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Research Article

A Crank-Nicolson Scheme for the Dirichlet-to-Neumann Semigroup

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The aim of this work is to study a semidiscrete Crank-Nicolson type scheme in order to approximate numerically the Dirichlet-to-Neumann semigroup. We construct an approximating family of operators for the Dirichlet-to-Neumann semigroup, which satisfies the assumptions of Chernoff's product formula, and consequently the Crank-Nicolson scheme converges to the exact solution. Finally, we write a P_1 finite element scheme for the problem, and we illustrate this convergence by means of a FreeFem++ implementation.

1. Introduction

Let Ω be a bounded smooth domain $\Omega \subset \mathbb{R}^n$ and let $\gamma(x) = [\gamma_{i,j}(x)]_{i,j=1}^n$ be a real-valued matrix function, which is known as the electrical conductivity matrix. The matrix $\gamma(x)$ is symmetric and smooth; that is, $\gamma_{ij} = \gamma_{ji} \in C^\infty(\overline{\Omega})$. Its positive eigenvalues are uniformly bounded; that is, there exists $0 < c_1 < c_2$ such that, for all $(x,\xi) \in \Omega \times \mathbb{R}^n$, one has $c_1\|\xi\|^2 \le \xi^T\gamma(x)\xi \le c_2\|\xi\|^2$, where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$.

In this paper, we denote by $X = L^2(\Omega)$, with boundary space $\partial X = L^2(\partial\Omega)$. It is well known that, for any $f \in H^{1/2}(\partial\Omega)$, the following Dirichlet problem,

$$\begin{aligned} \operatorname{div}(\gamma\nabla u) &= 0, \quad \text{in } \Omega, \\ u &= f, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

has a unique solution $u = L_\gamma f$ in $H^1(\Omega)$, which is called the γ-harmonic lifting of f; see [1] for more details.

Define the action of the Dirichlet-to-Neumann operator on f as the outward normal derivative of u on the boundary; that is,

$$\Lambda_\gamma(f) := (\nu\cdot\gamma\nabla u)\big|_{\partial\Omega}, \tag{2}$$

where ν is the outer normal vector to $\partial\Omega$ at $x \in \partial\Omega$.

This operator is defined on its domain:

$$D(\Lambda_\gamma) := \left\{ f \in H^{1/2}(\partial\Omega);\ \Lambda_\gamma f \in L^2(\partial\Omega) \right\}. \tag{3}$$

The Dirichlet-to-Neumann semigroup is the trace of the following boundary value problem:

$$(\text{BVP}) \begin{cases} \operatorname{div}(\gamma\nabla u(t,\cdot)) = 0, & \text{in } \Omega, \\ \partial_t u + \nu\cdot\gamma\nabla u = 0, & \text{on } \partial\Omega, \\ u(0,\cdot) = f, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

In [1], Lax has introduced a semigroup in ∂X, where Ω is the unit ball $B(0,1)$ and γ is the identity matrix. Let v be the solution of the problem:

$$\begin{aligned} \Delta v &= 0, \quad \text{in } \Omega, \\ Lv &= f, \quad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where L is the trace operator on $\partial\Omega$. The Lax semigroup is defined by

$$S(t) = LT(t)L_0, \tag{6}$$

where L_0 is the lifting operator and $T(t)v(x) := v(e^{-t}x)$.

Therefore, if $f \in \partial X$, then, for any $w \in \partial\Omega$,

$$(S(t) f)(w) = v\left(e^{-t} w\right). \tag{7}$$

It has been shown that the Lax semigroup is actually the Dirichlet-to-Neumann semigroup.

In [2], it is shown that Lax's representation cannot be generalized if Ω is not the unit ball. This was a motivation for many authors to use Chernoff's formula in order to approximate the solution.

