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Research Article

Asymptotic Behavior of Solutions of Free Boundary Problem with Logistic Reaction Term

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We study a free boundary problem for a reaction diffusion equation modeling the spreading of a biological or chemical species. In this model, the free boundary represents the spreading front of the species. We discuss the asymptotic behavior of bounded solutions and obtain a trichotomy result: spreading (the free boundary tends to $+\infty$ and the solution converges to a stationary solution defined on $[0+\infty)$), transition (the free boundary stays in a bounded interval and the solution converges to a stationary solution with positive compact support), and vanishing (the free boundary converges to 0 and the solution tends to 0 within a *finite* time).

1. Introduction

Consider the following free boundary problem:

$$\begin{gathered} u_t = u_{xx} + u(1-u), \quad 0 < x < h(t),\ t > 0, \\ u(t,0) = u(t,h(t)) = 0, \quad t > 0, \\ h'(t) = -\mu u_x(t,h(t)) - \mu\alpha, \quad t > 0, \\ h(0) = h_0, \qquad u(0,x) = u_0(x), \quad 0 \le x \le h_0, \end{gathered} \tag{1}$$

where $x = h(t)$ is a moving boundary to be determined together with $u(t,x)$ and $\alpha > 0$ is a given constant. The initial function u_0 belongs to $\mathscr{Y}(h_0)$ for some $h_0 > 0$, where

$$\begin{gathered} \mathscr{Y}(h_0) := \{\phi \in C^2([0,h_0]) : \phi(0) = \phi(h_0) = 0, \\ \phi(x) \ge (\not\equiv)\, 0 \text{ in } (0,h_0)\}. \end{gathered} \tag{2}$$

Recently, problem (1) with $\alpha = 0$ was studied by [1–3] and so forth. They used this model to describe the spreading of a new or invasive species; they used the free boundary $h(t)$ which represents the expanding front of the species whose density is represented by $u(t,x)$. They obtained a spreading-vanishing dichotomy result; namely, the species either spreads to the whole environment and stabilizes at the positive state 1 (i.e., $u \to 1$) or vanishes (i.e., $u \to 0$) as time goes to *infinity*. Such a result shows that problem (1) with $\alpha = 0$ has advantages comparing with the Cauchy problems (the Cauchy problems have hair-trigger effect: any positive solution which converges to a positive constant; cf. [4, 5]). In the last two years, [6] also studied the corresponding problem of (1) with $\alpha = 0$ in high dimension spaces.

In this paper, we mainly study problem (1) with $\alpha > 0$; such a boundary condition represents that there is a spreading resistant force at the front for some species. Intuitively, the presence of $\alpha > 0$ makes the solution more difficult to spread than the case where $\alpha = 0$. Indeed, $h'(t) > 0$ only if $u_x(t,h(t)) < -\alpha$. This boundary condition is widely used in many biological models. For example, it is often used in protocell models (cf. [7, 8]).

We give the following theorem whose proof is similar to that of [1, 2]. It suffices to repeat their arguments with obvious modification.

Theorem 1. *For any given $\gamma \in (0,1)$, there is a $T \in (0,+\infty)$ such that free boundary problem* (1) *has a solution*

$$(u,h) \in C^{((1+\gamma)/2),1+\gamma}\left(\overline{D}_T\right) \times C^{1+\gamma/2}([0,T]), \tag{3}$$

where $D_T := \{(t,x) \in \mathbb{R}^2 : x \in [0,h(t)], t \in (0,T]\}$, *and the solution can be extended to some interval* $(0,T_0)$ *with* $T_0 > T$ *as long as* $\inf_{0<t<T} h(t) > 0$.

Moreover, as in the proof of [9, Lemma 2.8], one can show that $h_\infty := \lim_{t\to T} h(t) \in [0,+\infty]$ exist.

