{\beta}}(x)$ as the initial metric. From Theorem 4.1, we know that the solution exists for all times and satisfies

$$R(x,t) \le \frac{C_8}{1+t} \qquad \text{on} \quad M\times[0,+\infty) \tag{6.2}$$

for some positive constant C_8.

Let

$$F(x,t) = \log\frac{\det\Big(g_{\alpha\overline{\beta}}(x,t)\Big)}{\det\Big(g_{\alpha\overline{\beta}}(x,0)\Big)}$$

be the function introduced in the proof of Proposition 2.1. Since

$$-\partial_\alpha\overline{\partial}_\beta \log \frac{\det\left(g_{\gamma\overline{\delta}}(\cdot,t)\right)}{\det\left(g_{\gamma\overline{\delta}}(\cdot,0)\right)} = R_{\alpha\overline{\beta}}(\cdot,t) - R_{\alpha\overline{\beta}}(\cdot,0),$$

after taking trace with the initial metric $g_{\alpha\overline{\beta}}(\cdot,0)$, we get

$$R(\cdot,0) = \triangle_0 F(\cdot,t) + g^{\alpha\overline{\beta}}(\cdot,0)R_{\alpha\overline{\beta}}(\cdot,t) \tag{6.3}$$

where $\triangle_0$ is the Laplace operator of the metric $g_{\alpha\overline{\beta}}(\cdot,0)$.

Since $(M, g_{\alpha\overline{\beta}}(\cdot,0))$ has positive Ricci curvature and maximal volume growth, it is well known (see [**32**]) that the Green function $G_0(x,y)$ of the initial metric $g_{\alpha\overline{\beta}}(\cdot,0)$ exists on M and satisfies the estimates

$$\frac{C_9^{-1}}{d_0^2(x,y)} \leq G_0(x,y) \leq \frac{C_9}{d_0^2(x,y)} \tag{6.4}$$

and

$$|\nabla_y G_0(x,y)|_0 \leq \frac{C_9}{d_0^3(x,y)} \tag{6.5}$$

for some positive constant C_9 depending only on C_1.

For any fixed $\overline{x}_0 \in M$ and any $\alpha > 0$, we denote

$$\Omega_\alpha = \{\, x \in M |\ G_0(\overline{x}_0, x) \geq \alpha\ \}.$$

Note that for $\alpha \geq \beta > 0$, we have $\Omega_\alpha \subset \Omega_\beta$. By (6.4), it is not hard to see

$$B_0\left(\overline{x}_0, \left(\frac{C_9^{-1}}{\alpha}\right)^{\frac{1}{2}}\right) \subset \Omega_\alpha \subset B_0\left(\overline{x}_0, \left(\frac{C_9}{\alpha}\right)^{\frac{1}{2}}\right). \tag{6.6}$$

Recall that F evolves by

$$\frac{\partial F(x,t)}{\partial t} = -R(x,t) \qquad \text{on} \quad M \times [0,+\infty).$$

Combining with (6.2), we obtain

$$0 \geq F(x,t) \geq -C_{10}\log(1+t) \qquad \text{on} \quad M \times [0,+\infty). \tag{6.7}$$

Multiplying (6.3) by $G_0(\overline{x}_0, x) - \alpha$ and integrating over Ω_α, we have

$$\begin{aligned}
(6.8)\qquad & \int_{\Omega_\alpha} R(x,0)\,(G_0(\overline{x}_0,x)-\alpha)\,dx \\
&= \int_{\Omega_\alpha} (\triangle_0 F(x,t))\,(G_0(\overline{x}_0,x)-\alpha)\,dx \\
&\quad + \int_{\Omega_\alpha} g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(\cdot,t)\,(G_0(\overline{x}_0,x)-\alpha)\,dx \\
&= -\int_{\partial\Omega_\alpha} F(x,t)\frac{\partial G_0(\overline{x}_0,x)}{\partial\nu}d\sigma - F(\overline{x}_0,t) \\
&\quad + \int_{\Omega_\alpha} g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(\cdot,t)\,(G_0(\overline{x}_0,x)-\alpha)\,dx \\
&\le C_{10}\left(1 + C_9^{\frac{5}{2}}\alpha^{\frac{3}{2}}\mathrm{Vol}_0\left(\partial\Omega_\alpha\right)\right)\log(1+t) \\
&\quad + \int_{\Omega_\alpha} g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(\cdot,t)G_0(\overline{x}_0,x)dx,
\end{aligned}$$

by (6.4) and (6.7). Here, we have used ν to denote the outer unit normal of $\partial\Omega_\alpha$.

From the coarea formula, we have

$$\begin{aligned}
\frac{1}{\alpha}\int_\alpha^{2\alpha} r^{\frac{3}{2}}\mathrm{Vol}_0\left(\partial\Omega_r\right)dr &\le 2^{\frac{3}{2}}\alpha^{\frac{1}{2}}\int_\alpha^{2\alpha}\int_{\partial\Omega_r}\left|\nabla G_0(\overline{x}_0,x)\right|_0 d\sigma\,|d\nu| \\
&\le 2^{\frac{3}{2}}C_9^{\frac{5}{2}}\alpha^2\mathrm{Vol}_0\left(\Omega_\alpha\right) \\
&\le 2^{\frac{3}{2}}C_9^{\frac{5}{2}}\alpha^2\mathrm{Vol}_0\left(B_0\left(\overline{x}_0,\left(\frac{C_9}{\alpha}\right)^{\frac{1}{2}}\right)\right) \\
&\le C_{11}
\end{aligned}$$

for some positive constant C_{11} by the standard volume comparison. Substitute this into (6.8) and integrate (6.8) from α to 2α, we get

$$\begin{aligned}
(6.9)\qquad & \int_{\Omega_{2\alpha}} R(x,0)\,(G_0(\overline{x}_0,x)-2\alpha)\,dx \le C_{10}\left(1 + C_9^{\frac{5}{2}}C_{11}\right)\log(1+t) \\
&\qquad + \int_{\Omega_\alpha} g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(x,t)G_0(\overline{x}_0,x)dx.
\end{aligned}$$

It is easy to see that

$$\int_{\Omega_{4\alpha}} R(x,0)G_0(\overline{x}_0,x)dx \leq 2\int_{\Omega_{2\alpha}} R(x,0)\left(G_0(\overline{x}_0,x)-2\alpha\right)dx$$

and by the equation Ricci flow (2.1), we also have

$$\begin{aligned}
&\int_0^t \int_{\Omega_\alpha} g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(x,t)G_0(\overline{x}_0,x)dxdt \\
&\quad = \int_{\Omega_\alpha} g^{\alpha\overline{\beta}}(x,0)\left(g_{\alpha\overline{\beta}}(x,0)-g_{\alpha\overline{\beta}}(x,t)\right)G_0(\overline{x}_0,x)dx \\
&\quad \leq 2\int_{\Omega_\alpha} G_0(\overline{x}_0,x)dx.
\end{aligned}$$

Thus, by integrating (6.9) in time from 0 to t and combining the above two inequalities, we get for any $t>0$,

$$\begin{aligned}
&\int_{\Omega_{4\alpha}} R(x,0)G_0(\overline{x}_0,x)dx \\
&\quad \leq 2C_{10}\left(1+C_9^{\frac{5}{2}}C_{11}\right)\log(1+t)+\frac{4}{t}\int_{\Omega_\alpha} G_0(\overline{x}_0,x)dx.
\end{aligned}$$

Finally, substituting (6.4) and (6.6) into the above inequality, we see that there exists some positive constant C_{12} such that for any $\overline{x}_0 \in M$, $t>0$ and $r>0$,

$$\int_{B_0(\overline{x}_0,r)} R(x,0)\frac{1}{d^2(\overline{x}_0,x)}dx \leq C_{12}\left(\log(1+t)+\frac{r^2}{t}\right). \tag{6.10}$$

Choose $t=r^2$, and we get the desired estimate. q.e.d.

Now, we can use the estimate (6.1) to solve the following Poincaré–Lelong equation on M

$$\sqrt{-1}\partial\overline{\partial}u = \mathrm{Ric} \tag{6.11}$$

to get the strictly plurisubharmonic function mentioned at the beginning of this section.

As in [**23**] or [**28**], we first study the corresponding Poisson equation on M

$$\triangle u = R. \tag{6.12}$$

After we solve the Poisson equation (6.12) with a solution of logarithmic growth, we will see that it is indeed a solution of the Poincaré–Lelong equation with logarithmic growth.

To solve (6.12), we first construct a family of approximate solutions u_r as follows.

