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Research Article

Stochastic Current of Bifractional Brownian Motion

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We study the regularity of stochastic current defined as Skorohod integral with respect to bifractional Brownian motion through Malliavin calculus. Moreover, we similarly derive some results in the case of multidimensional multiparameter. Finally, we consider stochastic current of bifractional Brownian motion as a distribution in Watanabe spaces.

1. Introduction

The fractional Brownian motion was first introduced within a Hilbert space framework by Kolmogorov in [1]. It was further studied by Mandelbrot and Van Ness in [2], who provided a stochastic integral representation of this process in terms of a standard Brownian motion in 1968. In recent years, fractional Brownian motion has become an intense object in stochastic analysis and related fields for the moment, due to its interesting properties, such as self-similarity, and its applications in various scientific areas. However, when Hurst parameters $H \neq 1/2$, fractional Brownian motion is neither a semimartingale nor a Markovian process. The techniques used in Brownian motion cannot be directly applied.

Nevertheless, every fractional Brownian motion has its limits in modelling certain phenomena. In order to fit better in concrete situations, several authors have recently introduced some generalized fractional Brownian motions. For instance, we mention subfractional Brownian motion (see [3, 4]) and bifractional Brownian motion (see [5, 6]).

The concept of current comes from geometric measure theory. The simplest is the functional

$$\varphi \longrightarrow \int_0^T \left\langle \varphi\left(\gamma\left(t\right)\right), \gamma(t)' \right\rangle_{R^d} dt, \tag{1}$$

where $\varphi : R^d \rightarrow R^d$ and $\gamma(t)$ is a rectifiable curve. This functional $\xi(x)$ can be defined by

$$\xi\left(x\right) = \int_0^T \delta\left(x - \gamma\left(t\right)\right) \gamma(t)' dt, \tag{2}$$

where $\delta(x)$ is a Dirac function (see [7]). If we want to simulate this current, we need to replace the deterministic curve $\gamma(t)$ with stochastic process X_t. At the same time, the stochastic integral must be properly interpreted. Recently, people pay attentions to the research on stochastic current. Give the following map:

$$\varphi \longrightarrow I\left(\varphi\right) = \int_0^T \left\langle \varphi\left(X_t\right), dX_t \right\rangle, \tag{3}$$

where φ is a vector function on R^d which belongs to some Banach spaces V, X_t is a stochastic process, and the integral is some version of a stochastic integral defined through regularization. Stochastic current is a continuous version of the mapping; that is, stochastic current is regarded as a stochastic element of the dual space of V in [8].

The problem of stochastic current is motivated by the study of fluidodynamical models. In [9], in the study of the energy of a vortex filament naturally appear some stochastic double integrals related to Wiener process

$$\int_{[0,T]^2} f\left(X_s - X_t\right) dX_s\, dX_t, \tag{4}$$

where $f(x) = K_\alpha(x)$ is the kernel of the pseudodifferential operator $(1 - \Delta)^{-\alpha}$. In the recent years, some results of stochastic currents of Gaussian processes have been obtained through different stochastic integrals in [7, 8, 10]. For example, Flandoli and Tudor [7] have studied the existence and regularity of stochastic currents through Malliavin calculus, where the integrals are defined as Skorohod integrals with respect to the Brownian motion and fractional Brownian motion, respectively. In [10] authors have shown the Sobolev regularity of the stochastic current, which is associated with the pathwise integral.

Recall that the bifractional Brownian motion $B^{H,K}$ is a centered Gaussian process with covariance function

$$R_{H,K}(t,s)=\frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K-|t-s|^{2HK}\right), \quad (5)$$

where parameters $H\in(0,1)$ and $K\in(0,1]$. It is well known that, when $K=1$, bifractional Brownian motion is a fractional Brownian motion. Since bifractional Brownian motion seems to be more flexible and more complex model than fractional Brownian motion, it seems desirable to extend the stochastic current of fractional Brownian motion to the case of bifractional Brownian motion. For this aim, motivated by [7, 11], we use Malliavin calculus and multiple integrals to discuss the stochastic current defined as divergence integral with respect to bifractional Brownian motion. Let us compare our results with the analogous ones from the case of fractional Brownian stochastic current. Note that the regularity condition of bifractional Brownian current does not depend on parameters H and K, while the situation is different in the case of fractional Brownian motion. On the other hand, because the problems of bifractional Brownian motion are more complex, we need some useful techniques to deal with bifractional Brownian current.

The paper is organized as follows. In Section 2, we provide some background materials from bifractional Brownian motion. In Section 3, we firstly consider the regularity of stochastic current of bifractional Brownian motion with respect to x. Lastly, we regard stochastic current of bifractional Brownian motion as a distribution in Watanabe spaces.

2. Bifractional Brownian Motion

In this section, we briefly recall some notations and facts of bifractional Brownian motion, and for details see [5, 6, 11].

A bifractional Brownian motion $B^{H,K}$ is a center Gaussian process with variance

$$R_{H,K}(t,s)=\frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K-|t-s|^{2HK}\right), \quad (6)$$

where parameters $H\in(0,1)$ and $K\in(0,1]$. In the case $K=1$ we retrieve the fractional Brownian motion, while in the case $K=1$ and $H=1/2$ bifractional Brownian motion corresponds to the Brownian motion.

Let $\mathscr{H}$ be a Hilbert space. $\mathscr{H}$ is defined as the completion of the linear space generated by the $\mathbf{I}_{[0,t]}$, $t\in[0,T]$ with respect to the inner product

$$\begin{aligned}\langle \mathbf{I}_{[0,t]},\mathbf{I}_{[0,s]}\rangle_{\mathscr{H}} &= R_{H,K}(t,s)\\ &=\iint_0^T \mathbf{I}_{[0,t]}(u)\,\mathbf{I}_{[0,s]}(v)\,\frac{\partial^2 R_{H,K}(u,v)}{\partial u\partial v}du\,dv.\end{aligned} \quad (7)$$

Sometimes working with the space $\mathscr{H}$ is not convenient, because this space also contains distributions (see [11]) and the norm in this space is not always tractable. We always use the subspace $\mathscr{H}_1$ of $\mathscr{H}$, which is defined as the set of measurable functions f on $[0,T]$ with

$$\|f\|^2_{\mathscr{H}_1}\equiv\iint_0^T|f(u)||f(v)|\left|\frac{\partial^2 R_{H,K}(u,v)}{\partial u\partial v}\right|du\,dv<\infty. \quad (8)$$

We can prove that $\mathscr{H}_1$ is a Banach space for the norm $\|\cdot\|_{\mathscr{H}_1}$. At the same time, we have

$$L^2([0,T])\subset\mathscr{H}_1\subset\mathscr{H}. \quad (9)$$

Denote multiple stochastic integrals by $I_n(f_n)$ with respect to $B^{H,K}$, where $f_n\in\mathscr{H}^{\otimes n}$. For each $F\in L^2([0,T])$, F has chaos expansion $F=\sum_{n=0}^{\infty}I_n(f_n)$. Let L be Ornstein-Uhlenbeck operator

$$LF=-\sum_{n=0}^{\infty}nI_n(f_n). \quad (10)$$

For each $p\in(1,\infty)$ and $\alpha\in\mathbb{R}$, define Sobolev-Watanabe $D^{\alpha,p}$ as the closure of the set of polynomial variables with respect to the norm

$$\|F\|_{\alpha,p}=\left\|(Id-L)^{\alpha/2}\right\|_{L^p(\Omega)}, \quad (11)$$

where I denotes the identity. Malliavin derivative operator D is defined as follows:

$$D_t(I_n(f_n))=nI_{n-1}(f_n(\cdot,t)). \quad (12)$$

It is well known that stochastic variable F belongs to $D^{\alpha,2}$ if and only if

$$\sum_{n=0}^{\infty}(1+n)^{\alpha}\|I_n(f_n)\|^2_{L^2(\Omega)}<\infty. \quad (13)$$

The adjoint of D is always called the divergence integral (or Skorohod integral). For adapted integrands, the divergence integral coincides with the classical Itô integral. Hence the divergence integral is called generalized Itô integral. If u is a stochastic process, it has the following chaos expansion:

$$u_s=\sum_{n\geq0}I_n(f_n(\cdot,s)), \quad (14)$$

where $f_n(\cdot,s)\in\mathscr{H}^{\otimes(n+1)}$. Skorohod integral of u is defined as

$$\int_0^T u_s dB_s^{H,K}=\sum_{n\geq0}I_{n+1}\left(f_n(\cdot,s)^{(s)}\right), \quad (15)$$

where $f_n^{(s)}$ denotes the symmetrization of f_n with respect to $n+1$ variables.