In [3], Emamirad and Shariftabar proved that the Euler Explicit scheme

$$\begin{aligned} &\operatorname{div}\left(\gamma \nabla u^{m+1}\right) = 0, \quad \text{in } \Omega, \\ &\frac{1}{\Delta t}\left(u^{m+1} - u^{m}\right) + \frac{\partial u^{m}}{\partial \nu_{\gamma}} = 0, \quad \text{on } \partial\Omega, \\ &u^{0} = f, \quad \text{on } \partial\Omega \end{aligned} \tag{8}$$

converges to the solution of the (BVP).

Later, Cherif et al. proposed, in [4], the following Implicit Euler Scheme:

$$\begin{aligned} &\operatorname{div}\left(\gamma \nabla u^{m+1}\right) = 0, \quad \text{in } \Omega, \\ &\frac{1}{\Delta t}\left(u^{m+1} - u^{m}\right) + \frac{\partial u^{m+1}}{\partial \nu_{\gamma}} = 0, \quad \text{on } \partial\Omega, \\ &u^{0} = f, \quad \text{on } \partial\Omega, \end{aligned} \tag{9}$$

which converges to the solution of the (BVP).

After studying the implicit and explicit schemes, by the same strategy, we are interested in studying the convergence of the Crank-Nicolson type scheme. This paper is divided into 4 sections, as follows.

First, in Section 2, we introduce the Crank-Nicolson type scheme for evolution equations.

In Section 3, we prove the convergence of a Crank-Nicholson type scheme for the (BVP), by constructing an approximating family, and use Chernoff's product formula in order to prove the convergence in $L^2(\partial\Omega)$.

Finally, in Section 4, we present a numerical implementation of the Crank-Nicolson type scheme in the computational framework of the finite element method.

2. The Crank-Nicolson Scheme

In numerical analysis, the Crank-Nicolson method is a finite difference method used for numerically solving the heat equation and similar partial differential equations. It is a second-order method in time and it is numerically stable. The method was developed by John Crank and Phyllis Nicolson in the mid-20th century.

Let A be a m-accretive operator on the Hilbert space H defined as $A : D(A) \to H$.

Consider the evolution problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} + Au = 0, \quad \text{for } 0 < t < T, \\ &u(0, x) = u_0(x). \end{aligned} \tag{10}$$

We approximate the time derivative by using the finite difference method as follows:

$$\begin{aligned} &u(n\Delta t) \approx u^{n} \\ &\frac{\partial u}{\partial t}(n\Delta t) = \frac{1}{\Delta t}\left(u\left(n+1\right)\Delta t - u\left(n\Delta t\right)\right) \\ &\qquad \approx \frac{1}{\Delta t}\left(u^{n+1} - u^{n}\right), \quad \text{for } n = 0, \ldots, \left[\frac{T}{\Delta t}\right]. \end{aligned} \tag{11}$$

Crank-Nicolson introduced the scheme:

$$\begin{aligned} &\frac{u^{n+1} - u^{n}}{\Delta t} + \frac{1}{2} A u^{n} + \frac{1}{2} A u^{n+1} = 0, \\ &\qquad \text{for } n = 0, \ldots, \left[\frac{T}{\Delta t}\right], \\ &u_0 \quad \text{given}. \end{aligned} \tag{12}$$

The Chernoff product formula holds not only in the case of semigroup of contractions but also if the semigroup is bounded, so that the Crank-Nicholson scheme for the (BVP) can be written as

$$(\mathrm{CNS}) \begin{cases} \operatorname{div}\left(\gamma \nabla u^{m+1}\right) + \operatorname{div}\left(\gamma \nabla u^{m}\right) = 0, & \text{in } \Omega, \\ \dfrac{1}{\Delta t}\left(u^{m+1} - u^{m}\right) + \dfrac{1}{2}\left(\dfrac{\partial u^{m}}{\partial \nu_{\gamma}} + \dfrac{\partial u^{m+1}}{\partial \nu_{\gamma}}\right) = 0, & \text{on } \partial\Omega, \\ u^{0} = f, & \text{on } \partial\Omega. \end{cases} \tag{13}$$

3. Convergence of the Crank-Nicholson Type Scheme

In order to show the convergence of the (CNS), let us recall the following Chernoff theorem.