For a fixed $x_0 \in M$ and $r > 0$, define $u_r(x)$ on $B(x_0, r)$ by

$$u_r(x) = \int_{B(x_0,r)} \left(G(x_0, y) - G(x, y)\right) R(y)dy,$$

where $G(x, y)$ is the Green function of the metric $g_{\alpha\overline{\beta}}$ on M. It is clear that

$$u_r(x_0) = 0 \qquad \text{and} \qquad \triangle\, u_r(x) = R(x) \quad \text{on} \quad B(x_0, r).$$

For $x \in B(x_0, \frac{r}{2})$, we write

$$\begin{aligned} u_r(x) &= \left(\int_{B(x_0,r)\setminus B(x_0,2d(x,x_0))} + \int_{B(x_0,2d(x,x_0))} \right) \\ &\qquad \cdot \left(G(x_0, y) - G(x, y)\right) R(y)dy \\ &:= I_1 + I_2. \end{aligned}$$

From (6.1), we see that

$$|I_2| \le C_{13} \log\left(2 + d(x, x_0)\right) \qquad \text{on} \quad B\left(x_0, \frac{r}{2}\right) \tag{6.13}$$

for some positive constant C_{13} independent of x_0, x and r.

To estimate I_1, we get from (6.5) that for $y \in B(x_0, r)\setminus B(x_0, 2d(x, x_0))$,

$$\begin{aligned} |G(x_0, y) - G(x, y)| &\le d(x, x_0) \cdot \sup_{z\in B(x_0,d(x,x_0))} |\nabla_z G(z, y)| \\ &\le C_9 d(x, x_0) \cdot \sup_{z\in B(x_0,d(x,x_0))} \frac{1}{d^3(z, y)} \\ &\le 8C_9 \frac{d(x, x_0)}{d^3(y, x_0)}. \end{aligned}$$

Thus, by (6.1), we have

$$
\begin{aligned}
|I_1| &\le 8C_9 d(x,x_0)\int_{B(x_0,r)\backslash B(x_0,2d(x,x_0))}\frac{R(y)}{d^3(y,x_0)}dy \\
&\le 8C_9 d(x,x_0)\sum_{k=1}^{\infty}\frac{1}{2^k d(x,x_0)} \\
&\quad\cdot\int_{B(x_0,2^{k+1}d(x,x_0))\backslash B(x_0,2^k d(x,x_0))}\frac{R(y)}{d^2(y,x_0)}dy \\
&\le 8C_9 C\sum_{k=1}^{\infty}\frac{1}{2^k}\log\left(2+2^{k+1}d(x,x_0)\right) \\
&\le C_{14}\log\left(2+d(x,x_0)\right)
\end{aligned}
\tag{6.14}
$$

for some positive constant C_{14}.

Hence, by combining (6.13) and (6.14), we deduce

$$
|u_r(x)| \le (C_{13}+C_{14})\log\left(2+d(x,x_0)\right) \tag{6.15}
$$

for any $r \ge 2d(x,x_0)$.

On the other hand, by taking the derivative of $u_r(x)$, we get

$$
\begin{aligned}
&|\nabla u_r(x)| \\
&\quad\le C_9\int_M\frac{R(y)}{d^3(x,y)}dy \\
&\quad\le C_9\left(\int_{B(x,1)}\frac{R(y)}{d^3(x,y)}dy+\sum_{k=1}^{\infty}\frac{1}{2^{k-1}}\int_{B(x,2^k)\backslash B(x,2^{k-1})}\frac{R(y)}{d^2(x,y)}dy\right) \\
&\quad\le C_9\left(C_{15}+\sum_{k=1}^{\infty}\frac{1}{2^{k-1}}C\log\left(2+2^k\right)\right) \\
&\quad= C_{16}.
\end{aligned}
\tag{6.16}
$$

Here, we have used (6.1) and (6.5), C_{15} and C_{16} are positive constants independent of r. Therefore, it follows from the Schauder theory of elliptic equations that there exists a sequence of $r_j \to +\infty$ such that $u_{r_j}(x)$ converges uniformly on compact subset of M to a smooth function

u satisfying

$$\text{(6.17)} \qquad \begin{cases} u(x_0) = 0 \quad \text{and} \quad \triangle u = R & \text{on } M, \\ |u(x)| \le (C_{13} + C_{14}) \log(2 + d(x, x_0)) & \text{for } x \in M, \\ |\nabla u(x)| \le C_{16} & \text{for } x \in M. \end{cases}$$

Thus, we have obtained a solution u of logarithmic growth to the Poisson equation (6.12) on M. In the following, we prove that u is actually a solution of the Poincaré–Lelong equation (6.11).

Recall the Bochner identity, with $\triangle u = R$

$$\text{(6.18)} \qquad \begin{aligned} \frac{1}{2} \triangle |\nabla u|^2 &= \left|\nabla^2 u\right|^2 + \langle \nabla u, \nabla R \rangle + \text{Ric}\,(\nabla u, \nabla u) \\ &\ge \left|\nabla^2 u\right|^2 + \langle \nabla u, \nabla R \rangle . \end{aligned}$$

For any $r > 0$ and any $\overline{x}_0 \in M$, by multiplying (6.18) by the cutoff function in (2.7) and integrating by parts, we get

$$\begin{aligned} \int_M \left|\nabla^2 u\right|^2 \varphi dx &\le \frac{1}{2} \int_M |\nabla u|^2 \cdot |\triangle \varphi| \, dx + \int_M \left|\nabla^2 u\right| \cdot R\varphi dx \\ &\quad + \int_M |\nabla u| \cdot R \cdot |\nabla \varphi| \, dx \\ &\le \frac{C_{16}^2}{2} \cdot \frac{C_3}{r^2} \int_M \varphi dx + \frac{1}{2} \int_M \left|\nabla^2 u\right|^2 \varphi dx \\ &\quad + \frac{1}{2} \int_M R^2 \varphi dx + C_{16} \cdot \left(\sup_M R\right) \cdot \frac{C_3}{r} \int_M \varphi dx. \end{aligned}$$

Thus,

$$\text{(6.19)} \qquad \begin{aligned} &\int_M \left|\nabla^2 u\right|^2 \varphi dx \\ &\quad \le \left(C_3 C_{16}^2 \cdot \frac{1}{r^2} + 2 C_{16} C_3 \left(\sup_M R\right) \frac{1}{r} \right) \int_M \varphi dx + \int_M R^2 \varphi dx. \end{aligned}$$

By (6.1), (2.7) and the standard volume comparison, we have

(6.20)
$$\begin{aligned}
&\int_M R^2\varphi dx\\
&\quad\le\left(\sup_M R\right)\int_M R(x)e^{-\left(1+\frac{d(x,\overline{x}_0)}{r}\right)}dx\\
&\quad\le\left(\sup_M R\right)\\
&\qquad\cdot\left(\int_{B(\overline{x}_0,r)}R(x)dx+\sum_{k=0}^{\infty}e^{-2^{k-1}}\int_{B(\overline{x}_0,2^{k+1}r)\setminus B(\overline{x}_0,2^k r)}R(x)dx\right)\\
&\quad\le C_{17}r^2\log(2+r)
\end{aligned}$$

and

$$\begin{aligned}
\int_M\varphi dx&\le\int_M e^{-\left(1+\frac{d(x,\overline{x}_0)}{r}\right)}dx\\
&\le\int_{B(\overline{x}_0,r)}dx+\sum_{k=0}^{\infty}e^{-2^{k-1}}\int_{B(\overline{x}_0,2^{k+1}r)\setminus B(\overline{x}_0,2^k r)}dx\\
&\le C_{17}r^4
\end{aligned}\tag{6.21}$$

for some positive constant C_{17} independent of r and $\overline{x}_0$.

Substituting these two inequalities into (6.19), we have

$$\frac{1}{r^4}\int_{B(\overline{x}_0,r)}\left|\nabla^2u\right|^2dx\le C_{18}\left(\frac{1}{r^2}+\frac{1}{r}+\frac{\log(2+r)}{r^2}\right)\tag{6.22}$$

for some positive constant C_{18} independent of r and $\overline{x}_0$.

Since the holomorphic bisectional curvature of $g_{\alpha\overline{\beta}}$ is positive, it was shown in [**23**] that the function $\left|\sqrt{-1}\partial\overline{\partial}u-\mathrm{Ric}\right|^2$ is subharmonic on M. Then, by the mean value inequality and (6.20), (6.22), we have

$$\begin{aligned}
\left|\sqrt{-1}\partial\overline{\partial}u-\mathrm{Ric}\,\right|^2(\overline{x}_0)&\le\frac{C_{19}}{r^4}\int_{B(\overline{x}_0,r)}\left|\sqrt{-1}\partial\overline{\partial}u-\mathrm{Ric}\,\right|^2(x)dx\\
&\le\frac{2C_{19}}{r^4}\int_{B(\overline{x}_0,r)}\left(\left|\nabla^2u\right|^2+R^2\right)dx\\
&\le C_{20}\left(\frac{1}{r^2}+\frac{1}{r}+\frac{\log(2+r)}{r^2}\right)
\end{aligned}$$

for some positive constants $C_{19},\ C_{20}$ independent of r and $\overline{x}_0$. Since $\overline{x}_0 \in M$ and $r > 0$ are arbitrary, by letting $r \to +\infty$ we know that

$$\sqrt{-1}\partial\overline{\partial}u = \text{Ric} \qquad \text{on} \quad M.$$

In summary, we have proved the following result.