3. Stochastic Current of Bifractional Brownian Motion

3.1. Stochastic Current of One-Dimensional Case with respect to x. In this section, we give stochastic current of bifractional Brownian motion as follows:

$$\xi(x) = \int_{[0,T]^N} \delta\left(x - B_s^{H,K}\right) dB_s^{H,K}, \tag{16}$$

where the integral is a Skorohod integral, $x \in \mathbb{R}$, and $T > 0$.

Put

$$\begin{aligned}\beta_n^x(s) &= \frac{p_{s^{2HK}}(x)}{\left[s^{2HK}\right]^{n/2}} H_n\left(\frac{x}{s^{HK}}\right) \\ &= \left(R_{H,K}(s)\right)^{-(n/2)} p_{R_{H,K}(s)}(x) H_n\left(\frac{x}{R_{H,K}(s)^{1/2}}\right),\end{aligned} \tag{17}$$

where $p_{s^{2HK}}(x)$ is a Gaussian kernel function of variance s^{2HK} and $H_n(x)$ is the Hermite polynomial of degree n.

By Lemma 3.1 in [7], the following lemma is obtained. Indeed, the lemma can be regarded as a version in the case of bifractional Brownian motion.

Lemma 1. *Use $\beta_n^{\hat{x}}(s)$ to denote the Fourier transform of the function $x \to \beta_n^x(s)$; then*

$$\beta_n^{\hat{x}}(s) = \exp\left\{-\frac{x^2}{2} R_{H,K}(s)\right\} \frac{(-i)^n x^n}{n!}. \tag{18}$$

Applying Lemma 1 and as in [7], we can obtain the stochastic current of bifractional Brownian motion.

Theorem 2. *Let $B^{H,K}$ be a bifractional Brownian motion with Hurst parameters $H \in (0,1)$, $K \in (0,1]$ satisfying $2HK > 1$ and let $\xi(x)$ be given by (16). Then, for each $\omega \in \Omega$ and when $r > 1/2$, $\xi(x)$ belongs to the negative Sobolev space $H^{-r}(\mathbb{R};\mathbb{R})$.*

Proof. By the chaos expansion of $\delta(x - B^{H,K})$ (see [11] or [5]), we have

$$\begin{aligned}\delta\left(x - B^{H,K}\right) &= \sum_{n\geq 0} \frac{p_{s^{2HK}}(x)}{s^{nHK}} H_n\left(\frac{x}{s^{2HK}}\right) I_n^{B^{H,K}}\left(\mathbf{I}_{[0,s]}^{\otimes n}(\cdot)\right) \\ &= \sum_{n\geq 0} \beta_n^x(s) I_n^{B^{H,K}}\left(\mathbf{I}_{[0,s]}^{\otimes n}(\cdot)\right),\end{aligned} \tag{19}$$

where $I_n^{B^{H,K}}$ denotes multiple stochastic integrals with respect to bifractional Brownian motion $B^{H,K}$.

Let us consider the Fourier transform of $\delta(x - B^{H,K})$:

$$\hat{\delta}\left(x - B^{H,K}\right) = \sum_{n\geq 0} \beta_n^{\hat{x}}(s) I_n^{B^{H,K}}\left(\mathbf{I}_{[0,s]}^{\otimes n}(\cdot)\right). \tag{20}$$

By Lemma 1, we obtain

$$\beta_n^{\hat{x}}(s) = \exp\left\{-\frac{x^2}{2} s^{2HK}\right\} \frac{(-ix)^n}{n!}. \tag{21}$$

Hence

$$\hat{\xi}(x) = \sum_{n\geq 0} \frac{(-ix)^n}{n!} I_{n+1}^{B^{H,K}}\left(\left(\exp\left\{-\frac{x^2}{2} s^{2HK}\right\} \mathbf{I}_{[0,s]}^{\otimes n}(\cdot)\right)^{(s)}\right), \tag{22}$$

where (s) denotes the symmetrization with respect to $n+1$ variables.

By the definition of $\|\cdot\|_{H^{-r}(\mathbb{R};\mathbb{R})}$ and taking advantage of (22), we get

$$\begin{aligned}E\|\xi(x)\|^2_{H^{-r}(\mathbb{R};\mathbb{R})} &= E\left[\int_{\mathbb{R}} \left(1+x^2\right)^{-r} \left|\hat{\xi}(x)\right|^2 dx\right] \\ &\leq \int_{\mathbb{R}} \left(1+x^2\right)^{-r} \sum_{n\geq 0} \frac{x^{2n}}{(n!)^2} (n+1)! \\ &\quad \times \left\|\left(\exp\left\{-\frac{x^2}{2} s^{2HK}\right\} \mathbf{I}_{[0,s]}^{\otimes n}(\cdot)\right)^{(s)}\right\|^2_{\mathscr{H}^{\otimes(n+1)}} dx.\end{aligned} \tag{23}$$

From the following fact:

$$\begin{aligned}&\left(\exp\left\{-\frac{x^2}{2} s^{2HK}\right\} \mathbf{I}_{[0,s]}^{\otimes n}(\cdot)\right)^{(s)} (t_1, \ldots, t_{n+1}) \\ &\quad = \frac{1}{n+1} \sum_{i=1}^{n+1} \exp\left\{-\frac{x^2}{2} t_i^{2HK}\right\} \mathbf{I}_{[0,t_i]}^{\otimes n}\left(t_1, \ldots, \hat{t}_i, \ldots, t_{n+1}\right),\end{aligned} \tag{24}$$