The main purpose of this paper is to study the asymptotic behavior of bounded solutions of (1) and obtain trichotomy result. We will prove that, for a solution (u,h) of (1), one has either

(i) spreading: $h_\infty = +\infty$ and

$$\lim_{t\to\infty} u(t,x) = w(x) \quad \text{locally uniformly in } (0,+\infty), \tag{4}$$

where w is the unique positive solution of

$$\begin{gathered} q'' + q(1-q) = 0, \quad x > 0, \\ q(0) = 0, \end{gathered} \tag{5}$$

or

(ii) vanishing: $\lim_{t\to T} h(t) = 0$ and

$$T < +\infty, \quad \lim_{t\to T} \max_{0\le x\le h(t)} u(t,x) = 0 \tag{6}$$

or

(iii) transition: $0 < h_\infty < +\infty$ and

$$\lim_{t\to\infty} u(t,\cdot) = v(\cdot) \quad \text{locally uniformly in } (0,h_\infty), \tag{7}$$

where v is the solution of

$$\begin{gathered} v'' + v(1-v) = 0, \quad x \in (0,h_\infty), \\ v(0) = v(h_\infty) = 0, \qquad -v'(h_\infty) = \alpha. \end{gathered} \tag{8}$$

Remark 2. Comparing with the results in [1–3], the phenomenon (iii) is a new one, since it does not happen in case $\alpha = 0$.

Remark 3. (ii) shows that vanishing happens in *a finite* time and the free boundary converges to the *point* 0; those phenomena are also new and do not happen in case $\alpha = 0$.

2. Asymptotic Behavior of Solutions

In this section, we study the asymptotic behavior of solutions and obtain trichotomy result when $\alpha < \sqrt{3}/3$; namely, the solution of (1) is either vanishing (Theorem 6) or transition (Theorem 7) or spreading (Theorem 10). Then, we prove that only vanishing happens if $\alpha \ge \sqrt{3}/3$ (Theorem 11) for the completeness of the paper.

We first prepare the following comparison theorems which can be proved similarly as in [2, Lemma 3.5].

Lemma 4. *Suppose that* $T \in (0,\infty)$, $\overline{h} \in C^1([0,T])$, *and* $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \le T, 0 < x < \overline{h}(t)\}$ *and*

$$\begin{gathered} \overline{u}_t \ge \overline{u}_{xx} + \overline{u}(1-\overline{u}), \quad 0 < t \le T,\ 0 < x < \overline{h}(t), \\ \overline{u}(t,0) \ge 0, \qquad \overline{u}(t,\overline{h}(t)) = 0, \quad 0 < t \le T, \\ \overline{h}'(t) \ge -\mu\overline{u}(t,\overline{h}(t)) - \mu\alpha, \quad 0 < t \le T. \end{gathered} \tag{9}$$

If $h_0 \le \overline{h}(0)$ *and* $u_0(x) \le \overline{u}(0,x)$ *in* $[0,h_0]$ *and if* (u,h) *is a solution of* (1), *then*

$$h(t) \le \overline{h}(t), \qquad u(x,t) \le \overline{u}(x,t) \quad \text{for } t \in (0,T],\ x \in (0,h(t)). \tag{10}$$

Remark 5. The pair $(\overline{u},\overline{h})$ is usually called an upper solution of problem (1) and one can define a lower solution by revising all the inequalities.

Theorem 6. *Let* (u,h) *be a solution of* (1) *on* $[0,T^*)$. *If* $\lim_{t\to T^*} h(t) = 0$, *then* $T^* < +\infty$ *and*

$$\lim_{t\to T^*} \max_{0\le x\le h(t)} u(t,x) = 0. \tag{11}$$

Proof. By [2, 10], one can prove that there exists a constant C_1 such that $u(t,x) \le C_1$. In order to prove that u converges to 0, we need to construct the function

$$U(t,x) := C_1\left[2M(h(t)-x) - M^2(h(t)-x)^2\right] \tag{12}$$

over the region

$$Q := \left\{(t,x) : 0 < t < T^*, \max\left\{h(t) - M^{-1}, 0\right\} < x < h(t)\right\}, \tag{13}$$

where

$$M := \max\left\{\frac{\alpha + \sqrt{\alpha^2+2}}{2}, \frac{4\|u_0\|_{C^1([-h_0,h_0])}}{3C_1}\right\}. \tag{14}$$