Proposition 6.2. *Suppose $(M, g_{\alpha\overline{\beta}})$ is a complete non-compact Kähler surface satisfying all the assumptions in the Main Theorem. Then, there exists a strictly plurisubharmonic function $u(x)$ on M satisfying the Poincaré–Lelong equation (6.11) with the estimate*

$$|u(x)| \leq C \log(2 + d(x, x_0)) \qquad \textit{for all } x \in M$$

for some positive constant C.

7. Uniform estimates on multiplicity and the number of components of an "algebraic" divisor

Let $(M, g_{\alpha\overline{\beta}})$ be a complete non-compact Kähler surface satisfying all the assumptions in the Main Theorem. In this section, we will consider the algebra $P(M)$ of holomorphic functions of polynomial growth on M. We first construct $f_1,\ f_2$ in $P(M)$ which are algebraically independent over $\mathbf{C}$.

In the previous section, by solving the Poincaré–Lelong equation, we have obtained a strictly plurisubharmonic function u on M of logarithmic growth. As shown in [**24**], the existence of non-trivial functions in the algebra $P(M)$ then follows readily from the L^2 estimates of the $\overline{\partial}$ operator on complete Kähler manifold of Andreotti–Vesentini [**1**] and Hörmander [**19**]. For completeness, we give the proof as follows.

Let $x \in M$ and $\left\{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 < 1\right\}$ be local holomorphic coordinates at x with $z_1(x) = z_2(x) = 0$. Let η be a smooth cut-off function on $\mathbf{C^2}$ with $\operatorname{Supp}\eta \subset\subset \left\{|z_1|^2 + |z_2|^2 < 1\right\}$ and $\eta \equiv 1$ on $\left\{|z_1|^2 + |z_2|^2 < \frac{1}{4}\right\}$. Then, the function

$$\eta \log|z| = \eta\,(z_1, z_2) \log\left(|z_1|^2 + |z_2|^2\right)^{\frac{1}{2}}$$

is globally defined on M and is smooth except at x. Furthermore, the $(1, 1)$ form $\partial\overline{\partial}\,(\eta \log|z|)$ is bounded from below. Since u is strictly plurisubharmonic, we can choose a sufficiently large positive constant C such that

$$v = Cu + 6\eta\,\log|\,z|$$

is strictly plurisubharmonic on M. Then, for any non-zero tangent vector ξ of type $(1,0)$ on M, we have

$$\left\langle \sqrt{-1}\partial\overline{\partial}v + \mathrm{Ric},\ \xi \wedge \overline{\xi} \right\rangle > 0.$$

Now, $\overline{\partial}\,(\eta z_i)$, i=1,2 , is a $\overline{\partial}$ closed $(0,1)$ form on the complete Kähler manifold M. Using the standard L^2-estimates of $\overline{\partial}$ operator (cf. Theorem 2.1 in [**24**]), there exists a smooth function u_i such that

$$\overline{\partial}u_i = \overline{\partial}(\eta z_i), \qquad i = 1, 2$$

and

$$\int_M |u_i|^2\, e^{-v}dx \leq \frac{1}{c}\int_M \left|\overline{\partial}(\eta z_i)\right|^2 e^{-v}dx,$$

where c is a positive constant satisfying

$$\left\langle \sqrt{-1}\partial\overline{\partial}v + \mathrm{Ric}, \xi \wedge \overline{\xi} \right\rangle \geq c|\xi|^2,$$

whenever ξ is a tangent vector on $\mathrm{Supp}\,\eta$. First of all, this estimate implies that u_i is of polynomial growth by standard elliptic estimates, as the weight function v is of logarithmic growth. Secondly, because of the singularity of $6\log|z|$ at x, it forces the function u_i and its first order derivative to vanish at x. Therefore, the holomorphic functions $f_1 = u_1 - \eta z_1$ and $f_2 = u_2 - \eta z_2$ define a local biholomorphism at x. Clearly, they are algebraically independent over $\mathbf{C}$. This concludes our construction.

For later use, we also point out here that, as a consequence of the above argument, the algebra $P(M)$ separates points on M. In other words, for any $x_1, x_2 \in M$ with $x_1 \neq x_2$, there exists $f \in P(M)$ such that $f(x_1) \neq f(x_2)$.

Before we can state our main result in this section, we need the following definition. For a holomorphic function $f \in P(M)$, we define the degree of f, $\deg(f)$, to be the infimum of all q for which the following inequality holds

$$|f(x)| \leq C(q)\,(1 + d^q(x, x_0)) \qquad \text{for all } x \in M,$$

where x_0 is some fixed point in M and $C(q)$ is some positive constant depending on q.

Our main result in this section is the following uniform bound on the multiplicity of the zero divisor of a function $f \in P(M)$ by its degree.

Proposition 7.1. *Let $(M, g_{\alpha\overline{\beta}})$ be a complete non-compact Kähler surface as above. For $f \in P(M)$, let*

$$[V] = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2$$

be the zero divisor, counting multiplicity, determined by f. Then, there exists a positive constant C, independent of f, such that

$$mult([V], x) \le C\deg(f)$$

holds for all $x \in M$.

Proof. Recall that the Ricci flow (2.1) with $g_{\alpha\overline{\beta}}(x)$ as initial metric has a solution $g_{\alpha\overline{\beta}}(x,t)$ for all times $t \in [0, +\infty)$ and satisfies the following estimates

$$R(x,t) \le \frac{C}{1+t} \tag{7.1}$$

and

$$\mathrm{inj}\left(M, g_{\alpha\overline{\beta}}(\cdot,t)\right) \ge C_7(1+t)^{\frac{1}{2}} \tag{7.2}$$

on $M \times [0, +\infty)$.

Let d_t be the distance function from an arbitrary fixed point $\overline{x}_0 \in M$ with respect to the metric $g_{\alpha\overline{\beta}}(\cdot,t)$. By the standard Hessian comparison theorem (see [**32**]), we have, for any unit real vector v orthogonal to the radial direction $\partial/\partial d_t$,

$$\frac{\sqrt{\frac{\alpha_1}{t}}d_t}{\tan\left(\sqrt{\frac{\alpha_1}{t}}d_t\right)} \le \mathrm{Hess}\left(d_t^2\right)(v,v) \le \frac{\sqrt{\frac{\alpha_1}{t}}d_t}{\tanh\left(\sqrt{\frac{\alpha_1}{t}}d_t\right)}, \quad \text{when} \quad d_t \le \frac{\pi}{4}\sqrt{\frac{t}{\alpha_1}}.$$

Here, α_1 is some positive constant depending only on the constants C and C_7 in (7.1) and (7.2). Hence, for any unit vector $\widetilde{v}$, we have

$$\begin{aligned}\frac{\sqrt{\frac{\alpha_1}{t}}d_t}{\tan\left(\sqrt{\frac{\alpha_1}{t}}d_t\right)} &\le \mathrm{Hess}\left(d_t^2\right)(\widetilde{v},\widetilde{v}) + \mathrm{Hess}\left(d_t^2\right)(J\widetilde{v},J\widetilde{v}) \\ &\le \frac{2\sqrt{\frac{\alpha_1}{t}}d_t}{\tanh\left(\sqrt{\frac{\alpha_1}{t}}d_t\right)},\end{aligned}$$

whenever $d_t \leq \frac{\pi}{4}\sqrt{\frac{t}{\alpha_1}}$. Since M is Kähler, the above expression is equivalent to

$$\begin{aligned}\frac{\sqrt{\frac{\alpha_1}{t}}d_t}{\tan\left(\sqrt{\frac{\alpha_1}{t}}d_t\right)}\omega_t &\leq \sqrt{-1}\partial\overline{\partial}d_t^2 \\ &\leq \frac{2\sqrt{\frac{\alpha_1}{t}}d_t}{\tanh\left(\sqrt{\frac{\alpha_1}{t}}d_t\right)}\omega_t.\end{aligned}$$

In particular, we have

$$\frac{1}{2}\omega_t \leq \sqrt{-1}\partial\overline{\partial}d_t^2 \leq 4\omega_t, \qquad \text{whenever} \quad d_t \leq \frac{\pi}{4}\sqrt{\frac{t}{\alpha_1}}. \tag{7.3}$$

Here, ω_t is the Kähler form of the metric $g_{\alpha\overline{\beta}}(\cdot,t)$.

