

Research Article

New Results on Robust Stability and Stabilization of Linear Discrete-Time Stochastic Systems with Convex Polytopic Uncertainties

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This paper addresses the robust stability for a class of linear discrete-time stochastic systems with convex polytopic uncertainties. The system to be considered is subject to both interval time-varying delays and convex polytopic type uncertainties. Based on the augmented parameter-dependent Lyapunov-Krasovskii functional, new delay-dependent conditions for the robust stability are established in terms of linear matrix inequalities. An application to robust stabilization of linear discrete-time stochastic control systems is given. Numerical examples are included to illustrate the effectiveness of our results.

1. Introduction

In the past decades, the problem of stability for neutral differential systems, which have delays in both its state and the derivatives of its states, has been widely investigated by many researchers. Such systems are often encountered in engineering, biology, and economics. The existence of time delay is frequently a source of instability or poor performances in the systems. Recently, some stability criteria for neutral system with time delay have been given in [1–8] and the references therein. Some delay-dependent stability criteria for discrete-time systems with time-varying delay are investigated in [2, 6, 9–11], where the discrete Lyapunov functional method is employed to prove stability conditions in terms of linear matrix inequalities (LMIs). A number of research works for dealing with asymptotic stability problem for discrete systems with interval time-varying delays have been presented in [12–24]. Theoretically, stability analysis of the systems with time-varying delays is more complicated, especially for the case where the system matrices belong to some convex polytope. In this case, the parameter-dependent Lyapunov-Krasovskii functionals are constructed as the convex combination of a set of functions assures the robust

stability of the nominal systems, and the stability conditions must be solved upon a grid on the parameter space, which results in testing a finite number of linear matrix inequalities (LMIs) [11, 25, 26]. To the best of the authors' knowledge, the stability for linear discrete-time systems with both time-varying delays and polytopic uncertainties has not been fully investigated. The papers [27, 28] propose sufficient conditions for robust stability of discrete and continuous polytopic systems without time delays. More recently, combining the ideas in [25, 26], improved conditions for \mathcal{D} -stability and \mathcal{D} -stabilization of linear polytopic delay-difference equations with constant delays have been proposed in [29]. To the best of our knowledge, the stability and stabilization of linear discrete-time stochastic systems with convex polytopic uncertainties, nondifferentiable time-varying delays has not been fully studied yet (see, e.g., [1, 3–11, 13–36] and the references therein), which are important in both theories and applications. This motivates our research.

In this paper, we consider polytopic discrete-time stochastic equations with interval time-varying delays. By using the parameter-dependent Lyapunov-Krasovskii functional combined with LMI techniques, we propose new criteria for the robust stability of the stochastic system. The

delay-dependent stability conditions are formulated in terms of LMIs, being thus solvable by the numeric technology available in the literature to date. The result is applied to robust stabilization of linear discrete-time stochastic control systems. Compared to other results, our result has its own advantages. First, it deals with the delay-difference stochastic system, where the state-space data belong to the convex polytope of uncertainties and the rate of change of the state depends not only on the current state of the systems, but also its state at some times in the past. Second, the time delay is assumed to be a time-varying function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available. Third, our approach allows us to apply in robust stabilization of the linear discrete-time stochastic system subjected to polytopic uncertainties and external controls. Therefore, our results are more general than the related previous results.

The paper is organized as follows. Section 2 introduces the main notations, definitions, and some lemmas needed for the development of the main results. In Section 3, sufficient conditions are derived for robust stability, stabilization of discrete-time stochastic systems with interval time-varying delays, and polytopic uncertainties. They are followed by some remarks. Illustrative examples are given in Section 4.

2. Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all real nonnegative numbers; R^n denotes the n -dimensional space with the scalar product $\langle \cdot, \cdot \rangle$ and the vector norm $\| \cdot \|$; $R^{n \times r}$ denotes the space of all real matrices of $(n \times r)$ -dimension. A^T denotes the transpose of A ; a matrix A is symmetric if $A = A^T$, and a matrix I is the identity matrix of appropriate dimension.

Matrix A is semipositive definite ($A \geq 0$) if $\langle Ax, x \rangle \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $\langle Ax, x \rangle > 0$ for all $x \neq 0$; $A \geq B$ means that $A - B \geq 0$.

Consider delay-difference stochastic systems with polytopic uncertainties of the form

$$\begin{aligned} x(k+1) &= A(\xi)x(k) + D(\xi)x(k-h(k)) \\ &\quad + \sigma(x(k), x(k-h(k)), k)\omega(k), \\ &\quad k = 0, 1, 2, \dots, \\ x(k) &= v_k, \quad k = -h_2, -h_2 + 1, \dots, 0, \end{aligned} \quad (1)$$

where $x(k) \in R^n$ is the state (Figures 1 and 2), and the system matrices are subjected to uncertainties and belong to the polytope Ω given by

$$\Omega = \left\{ [A, D](\xi) := \sum_{i=1}^p \xi_i [A_i, D_i], \sum_{i=1}^p \xi_i = 1, \xi_i \geq 0 \right\}, \quad (2)$$

where $A_i, D_i, i = 1, 2, \dots, p$, are given constant matrices with appropriate dimensions, $\omega(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$\begin{aligned} E[\omega(k)] &= 0, \quad E[\omega^2(k)] = 1, \\ E[\omega(i)\omega(j)] &= 0 \quad (i \neq j), \end{aligned} \quad (3)$$