we show that

$$\begin{aligned}E\|\xi(x)\|^2_{H^{-r}(\mathbb{R};\mathbb{R})} &\leq \int_{\mathbb{R}} \left(1+x^2\right)^{-r} \sum_{n\geq 0} \frac{x^{2n}}{(n!)^2} (n+1)! \\ &\quad \times \iint_{[0,T]^{n+1}} \sum_{i,j=1}^{n} \frac{1}{(n+1)^2} \exp\left\{-\frac{x^2}{2} s_i^{2HK}\right\} \\ &\quad \cdot \exp\left\{-\frac{x^2}{2} t_j^{2HK}\right\} \mathbf{I}_{[0,s_i]}^{\otimes n}\left(s_1, \ldots, \hat{s}_i, \ldots, s_{n+1}\right) \\ &\quad \times \mathbf{I}_{[0,t_j]}^{\otimes n}\left(t_1, \ldots, \hat{t}_j, \ldots, t_{n+1}\right) \\ &\quad \cdot \prod_{q=1}^{n+1} \frac{\partial^2}{\partial s_q \partial t_q} R_{H,K} \\ &\quad \times \left(s_q, t_q\right) ds_1 \cdots ds_{n+1}\, dt_1 \cdots dt_{n+1}\, dx \\ &= \int_{\mathbb{R}} \left(1+x^2\right)^{-r} \sum_{n\geq 0} \frac{x^{2n}}{(n+1)!} \\ &\quad \times \sum_{i=1}^{n+1} \iint_{[0,T]^{n+1}} \exp\left\{-\frac{x^2}{2} s_i^{2HK}\right\} \\ &\quad \cdot \exp\left\{-\frac{x^2}{2} t_i^{2HK}\right\} \mathbf{I}_{[0,s_i]}^{\otimes n}\left(s_1, \ldots, \hat{s}_i, \ldots, s_{n+1}\right) \\ &\quad \times \mathbf{I}_{[0,t_i]}^{\otimes n}\left(t_1, \ldots, \hat{t}_i, \ldots, t_{n+1}\right) \\ &\quad \cdot \prod_{q=1}^{n+1} \frac{\partial^2}{\partial s_q \partial t_q} R_{H,K}\left(s_q, t_q\right) ds_1 \cdots ds_{n+1} dt_1 \cdots dt_{n+1} dx\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}}\left(1+x^2\right)^{-r}\sum_{n\geq 0}\frac{x^{2n}}{(n+1)!}\\
&\quad\times\sum_{i,j=1,i\neq j}^{n+1}\iint_{[0,T]^{n+1}}\exp\left\{-\frac{x^2}{2}s_i^{2HK}\right\}\\
&\qquad\cdot\exp\left\{-\frac{x^2}{2}t_j^{2HK}\right\}\\
&\qquad\times\mathbf{I}_{[0,s_i]}^{\otimes n}\left(s_1,\ldots,\widehat{s}_i,\ldots,s_{n+1}\right)\\
&\qquad\times\mathbf{I}_{[0,t_j]}^{\otimes n}\left(t_1,\ldots,\widehat{t}_j,\ldots,t_{n+1}\right)\\
&\qquad\cdot\prod_{q=1}^{n+1}\frac{\partial^2}{\partial s_q\partial t_q}\\
&\qquad\times R_{H,K}\left(s_q,t_q\right)ds_1\\
&\qquad\cdots ds_{n+1}dt_1\cdots dt_{n+1}dx\\
&\equiv\Delta_1+\Delta_2.
\end{aligned}
\tag{25}
$$

Firstly, we turn to estimate Δ_1. Using the similar methods in [7, 11] and the following fact:

$$
\iint_0^T\mathbf{I}_{[0,t]}(u)\,\mathbf{I}_{[0,s]}(v)\,\frac{\partial^2}{\partial u\partial v}R_{H,K}(u,v)\,du\,dv=R_{H,K}(t,s),
\tag{26}
$$

we find

$$
\begin{aligned}
\Delta_1&=\int_{\mathbb{R}}\left(1+x^2\right)^{-r}\sum_{n\geq 0}\frac{x^{2n}}{(n+1)!}\\
&\quad\times\sum_{i=1}^{n+1}\iint_{[0,T]^{n+1}}\exp\left\{-\frac{x^2}{2}s_1^{2HK}\right\}\\
&\qquad\cdot\exp\left\{-\frac{x^2}{2}t_1^{2HK}\right\}\mathbf{I}_{[0,s_1]}^{\otimes n}\left(s_2,\ldots,\widehat{s}_i,\ldots,s_{n+1}\right)\\
&\qquad\times\mathbf{I}_{[0,t_1]}^{\otimes n}\left(t_2,\ldots,\widehat{t}_j,\ldots,t_{n+1}\right)\\
&\qquad\cdot\prod_{q=2}^{n+1}\frac{\partial^2}{\partial s_q\partial t_q}R_{H,K}\left(s_q,t_q\right)\frac{\partial^2}{\partial s_1\partial t_1}\\
&\qquad\times R_{H,K}\left(s_1,t_1\right)ds_1\cdots ds_{n+1}\,dt_1\cdots dt_{n+1}\,dx\\
&=\int_{\mathbb{R}}\left(1+x^2\right)^{-r}\\
&\quad\times\sum_{n\geq 0}\frac{x^{2n}}{n!}\iint_{[0,T]}\exp\left\{-\frac{x^2}{2}s_1^{2HK}\right\}\\
&\qquad\cdot\exp\left\{-\frac{x^2}{2}t_1^{2HK}\right\}R_{H,K}^n\left(s_1,t_1\right)\frac{\partial^2}{\partial s_1\partial t_1}\\
&\qquad\times R_{H,K}\left(s_1,t_1\right)ds_1\,dt_1\,dx\\
&=\int_{\mathbb{R}}\left(1+x^2\right)^{-r}\iint_{[0,T]}\exp\left\{-\frac{x^2}{2}s_1^{2HK}\right\}\exp\left\{-\frac{x^2}{2}t_1^{2HK}\right\}\\
&\qquad\cdot\sum_{n\geq 0}\frac{x^{2n}}{n!}R_{H,K}^n\left(s_1,t_1\right)\frac{\partial^2}{\partial s_1\partial t_1}\\
&\qquad\times R_{H,K}\left(s_1,t_1\right)ds_1\,dt_1\,dx\\
&=\int_{\mathbb{R}}\left(1+x^2\right)^{-r}\iint_{[0,T]}\exp\left\{-\frac{x^2}{2}s_1^{2HK}\right\}\exp\left\{-\frac{x^2}{2}t_1^{2HK}\right\}\\
&\qquad\cdot\exp\left\{x^2R_{H,K}\left(s_1,t_1\right)\right\}\\
&\qquad\times\frac{\partial^2}{\partial s_1\partial t_1}R_{H,K}\left(s_1,t_1\right)ds_1\,dt_1\,dx,
\end{aligned}
\tag{27}
$$

where the last equality is established due to Taylor expansion formula of exponential function.

Since

$$
2R_{H,K}\left(s_1,t_1\right)-s_1^{2HK}-t_1^{2HK}=-E\left[\left(B_{s_1}^{H,K}-B_{t_1}^{H,K}\right)^2\right],
\tag{28}
$$

we get

$$
\begin{aligned}
&\exp\left\{\frac{x^2}{2}\left(2R_{H,K}\left(s_1,t_1\right)-s_1^{2HK}-t_1^{2HK}\right)\right\}\\
&\quad=\exp\left\{-\frac{x^2}{2}E\left[\left(B_{s_1}^{H,K}-B_{t_1}^{H,K}\right)^2\right]\right\}.
\end{aligned}
\tag{29}
$$

By [5], for each $s_1,t_1\in[0,T]$, we have

$$
2^{-K}\left|s_1-t_1\right|^{2HK}\leq E\left[\left(B_{s_1}^{H,K}-B_{t_1}^{H,K}\right)^2\right]\leq 2^{1-K}\left|s_1-t_1\right|^{2HK},
\tag{30}
$$

which implies that

$$
E\left[\left(B_{s_1}^{H,K}-B_{t_1}^{H,K}\right)^2\right]\geq 0.
\tag{31}
$$

Use the change of variables $y=x\{E[(B_{s_1}^{H,K}-B_{t_1}^{H,K})^2]\}^{1/2}$. Furthermore, we have

$$
\begin{aligned}
dx&=\left(s_1^{2HK}+t_1^{2HK}-\frac{1}{2^{K-1}}\right.\\
&\qquad\left.\times\left[\left(s_1^{2H}+t_1^{2H}\right)^K\left|s_1-t_1\right|^{2HK}\right]\right)^{-1/2}dy,\\
\left(1+x^2\right)^{-r}&=\left\{1+\left(s_1^{2HK}+t_1^{2HK}-\frac{1}{2^{K-1}}\right.\right.\\
&\qquad\left.\left.\times\left[\left(s_1^{2H}+t_1^{2H}\right)^K-\left|s_1-t_1\right|^{2HK}\right]\right)^{-1}y^2\right\}^{-r}.
\end{aligned}
\tag{32}
$$

On the other hand, by [11], there exists a constant $C_{3,1}(H,K)$ depending on H and K such that

$$
\left|\frac{\partial^2}{\partial s_1\partial t_1}R_{H,K}\left(s_1,t_1\right)\right|\leq C_{3,1}(H,K)\left(s_1t_1\right)^{HK-1}.
\tag{33}
$$