Theorem 1 (Chernoff's product formula). *Let X be a Banach space, real or complex, and let $\{V(t); t \geq 0\}$ be a family of contractions on X with $V(0) = I$.*

Assume that there exists $A : D(A) \subseteq X \to X$ which generates a C_0-semigroup of contractions $\{S(t); t \geq 0\}$ and that for all $x \in D(A)$ there exists

$$\lim_{h \to 0} \frac{1}{h}\left(V(h) - I\right) x = Ax. \tag{14}$$

Then, for each x in X, one has $\lim_{n \to \infty} V^{n}(t/n) x = S(t) x$.

Remark 2. Note that Chernoff's product formula holds not only in the case of semigroup of contractions but also if the semigroup is bounded (see [5]).

This theorem was used to prove the convergence of both implicit and explicit schemes.

Now, in order to prove the convergence of the Crank-Nicolson scheme, we will show that the operator $Z(t)$, defined below, satisfies the 3 conditions of Chernoff's theorem.

For any $x \in \partial\Omega$, we have

$$\frac{\partial u^{m+1}}{\partial \nu_\gamma} = \frac{1}{\Delta x}\left[u^{m+1}(x) - u^{m+1}(x - \Delta x \gamma(x)\nu)\right],$$
$$\frac{\partial u^{m}}{\partial \nu_\gamma} = \frac{1}{\Delta x}\left[u^{m}(x) - u^{m}(x - \Delta x \gamma(x)\nu)\right]. \tag{15}$$

Using in (CNS) the approximation given in (15), we get

$$\left[1 + \frac{1}{2}\frac{\Delta t}{\Delta x}\right]u^{m+1}(x) - \left[1 - \frac{1}{2}\frac{\Delta t}{\Delta x}\right]u^{m}(x) - \frac{1}{2}\frac{\Delta t}{\Delta x}u^{m+1}(x - \Delta x\gamma(x)\nu) - \frac{1}{2}\frac{\Delta t}{\Delta x}u^{m+1}(x - \Delta x\gamma(\nu)) = 0. \tag{16}$$

Taking $\alpha = \Delta t/\Delta x$, we get

$$\left[1 + \frac{\alpha}{2}\right]u^{m+1}(x) - \frac{\alpha}{2}u^{m+1}(x - \Delta x\gamma(x)\nu) = \left[1 - \frac{\alpha}{2}\right]u^{m}(x) + \frac{\alpha}{2}u^{m+1}(x - \Delta x\gamma(x)\nu) \tag{17}$$

which can be written as

$$W(\Delta t)\, u^{m+1} = V(\Delta t)\, u^{m}, \tag{18}$$

where

$$W(\Delta t) f = \left[1 + \frac{\alpha}{2}\right]u - \frac{\alpha}{2}u\left(x - \alpha^{-1}t\gamma(x)\nu\right),$$
$$V(\Delta t) f = \left[1 - \frac{\alpha}{2}\right]u + \frac{\alpha}{2}u\left(x - \alpha^{-1}t\gamma(x)\nu\right). \tag{19}$$

Now, we define the family $Q(t)$ as the inverse of $W(t)$ in the following sense:

$$W(\Delta t)\, Q(\Delta t)\, f(x) = f(x), \tag{20}$$

and we show that $Z(t) = Q(\Delta t)V(\Delta t)$ satisfies all the assumptions of Chernoff's product formula.

(i) $Z(0)f = f$. For any $x \in \partial\Omega$, we have

$$W(0) f(x) = \left(1 + \frac{\alpha}{2}\right)u(x) - \frac{\alpha}{2}u(x) = u(x) = f(x) \tag{21}$$

so, $f(x) = W(0)[Q(0)f(x)] = Q(0)f(x)$.