We next claim that(we are grateful to Professor L.F. Tam for this suggestion.)

$$\sqrt{-1}\partial\overline{\partial}\log\tan\left(\sqrt{\frac{\alpha_1}{t}}\frac{d_t}{2}\right) \geq 0, \qquad \text{whenever} \quad d_t \leq \frac{\pi}{4}\sqrt{\frac{t}{\alpha_1}}. \tag{7.4}$$

In fact, after recalling, we may assume that the sectional curvature of $g_{\alpha\overline{\beta}}(\cdot,t)$ is less than 1 and $\sqrt{\frac{\alpha_1}{t}} = 1$. Then, by the standard Hessian comparison, we have

$$\operatorname{Hess}(d_t)(v,v) \geq \frac{1}{\tan d_t}\left(|v|_t^2 - \left\langle v, \frac{\partial}{\partial d_t}\right\rangle_t^2\right)$$

for any vector v and $d_t \leq \frac{\pi}{4}$. Thus, by a direct computation,

$$\begin{aligned}&\operatorname{Hess}\left(\log\tan\left(\frac{d_t}{2}\right)\right)(v,v) + \operatorname{Hess}\left(\log\tan\left(\frac{d_t}{2}\right)\right)(Jv,Jv) \\ &\geq \frac{1}{(\tan d_t)\tan\left(\frac{d_t}{2}\right)}\left(1+\tan d_t\right)|v|_t^2 \\ &\geq 0,\end{aligned}$$

which is our claim (7.4).

Now for any $0 < b < a < \frac{\pi}{8}\sqrt{\frac{t}{\alpha_1}}$, it follows from Stoke's theorem that

$$\begin{aligned} 0 &\le \sqrt{-1} \int\limits_{\{b \le d_t \le a\}} [V] \wedge \partial\overline{\partial} \log \tan\left(\sqrt{\frac{\alpha_1}{t}}\frac{d_t}{2}\right) \\ &= \sqrt{-1} \int\limits_{\{d_t = a\}} [V] \wedge \overline{\partial} \log \tan\left(\sqrt{\frac{\alpha_1}{t}}\frac{d_t}{2}\right) \\ &\quad - \sqrt{-1} \int\limits_{\{d_t = b\}} [V] \wedge \overline{\partial} \log \tan\left(\sqrt{\frac{\alpha_1}{t}}\frac{d_t}{2}\right). \end{aligned}$$

Then, it is not hard to see that for $0 < b < a < \frac{\pi}{8}\sqrt{\frac{t}{\alpha_1}}$,

$$\frac{\sqrt{-1}}{a^2} \int\limits_{\{d_t = a\}} [V] \wedge \overline{\partial}\left(d_t^2\right) \ge \frac{1}{2} \cdot \frac{\sqrt{-1}}{b^2} \int\limits_{\{d_t = b\}} [V] \wedge \overline{\partial}\left(d_t^2\right). \tag{7.5}$$

Using Stoke's theorem on the right-hand side of (7.5) and letting $b \to 0$, it follows from the inequality of Bishop–Lelong that

$$\frac{\sqrt{-1}}{a^2} \int\limits_{\{d_t = a\}} [V] \wedge \overline{\partial}\left(d_t^2\right) \ge \alpha_2 \mathrm{mult}\left([V], \overline{x}_0\right) \tag{7.6}$$

for some positive absolute constant α_2.

Then, by (7.3), (7.5), (7.6) and Stoke's theorem, we have

(7.7)

$$\begin{aligned} &\frac{1}{a^2} \int_{B_t(\overline{x}_0, a) \setminus B_t(\overline{x}_0, \frac{a}{2})} [V] \wedge \omega_t \\ &\quad \ge \frac{1}{4a^2} \int_{B_t(\overline{x}_0, a) \setminus B_t(\overline{x}_0, \frac{a}{2})} [V] \wedge \sqrt{-1}\partial\overline{\partial}\left(d_t^2\right) \\ &\quad = \frac{\sqrt{-1}}{4}\left(\frac{1}{a^2}\int_{\{d_t = a\}} [V] \wedge \overline{\partial}\left(d_t^2\right) - \frac{1}{4 \cdot \left(\frac{a}{2}\right)^2}\int_{\{d_t = \frac{a}{2}\}} [V] \wedge \overline{\partial}\left(d_t^2\right)\right) \\ &\quad = \frac{\sqrt{-1}}{8}\left(\frac{1}{a^2}\int_{\{d_t = a\}} [V] \wedge \overline{\partial}\left(d_t^2\right) - \frac{1}{2 \cdot \left(\frac{a}{2}\right)^2}\int_{\{d_t = \frac{a}{2}\}} [V] \wedge \overline{\partial}\left(d_t^2\right)\right) \\ &\qquad + \frac{\sqrt{-1}}{8a^2}\int_{\{d_t = a\}} [V] \wedge \overline{\partial}\left(d_t^2\right) \end{aligned}$$

$$\geq \frac{\sqrt{-1}}{8a^2}\int_{\{d_t=a\}}[V]\wedge\overline{\partial}\left(d_t^2\right)$$
$$\geq \frac{\alpha_2}{8}\text{mult}\left([V],\overline{x}_0\right)$$

for $0<a<\frac{\pi}{8}\sqrt{\frac{t}{\alpha_1}}$.

For the function $f\in P(M)$, let $\widetilde{x}_0$ be a point close to $\overline{x}_0$ such that $f(\widetilde{x}_0)\neq 0$. By definition, for any $\delta>0$, there exists a constant $C(\delta)>0$ such that

$$|f(x)|\leq C(\delta)\left(1+d_0^{\deg(f)+\delta}(x,\widetilde{x}_0)\right)\qquad \text{on}\quad M. \tag{7.8}$$

By equation (2.1) and estimate (7.1), we have

$$\frac{\partial g_{\alpha\overline{\beta}}(\cdot,t)}{\partial t}\geq -R(\cdot,t)g_{\alpha\overline{\beta}}(\cdot,t)$$
$$\geq -\frac{C}{1+t}g_{\alpha\overline{\beta}}(\cdot,t),$$

which implies that

$$g_{\alpha\overline{\beta}}(\cdot,0)\leq(1+t)^C g_{\alpha\overline{\beta}}(\cdot,t)\qquad\text{for any}\quad t>0.$$

Hence, (7.8) becomes

$$|f(x)|\leq C(\delta)\left\{1+\left[(1+t)^{\frac{C}{2}}d_t(x,\widetilde{x}_0)\right]^{\deg(f)+\delta}\right\}\qquad\text{on}\quad M. \tag{7.9}$$

We now fix $t=\frac{\alpha_1}{\pi^2}4^{K+8}$ for each positive interger K. Set

$$v_K(x)=\int_{B_t(\widetilde{x}_0,2^K)}-G_t^{(K)}(x,y)\triangle_t\log|f(y)|^2\cdot\omega_t^2(y),$$

where $G_t^{(K)}$ is the positive Green function with value zero on the boundary $\partial B_t(\widetilde{x}_0,2^K)$ with respect to the metric $g_{\alpha\overline{\beta}}(\cdot,t)$. The function $\log|f|^2-v_K$ is then harmonic on $B_t(\widetilde{x}_0,2^K)$. From the maximum principle and (7.9), we have

$$\log\left(|f(\widetilde{x}_0)|^2\right)-v_K(\widetilde{x}_0)\leq\sup_{x\in\partial B_t(\widetilde{x}_0,2^K)}\log|f(x)|^2 \tag{7.10}$$
$$\leq C_{19}K\left(\deg(f)+\delta\right)+C'(\delta)$$

for some positive constants C_{19}, $C'(\delta)$ independent of K, f, and $\overline{x}_0$.

On the other hand, since the volume growth condition (i) is preserved for all times, by virtue of (6.4) (cf. Proposition 1.1 in [**24**]), we have

$$
\begin{aligned}
-v_K(\widetilde{x}_0) &\geq \frac{1}{C_9}\int_{B_t(\widetilde{x}_0,2^K)} \frac{1}{d_t^2(x,\widetilde{x}_0)} \triangle_t \log|f(x)|^2 \cdot \omega_t^2(x) \\
&\geq \frac{1}{C_9}\sum_{j=1}^{K}\left(\frac{1}{2^j}\right)^2 \int_{B_t(\widetilde{x}_0,2^j)\setminus B_t(\widetilde{x}_0,2^{j-1})} \triangle_t \log|f(x)|^2 \cdot \omega_t^2(x).
\end{aligned}
$$

Then, by (7.7) and the fact that $\widetilde{x}_0$ is arbitrarily close to $\overline{x}_0$,

$$
-v_K(\widetilde{x}_0) \geq C_{20}\, K \,\mathrm{mult}\,([V], \overline{x}_0) \tag{7.11}
$$

for some positive constant C_{20} independent of K, f and $\overline{x}_0$.