Putting (32)-(33) into (27), calculate

$$
\begin{aligned}
\Delta_1\leq C_{3,1}(H,K)\iint_0^T&\left(s_1t_1\right)^{HK-1}\\
&\times\left(s_1^{2HK}+t_1^{2HK}-\frac{1}{2^{K-1}}\right.\\
&\qquad\left.\times\left[\left(s_1^{2H}+t_1^{2H}\right)^K-\left|s_1-t_1\right|^{2HK}\right]\right)^{r-1/2}
\end{aligned}
$$

$$\times \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{2}\right\} \cdot \left(s_1^{2HK} + t_1^{2HK} - \frac{1}{2^{K-1}} \cdot \left[\left(s_1^{2H} + t_1^{2H}\right)^K - \left|s_1 - t_1\right|^{2HK}\right] + y^2\right)^{-r} dy\, ds_1\, dt_1$$

$$\leq C_{3,1}(H,K) \times \iint_0^T (s_1 t_1)^{HK-1} \left(s_1^{2HK} + t_1^{2HK} - \frac{1}{2^{K-1}} \times \left[\left(s_1^{2H} + t_1^{2H}\right)^K - \left|s_1 - t_1\right|^{2HK}\right]\right)^{r-1/2} \times \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{2}\right\} y^{-2r} dy\, ds_1\, dt_1. \tag{34}$$

When $HK - 1 + 2HK(r - 1/2) > -1$, that is, $r > 0$, (34) is finite.

Secondly, using the similar estimation method in the first part, consider the estimation of Δ_2 as follows:

$$\begin{aligned}\Delta_2 &= \int_{\mathbb{R}} \left(1 + x^2\right)^{-r} \sum_{n\geq 1} \frac{x^{2n}}{(n+1)!} n(n+1) \times \iint_{[0,T]^{n+1}} \exp\left\{-\frac{x^2}{2} s_1^{2HK}\right\} \cdot \exp\left\{-\frac{x^2}{2} t_2^{2HK}\right\} \mathbf{I}_{[0,s_1]}^{\otimes n}(s_2, \ldots, s_{n+1}) \\ &\quad \times \mathbf{I}_{[0,t_2]}^{\otimes n}(t_1, t_3, \ldots, t_{n+1}) \cdot \prod_{q=1}^{n+1} \frac{\partial^2}{\partial s_q \partial t_q} \times R_{H,K}\left(s_q, t_q\right) ds_1 \cdots ds_{n+1}\, dt_1 \cdots dt_{n+1}\, dx \\ &= \int_{\mathbb{R}} \left(1 + x^2\right)^{-r} \sum_{n\geq 1} \frac{x^{2n}}{(n-1)!} \times \iiiint_0^T \exp\left\{-\frac{x^2}{2} s_1^{2HK}\right\} \exp\left\{-\frac{x^2}{2} t_2^{2HK}\right\} \cdot \mathbf{I}_{[0,s_1]}(s_2)\, \mathbf{I}_{[0,t_2]}(t_1) \frac{\partial^2}{\partial s_1 \partial t_1} \\ &\quad \times R_{H,K}(s_1, t_1) \frac{\partial^2}{\partial s_2 \partial t_2} R_{H,K}(s_2, t_2) \cdot R_{H,K}^{n-1}(s_1, t_2)\, ds_1\, ds_2\, dt_1\, dt_2\, dx.\end{aligned} \tag{35}$$

By Taylor expansion formula, the following equality is obvious:

$$\begin{aligned}\Delta_2 &= \int_{\mathbb{R}} \left(1 + x^2\right)^{-r} x^2 \times \iiiint_0^T \exp\left\{-\frac{x^2}{2}\left(s_1^{2HK} + t_2^{2HK} - R_{H,K}(s_1, t_2)\right)\right\} \cdot \mathbf{I}_{[0,s_1]}(s_2)\, \mathbf{I}_{[0,t_2]}(t_1) \frac{\partial^2}{\partial s_1 \partial t_1} R_{H,K}(s_1, t_1) \\ &\quad \times \frac{\partial^2}{\partial s_2 \partial t_2} R_{H,K}(s_2, t_2) \cdot R_{H,K}^{n-1}(s_1, t_2)\, ds_1\, ds_2\, dt_1\, dt_2\, dx.\end{aligned} \tag{36}$$

Recalling some results in [11], there exist parameters H, K and a constant $C_{3,2}(H,K)$ depending on H and K such that

$$\left|\frac{\partial^2}{\partial s_2 \partial t_2} R_{H,K}(s_2, t_2)\right| \leq C_{3,2}(H,K)(s_2 t_2)^{HK-1}. \tag{37}$$

Thus, by (33) and (37), we obtain

$$\begin{aligned}\Delta_2 &\leq C_{3,1}(H,K)\, C_{3,2}(H,K) \times \int_{\mathbb{R}} \left(1 + x^2\right)^{-r} \times \iiiint_0^T x^2 \cdot \exp\left\{-\frac{x^2}{2}\left[s_1^{2HK} + t_2^{2HK} - \frac{1}{2^{K-1}} \times \left(\left(s_1^{2H} + t_2^{2H}\right)^K - \left|s_1 - t_2\right|^{2HK}\right)\right]\right\} \\ &\quad \cdot (s_1 t_1)^{HK-1} (s_2 t_2)^{HK-1} \mathbf{I}_{[0,s_1]}(s_2) \times \mathbf{I}_{[0,t_2]}(t_1)\, ds_1\, ds_2\, dt_1\, dt_2\, dx.\end{aligned} \tag{38}$$

Calculate

$$\begin{aligned}&\iiiint_0^T \exp\left\{-\frac{x^2}{2}\left[s_1^{2HK} + t_2^{2HK} - \frac{1}{2^{K-1}} \times \left(\left(s_1^{2H} + t_2^{2H}\right)^K - \left|s_1 - t_2\right|^{2HK}\right)\right]\right\} \cdot (s_1 t_1)^{HK-1} (s_2 t_2)^{HK-1} \mathbf{I}_{[0,s_1]}(s_2) \times \mathbf{I}_{[0,t_2]}(t_1)\, ds_1\, ds_2\, dt_1\, dt_2 \\ &= \iint_0^T \exp\left\{-\frac{x^2}{2}\left[s_1^{2HK} + t_2^{2HK} - \frac{1}{2^{K-1}} \times \left(\left(s_1^{2H} + t_2^{2H}\right)^K - \left|s_1 - t_2\right|^{2HK}\right)\right]\right\} \\ &\quad \cdot (s_1 t_2)^{HK-1} \left(\int_0^T s_2^{HK-1} \mathbf{I}_{[0,s_1]}(s_2)\, ds_2\right) \times \left(\int_0^T t_1^{HK-1} \mathbf{I}_{[0,t_2]}(t_1)\, dt_1\right) ds_1\, dt_2\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(HK)^2} \\
&\quad \times \iint_0^T \exp\left\{-\frac{x^2}{2}\left[s_1^{2HK} + t_2^{2HK} - \frac{1}{2^{K-1}}\right.\right. \\
&\qquad\qquad \left.\left.\times\left(\left(s_1^{2H} + t_2^{2H}\right)^K - \left|s_1 - t_2\right|^{2HK}\right)\right]\right\} \\
&\quad \times (s_1 t_2)^{2HK-1} ds_1\, dt_2.
\end{aligned} \tag{39}
$$