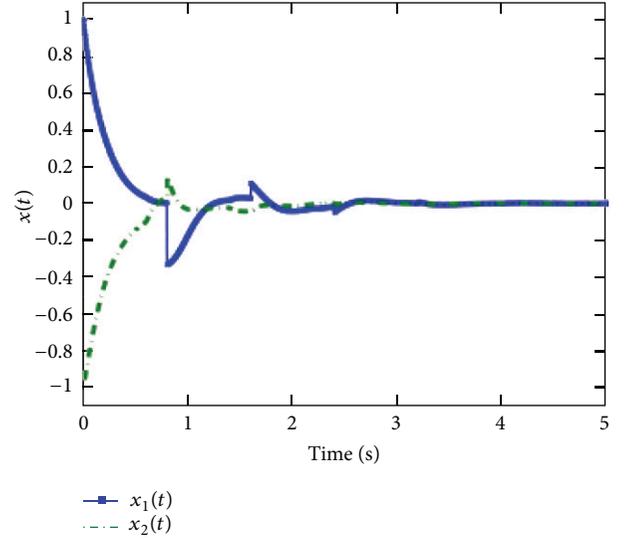


FIGURE 1: The simulation of the solutions $x_1(k)$ and $x_2(k)$ with the initial condition $\phi(k) = [10 \ 5]^T, k \in [0, 10]$.

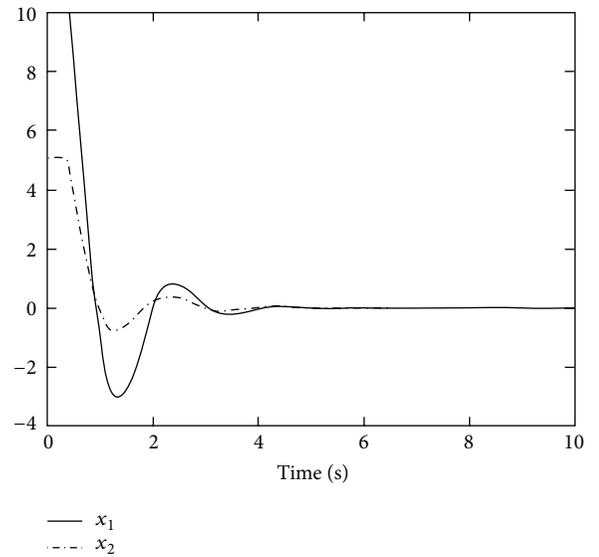


FIGURE 2: The simulation of the solutions $x_1(k)$ and $x_2(k)$ with the initial condition $\phi(k) = [10 \ 5]^T, k \in [0, 10]$.

and $\sigma: R^n \times R^n \times R \rightarrow R^n$ is the continuous function and is assumed to satisfy that

$$\begin{aligned} &\sigma^T(x(k), x(k-h(k)), k)\sigma(x(k), x(k-h(k)), k) \\ &\leq \rho_1 x^T(k)x(k) + \rho_2 x^T(k-h(k))x(k-h(k)), \quad (4) \\ &x(k), x(k-h(k)) \in R^n, \end{aligned}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are known constant scalars.

For simplicity, we denote $\sigma(x(k), x(k-h(k)), k)$ by σ , respectively.

The time-varying function $h(k)$ satisfies the condition:

$$0 < h_1 \leq h(k) \leq h_2, \quad \forall k = 0, 1, 2, \dots \quad (5)$$

Remark 1. It is worth noting that the time delay is a time-varying function belonging to a given interval, which allows the time delay to be a fast time-varying function, and the lower bound is not restricted to being zero as considered in [2, 6, 9–11, 18–24, 30–33].

Definition 2. The system (1) is robustly stable in the mean square if there exists a positive definite scalar function $V(k, x(k)): R^n \times R^n \rightarrow R$ such that

$$\begin{aligned} E[\Delta V(k, x(k))] \\ = E[V(k+1, x(k+1)) - V(k, x(k))] < 0, \end{aligned} \quad (6)$$

along any trajectory of zero solution of the system (1) for all uncertainties in Ω .

