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THE METRIC FIBRATIONS OF EUCLIDEAN SPACE

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Abstract

The purpose of this note is to complete the classification of metric fibrations in Euclidean space begun in [1]. Building on our techniques there, we show that regardless of dimension, the fibers are always the orbits of a free isometric group action by generalized glide rotations. A key ingredient of the argument is the fact that in the global setting, these fibrations satisfy a strong algebraic rigidity.

1. The fiber over a soul and the main result

We begin by recalling some general facts concerning metric fibrations $\pi : \mathbb{R}^{n+k} \to M^n$ that were established in [1]. Notationwise, X, Y, Z will denote local horizontal fields, T, U, V vertical ones, and lower-case letters refer to individual vectors. We write $e = e^h + e^v \in \mathcal{H} \oplus \mathcal{V}$ for the decomposition of $e \in T\mathbb{R}^{n+k}$ into its horizontal and vertical parts. Thus, the integrability tensor A and the second fundamental tensor S are given by

$$A_X Y = \tfrac{1}{2}[X, Y]^v = \overset{v}{\nabla}_X Y, \qquad S_X U = -\overset{v}{\nabla}_U X.$$

M has nonnegative sectional curvature by O'Neill's formula, and is diffeomorphic to $\mathbb{R}^n$ since the fibers of the fibration are connected. In particular, any soul of M consists of a single point. The fiber F over a soul is a totally geodesic affine subspace of Euclidean space, and up to congruence, $F = \mathbb{R}^k \times 0 \subset \mathbb{R}^k \times \mathbb{R}^n$.

The normal bundle ν of F has two Riemannian connections relevant to the present situation: One is the usual connection $\overset{h}{\nabla}$, which is just

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the horizontal component of the Euclidean one. The other is the Bott connection $\overset{B}{\nabla}$ for which the basic fields along F are parallel sections of ν. The connection difference form $\Omega = \overset{h}{\nabla} - \overset{B}{\nabla}$ is then the 1-form on F with values in the skew-symmetric endomorphism bundle of ν given by

$$\Omega(U)X = -A_X^* U,$$

where A_X^* denotes the pointwise adjoint of A_X. When X, Y are basic, one always has

$$d(A_X Y)^\flat(U,V) = \langle d\Omega(U,V)X, Y\rangle \tag{1.1}$$

for the 1-form $(A_X Y)^\flat$ metrically dual to $A_X Y$.

Our goal is to establish that Ω is Bott-closed, or equivalently, that each integrability field $A_X Y$ is parallel on F for basic X, Y. The following main result is then an immediate consequence of [1, Theorem 2.6]:

Theorem. *Let $\pi : \mathbb{R}^{n+k} \to M^n$ be a metric fibration of Euclidean space with connected fibers. Then*

1. *The fiber F over a soul of M is an affine subspace of Euclidean space, which, up to congruence, may be taken to be $F = \mathbb{R}^k \times 0$.*

2. *The connection difference form Ω along the normal bundle of F induces a Lie algebra homomorphism $\Omega : \mathbb{R}^k \to \mathfrak{so}(n)$, and π is the orbit fibration of the free isometric group action ψ of $\mathbb{R}^k$ on $\mathbb{R}^{n+k} = \mathbb{R}^k \times \mathbb{R}^n$ given by*

$$\psi(v)(u,x) = (u+v, \phi(v)x), \qquad u, v \in \mathbb{R}^k, \quad x \in \mathbb{R}^n,$$

where $\phi : \mathbb{R}^k \to SO(n)$ is the representation of $\mathbb{R}^k$ induced by Ω.

2. Polynomial growth of the holonomy form

The *mean curvature form* of the fibration is the horizontal 1-form κ on $\mathbb{R}^{n+k}$ given by $\kappa(E) = \operatorname{tr} S_{E^h}$. By [1, Corollary 2.3], every metric fibration of Euclidean space is *taut*; i.e., κ is basic and exact. Let f denote the function on $\mathbb{R}^{n+k}$ that vanishes on F and satisfies $df = \kappa$ (observe that f is constant along fibers since κ is basic), and set $V =$

e^{-f}. Define the *holonomy form* ω to be the k-form $\omega := V\tau$, where τ is the vertical volume form of the fibers of π; i.e., τ is the k-form on $\mathbb{R}^{n+k}$ whose metric dual at a point p is given by

$$\tau^\sharp(p) = u_1 \wedge \cdots \wedge u_k,$$

where $u_1, \dots, u_k$ denotes any oriented orthonormal basis of the tangent space to the fiber at p. It is well known that in general, the Lie derivative of τ in horizontal directions X satisfies

$$L_X\tau = -\kappa(X)\,\tau \tag{2.1}$$

vertically. Now let $E_1, \cdots, E_k$ be an oriented orthonormal basis of parallel vector fields on F, and extend them smoothly to all of $\mathbb{R}^{n+k}$ by setting

$$U_i(a,y) := E_i(a,0) - A_y^* E_i(a,0), \qquad (a,y) \in \mathbb{R}^k \times \mathbb{R}^n.$$