Use the change of variables $y = [s_1^{2HK} + t_2^{2HK} - (1/2^{K-1})((s_1^{2H} + t_2^{2H})^K - |s_1 - t_2|^{2HK})]^{1/2} x$. Hence

$$
\begin{aligned}
\Delta_2 &\le C_{3,3}(H,K) \\
&\quad \times \iint_0^T (s_1 t_2)^{2HK-1}\left(s_1^{2HK} + t_2^{2HK} - \frac{1}{2^{K-1}}\right. \\
&\qquad \times\left[\left(s_1^{2H} + t_2^{2H}\right)^K\right. \\
&\qquad \left.\left. - \left|s_1 - t_2\right|^{2HK}\right]\right)^{r-3/2} \\
&\quad \times \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{2}\right\} \\
&\quad \cdot\left(s_1^{2HK} + t_2^{2HK} - \frac{1}{2^{K-1}}\right. \\
&\qquad \times\left[\left(s_1^{2H} + t_2^{2H}\right)^K\right. \\
&\qquad \left. - \left|s_1 - t_2\right|^{2HK}\right] + y^2\Big)^{-r} dy\, ds_1\, dt_2 \\
&\le C_{3,3}(H,K) \iint_0^T (s_1 t_2)^{2HK-1} \\
&\qquad \times\left(s_1^{2HK} + t_2^{2HK} - \frac{1}{2^{K-1}}\right. \\
&\qquad \times\left[\left(s_1^{2H} + t_2^{2H}\right)^K\right. \\
&\qquad \left.\left. - \left|s_1 - t_2\right|^{2HK}\right]\right)^{r-3/2} \\
&\qquad \times \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{2}\right\} y^{-2r} dy\, ds_1\, dt_2,
\end{aligned} \tag{40}
$$

where $C_{3,3}(H,K) = C_{3,1}(H,K) C_{3,2}(H,K)$.

When $2HK - 1 + 2HK(r - 3/2) > -1$, (40) is finite. In other words, when $r > 1/2$, (40) is finite.

From what we have said above we can draw a conclusion that when $r > 1/2$,

$$
E\|\xi\|^2_{H^r(\mathbb{R};\mathbb{R})} < \infty. \tag{41}
$$

□

From Theorem 2 we see that when $r > 1/2$, the mapping $\xi(x) = \int_{[0,T]^N} \delta(x - B_s^{H,K}) dB_s^{H,K}$ belongs to negative Sobolev space $H^{-r}(\mathbb{R};\mathbb{R})$. Note that the regularity condition in Theorem 2 does not depend on H and K. The condition is interesting, because the condition of $\xi(x) = \int_{[0,T]^N} \delta(x - B_s) dB_s$ which belongs to negative Sobolev space is also $r > 1/2$, where B_s is the Brownian motion (see [7]). In other words, they have the same regularity condition. However, the situation is different in the case of fractional Brownian motion, because the regularity condition of fractional Brownian stochastic current is $r > 1/2H - 1/2$, which is dependent on Hurst parameter H.

3.2. Stochastic Current of d-Dimensional Case with respect to x. As in [7], we can extend stochastic current of one-dimensional bifractional Brownian motion to the case of d-dimensional bifractional Brownian motion.

Let $B^{H,K}$ be the vector valued bifractional Brownian motion; that is, $B^{H,K} = (B^{H_1,K_1}, \ldots, B^{H_d,K_d})$, where B^{H_i,K_i} are independent one-dimensional bifractional Brownian sheet. In this part, we consider $\xi(x)$ as follows:

$$
\begin{aligned}
\xi(x) = \Bigg(&\int_{[0,T]^N} \delta\left(x - B_s^{H,K}\right) dB_s^{H_1,K_1}, \ldots, \\
&\int_{[0,T]^N} \delta\left(x - B_s^{H,K}\right) dB_s^{H_d,K_d}\Bigg),
\end{aligned} \tag{42}
$$

where the integrals are Skorohod integrals with respect to bifractional Brownian motion.

Denote

$$
R_{H,K}^l(t,s) = \prod_{i=1}^N R_{H,K}^l(t_i, s_i) = \left\langle \mathbf{I}_{[\mathbf{0},\mathbf{t}]}, \mathbf{I}_{[\mathbf{0},\mathbf{s}]} \right\rangle_{\mathscr{H}_{H_l}}. \tag{43}
$$

Theorem 3. *Let $B^{H,K}$ be d-dimensional bifractional Brownian motion with parameters H_i and K_i satisfying $2H_iK_i > 1$ $(i = 1, \ldots, d)$ and let $\xi(x)$ be given by (42); then, for each $\omega \in \Omega$ and when $r > d/2 - 1$, $\xi(x)$ belongs to Sobolev space $H^{-r}(\mathbb{R};\mathbb{R})$.*

Proof. Denote $\xi_l(x)$ by

$$
\begin{aligned}
\xi_l(x) &= \int_{[0,T]^N} \delta\left(x - B_s^{H,K}\right) dB_s^{H_l,K_l} \\
&= \int_{[0,T]^N} \delta_l\left(x - B_s^{H,K}\right) \delta\left(x_l - B_s^{H_l,K_l}\right) dB_s^{H_l,K_l} \\
&= \int_{[0,T]^N} \delta_l\left(x - B_s^{H,K}\right) \\
&\qquad \times \sum_{n_l \ge 0} \beta_{n_l}^{x_l}(s)\, I_{n_l}^{B^{H_l,K_l}}\left(\mathbf{I}_{[\mathbf{0},\mathbf{s}]}^{\otimes n_l}(\cdot)\right) dB_s^{H_l,K_l},
\end{aligned} \tag{44}
$$

where $\delta_l(x - B_s^{H,K}) = \prod_{q=1, q\ne l}^d \delta(x_q - B_s^{H_q,K_q})$.

Calculate the Fourier transform of (44) as follows:

$$
\begin{aligned}
\widehat{\xi}_l(x) = \sum_{n_l \ge 0} I_{n_l+1}^{B^{H_l,K_l}}\Bigg(\Bigg(&\exp\left\{-i \sum_{r=1, r\ne l}^d x_r B_s^{H_r,K_r}\right\} \\
&\times \frac{(-i)^{n_l} x_l^{n_l}}{n_l!} \\
&\times \exp\left\{-\frac{x^2}{2}|s|^{2H_lK_l}\right\} \mathbf{I}_{[0,s]}^{\otimes n_l}(\cdot)\Bigg)^{(s)}\Bigg).
\end{aligned} \tag{45}
$$

According to the definition of the normal $\|\xi(x)\|^2_{H^{-r}(\mathbb{R};\mathbb{R})}$ and using Euler formula, we can prove that

$$\begin{aligned}
E\left|\widehat{\xi}_l(x)\right|^2
&= \sum_{n_l\geq 0}\frac{x_l^{2n_l}}{(n_l+1)!} \\
&\quad\times\sum_{i,j=1}^{n_l+1}\iint_{[0,T]^{N(n_l+1)}}\prod_{q=1}^{n_l+1}\frac{\partial^2}{\partial s\partial t}R^q_{H,K}(s,t) \\
&\qquad\cdot\cos\left(\sum_{r=1,r\neq l}^{d}x_r B^{H_r,K_r}_{s^i}\right)\cos\left(\sum_{r=1,r\neq l}^{d}x_r B^{H_r,K_r}_{t^j}\right) \\
&\qquad\times\exp\left\{-\frac{x_l^2}{2}\left|s^i\right|^{2H_lK_l}\right\}\cdot\exp\left\{-\frac{x_l^2}{2}\left|t^j\right|^{2H_lK_l}\right\} \\
&\qquad\times\mathbf{I}^{\otimes n_l}_{[0,s^i]}\left(s^1,\ldots,\widehat{s}^i,\ldots,s^{n_l+1}\right) \\
&\qquad\times\mathbf{I}^{\otimes n_l}_{[0,t^j]}\left(t^1,\ldots,\widehat{t}^j,\ldots,t^{n_l+1}\right)ds\,dt \\
&\quad+\sum_{n_l\geq 0}\frac{x_l^{2n_l}}{(n_l+1)!} \\
&\quad\times\sum_{i,j=1}^{n_l+1}\iint_{[0,T]^{N(n_l+1)}}\prod_{q=1}^{n_l+1}\frac{\partial^2}{\partial s\partial t}R^q_{H,K}(s,t) \\
&\qquad\cdot\sin\left(\sum_{r=1,r\neq l}^{d}x_r B^{H_r,K_r}_{s^i}\right)\sin\left(\sum_{r=1,r\neq l}^{d}x_r B^{H_r,K_r}_{t^j}\right) \\
&\qquad\times\exp\left\{-\frac{x_l^2}{2}\left|s^i\right|^{2H_lK_l}\right\} \\
&\qquad\cdot\exp\left\{-\frac{x_l^2}{2}\left|t^j\right|^{2H_lK_l}\right\}\mathbf{I}^{\otimes n_l}_{[0,s^i]}\left(s^1,\ldots,\widehat{s}^i,\ldots,s^{n_l+1}\right) \\
&\qquad\times\mathbf{I}^{\otimes n_l}_{[0,t^j]}\left(t^1,\ldots,\widehat{t}^j,\ldots,t^{n_l+1}\right)ds\,dt,
\end{aligned}\tag{46}$$