Clearly $0 \le U \le C_1$ in Q. By the definitions of U and M, we have

$$U_t - U_{xx} - U(1-U) \ge C_1\left(2M^2 - 2M\alpha - 1\right) \ge 0 \quad \text{in } Q. \tag{15}$$

Moreover,

$$\begin{gathered} U(t,h(t)) = u(t,h(t)) = 0 \quad \text{for } t \in (0,T^*), \\ U(t,0) > 0 = u(t,0) \quad \text{when } h(t) < M^{-1}. \end{gathered} \tag{16}$$

Therefore, $u(t,x) \le U(t,x)$ in Q by the comparison principle Lemma 4. Note that $\lim_{t\to T^*} h(t) = 0$; then there exists $T_1 < T^*$ such that $h(t) - M^{-1} < 0$ for $t > T_1$. Therefore, $u(t,x) \le U(t,x)$ for $t > T_1$ and $x \in [0,h(t)]$. For such t and x, we have

$$U(t,x) \le 2MC_1 h(t) \longrightarrow 0 \quad \text{as } t \longrightarrow T^*; \tag{17}$$

it follows that

$$\|u(t,\cdot)\|_{L^\infty([0,h(t)])} \longrightarrow 0 \quad \text{as } t \longrightarrow T^*. \tag{18}$$

We now prove that $T^* < +\infty$. By $\lim_{t\to T^*} h(t) = 0$, there is some $L_* > 0$ such that

$$h(t) \le L_* \quad \text{for } t \in [0, T^*). \tag{19}$$

Set $L := 2(1 + L_*)$ and

$$\xi_0(x) := \frac{2\varepsilon}{L^2}\left(L^2 - x^2\right), \tag{20}$$

where $\varepsilon > 0$ is small such that

$$8\left(\alpha + \sqrt{\alpha^2+2}\right)\varepsilon \le \alpha, \qquad 32\varepsilon \le \alpha. \tag{21}$$

Consider the problem

$$\begin{aligned} &\xi_t = \xi_{xx} + 2\xi\left(1 - \frac{\xi}{2\varepsilon}\right), \quad 0 < x < \overline{h}(t),\ t > 0, \\ &\xi(t,0) = \xi\left(t, \overline{h}(t)\right) = 0, \quad t > 0, \\ &\overline{h}'(t) = -\mu\xi_x\left(t, \overline{h}(t)\right) - \mu\alpha, \quad t > 0, \\ &\overline{h}(0) = L, \qquad \xi(0,x) = \xi_0(x), \quad 0 \le x \le L. \end{aligned} \tag{22}$$

It is obvious that $\xi(t,x) \le 2\varepsilon$ for all $t \ge 0$. Construct a function

$$U^\varepsilon(t,x) := 2\varepsilon\left[2M\left(\overline{h}(t) - x\right) - M^2\left(\overline{h}(t) - x\right)^2\right] \tag{23}$$

over $\overline{Q} := \{(t,x) : t > 0, \max\{0, \overline{h}(t) - M^{-1}\} \le x \le \overline{h}(t)\}$, where $M := \max\{\alpha + \sqrt{\alpha^2+2}, 4\}$. Then $U^\varepsilon(t,x)$ is an upper solution of (22) over $\overline{Q}$ and so

$$-\xi_x\left(t, \overline{h}(t)\right) \le -U_x^\varepsilon\left(t, \overline{h}(t)\right) = 4M\varepsilon \le \frac{\alpha}{2}. \tag{24}$$

Therefore, $\overline{h}'(t) \le -\alpha\mu/2$. Thus, $\overline{h}(t) \to 0$ as $t \to \overline{T}^* \le 2L/\alpha\mu$.