Therefore, by combining (7.10) and (7.11) and letting $K \to +\infty$ and then $\delta \to 0$, we obtain

$$
\mathrm{mult}\,([V], \overline{x}_0) \leq C_{21}\deg(f),
$$

where C_{21} is some positive constant independent of f and $\overline{x}_0$. q.e.d.

A modified version of the proof of Proposition 7.1 gives the uniform bound on the number of irreducible components of $[V]$.

Proposition 7.2. *Suppose $(M, g_{\alpha\overline{\beta}})$ is a complete non-compact Kähler surface as assumed in Proposition 7.1. Let f be a holomorphic function of polynomial growth,*

$$
[V] = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2
$$

be the corresponding zero divisor determined by f. Then, the number of irreducible components of $[V]$ is not bigger than $C\deg(f)$ for the same positive constant C as in Proposition 7.1.

Proof. Let $g_{\alpha\overline{\beta}}(\cdot,t)$ be the evolving metric to the Ricci flow with $g_{\alpha\overline{\beta}}(\cdot)$ as the initial metric and $[V_1]$, $[V_2]$, $\ldots$, $[V_l]$ be any l distinct irreducible components of $[V]$. Fix a constant $a > 0$ such that the intersection of the smooth points of $[V_i]$ with $B_0(\overline{x}_0, a)$ is non-empty for each $0 \leq i \leq l$.

Choose $t = \frac{\alpha_1}{\pi^2}4^{K+8}a^2$ for each positive integer K. As the manifold M is Stein by Theorem 4.1, each $[V_i]$ must be non-compact. Hence, for $j = 1, 2, \ldots, K$, we have

$$
[V_i] \cap \left(B_t(\overline{x}_0, 2^j a)\setminus B_t(\overline{x}_0, 2^{j-1}a)\right) \neq \emptyset
$$

and there exists a point $x_j \in [V_i]$ with $d_t(x_j, \overline{x}_0) = \frac{3}{2}2^{j-1}a$ in the middle of $B_t(\overline{x}_0, 2^j a) \setminus B_t(\overline{x}_0, 2^{j-1}a)$. The triangle inequality says

$$B_t(x_j, 2^{j-2}a) \subset \left(B_t(\overline{x}_0, 2^j a) \setminus B_t(\overline{x}_0, 2^{j-1}a)\right).$$

Applying a slight variant of (7.7) to $[V_i]$, we have

$$\begin{aligned}\frac{1}{(2^{j-2}a)^2}\int_{B_t(x_j,2^{j-2}a)}[V_i]\wedge\omega_t &\geq \frac{\alpha_2}{8}\text{mult}\left([V_i], x_j\right)\\ &\geq \frac{\alpha_2}{8}.\end{aligned}$$

Since $\sum_{i=1}^{l}[V_i]$ is only a part of the divisor $[V]$, we get

$$\frac{1}{(2^{j-2}a)^2}\int_{B_t(\overline{x}_0,2^j a)\setminus B_t(\overline{x}_0,2^{j-1}a)} \triangle_t \log|f(x)|^2\cdot\omega_t^2(x) \geq \frac{\alpha_2}{8}\,l.$$

The subsequent argument is then exactly as in the proof of Proposition 7.1. In the end, we have

$$C_{20}K\cdot l \leq -\log\left(\left|f\left(\widetilde{x}_0\right)\right|^2\right) + C_{19}K\left(\deg(f)+\delta\right) + C'(\delta).$$

Letting $K\to+\infty$ and then $\delta\to 0$, we get the desired estimate. q.e.d.

8. Proof of the main theorem

In this section, we will basically follow the approach of Mok [**24**], [**26**] to accomplish the proof of the main theorem. Let M be a Kähler surface as assumed in the Main Theorem. Recall that $P(M)$ stands for the algebra of holomorphic functions of polynomial growth on M. Let $R(M)$ be the quotient field of $P(M)$. By an abuse of terminology, we will call it the field of rational functions on M.

In Section 7, we showed that there exist two functions $f_1,\ f_2 \in P(M)$ giving local holomorphic coordinates at any given point $x\in M$, and that the algebra $P(M)$ separates points on M. Moreover, we obtained the following basic multiplicity estimate

$$\text{mult}\left([V], x\right) \leq C\deg(f) \tag{8.1}$$

for all $x\in M$ and $f\in P(M)$, where

$$[V] = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2$$

is the zero divisor of f and C is a constant independent of f and x. Thus, by combining these facts with the classical arguments of Poincaré and Siegel, we have (cf. the proof of Proposition 5.1 in [24])

$$\dim_{\mathbf{C}} H_p \leq 10^3 C p^2, \tag{8.2}$$

where H_p denotes the vector space of all holomorphic functions with degree $\leq p$, and the field of rational functions $R(M)$ is a finite extension field over $\mathbf{C}(f_1, f_2)$ for some algebraically independent holomorphic functions $f_1,\ f_2 \in P(M)$ over $\mathbf{C}$. By the primitive element theorem, we can then write

$$R(M) = \mathbf{C}\left(f_1, f_2, \frac{f_3}{f_4}\right)$$

for some $f_3,\ f_4 \in P(M)$.

Now, consider the mapping $F : M \to \mathbf{C}^4$ defined by

$$F = (f_1, f_2, f_3, f_4)\,.$$

Since $R(M)$ is a finite extension field of $\mathbf{C}(f_1\ f_2)$, f_3 and f_4 satisfy equations of the form

$$f_3^p + \sum_{j=0}^{p-1} P_j(f_1, f_2) f_3^j = 0,$$

$$f_4^q + \sum_{j=0}^{q-1} Q_j(f_1, f_2) f_4^j = 0,$$

where $P_j(w_1, w_2)$, $Q_j(w_1, w_2)$ are rational functions of $w_1,\ w_2$. After clearing denominators, we see that f_1, f_2, f_3, f_4 satisfy polynomial equations

$$P(f_1, f_2, f_3, f_4) = 0 \qquad \text{and} \qquad Q(f_1, f_2, f_3, f_4) = 0.$$

Let Z_0 be the subvariety of $\mathbf{C^4}$ defined by

$$Z_0 = \left\{ (w_1, w_2, w_3, w_4) \in \mathbf{C^4} \;\middle|\; \begin{array}{c} P(w_1, w_2, w_3, w_4) = 0 \\ Q(w_1, w_2, w_3, w_4) = 0 \end{array} \right\},$$

and let Z be the connected component of Z_0 containing $F(M)$. It is clear that $\dim_{\mathbf{C}} Z = 2$.

In the following, we will show that F is an "almost injective" and "almost surjective" map to Z and we can desingularize F to obtain a biholomorphic map from M onto a quasi-affine algebraic variety by adjoining a finite number of holomorphic functions of polynomial growth.

First of all, we claim that Z is irreducible and F is "almost injective", i.e., there exists a subvariety V of M such that $F|_{M\backslash V} : M\backslash V \to Z$ is an injective locally biholomorphic mapping. Indeed, take V to be the union of $F^{-1}(\mathrm{Sing}\,(Z))$ and the branching locus of F, here Sing(Z) denotes the singular set of Z. It is clear that F is locally biholomorphic on $M\backslash V$. That F is also injective there follows from the fact that $P(M)$ separates points and $f_1, \dots, f_4$ generate $P(M)$. To see the irreducibility of Z, note that $M\backslash F^{-1}(\mathrm{Sing}\,(Z))$ is connected and hence $\overline{F(M\backslash F^{-1}(\mathrm{Sing}\,(Z))}$ is irreducible (as its set of smooth points is connected). Since $F(M) \subset \overline{F(M\backslash F^{-1}(\mathrm{Sing}\,(Z))}$, by the definition of Z, it must be irreducible.

Next, we come to the "almost surjectivity" of F, i.e., there exists an algebraic subvariety T of Z such that $F(M)$ contains $Z\backslash T$. The method of Mok [**24**] in proving the almost surjectivity of F is to solve an ideal problem for each $x \in Z\backslash T_0$ missed by F, where T_0 is some fixed algebraic subvariety of Z containing the singular set of Z. The solution of the ideal problem gives a holomorphic function $f_x \in P(M)$ with degree bounded independent of x which will correspond to a rational function on $\mathbf{C}^4$ with pole set passes through x. Then, the almost surjectivity of F follows. Otherwise, one could select an infinite number of linearly independent f_x's contradicting the finite dimensionality of the space of holomorphic functions with polynomial growth of some fixed degree, c.f. (8.2).