More precisely, $U_i(a,y) = \|\,[E_i(a,0) - A^*_{I_{(a,0)}y}E_i(a,0)]$, with $\|$ denoting parallel translation from $(a,0)$ to (a,y), and $I_{(a,y)}$ the canonical isomorphism of $\mathbb{R}^{n+k}$ with its tangent space at (a,y). In order to avoid cumbersome notation, we shall from now on just assume these identifications. Observe that for horizontal lines γ originating at F, $U_i \circ \gamma$ is the holonomy Jacobi field along γ which equals E_i at $\gamma(0)$, see [1].

Lemma 2.2. $\omega^\sharp = U_1 \wedge \cdots \wedge U_k$.

Proof. We must show that $V = \tau(U_1, \dots, U_k)$. Both functions are by definition constant equal to 1 on F. Next, observe that that if X is the tangent field of a horizontal geodesic from F, then $XV = -V\kappa(X)$, whereas

$$\begin{aligned} X(\tau(U_1,\dots,U_k)) &= L_X(\tau(U_1,\dots,U_k)) = (L_X\tau)(U_1,\dots,U_k) \\ &= -\tau(U_1,\dots,U_k)\kappa(X) \end{aligned}$$

by (2.1). Here we have used the fact that $L_XU_i = 0$. The lemma clearly follows. q.e.d.

Lemma 2.3. *The form $U_1 \wedge \cdots \wedge U_k$ is polynomial of degree at most k on every horizontal affine subspace.*

Proof. Notice that the holonomy fields U_i are *a priori* linear only along each affine subspace $a \times \mathbb{R}^n$ orthogonal to F. It will later become apparent that they are in fact global Killing fields generating the isometric group action.

Let $p \in \mathbb{R}^{n+k}$, and q a point on the horizontal space H through p. By Lemma 2.2, $\wedge_i U_i$ is holonomy invariant, so that

$$\wedge_i U_i(q) = \wedge_i [U_i(p) - (A^*_{q-p} + S_{q-p})U_i(p)].$$

Thus, by translating the origin to p, it suffices to show that the map $x \mapsto \wedge_i(E_i - A^*_x E_i - S_x E_i)$ is polynomial of degree at most k in x. But this follows from the fact that $x \mapsto A^*_x E + S_x E$ is a linear map. q.e.d.

Lemma 2.4. *For any* $(a, 0)$ *and* $(0, x)$ *in* $\mathbb{R}^k \times \mathbb{R}^n$, $U_1 \wedge \cdots \wedge U_k$ *is polynomial in* x *on every affine line through* $(a, 0)$ *in directions of the image of* A_x.

Proof. We show that if f is a component of $\wedge_i U_i$, then all derivatives of f of sufficiently high order vanish in directions $A_x y$. The result then follows from Taylor's expansion. Notice that it is actually sufficient to establish this for directions $(A_x y, y)$ (since the derivatives of order $> k$ in directions $(0, y)$ vanish by Lemma 2.3). Using Lemma 2.3 once more, it remains to show that both $(a, 0)$ and $(a + A_x y, y)$ belong to a common horizontal affine subspace. We claim, in fact, that they both belong to the horizontal space through (a, x): Clearly, $(a, 0)$ does; as to the other point, just observe that $(a + A_x y, y) - (a, x) = (A_x y, y - x)$ is orthogonal to the vertical space at (a, x), since

$$\langle (A_x y, y - x), (u, -A^*_x u) \rangle = \langle A_x y, u \rangle - \langle y - x, A^*_x u \rangle = 0.$$

q.e.d.

3. Constancy of integrability fields

In this section, we use the polynomial behavior of the holonomy form to deduce that each integrability field $A_X Y$ is parallel along the totally geodesic fiber F. Before getting into the details of the argument, we provide a brief outline of the strategy involved, which relies on the following splitting principle: The fiber $F = \mathbb{R}^k \times 0 \subset \mathbb{R}^k \times \mathbb{R}^n$ splits isometrically as $\mathbb{R}^l \times \mathbb{R}^{k-l}$ with the kernel of A^* tangent to the first factor, and the image of A tangent to the second. This kernel extends to the whole ambient space via parallel transport, and corresponds to the translational part of the representation. In other words, the fibration $\mathbb{R}^{n+k} \to M^n$ factors as an orthogonal projection $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k-l}$ followed by a fibration $\pi' : \mathbb{R}^{n+k-l} \to M^n$ which is *weakly substantial* in the sense that the image of the A-tensor spans the whole fiber. π'

thus measures the twisting or rotational part of the representation. The splitting itself is in turn due to a kind of maximum principle: We establish that the polynomial holonomy form has bounded, and therefore parallel derivative.