where $|s^i|^{2H_lK_l} = \prod_{j=1}^N(s^i_j)^{2H_lK_l}$, $\mathbf{I}_{[0,s^i]} = \prod_{j=1}^N\mathbf{I}_{[0,s^i_j]}$ and $R^q_{H,K}(s,t) = R_{H_q,K_q}(s^q,t^q)$.

By the bound of $\sin(x)$ and $\cos(x)$, we can get the following inequality:

$$\begin{aligned}
E\left|\widehat{\xi}_l(x)\right|^2
&\leq C_{3,4}\sum_{n_l\geq 0}\frac{x_l^{2n_l}}{(n_l+1)!} \\
&\quad\times\sum_{i,j=1}^{n_l+1}\iint_{[0,T]^{N(n_l+1)}}\prod_{q=1}^{n_l+1}\frac{\partial^2}{\partial s\partial t}R^q_{H,K}(s,t) \\
&\qquad\cdot\exp\left\{-\frac{x_l^2}{2}\left|s^i\right|^{2H_lK_l}\right\} \\
&\qquad\times\exp\left\{-\frac{x_l^2}{2}\left|t^j\right|^{2H_lK_l}\right\} \\
&\qquad\cdot\mathbf{I}^{\otimes n_l}_{[0,s^i]}\left(s^1,\ldots,\widehat{s}^i,\ldots,s^{n_l+1}\right) \\
&\qquad\times\mathbf{I}^{\otimes n_l}_{[0,t^j]}\left(t^1,\ldots,\widehat{t}^j,\ldots,t^{n_l+1}\right)ds\,dt \\
&\quad+C_{3,5}\sum_{n_l\geq 0}\frac{x_l^{2n_l}}{(n_l+1)!} \\
&\quad\times\sum_{i,j=1}^{n_l+1}\iint_{[0,T]^{N(n_l+1)}}\prod_{q=1}^{n_l+1}\frac{\partial^2}{\partial s\partial t}R^q_{H,K}(s,t) \\
&\qquad\cdot\exp\left\{-\frac{x_l^2}{2}\left|s^i\right|^{2H_lK_l}\right\}\exp\left\{-\frac{x_l^2}{2}\left|t^j\right|^{2H_lK_l}\right\} \\
&\qquad\cdot\mathbf{I}^{\otimes n_l}_{[0,s^i]}\left(s^1,\ldots,\widehat{s}^i,\ldots,s^{n_l+1}\right) \\
&\qquad\times\mathbf{I}^{\otimes n_l}_{[0,t^j]}\left(t^1,\ldots,\widehat{t}^j,\ldots,t^{n_l+1}\right)ds\,dt \\
&\equiv\Delta_{3,l}+\Delta_{4,l},
\end{aligned}\tag{47}$$

where $C_{3,4}$ and $C_{3,5}$ are both constants.

By the same estimation techniques of Δ_2 in Theorem 2, we can obtain the estimation of $\Delta_{4,l}$. Here we need to discuss the estimation of $\Delta_{3,l}$.

Applying $\iint_0^T\mathbf{I}_{[0,t]}(u)\mathbf{I}_{[0,s]}(v)(\partial^2/\partial u\partial v)R_{H,K}(u,v)dudv = R_{H,K}(t,s)$ again, we can write

$$\begin{aligned}
\Delta_{3,l} &= C_{3,4}\sum_{n_l\geq 0}\frac{x_l^{2n_l}}{n_l!} \\
&\quad\times\iint_{[0,T]^{N(n_l+1)}}\prod_{q=1}^{n_l+1}\frac{\partial^2}{\partial s^1\partial t^1}R^q_{H,K}\left(s^1,t^1\right) \\
&\qquad\cdot\exp\left\{-\frac{x_l^2}{2}\left|s^1\right|^{2H_lK_l}\right\}\exp\left\{-\frac{x_l^2}{2}\left|t^1\right|^{2H_lK_l}\right\} \\
&\qquad\cdot\mathbf{I}^{\otimes n_l}_{[0,s^1]}\left(s^2,\ldots,s^{n_l+1}\right) \\
&\qquad\times\mathbf{I}^{\otimes n_l}_{[0,t^1]}\left(t^2,\ldots,t^{n_l+1}\right)ds\,dt \\
&= C_{3,4}\sum_{n_l\geq 0}\frac{x_l^{2n_l}}{n_l!} \\
&\quad\times\iint_{[0,T]^N}\left(R^l_{H,K}\left(s^1,t^1\right)\right)^{n_l}\cdot\frac{\partial^2}{\partial s\partial t}R^l_{H,K}\left(s^1,t^1\right) \\
&\qquad\times\exp\left\{-\frac{x_l^2}{2}\left|s^1\right|^{2H_lK_l}\right\}\exp\left\{-\frac{x_l^2}{2}\left|t^1\right|^{2H_lK_l}\right\}ds^1dt^1 \\
&= C_{3,4}\iint_{[0,T]^N}\frac{\partial^2}{\partial s^1\partial t^1}R^l_{H,K}\left(s^1,t^1\right) \\
&\qquad\cdot\exp\left\{\frac{x_l^2}{2}\left(2R^l_{H,K}\left(s^1,t^1\right)\right.\right. \\
&\qquad\qquad\left.\left.-\left|s^1\right|^{2H_lK_l}-\left|t^1\right|^{2H_lK_l}\right)\right\}ds^1\,dt^1.
\end{aligned}\tag{48}$$

According to [11], there exists a constant $C_{3,6}(H,K,l)$ depending upon H, K, and l such that

$$\left|\frac{\partial^2}{\partial s^1\partial t^1}R^l_{H,K}\left(s^1,t^1\right)\right|\leq C_{3,6}(H,K,l)\left(s^1t^1\right)^{H_lK_l-1}.\tag{49}$$

Therefore

$$\int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-r}\Delta_{3,l}dx \le C_{3,6}(H,K,l) \times \int_{\mathbb{R}^d}\iint_{[0,T]^N}\left(s^1t^1\right)^{H_lK_l-1} \cdot \exp\left\{-\frac{x^2}{2}E\left[\left(B^{H_l,K_l}_{s^1}-B^{H_l,K_l}_{t^1}\right)^2\right]\right\}ds^1\,dt^1\,dx. \tag{50}$$

Use the change of variables $y_l = x_l\{E[(B^{H_l,K_l}_{s^1}-B^{H_l,K_l}_{t^1})^2]\}^{1/2}$. Thus

$$\begin{aligned}\int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-r}\Delta_{3,l}dx &\le C_{3,7}(H,K,l) \times \iint_{[0,T]^N}\left(s^1t^1\right)^{H_lK_l-1}\left\{E\left[\left(B^{H_l,K_l}_{s^1}-B^{H_l,K_l}_{t^1}\right)^2\right]\right\}^{r-d/2} \\ &\quad \cdot \int_{\mathbb{R}^d}\exp\left\{-\frac{y_l^2}{2}\right\} \times \left(E\left[\left(B^{H_l,K_l}_{s^1}-B^{H_l,K_l}_{t^1}\right)^2\right]+y_l^2\right)^{-r}ds^1\,dt^1\,dy,\end{aligned} \tag{51}$$

where $y=(y_1,\ldots,y_d)$.