On the other hand, (18) implies that there exists some $T_0 \in (0, T^*)$ such that $u(t,x) \le \varepsilon$ for all $x \in [0, h(t)]$ and $t > T_0$. Clearly $\xi_0(x) \ge u(T_0, x)$ for $x \in [0, h(T_0)]$. By the comparison principle, we have $h(t + T_0) \le \overline{h}(t)$, and so T^* cannot be ∞. □

Theorem 7. *Assume that $0 < \alpha < \sqrt{3}/3$. Let (u,h) be a solution of (1). If $0 < h_\infty < +\infty$, then*

$$\begin{aligned} &h_\infty = L_\alpha, \\ &\lim_{t\to\infty} u(t,\cdot) = v_\alpha(\cdot) \quad \text{locally uniformly in } (0, h_\infty), \end{aligned} \tag{25}$$

where v_α is a unique positive solution of

$$\begin{aligned} &v'' + v(1 - v) = 0, \quad 0 < x < L_\alpha, \\ &v(0) = v(L_\alpha) = 0, \qquad v'(0) = -v'(L_\alpha) = \alpha, \end{aligned} \tag{26}$$

where

$$L_\alpha := 2\int_0^B \frac{dr}{\sqrt{\alpha^2 - r^2 + (2/3)r^3}} \tag{27}$$

with $B \in (0,1)$ given by $\alpha^2 = 2\int_0^B s(1-s)ds$.

Remark 8. This is a new phenomenon. It never happens when $\alpha = 0$. Moreover, by the phase plane method, one can prove that $v_\alpha \to 0$ and $L_\alpha \to \pi$ as $\alpha \to 0$. This conclusion gives an explanation of Lemma 3.1 in [2]; that is, vanishing happens if $h_\infty \le \pi$.

Remark 9. It is easily seen that (26) has no positive solution when $\alpha \ge 2\int_0^1 s(1-s)ds = \sqrt{3}/3$.

Proof of Theorem 7. For any $\varepsilon > 0$, there exists $t^* > 0$ such that $h_\infty - \varepsilon < h(t) < h_\infty + \varepsilon$ for $t > t^*$. Let $\overline{u}_0(x)$ be a function defined on $(0, h_\infty + \varepsilon)$ and satisfies

$$\begin{aligned} &\overline{u}_0(x) \ge u(t^*, x) \quad \text{for } x \in (0, h_\infty), \\ &\overline{u}_0(0) = \overline{u}_0(h_\infty + \varepsilon) = 0. \end{aligned} \tag{28}$$

By the comparison principle we have $u(t,x) \le \overline{u}(t,x)$ in $(t^*, \infty) \times (0, h(t))$, where $\overline{u}(t,x)$ is the solution of

$$\begin{aligned} &\overline{u}_t = \overline{u}_{xx} + \overline{u}(1 - \overline{u}), \quad t > t^*,\ 0 < x < h_\infty + \varepsilon, \\ &\overline{u}(t,0) = \overline{u}(t, h_\infty + \varepsilon) = 0, \quad t > t^*, \\ &\overline{u}(t^*, x) = \overline{u}_0(x), \quad 0 < x < h_\infty + \varepsilon. \end{aligned} \tag{29}$$

It is well known that

(i) $\overline{u} \to 0$ as $t \to \infty$ if $h_\infty + \varepsilon \le \pi$; or

(ii) $\overline{u} \to \overline{u}_\varepsilon^*$ as $t \to \infty$ if $h_\infty + \varepsilon > \pi$,

where $\overline{u}_\varepsilon^*$ is a positive function. More precisely, when $h_\infty + \varepsilon > \pi$, it follows from [11, Corollary 3.4] that $\overline{u}_\varepsilon^*$ is the unique positive solution of

$$\begin{aligned} &(\overline{u}_\varepsilon^*)'' + \overline{u}_\varepsilon^*(1 - \overline{u}_\varepsilon^*) = 0, \quad 0 < x < h_\infty + \varepsilon, \\ &\overline{u}_\varepsilon^*(h_\infty + \varepsilon) = \overline{u}_\varepsilon^*(0) = 0. \end{aligned} \tag{30}$$