In [**24**], Mok used the solution u of the Poincaré–Lelong equation as the weight function in the Skoda's estimates for solving the ideal problem. In his case, because of his curvature quadratic decay condition, the growth of u is bounded both from above and below by the logarithm of the distance function on M. This does not work in our case because we do not have the luxury of the lower bound of u. However, thanks to the Steinness of M by Theorem 5.1, we can adapt the argument of Mok in [**26**] to choose another weight function by resorting to Oka's theory of pseudoconvex Riemann domains.

Before we carry out the above procedures in proving the almost surjectivity of F, we first need to construct a non-trivial holomorphic $(2,0)$ vector field of polynomial growth on M.

Consider the anticanonical line bundle, $\mathbf{K}^{-1}$, on M equipped with the induced Hermitian metric, its curvature form $\Omega(\mathbf{K}^{-1})$ is then simply the Ricci form of M. Let u be the strictly plurisubharmonic function of logarithmic growth obtained in Proposition 6.2. For any given point

$\overline{x}_0 \in M$, let $\{z_1, z_2\}$ be local holomorphic coordinates at $\overline{x}_0$. Choose a smooth cutoff function η supporting in this local holomorphic coordinate chart with value 1 in a neighborhood of $\overline{x}_0$. We study the following $\overline{\partial}$ equation for the sections of $\mathbf{K}^{-1}$ on M,

$$\overline{\partial} S = \overline{\partial}\left(\eta \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}\right). \tag{8.3}$$

Clearly, we can choose $k > 0$ large enough such that

$$k\sqrt{-1}\partial\overline{\partial}u + \Omega(\mathbf{K}^{-1}) + 3\sqrt{-1}\partial\overline{\partial}\left(\eta \log\left(|z_1|^2 + |z_2|^2\right)\right) > 0.$$

Then, by the standard L^2 estimate of $\overline{\partial}$ operator on Hermitian holomorphic line bundles (cf. Theorem 1.2 in [**26**]), Equation (8.3) has a smooth solution $S(x)$ satisfying the estimate

$$\begin{aligned} &\int_M |S|^2 e^{-ku-3\eta\log\left(|z_1|^2+|z_2|^2\right)}\omega^2 \\ &\quad \le C \int_M \left|\overline{\partial}\left(\eta\frac{\partial}{\partial z_1}\wedge\frac{\partial}{\partial z_2}\right)\right|^2 e^{-ku-3\eta\log\left(|z_1|^2+|z_2|^2\right)}\omega^2 \\ &\quad < +\infty \end{aligned} \tag{8.4}$$

for some positive constant C. Recall the Poincaré–Lelong equation for the section $S(x)$ of the anticanonical line bundle $\mathbf{K^{-1}}$,

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|S|^2 = [V] - \frac{1}{2\pi}\mathrm{Ric} \qquad \text{on} \quad M,$$

where $[V]$ is the zero divisor of $S(x)$ (cf. [**26**]). Thus, $\log|S|^2 + u$ is subharmonic and so is $|S|^2 e^u = \exp(\log|S|^2 + u)$. Since M has positive Ricci curvature and maximal volume growth, we can apply the mean value inequality of subharmonic functions (8.4), and the fact that u has logarithmic growth to show that $S(x)$ is of polynomial growth. Set

$$v = \eta\left(\frac{\partial}{\partial z_1}\wedge\frac{\partial}{\partial z_2}\right) - S\ .$$

Then, v is a non-trivial holomorphic $(2,0)$ vector field over M with polynomial growth we desired.

Now, for any $f_i, f_j \in \{f_1, f_2, f_3, f_4\}$ with $df_i \wedge df_j \not\equiv 0$, we can choose the point $\overline{x}_0$ in the above construction of v so that the holomorphic function f_{ij} defined by

$$f_{ij} = \langle v, df_i \wedge df_j\rangle \tag{8.5}$$

is a non-trivial holomorphic function of polynomial growth. Here, we have used the fact that $\|df_i \wedge df_j\|$ grows at most polynomially by the gradient estimate of harmonic functions of Yau [40]. It is obvious that the zero divisor of $df_i \wedge df_j$ is contained in the zero divisor of f_{ij}, for which we denote by V_0. Since M is Stein, the same is also true for $M \backslash V_0$.

Denote by $\pi_{ij} : Z \to \mathbf{C^2}$ the projection map given by $(w_1, w_2, w_3, w_4) \mapsto (w_i, w_j)$. Then, the map

$$\rho = \pi_{ij} \circ F : M \backslash V_0 \to \mathbf{C^2}$$

realises the Stein manifold $M \backslash V_0$ as a Riemann domain of holomorphy over $\mathbf{C^2}$. Let $\delta(x)$ be the Euclidean distance to the boundary as in Oka [30]. Then, $-\log \delta$ is a plurisubharmonic function on $M \backslash V_0$ by a theorem of Oka [30]. $\delta(x)$ will be used in the weight function of the Skoda's estimate mentioned above. It is essential to estimate it from below in terms of the intrinsic distance $d(x, x_0)$ on M.

Lemma 8.1. *There exist positive constants p and C such that*

$$\delta(x) \geq C \, |f_{ij}(x)|^2 \, (d(x, x_0) + 1)^{-p} \, .$$

Proof. Let $v_i,\ v_j$ be two holomorphic vector fields on $M \backslash V_0$ defined by

$$\langle v_k, df_l \rangle = \delta_{kl}, \qquad k, l = i, j.$$

By the Cramer's rule, we have

$$\begin{aligned} |v_k| \leq \frac{|df_i| + |df_j|}{|df_i \wedge df_j|} &\leq \frac{|v| \, (|df_i| + |df_j|)}{|f_{ij}|} \\ &\leq C_{22} \frac{(d(x, x_0) + 1)^{k_1}}{|f_{ij}(x)|} \quad \text{on} \quad M \backslash V_0, \end{aligned}$$

for $k = i, j$ and some positive constants C_{22}, k_1.

Since f_{ij} is of polynomial growth, $|\nabla f_{ij}|$ is also of polynomial growth by the gradient estimate of Yau, i.e.,

$$\max \{ f_{ij}(x), |\nabla f_{ij}(x)| \} \leq C_{23} \, (d(x, x_0) + 1)^{k_2} \qquad \text{on} \quad M \ ,$$

for some positive constants C_{23} and k_2. Take $x \in M\backslash V_0$, then for any $y \in B\left(x, |f_{ij}(x)| \Big/ 3C_{23}\left(d(x,x_0)+1\right)^{k_2}\right)$, we have

$$
\begin{aligned}
(8.6) \quad |f_{ij}(y)| &\geq |f_{ij}(x)| - C_{23}\left(d(x,x_0)+2\right)^{k_2} \cdot \frac{|f_{ij}(x)|}{3C_{23}\left(d(x,x_0)+1\right)^{k_2}} \\
&\geq \frac{1}{2}\left|f_{ij}(x)\right|.
\end{aligned}
$$

This implies

$$
B\left(x, |f_{ij}(x)| \Big/ 3C_{23}\left(d(x,x_0)+1\right)^{k_2}\right) \subset M\backslash V_0
$$

and

$$
(8.7) \qquad |v_k(y)| \leq 2C_{22}\frac{\left(d(x,x_0)+1\right)^{k_1}}{|f_{ij}(x)|},
$$

for all $y \in B\left(x, |f_{ij}(x)| \Big/ 3C_{23}\left(d(x,x_0)+1\right)^{k_2}\right)$, $k = i, j$.

By the definition of $\delta(x)$, it suffices to prove

$$
\begin{aligned}
(8.8) \quad &\rho\left(B\left(x, |f_{ij}(x)| \Big/ 6C_{23}\left(d(x,x_0)+1\right)^{k_2}\right)\right) \\
&\supset B_{\mathbf{C}^2}\left(\rho(x), C_{24}\left|f_{ij}(x)\right|^2 \Big/ \left(d(x,x_0)+1\right)^{k_1+k_2}\right),
\end{aligned}
$$

for some positive constant C_{24}. Here, $B_{\mathbf{C}^2}(a,r)$ denotes the Euclidean ball in $\mathbf{C}^2$ with center a and radius r.