On the other hand, by [6] for arbitrary $\varepsilon \ge 0$, $s^1,t^1 \in [\varepsilon,1]^N$, there exists a constant $C_{3,8}$ such that

$$E\left[\left(B^{H_l,K_l}_{s^1}-B^{H_l,K_l}_{t^1}\right)^2\right] \le C_{3,8}\sum_{j=1}^{N}\left|s^1_j-t^1_j\right|^{2H_lK_l}. \tag{52}$$

In order to be simple, here we only consider the case of $T=1$.

Comparing (51) with (52), we find that

$$\begin{aligned}\int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-r}\Delta_{3,l}dx &\le C_{3,9}\iint_{[0,T]^N}\left(s^1t^1\right)^{H_lK_l-1} \cdot \left(\sum_{j=1}^{N}\left|s^1_j-t^1_j\right|^{2H_jK_j}\right)^{r-d/2} \\ &\quad \times \int_{\mathbb{R}^d}\exp\left\{-\frac{y_l^2}{2}\right\}y^{-2r}dy\,ds\,dt.\end{aligned} \tag{53}$$

When $H_lK_l-1+2H_lK_l(r-d/2) > -1$, that is, $r > d/2-1$, (53) is finite. That is to say, when $r > d/2-1$, there is

$$\begin{aligned}\int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-r}E\left[\left|\widehat{\xi}(x)\right|^2\right]dx &= \int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-r}E\left[\left|\widehat{\xi}_1(x)\right|^2+\cdots+\left|\widehat{\xi}_d(x)\right|^2\right]dx \\ &< \infty.\end{aligned} \tag{54}$$

□

It is interesting to contrast Theorems 2 and 3 with Propositions 3 and 4 in [7]. Parameters H and K of bifractional Brownian motion do not affect the regularity condition of bifractional Brownian current. However, the regularity condition of stochastic current of one-dimensional fractional Brownian motion is different from the case of d-dimensional setting (see [7]). In other words, Hurst parameters H of fractional Brownian motion have influence on fractional Brownian currents in the case of different dimension.

3.3. Stochastic Current of d-Dimensional Bifractional Brownian Motion with respect to ω. Let $B^{H,K}$ be vector valued bifractional Brownian motion where vectors $H=(H_1,\ldots,H_d)\in(0,1)^d$ and $K=(K_1,\ldots,K_d)\in(0,1]^d$; that is, $B^{H,K}=(B^{H_1,K_1},\ldots,B^{H_d,K_d})$. In this part, $\xi(x)$ is given by

$$\xi(x)=\int_0^T\delta\left(x-B^{H,K}_s\right)dB^{H,K}_s, \tag{55}$$

where

$$\begin{aligned}\delta\left(x-B^{H,K}_s\right) &= \prod_{i=1}^{d}\delta\left(x_i-B^{H_i,K_i}_s\right) = \sum_{n=(n_1,\ldots,n_d)}\beta_n(s,x)\,I^{B^{H_i,K_i}}_n\left(\mathbf{I}^{\otimes|n|}_{[0,s]}(\cdot)\right), \\ \beta_n(s,x) &= \prod_{i=1}^{d}\frac{1}{R^i_{H,K}(s)^{n_i/2}}p_{R^i_{H,K}(x_i)}H_{n_i}\left(\frac{x_i}{R^i_{H,K}(s)^{1/2}}\right), \\ R^i_{H,K}(s) &= R^i_{H,K}(s,s)=s^{2H_iK_i}, \\ I^{B^{H,K}}_n\left(\mathbf{I}^{\otimes|n|}_{[0,s]}(\cdot)\right) &= \prod_{i=1}^{d}I^{B^{H_i,K_i}}_{n_i}\left(\mathbf{I}^{\otimes n_i}_{[0,s]}(\cdot)\right).\end{aligned} \tag{56}$$

Using the chaos expansion of divergent integral with respect to bifractional Brownian motion, we can obtain the following expression:

$$\begin{aligned}\xi_i(x) &= \int_0^T\delta\left(x-B^{H,K}_s\right)dB^{H_i,K_i}_s \\ &= \sum_{n=(n_1,\ldots,n_d)}I^{B^{H_i,K_i}}_{n_i+1}\left[\left(\beta_n(s,x)\,\mathbf{I}^{\otimes n_i}_{[0,s]}\left(s_1,\ldots,s_{n_i}\right)\mathbf{I}_{[0,T]}(s) \cdot \prod_{j=1,j\neq i}^{d}I^{B^{H_j,K_j}}_{n_j}\left(\mathbf{I}^{\otimes n_j}_{[0,s]}(\cdot)\right)\right)^{(s)}\right].\end{aligned} \tag{57}$$

Theorem 4. *Let $B^{H,K}$ be a bifractional Brownian motion with parameters $H=(H_1,\ldots,H_d)\in(0,1)^d$ and $K=(K_1,\ldots,K_d)\in(0,1]^d$ satisfying $2H_iK_i>1$ $(i=1,\ldots,d)$. If $\xi(x)$ is given by (55), then, for every $x\in\mathbb{R}^d$ and $\alpha < 1/2(HK)^*-d/2$, $\xi(x)$ is one member of Sobolev-Watanabe spaces $D^{\alpha-1,2}$, where $(HK)^*=\max\{H_1K_1,\ldots,H_dK_d\}$.*

Proof. In order to be convenient, we always replace the normal $\|\cdot\|_{\mathscr{H}^{\otimes(n+1)}}$ with the normal $\|\cdot\|_{\mathscr{H}_1^{\otimes(n+1)}}$. Using the chaos expansion of $\delta(x-B^{H,K})$ and by the definition of the normal of $\|\xi(x)\|^2_{2,\alpha-1}$, we verify that

$$\begin{aligned}
&\|\xi_i(x)\|^2_{2,\alpha-1}\\
&\quad\le\sum_{m\ge1}(m+1)^{\alpha-1}\sum_{|n|=n_1+\cdots+n_d=m-1}(n_i+1)!\\
&\qquad\cdot\iint_{[0,T]^{n_i+1}}\sum_{l,k=1}^{n_i+1}\beta_n(s_l)\,\beta_n(t_k)\,\mathbf{I}_{[0,T]}(s_l)\,\mathbf{I}_{[0,T]}(t_k)\\
&\qquad\quad\cdot\mathbf{I}^{\otimes n_i}_{[0,s_l]}\left(s_1,\ldots,\widehat{s}_l,\ldots,s_{n_i+1}\right)\\
&\qquad\quad\times\mathbf{I}^{\otimes n_i}_{[0,t_k]}\left(t_1,\ldots,\widehat{t}_k,\ldots,t_{n_i+1}\right)\\
&\qquad\quad\cdot\prod_{j=1,j\ne i}^{d}n_j!R^j_{H,K}(s_l,t_k)^{n_j}\\
&\qquad\quad\times\prod_{q=1}^{n_i+1}\frac{\partial^2}{\partial s_q\partial t_q}R^i_{H,K}\left(s_q,t_q\right)dt_1\cdots dt_{n_i+1}\,ds_1\cdots ds_{n_i+1}.
\end{aligned}\tag{58}$$