Hence,

$$\lim_{t\to\infty} u(t,x) = 0, \quad \text{or } \limsup_{t\to\infty} u(t,x) \le \overline{u}_\varepsilon^*. \tag{31}$$

Similarly,

$$\liminf_{t\to\infty} u(t,x) \ge \underline{u}_\varepsilon^*(x) \quad \text{when } h_\infty - \varepsilon > \pi, \tag{32}$$

where $\underline{u}_\varepsilon^*(x)$ is a positive solution of

$$\begin{aligned} &(\underline{u}_\varepsilon^*)'' + \underline{u}_\varepsilon^*(1 - \underline{u}_\varepsilon^*) = 0, \quad 0 < x < h_\infty - \varepsilon, \\ &\underline{u}_\varepsilon^*(h_\infty - \varepsilon) = \underline{u}_\varepsilon^*(0) = 0. \end{aligned} \tag{33}$$

We conclude from (31) and (32) that

$$\lim_{t\to\infty} u(t,x) = 0 \quad \text{if } h_\infty \le \pi, \tag{34}$$

or when $h_\infty > \pi$,

$$\lim_{t\to\infty} u(t,x) = u^*(x) \quad \text{locally uniformly in } (0,h_\infty), \tag{35}$$

where $u^*(x)$ is the unique positive solution of

$$(u^*)'' + u^*(1-u^*) = 0, \quad 0 < x < h_\infty, \qquad u^*(h_\infty) = u^*(0) = 0. \tag{36}$$

We now show that $\lim_{t\to\infty} u(t,x) = 0$ is impossible when $h_\infty > 0$. Suppose that this does not hold; there exists L_0 such that $h(t) \le L_0$. Then using the approach of proving $T^* < +\infty$ in Theorem 7, we can show that $\lim_{t\to T} h(t) = 0$ for some $0 < T < +\infty$; this contradicts the assumption $h_\infty > 0$. Hence, $\lim_{t\to\infty} u(t,x) = u^*(x)$, locally uniformly in $(0,h_\infty)$; we next prove that $u^*(x) = v_\alpha(x)$.

Make a change of the variable x to reduce $[0,h(t)]$ to the fixed interval $[0,h_0]$ and use L^p estimates as well as Sobolev embedding theorems on the reduced equation with Dirichlet boundary conditions to conclude that

$$\|u(t,\cdot) - u^*(\cdot)\|_{C^{1+(\gamma/2)}([0,h(t)])} \longrightarrow 0 \quad (t \longrightarrow \infty) \tag{37}$$

for some $\gamma > 0$. It follows that $h'(t) = -\mu u_x(t,h(t)) - \mu\alpha \to -\mu(u^*)'(h_\infty) - \mu\alpha$ as $t \to \infty$. Hence, we conclude that $(0,h_\infty)$ is not a finite interval unless $-(u^*)'(h_\infty) = \alpha$. □

Theorem 10. *Let (u,h) be a solution of* (1). *If $h_\infty = +\infty$, then*

$$\lim_{t\to\infty} u(t,x) = w(x) \quad \text{locally uniformly in } [0,+\infty), \tag{38}$$

where w is the unique positive solution of

$$q'' + q(1-q) = 0, \quad x > 0, \qquad q(0) = 0. \tag{39}$$

Proof. Choose a bounded continuous function $W_0(x) \ge u_0(x)$ for $x \in [0,h_0]$ and $W_0 \ge 0$ for $x \in [0,+\infty)$. Let $W(t,x)$ be the unique solution of

$$\begin{gathered} W_t = W_{xx} + W(1-W), \quad t > 0,\ x > 0, \\ W(t,0) = 0, \quad t > 0, \\ W(0,x) = W_0(x), \quad x > 0. \end{gathered} \tag{40}$$