Moreover,

$$V(0) f(x) = \left(1 - \frac{\alpha}{2}\right)u(x) + \frac{\alpha}{2}u(x) = u(x) = f(x) \tag{22}$$

so,

$$V(0) f(x) = f(x). \tag{23}$$

Using (6) and (7), we get

$$W(0) f(x) = V(0) f(x). \tag{24}$$

Therefore, $f(x) = Q(0)V(0)f(x) = Z(0)f(x)$.

(ii) $Z'(0)f = -\Lambda_\gamma f$. The derivative of $W(t)$ with respect to t is

$$W'(t) f = -\frac{\alpha}{2}\left(-\alpha^{-1}\gamma\nu\right)\cdot\nabla u\left(\nu - \alpha^{-1}t\gamma\nu\right). \tag{25}$$

At point $t = 0$, we have $W'(0)f = 1/2\Lambda_\gamma f$.

Similarly, the derivative of $V(t)$ with respect to t is

$$V'(t) f = \frac{\alpha}{2}\left(-\alpha^{-1}\gamma\nu\right)\cdot\nabla u\left(\nu - \alpha^{-1}t\gamma\nu\right). \tag{26}$$

At point $t = 0$, we have $V'(0)f = -(1/2)\Lambda_\gamma f$.

But $W(t)Q(t)f = f$, so $W'(t)Q(0)f + W(0)Q'(0)f = 0$, and then $Q'(0)f = -(1/2)\Lambda_\gamma f$.

Now, the derivative of $Z(t)$ with respect to t is

$$Z'(t) f = Q'(t)V(t) f + Q(t)V'(t) f. \tag{27}$$

At $t = 0$, $Z'(0)f = Q'(0)V(0)f + Q(0)V'(0)f = -\Lambda_\gamma f$.

Therefore,

$$Z'(0) f = \Lambda_\gamma f. \tag{28}$$

(iii) $Z(t)$ *Is a Contraction*. In fact,

$$\|W(t) f\|^2_{L^2(\partial\Omega)} = \int_{\partial\Omega}\left|u(x) + \frac{\alpha}{2}\left(u(x) - u\left(x - \alpha^{-1}t\gamma\nu\right)\right)\right|^2 d\sigma \tag{29}$$

but $(u(x) - u(x - \alpha^{-1}t\gamma\nu))/\alpha^{-1}$ is an upper bound of the normal derivative $\partial u/\partial\nu_\gamma$, so that this term is positive according to Hopf's lemma, and, consequently,

$$\int_{\partial\Omega}\left|u(x) + \frac{\alpha}{2}\left(u(x) - u\left(x - \alpha^{-1}t\gamma\nu\right)\right)\right|^2 d\sigma \geq \int_{\partial\Omega}|u(x)|^2 d\sigma = \|f\|^2_{L^2(\partial\Omega)} \tag{30}$$

which implies that

$$\|W(t) f\|_{L^2(\partial\Omega)} \geq \|f\|_{L^2(\partial\Omega)}. \tag{31}$$

Now

$$\|f\|_{L^2(\partial\Omega)} = \|W(t)Q(t) f\|_{L^2(\partial\Omega)} \geq \|Q(t) f\|_{L^2(\partial\Omega)}. \tag{32}$$

Thus, $Q(t)$ is of contraction.

Similarly $V(t)$ is also of contraction.

Finally,

$$\|Z(t) f\|_{L^2(\partial\Omega)} = \|Q(t)V(t) f\|_{L^2(\partial\Omega)} \leq \|Q(t) f\|_{L^2(\partial\Omega)} \leq \|f\|_{L^2(\partial\Omega)}. \tag{33}$$

Therefore, $Z(t)$ is of contraction.

So, we have proved that $Z(t)$ satisfies all the assumptions of Chernoff's product formula, and, consequently, the Crank-Nicholson scheme converges to its exact solution.