Because $\alpha<0$, the following inequalities: $(m+2)^{\alpha-1}\le(m+1)^{\alpha-1}$ and $n_i+1\le m+1$, are obvious. Similarly to what is performed in [11] (or see [7]), we have

$$\begin{aligned}
\|\xi_i(x)\|^2_{2,\alpha-1}&\le\sum_{m\ge0}(m+1)^{\alpha}\sum_{|n|=n_1+\cdots+n_d=m}n_i!\\
&\quad\times\left[\left(1-\frac{1}{n_i+1}\right)\right.\\
&\qquad\times\iint_{[0,T]^2}\beta_n(s_1)\,\beta_n(t_2)\\
&\qquad\quad\cdot\mathbf{I}_{[0,T]}(s_1)\,\mathbf{I}_{[0,T]}(t_2)\\
&\qquad\quad\left.\times R^i_{H,K}(s_1,t_2)^{n_i-1}\right]\\
&\qquad\quad\times\prod_{j=1,j\ne i}^{d}n_j!R^j_{H,K}(s_1,t_2)^{n_j}\cdot\mathbf{I}_{[0,s_1]}(s_2)\\
&\qquad\quad\times\mathbf{I}_{[0,t_2]}(t_1)\,\frac{\partial^2}{\partial s_1\partial t_1}R^i_{H,K}(s_1,t_1)\\
&\qquad\quad\times\frac{\partial^2}{\partial s_2\partial t_2}R^i_{H,K}(s_2,t_2)\,ds_1\,ds_2\,dt_1\,dt_2\\
&\quad+\frac{1}{n_i+1}\\
&\quad\times\iint_{[0,T]^2}\beta_n(s_1)\,\beta_n(t_1)\,\mathbf{I}_{[0,T]}(s_1)\\
&\qquad\times\mathbf{I}_{[0,T]}(t_1)\,R^i_{H,K}(s_1,t_1)^{n_i-1}\\
&\qquad\cdot\prod_{j=1,j\ne i}^{d}n_j!R^j_{H,K}(s_1,t_1)^{n_j}\mathbf{I}_{[0,s_1]}(s_2)\\
&\qquad\times\mathbf{I}_{[0,t_1]}(t_2)\cdot\frac{\partial^2}{\partial s_1\partial t_1}R^i_{H,K}(s_1,t_1)\,\frac{\partial^2}{\partial s_2\partial t_2}\\
&\qquad\times R^i_{H,K}(s_2,t_2)\,ds_1\,ds_2\,dt_1\,dt_2\\
&\le\sum_{m\ge0}(m+1)^{\alpha}\sum_{|n|=n_1+\cdots+n_d=m}n_i!\\
&\quad\times\left[\left(1-\frac{1}{n_i+1}\right)\right.\\
&\qquad\times\int_{[0,T]^2}\beta_n(s_1)\,\beta_n(t_2)\\
&\qquad\quad\cdot R^i_{H,K}(s_1,t_2)^{n_i-1}\\
&\qquad\quad\times\prod_{j=1,j\ne i}^{d}n_j!R^j_{H,K}(s_1,t_2)^{n_j-1}\\
&\qquad\quad\cdot\frac{\partial}{\partial s_1}R^i_{H,K}(s_1,t_2)\\
&\qquad\quad\times\frac{\partial}{\partial t_2}R^i_{H,K}(s_1,t_2)\,ds_1\,dt_2\\
&\qquad+\frac{1}{n_i+1}\\
&\qquad\times\int_{[0,T]^2}\beta_n(s_1)\,\beta_n(t_1)\,R^i_{H,K}(s_1,t_1)^{n_i-1}\\
&\qquad\quad\cdot\prod_{j=1,j\ne i}^{d}n_j!R^j_{H,K}(s_1,t_1)^{n_j}\\
&\qquad\quad\left.\times\frac{\partial^2}{\partial s_1\partial t_1}R^i_{H,K}(s_1,t_1)\,ds_1\,dt_1\right].
\end{aligned}\tag{59}$$

According to (4.37) in [6], for $\beta\in[1/4,1/2)$, it follows that

$$\prod_{j=1}^{d}\beta_n(u)\le C_{3,10}\prod_{j=1}^{d}\frac{1}{\sqrt{n_j!}\left(n_j\vee1\right)^{((8\beta-1)/12)}},\tag{60}$$

where $C_{3,10}$ is a constant.

On the other hand, as in [11] and using the inequality $a^2+b^2\le2ab$, for $a,b\in\mathbb{R}_+$, there exists a constant $C_{3,11}(H,K)$ depending on H and K such that

$$\begin{gathered}
\left|\frac{\partial R^i_{H,K}(s,t)}{\partial t}\,\frac{\partial R^i_{H,K}(s,t)}{\partial s}\right|\le C_{3,11}(H,K)\,(st)^{2H_iK_i-1},\\
\left|\frac{\partial^2R^i_{H,K}(s,t)}{\partial s\partial t}\right|\le C_{3,11}(H,K)\,(st)^{H_iK_i-1},\\
\left|\frac{R^i_{H,K}(s,t)}{(st)^{H_iK_i}}\right|\le C_{3,11}(H,K).
\end{gathered}\tag{61}$$

Putting (60) and (61) into (59) and applying the self-similarity of the covariance $R(s,t)$, we have

$$\begin{aligned}
&\|\xi_i(x)\|^2_{2,\alpha-1}\\
&\quad\le C_{3,12}\sum_{m\ge0}(1+m)^{\alpha}\\
&\qquad\times\sum_{|n|=n_1+\cdots+n_d=m}\prod_{j=1}^{d}\frac{1}{\left(n_j\vee1\right)^{(8\beta-1)/6}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{[0,T]^2} \frac{R^i_{H,K}(s,t)^{n_i-1}}{(st)^{(n_i-1)H_iK_i}} (st)^{H_iK_i-1} \\
&\qquad \times \prod_{j=1,j\neq i}^{d} \frac{R^j_{H,K}(s,t)^{n_j}}{(st)^{n_jH_jK_j}} ds\, dt \\
&\leq C_{3,12} \sum_{m\geq 0} (1+m)^{\alpha} \\
&\quad \times \sum_{|n|=n_1+\cdots+n_d=m} \prod_{j=1}^{d} \frac{1}{\left(n_j \vee 1\right)^{(8\beta-1)/6}} \\
&\quad \cdot \int_0^T t^{2H_iK_i-1} dt \int_0^1 z^{H_iK_i-1} \left(\frac{R^i_{H_i,K_i}(1,z)}{z^{H_iK_i}} \right)^{n_i-1} \\
&\qquad\qquad \times \prod_{j=1,j\neq i}^{d} \frac{R^j_{H,K}(1,z)^{n_j}}{z^{H_jK_j}} dz,
\end{aligned}
\tag{62}
$$

where we use the change of variables $t = t$ and $z = s/r$ (see [6, 11]). According to (4.9) in [6], the bounded condition of integral part in (62) is proved.

Indeed,

$$
\int_0^1 \prod_{j=1}^{d} \left(\frac{R^j_{H,K}(1,z)}{z^{H_jK_j}} \right)^{n_j} dz \leq C_{3,13} m^{-1/2(HK)^*}, \tag{63}
$$

where $(HK)^* = \max\{H_1K_1, \ldots, H_dK_d\}$.

Therefore, when $\alpha < 1/2(HK)^* - d(1 - (8\beta - 1)/6)$,

$$
\begin{aligned}
&\left\|\xi_i(x)\right\|^2_{2,\alpha-1} \\
&\quad \leq C_{3,14} \sum_{m\geq 1} (1+m)^{\alpha} m^{-1/2(HK)^*-1+d(1-(8\beta-1)/6)} < \infty.
\end{aligned}
\tag{64}
$$

Since $\beta \in [1/4, 1/2)$, we may choose β to tend to 1/2. Hence, when $\alpha < 1/2(HK)^* - d/2$, (64) is finite. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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