Then the comparison principle theorem shows that $u(t,x) \le W(t,x)$ for $t > 0$, $x > 0$. Using [11, Lemma 3.4], we see that

$$\limsup_{t\to\infty} u(t,x) \le \lim_{t\to\infty} W(t,x) = w(x) \quad \text{for } x \in [0,+\infty). \tag{41}$$

On the other hand, since $h_\infty = +\infty$, for any large $l > \pi$, there is $\tau > 0$ such that $h(\tau) = l$ and $h(t) \ge l$ for all $t > \tau$. Let $\underline{u}_l(t,x)$ be the solution of the following problem:

$$\begin{gathered} \underline{u}_t = \underline{u}_{xx} + \underline{u}(1-\underline{u}), \quad t > \tau,\ 0 < x < l, \\ \underline{u}(t,0) = \underline{u}(t,l) = 0, \quad t > \tau, \\ \underline{u}(0,x) = \psi(x), \quad 0 < x < l, \end{gathered} \tag{42}$$

where ψ is a nonnegative continuous function satisfying $\psi(x) \le u(\tau,x)$ for $0 < x < l$. The comparison principle implies

$$\underline{u}_l(t,x) \le u(t,x) \quad \text{for } t > \tau,\ 0 \le x \le l. \tag{43}$$

By [11], one can obtain

$$\lim_{t\to\infty} \underline{u}_l(t,x) = v_l(x) \quad \text{uniformly in } [0,l], \tag{44}$$

where v_l is the positive solution of

$$v'' + v(1-v) = 0, \quad 0 < x < l, \qquad v(0) = v(l) = 0, \tag{45}$$

It is well known that $\lim_{l\to\infty} v_l(x) = w(x)$. Combining this with (43) and (44), we have

$$w(x) \le \liminf_{t\to\infty} u(t,x). \tag{46}$$

By (41) and (46), we have

$$\lim_{t\to\infty} u(t,x) = w(x). \tag{47}$$

□

Theorem 11. *Suppose that $\alpha \ge \sqrt{3}/3$ and (u,h) is a solution of* (1) *defined on some maximal existence interval $[0,T_*)$; then $T_* < +\infty$, u converges to 0 as $t \to T_*$, and $\lim_{t\to T_*} h(t) = 0$.*

Proof. The proof of this theorem is similar to [10]; it suffices to repeat their arguments with obvious modification. □

3. Example

In this section, we give some sufficient conditions for vanishing, spreading, and transition.

Example 1. Suppose that $\alpha < \sqrt{3}/3$. Let $h_0 > 0$ and $u_0(x) \in \mathscr{Y}(h_0)$; then the following properties hold:

(i) vanishing happens when $u_0(x) < v_\alpha(x)$;

(ii) spreading happens if $u_0(x) > v_\alpha(x)$ for $x \in [0,h_0]$;

(iii) transition happens if $u_0(x) \equiv v_\alpha(x)$ for $x \in [0,h_0]$.

Proof. (i) By [1], we see that $v_{\alpha_1}(x) < v_{\alpha_2}(x)$ for $\alpha_1 < \alpha_2$. Since $u_0(x) < v_\alpha(x)$, there is $\beta < \alpha$ such that $u_0(x) < v_\beta(x)$, by the comparison principle that $u(t,x) < v_\beta(x)$, so $h_\infty \ne +\infty$ and $h_\infty \ne L_\alpha$. It then follows from Theorem 6 that vanishing happens.

(ii) Let (u,h) be a solution of (1) with initial data $u_0(x)$; by the phase plane analysis, there is $\gamma > \alpha$ such that $u_0(x) > v_\gamma(x)$. It then follows from the comparison principle that $u(t,x) > v_\gamma(x)$, so Theorem 10 implies that $h_\infty = +\infty$ and spreading happens.

(iii) It follows from the comparison principle Lemma 4 that $u(t,x) \equiv v_\alpha(x)$ and $h(t) \equiv L_\alpha$ for all $t > 0$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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