Theorem 3. *The operator $Z(t)$ satisfies the Chernoff conditions; hence, the Crank-Nicolson method is convergent.*

4. Numerical Results

In the last section of this paper, we present the resolution of (CNS) in the particular case where γ is the unit matrix and Ω is the open unit disk. The point in dealing with this case is that the exact solution is known to be $v(e^{-t}x, e^{-t}y)$, where v is the solution of the Laplace equation $\Delta v = 0$, with $v = u_0$ on the boundary (u_0 is any regular function). Thus, we can compare this exact solution with the approximate solution obtained via our Crank-Nicolson scheme. For this test, we choose $u_0(x, y) = 0.5(x^2 - y^2) + y + 0.5$. The variational formulation of the (CNS) can be derived by multiplying both sides of the problem by a test function v:

$$\int_\Omega \gamma \Delta u^{m+1} v\, dx + \int_\Omega \gamma \Delta u^{m} v\, dx = 0. \tag{34}$$

Using Green's formula, we get

$$\begin{aligned} &-\int_\Omega \gamma \nabla u^{m+1} \nabla v\, dx - \int_{\partial\Omega} \frac{2}{\Delta t}\left[u^{m+1} - u^{m}\right] v\, d\sigma \\ &\quad - \int_\Omega \gamma \nabla u^{m} \nabla v\, dx = 0. \end{aligned} \tag{35}$$

Multiplying by Δt, we get

$$\begin{aligned} &\int_\Omega \Delta t \gamma \nabla u^{m+1} \nabla v\, dx + 2\int_{\partial\Omega} u^{m+1} v\, d\sigma \\ &\quad = -\int_\Omega \Delta t \gamma \nabla u^{m} \nabla v\, dx + 2\int_{\partial\Omega} u^{m} v\, d\sigma \end{aligned} \tag{36}$$

which is of the form

$$a\left(u^{m+1}, v\right) = l\left(v\right), \tag{37}$$

where the bilinear form

$$\begin{aligned} a\left(u^{m+1}, v\right) &= \int_\Omega \Delta t \gamma \nabla u^{m+1} \nabla v\, dx + 2\int_{\partial\Omega} u^{m+1} v\, d\sigma, \\ l\left(v\right) &= -\int_\Omega \Delta t \gamma \nabla u^{m} \nabla v\, dx + 2\int_{\partial\Omega} u^{m} v\, d\sigma \end{aligned} \tag{38}$$

is the linear form.

Using this variational formulation, the problem can be approximate via P_1-finite elements implemented in Freefem++.

The actual algorithm is as follows.

Choose an initial u_0 and, for a given Δt, where $t < N\Delta t$, we have the following:

for ($t = 0; t < N\Delta t$) do

(i) $u^{n+1} = Z(u^n)$

(ii) compute u^{n+1} via the variational problem

(iii) $n \leftarrow n + 1$

end do.

As an illustration, let us represent the exact (Figure 1) and the approximate solution (Figure 2), the L^2-error between the exact and the approximate solutions (Figure 3), and the decrement of this L^2-error according to the fineness of the meshing (Figure 4).

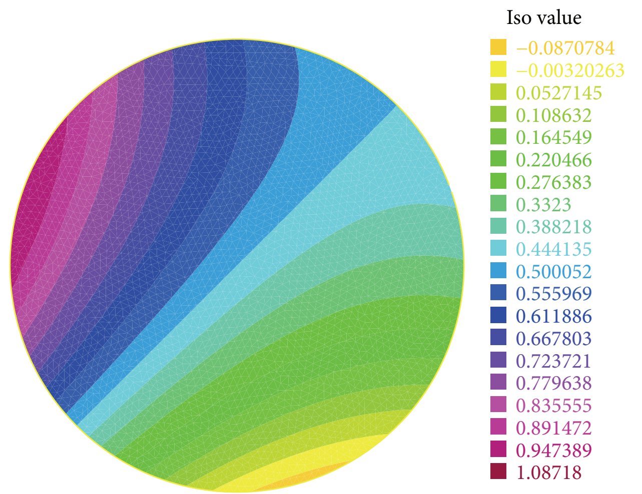

FIGURE 1: The exact solution.