Consider the real vector field

$$
\begin{aligned}
\xi &= \alpha_i\left(2\mathrm{Re}\ (v_i)\right) + \alpha_j\left(2\mathrm{Re}\ (v_j)\right) + \beta_i\left(2\mathrm{Im}\ (v_i)\right) + \beta_j\left(2\mathrm{Im}\ (v_j)\right) \\
&= \left(\alpha_i - \sqrt{-1}\beta_i\right)v_i + \left(\alpha_j - \sqrt{-1}\beta_j\right)v_j + \left(\alpha_i + \sqrt{-1}\beta_i\right)\overline{v}_i \\
&\quad + \left(\alpha_j + \sqrt{-1}\beta_j\right)\overline{v}_j
\end{aligned}
$$

with $|\alpha_i|^2 + |\alpha_j|^2 + |\beta_i|^2 + |\beta_j|^2 = 1$. Clearly, ξ also satisfies (8.7). Let $\gamma_\xi(\tau)$ be the integral curve in M defined by ξ and passes through x, i.e.,

$$
(8.9) \qquad \begin{cases} \dfrac{d\gamma_\xi(\tau)}{d\tau} = \xi \\ \gamma_\xi(0) = x. \end{cases}
$$

We have

$$\frac{d\left(f_i \circ \gamma_\xi(\tau)\right)}{d\tau} = \langle \xi, df_i \rangle = \alpha_i - \sqrt{-1}\beta_i,$$
$$\frac{d\left(f_j \circ \gamma_\xi(\tau)\right)}{d\tau} = \langle \xi, df_j \rangle = \alpha_j - \sqrt{-1}\beta_j,$$

and

$$|f_i \circ \gamma_\xi(\tau) - f_i(x)|^2 + |f_j \circ \gamma_\xi(\tau) - f_j(x)|^2 = \tau^2. \tag{8.10}$$

Note that (8.10) implies that $\gamma_\xi(\tau)$ cannot always stay in

$$B\left(x, |f_{ij}(x)| \Big/ 6C_{23}\left(d(x,x_0)+1\right)^{k_2}\right),$$

otherwise, $F = (f_1, f_2, f_3, f_4)$ would become unbounded in this ball. Denote by τ_0 the first time when $\gamma_\xi(\tau)$ touches the boundary

$$\partial B\left(x, |f_{ij}(x)| \Big/ 6C_{23}\left(d(x,x_0)+1\right)^{k_2}\right),$$

it is easy to see that

$$\begin{aligned}\frac{|f_{ij}(x)|}{6C_{23}\left(d(x,x_0)+1\right)^{k_2}} &\le \text{the length of } \gamma_\xi \text{ on } [0,\tau_0] \\ &\le 2C_{22}\int_0^{\tau_0} \frac{\left(d(x,x_0)+1\right)^{k_1}}{|f_{ij}(x)|} d\tau \qquad \text{(by (8.7))} \\ &= 2C_{22}\tau_0 \frac{\left(d(x,x_0)+1\right)^{k_1}}{|f_{ij}(x)|}.\end{aligned}$$

Thus,

$$\tau_0 \ge \frac{|f_{ij}(x)|^2}{2C_{22}C_{23}6\left(d(x,x_0)+1\right)^{k_1+k_2}}. \tag{8.11}$$

Note that the integral curve γ_ξ projects to straight line passing through $f(x)$ by ρ. Thus, when $(\alpha_i, \alpha_j, \beta_i, \beta_j)$ runs through the unit sphere in $\mathbf{C^2}$, the collection of integral curves γ_ξ inside

$$B\left(x, |f_{ij}(x)| \Big/ 6C_{23}\left(d(x,x_0)+1\right)^{k_2}\right)$$

will project, by ρ, onto the Euclidean ball

$$B_{\mathbf{C^2}}\left(\rho(x), |f_{ij}(x)|^2 \Big/ 2C_22C_26\left(d(x,x_0)+1\right)^{k_1+k_2}\right).$$

This proves (8.8) and hence the lemma. q.e.d.

Now, we are ready to prove the almost surjectivity of the holomorphic map $F : M \to \mathbf{C^4}$. For each $1 \le i, j \le 4$, since f_{ij} is a holomorphic function of polynomial growth and $R(M)$ is generated by $f_1, \ldots, f_4$, we can write

$$f_{ij}(x) = H_{ij}\left(f_1(x), f_2(x), f_3(x), f_4(x)\right) \qquad \text{on} \quad M$$

for some rational function H_{ij} on $\mathbf{C^4}$. Let T_0 be the union of the singular set of Z and the zero and pole sets of all H_{ij}, $1 \le i, j \le 4$. For any $b \in Z \backslash (F(M) \cup T_0)$, there exist fixed $\{i, j\} \subset \{1, 2, 3, 4\}$ such that the projection $\pi_{ij} : Z \to \mathbf{C^2}$ is non-degenerate at b. Since Z is algebraic, the number of points contained in $\pi_{ij}^{-1} \circ \pi_{ij}(b)$ is less than some fixed integer K depending only on Z. By interpolation, there is a polynomial h_b of degree $\le K$ on $\mathbf{C^4}$ such that $h_b(b) = 1$, and $h_b(w) = 0$ for all $w \in (\pi_{ij}^{-1} \circ \pi_{ij}(b)) \backslash \{b\}$. We now solve on $M \backslash V_0$ the ideal problem with unknown holomorphic functions g_i and g_j,

$$(f_i - b_i)\, g_i + (f_j - b_j)\, g_j = (h_b \circ F)^4, \tag{8.12}$$

where $b = (b_1, b_2, b_3, b_4)$.

Let

$$\psi = -n_1 \log \delta + n_2 \log(1 + |f_i|^2 + |f_j|^2),$$

where the integers n_1, $n_2 > 0$ will be determined later. Clearly, ψ is a strictly plurisubharmonic function on $M \backslash V_0$. By the estimate of Skoda (cf. Theorem 1.3 in [**26**]), given any $\alpha > 1$, there exists a solution $\{g_i, g_j\}$ to (8.12) such that

$$\begin{aligned}\int\limits_{M\backslash V_0} &\frac{\left(|g_i|^2 + |g_j|^2\right) e^{-\psi}}{\left(|f_i - b_i|^2 + |f_j - b_j|^2\right)^{2\alpha}} \rho^* dV_E \\ &\le C_\alpha \int\limits_{M\backslash V_0} \frac{(h_b \circ F)^8\, e^{-\psi}}{\left(|f_i - b_i|^2 + |f_j - b_j|^2\right)^{2\alpha+1}} \rho^* dV_E,\end{aligned} \tag{8.13}$$

provided the right-hand side is finite. Recall that $\rho = \pi_{ij} \circ F$ and here

$$\rho^* dV_E = \pm \left(\frac{\sqrt{-1}}{2}\right)^2 df_i \wedge \overline{df}_i \wedge df_j \wedge \overline{df}_j$$

denotes the pull back of the Euclidean volume element of $\mathbf{C^4}$.

Let $\{\zeta_1, \zeta_2, \ldots, \zeta_m\} = \pi_{ij}^{-1} \circ \pi_{ij}(b)$ $(m < K)$ be the preimages of $\pi_{ij}(b)$ with $\zeta_1 = b$. And let U_k $(1 \le k \le m)$ be disjoint small neighborhoods of ζ_k $(1 \leq k \leq m)$. The integral on the right-hand side of (8.13) can be decomposed into three parts

$$\begin{aligned} RHS &= \left(\int_{F^{-1}(U_1)} + \sum_{k=2}^{m} \int_{F^{-1}(U_k)} + \int_{(M \setminus V_0) \setminus \cup_{k=1}^{m} F^{-1}(U_k)} \right) \\ &\quad \cdot \frac{(h_b \circ F)^8 \, e^{-\psi}}{\left(|f_i - b_i|^2 + |f_j - b_j|^2 \right)^{2\alpha+1}} \rho^* dV_E \\ &= I_1 + I_2 + I_3 \; . \end{aligned}$$

For I_1, since $h_b(b) = 1$ and $\delta(x) \leq \left(|f_i - b_i|^2 + |f_j - b_j|^2 \right)^{\frac{1}{2}}$, we can choose $n_1 \geq 2(2\alpha + 1)$ and U_1 small enough so that the integral I_1 is finite.

For I_2, since $h_b(\zeta_k) = 0$ for $2 \leq k \leq m$, we can choose α such that $2(2\alpha + 1) < 8$ (e.g., $\alpha = 1.4$). Then, the integral I_2 is also finite.