It will be necessary to first work with parallel horizontal fields along F rather than basic (Bott parallel) ones, and we shall denote the former by lowercase letters, reserving the uppercase notation for basic fields. For a point p in the fiber F, let $\mathcal{A}_p = \operatorname{span}\{U_p \mid U \in \mathcal{A}\}$, where $\mathcal{A}$ denotes the space of integrability fields spanned by all $A_X Y$ along F. The image of A is then the union of all $\mathcal{A}_p$ as p ranges over F. Notice also that the kernel of A^* consists of the union of all $\mathcal{A}_p^\perp$.

By the results in Section 2, the form $\omega^\sharp$ is polynomial along every affine plane passing through a point $(a, 0) \in F$ spanned by a horizontal x and a vertical u in the image of A. The same is then true for the derivative

$$\nabla_x \omega^\sharp = -\sum_i E_1 \wedge \cdots \wedge A_x^* E_i \wedge \cdots \wedge E_k$$

of $\omega^\sharp$ in direction x. If $A_x^* E_i \neq 0$, then the corresponding wedge product in the above expression is nonzero, since $A_x^* E_i$ is horizontal. But the E_l are parallel along F, and $A_x^* E_i$ is bounded in norm, so that each $A_x^* E_i$ must be parallel along the geodesic line $t \mapsto \gamma_u(t) = (a + tu, 0)$. Thus, for all x, y,

$$(A_x y \circ \gamma_u)' \equiv 0, \qquad u \in \operatorname{im} A, \tag{3.1}$$

and the image of A, though *a priori* not of constant rank, is totally geodesic along F, and thus consists of a disjoint union of affine subspaces. The same is true of its orthogonal complement $\ker A^*$: Given $u \in \ker A^*$, we claim that $\dot\gamma_u(t)$ belongs to the kernel for all t. To see this, consider the variation $V(t, s) = \exp_{su} tx$, which projects down to a variation $W = \pi \circ V$ on the quotient. The Jacobi field $Y(t) = W_* \partial_s|_{t,0}$ induced by W satisfies $Y(0) = 0$, and

$$Y'(0) = \pi_* \nabla_{\partial_t} (V_* \partial_s)^h|_{(0,0)} = -\pi_* \overset{h}{\nabla}_{\partial_t} (V_* \partial_s)^v|_{(0,0)} = \pi_* A_x^* u = 0.$$

Thus, Y is identically 0, or equivalently, the parallel field x is actually basic along γ_u, so that $A_x^* \dot\gamma_u = -(x \circ \gamma_u)' \equiv 0$. This establishes the claim.

Up to congruence, $\mathcal{A}_0$ is $0 \times \mathbb{R}^{k-l}$ for some integer l by (3.1). It follows that for any $(a, b) \in \mathbb{R}^l \times \mathbb{R}^{k-l} = F$, $\mathcal{A}_{(a,b)}^\perp = \ker A_{(a,b)}^* = \mathbb{R}^l \times b$, since $\mathcal{A}^{\perp}_{(0,b)} = \mathbb{R}^l \times b$: Indeed, $(a,b) \in \mathcal{A}^{\perp}_{(0,b)}$, so that $\mathcal{A}^{\perp}_{(0,b)} \subset \mathcal{A}^{\perp}_{(a,b)}$, and by symmetry, the reverse inclusion also holds. Thus, $\mathcal{A}_{(a,b)} = a \times \mathbb{R}^{k-l}$, and F splits isometrically as $\mathbb{R}^l \times \mathbb{R}^{k-l}$ with the kernel of A^* tangent to the first factor and the image of A tangent to the second. But the holonomy displacement of $\ker A^*$ along horizontal lines γ that intersect F is just parallel translation along γ, so that π factors as an orthogonal projection

$$\mathbb{R}^l \times \mathbb{R}^{n+k-l} \to 0 \times \mathbb{R}^{n+k-l}$$

followed by a Riemannian submersion $\pi' : \mathbb{R}^{n+k-l} \to M^n$. Furthermore, the latter is weakly substantial in that the totally geodesic fiber F' over the soul of M is spanned by the image of A. (3.1) then implies that each $A_x y$ is a parallel field along F', or equivalently, that the form Ω is parallel, and therefore also closed. One concludes from (1.1) that for *basic* X, Y, the integrability field $A_X Y$ is a gradient, and having constant norm, $A_X Y$ is parallel. As pointed out earlier, the main theorem now follows from [1, Theorem 2.6].

References

[1] D. Gromoll & G. Walschap, *Metric fibrations in Euclidean space*, Asian J. Math. **1** (1997) 716–728.

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