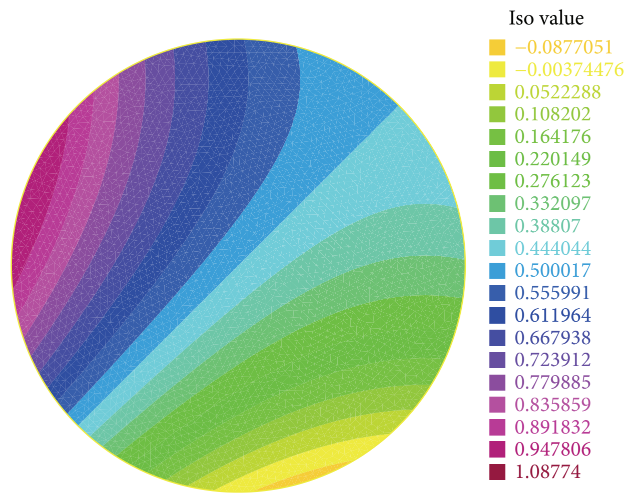

FIGURE 2: The numerical solution.

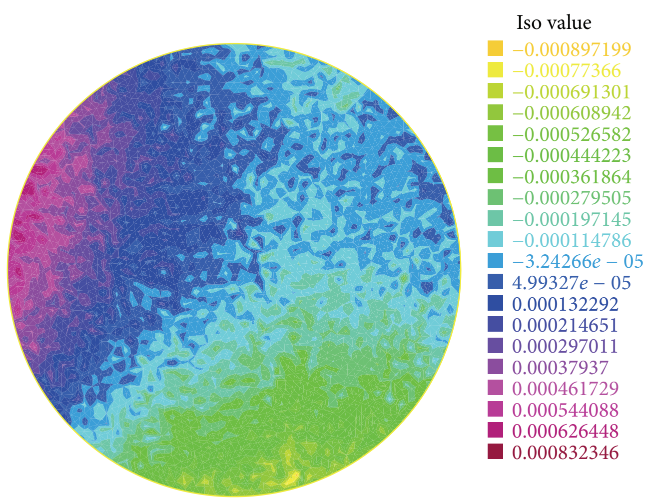

FIGURE 3: The difference between the exact and the numerical solutions.

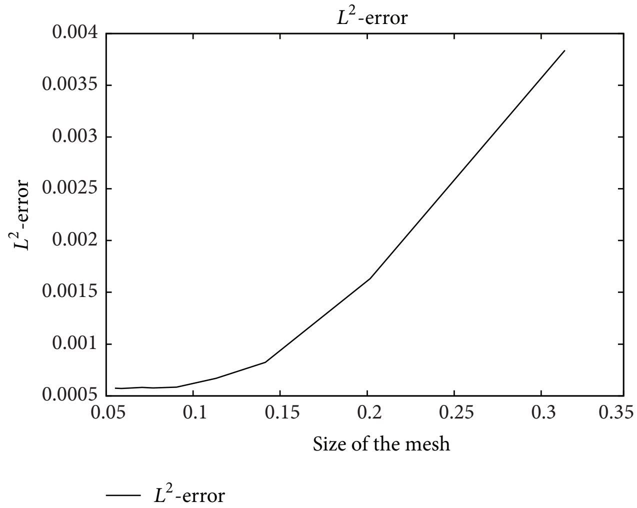

FIGURE 4: The L^2-error according to the fineness of the mesh.

5. Conclusion

As a conclusion, we have proved the convergence of the (CNS) by showing that the constructed family of operators satisfies the assumptions of Chernoff's product formula and consequently approximated numerically the Dirichlet-to-Neumann semigroup.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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