For I_3, we choose $n_2 \geq 10 + 8K + n_1$, where h_b is of degree $\leq K$. Then, I_3 can be estimated as

$$I_3 \leq C_{25} \int_{\mathbf{C^2}} \frac{1}{(1 + |w|^2)^{10}} dV_E < +\infty.$$

Hence, we have obtained a solution $\{g_i, g_j\}$ of the ideal problem (8.12) such that

$$\int_{M \setminus V_0} \frac{\left(|g_i|^2 + |g_j|^2 \right) e^{-\psi}}{\left(|f_i - b_i|^2 + |f_j - b_j|^2 \right)^{2\alpha}} \rho^* dV_E < +\infty. \tag{8.14}$$

Recall from Lemma 8.1 and (8.5), we have

$$\delta(x) \geq C \, |f_{ij}(x)|^2 \, (d(x, x_0) + 1)^{-p}$$

and

$$\begin{aligned} \rho^* dV_E &= \pm \left(\frac{\sqrt{-1}}{2} \right)^2 df_i \wedge df_j \wedge \overline{df}_i \wedge \overline{df}_j \\ &\geq \frac{|f_{ij}|^2}{|v \wedge \overline{v}|} \omega^2 \\ &\geq C_{26} \frac{|f_{ij}(x)|^2}{(d(x, x_0) + 1)^{k_3}} \omega^2 \end{aligned}$$

for some positive constants C_{26} and k_3. Substituting these two inequalities into (8.14), we get

$$(8.15)\qquad \int\limits_{M\backslash V_0} \frac{\left(|g_i|^2+|g_j|^2\right)|f_{ij}|^{2+2n_1}}{(d(x,x_0)+1)^{k_4}}\omega^2 < +\infty,$$

where k_4 is some positive constant independent of b and i, j. Then, both $g_i f_{ij}^{n_1+1}$ and $g_j f_{ij}^{n_1+1}$ are locally square integrable. They can thus be extended holomorphically from $M\backslash V_0$ to M. By the mean value inequality of subharmonic functions, we deduce also that they are of polynomial growth with degree bounded by some positive number k_5 independent of b.

Now, recall that $R(M) = \mathbf{C}(f_1,f_2,f_3/f_4)$, the holomorphic functions $g_i f_{ij}^{n_1+1}$ and $g_j f_{ij}^{n_1+1}$ are thus rational functions of f_1, f_2, f_3 and f_4. Hence, we can regard the equation (8.12) as an equation on the variety $Z \subset \mathbf{C}^4$, namely

$$(w_i-b_i)\,g_i f_{ij}^{n_1+1} + (w_j-b_j)\,g_j f_{ij}^{n_1+1} = H_{ij}^{n_1+1}\cdot h_b^4.$$

Since h_b is a polynomial with $h_b(b)=1$ and the point b lies outside of the zero and pole sets of H_{ij}, either $g_i f_{ij}^{n_1+1}$ or $g_j f_{ij}^{n_1+1}$, when regarded as rational function on $\mathbf{C}^4$, must have a pole at b. Denote this function by G^0. Thus, G^0 is a rational function on Z with $G^0(b)=\infty$ and $G^0\circ F$ is a holomorphic function on M with degree $\le k_5$. If $Z\backslash(F(M)\cup T_0\cup \text{pole sets of}\, G^0)$ is empty, then we are done. Otherwise, pick any $b_1 \in Z\backslash(F(M)\cup T_0 \cup \text{pole sets of}\, G^0)$ and repeat the same procedure to obtain a rational function G^1 on Z with $G^1(b_1)=\infty$ and $G^1\circ F$ a holomorphic function on M with degree $\le k_5$. Proceeding this way, we obtain a sequence of points $\{b,b_1,b_2,\ldots\}$ and rational functions $\{G^0,G^1,G^2,\ldots\}$ such that $G^k(b_k)=\infty$ and G^l regular at b_k for $l<k$. So, $\{G^0,G^1,G^2,\ldots\}$ must be linearly independent over $\mathbf{C}$. Moreover, all of $G^k\circ F$ are holomorphic functions with degree $\le k_5$. Hence, by (8.2), the above procedure must terminate in a finite number of steps. In other words, there exists an algebraic subvariety T of Z such that $F(M)\supset Z\backslash T$.

Moreover, F establishes a quasi embedding from M to a quasi–affine algebraic variety. Indeed, let $W=F^{-1}(T)$. By the definition of T_0 and the construction of T, we know that $W\supset V$, where V is the union of the branching locus of F and $F^{-1}(\mathrm{Sing}(Z))$, and W is the zero divisor of finitely many holomorphic functions of polynomial growth. Therefore, F maps $M\backslash W$ biholomorphically onto $Z\backslash T$.

Finally, to complete the proof of our Main Theorem, we have to show that the mapping F can be desingularized by adjoining a finite number of holomorphic functions of polynomial growth and taking normalization of the image.

We have constructed the mapping $F : M \to Z$ into an affine algebraic variety which maps $M\backslash W$ biholomorphically onto $Z\backslash T$. Now, we use normalization of the affine algebraic variety Z to resolve the codimension 1 singularities of F. Let $\mathrm{Reg}(Z)$ denote the Zariski dense subset of Z consisting of its regular points. It is well known that the normalization $\widetilde{Z}$ of Z can be obtained by taking $\widetilde{Z}$ to be the closure of the graph of $\{Q_1, Q_2, \ldots, Q_m\}$ on $\mathrm{Reg}(Z)$ where Q_i is a rational function which is holomorphic (or regular in the terminology of algebraic geometry) on $\mathrm{Reg}(Z)$. The lifting of $F : M \to Z$ to $\widetilde{F} : M \to Z$ is then given by $\{f_1, f_2, f_3, f_4, Q_1 \circ F, \ldots, Q_m \circ F\}$ where, as was shown in proposition 8.1 of Mok [**24**], for each i, $Q_i \circ F$ can be holomorphically extended to the whole manifold M as a holomorphic function of polynomial growth.

Write $F_0 = F : M \to Z$ and denote $\widetilde{F}_0 : M \to \widetilde{Z}$ the normalization of F_0. For any smooth point x on the subvariety W, by using the L^2 estimates of the $\overline{\partial}$ operator as in Section 7, one can find two holomorphic functions g_x^1, g_x^2 of polynomial growth which give local holomorphic coordinates at x. Adding g_x^1, g_x^2 to the map $\widetilde{F}_0$, we get a new map $F_1 = (\widetilde{F}_0, g_x^1, g_x^2) : M \to Z_1 \subset \mathbf{C}^{6+m}$, which is non-degenerate at x. Write the normalization of F_1 as $\widetilde{F}_1 : M \to \widetilde{Z}_1$ and continue in this way to get holomorphic mappings $F_i : M \to Z_i$ and their normalizations $\widetilde{F}_i : M \to \widetilde{Z}_i$ such that

$$\widetilde{W}_0 \underset{\neq}{\supset} \widetilde{W}_1 \underset{\neq}{\supset} \cdots \underset{\neq}{\supset} \widetilde{W}_i \underset{\neq}{\supset} \cdots ,$$

where $\widetilde{W}_i$ is the locus of ramification of $\widetilde{F}_i$.

Note that $\widetilde{W}_i$ contains no isolated point because $\widetilde{Z}_i$ is normal. Moreover, by Proposition 7.2, W has only finite number of irreducible components because W is the zero divisor of finitely many holomorphic functions of polynomial growth. This implies that the above procedure must terminate in a finite number of steps, say l. Thus, we get a biholomorphism $\widetilde{F}_l$ from M onto its image $\widetilde{F}_l(M) \subset \widetilde{Z}_l$. The argument in our proof of the almost surjectivity shows that $\widetilde{F}_l(M)$ can miss at most finitely many irreducible subvarieties of $\widetilde{Z}_l$, say $\widetilde{T}_1^{(l)}, \ldots, \widetilde{T}_q^{(l)}$. If $\widetilde{F}_l(M) \cap \widetilde{T}_i^{(l)} \neq \emptyset$, then it must intersect $\widetilde{T}_i^{(l)}$ in a non-empty open set because $\widetilde{F}_l$ is open. We arrange $\widetilde{T}_i^{(l)}$ so that $\widetilde{F}_l(M) \cap \widetilde{T}_i^{(l)} = \emptyset$ for $1 \leq i \leq p$ and $\widetilde{F}_l(M) \cap \widetilde{T}_i^{(l)} \neq \emptyset$ for $p+1 \leq i \leq q$. Note that$\widetilde{F}_l(M)$ is a Stein subset of $\widetilde{Z}_l$ because M is Stein by Theorem 5.1 and $\widetilde{F}_l$ maps M biholomorphically onto its image. By Hartog's extension theorem, every holomorphic function on $\widetilde{Z}_l \backslash \cup_{1 \leq i \leq q} \widetilde{T}_i^{(l)}$ extends to $\widetilde{Z}_l \backslash \cup_{1 \leq i \leq p} \widetilde{T}_i^{(l)}$. Hence, we get a biholomorphic map from M onto a quasi–affine algebraic variety. Finally, recall that a classical theorem of Ramanujam [**31**] in affine algebraic geometry says that an algebraic variety homeomorphic to $\mathbf{R}^4$ is biregular to $\mathbf{C}^2$. Combining this result of Ramanujam with Theorem 5.1, we deduce that M is actually biholomorphic to $\mathbf{C}^2$. Therefore, we have completed the proof of the Main Theorem.

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