Proposition 3. For real numbers $\xi_i \geq 0, i = 1, 2, \dots, p, \sum_{i=1}^p \xi_i = 1$, the following inequality holds:

$$(p-1) \sum_{i=1}^p \xi_i^2 - 2 \sum_{i=1}^{p-1} \sum_{j=i+1}^p \xi_i \xi_j \geq 0. \quad (7)$$

Proof. The proof is followed from completing the square

$$(p-1) \sum_{i=1}^p \xi_i^2 - 2 \sum_{i=1}^{p-1} \sum_{j=i+1}^p \xi_i \xi_j = \sum_{i=1}^{p-1} \sum_{j=i+1}^p (\xi_i - \xi_j)^2 \geq 0. \quad (8)$$

□

3. Main Results

3.1. Robust Stability. In this section, we present sufficient delay-dependent conditions for the robust stability of system (1). Let us set

$$\|x_k\| = \sup_{s \in [-h_2, 0]} \|x(k+s)\|,$$

$$\mathcal{M}_{ij}(P, Q, S_1, S_2) = \begin{pmatrix} (h_2 - h_1 + 1)Q_i - P_i - S_{1i}A_j - A_j^T S_{1i}^T + 2\rho_1 I & S_{1i} - S_{1i}A_j & -S_{1i}D_j - S_{2i}A_j \\ S_{1i}^T - A_j^T S_{1i}^T & P_i + S_{1i} + S_{1i}^T & S_{2i} - S_{1i}D_j \\ -D_j^T S_{1i}^T - A_j^T S_{2i}^T & S_{2i}^T - D_j^T S_{1i}^T & -Q_i - S_{2i}D_j - D_j^T S_{2i}^T + 2\rho_2 I \end{pmatrix}, \quad (9)$$

$$\mathcal{S} = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P(\xi) = \sum_{i=1}^p \xi_i P_i, \quad Q(\xi) = \sum_{i=1}^p \xi_i Q_i, \quad S_1(\xi) = \sum_{i=1}^p \xi_i S_{1i}, \quad S_2(\xi) = \sum_{i=1}^p \xi_i S_{2i}.$$

Theorem 4. The system (1) is robustly stable in the mean square if there exist symmetric matrices $P_i > 0, Q_i > 0, i = 1, 2, \dots, p$, and constant matrices $S \geq 0, S_{1i}, S_{2i}, i = 1, 2, \dots, p$, satisfying the following LMIs:

- (i) $\mathcal{M}_{ii}(P, Q, S_1, S_2) + \mathcal{S} < 0, i = 1, 2, \dots, p;$
- (ii) $\mathcal{M}_{ij}(P, Q, S_1, S_2) + \mathcal{M}_{ji}(P, Q, S_1, S_2) - (2/(p-1))\mathcal{S} < 0, i = 1, 2, \dots, p-1; j = i+1, \dots, p.$

Proof. Consider the following parameter-dependent Lyapunov-Krasovskii functional for system (1):

$$V(k) = V_1(k) + V_2(k) + V_3(k), \quad (10)$$

where

$$\begin{aligned} V_1(k) &= x(k)^T P(\xi) x(k), \\ V_2(k) &= \sum_{i=k-h(k)}^{k-1} x^T(i) Q(\xi) x(i), \\ V_3(k) &= \sum_{j=-h_2+2}^{-h_1+1} \sum_{l=k+j+1}^{k-1} x^T(l) Q(\xi) x(l). \end{aligned} \quad (11)$$

We can verify that

$$\lambda_1 \|x(k)\|^2 \leq V(k) \leq \lambda_2 \|x_k\|^2. \quad (12)$$

Let us set $z(k) = [x^T(k) \ x^T(k+1) \ x^T(k-h(k)) \ \omega^T(k)]$, and

$$G(\xi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P(\xi) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F(\xi) = \begin{pmatrix} P(\xi) & 0 & 0 & 0 \\ I & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}. \quad (13)$$

Then, with the difference of $V_1(k)$ along the solution of the system (1) and taking the mathematical expectation, we obtained

$$\begin{aligned} E[\Delta V_1(k)] \\ = E[x^T(k+1) P(\xi) x(k+1) - x^T(k) P(\xi) x(k)] \\ = E \left[z(k)^T G(\xi) z(k) - 2z^T(k) F^T(\xi) \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \end{pmatrix} \right], \end{aligned} \quad (14)$$

because of

$$\begin{aligned} z^T(k) G(\xi) z(k) &= x(k+1)^T P(\xi) x(k+1), \\ 2z^T(k) F^T(\xi) \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \end{pmatrix} &= x^T(k) P(\xi) x(k). \end{aligned} \quad (15)$$

Using the expression of system (1)

$$\begin{aligned} 0 &= -S_1(\xi) x(k+1) + S_1(\xi) A(\xi) x(k) \\ &\quad + S_1(\xi) D(\xi) x(k-h(k)) + S_1(\xi) \sigma \omega(k), \end{aligned}$$

$$\begin{aligned} 0 &= -S_2(\xi) x(k+1) + S_2(\xi) A(\xi) x(k) \\ &\quad + S_2(\xi) D(\xi) x(k-h(k)) + S_2(\xi) \sigma \omega(k), \\ 0 &= -\sigma^T x(k+1) + \sigma^T A(\xi) x(k) \\ &\quad + \sigma^T D(\xi) x(k-h(k)) + \sigma^T \sigma \omega(k), \end{aligned} \quad (16)$$

we have

$$\begin{aligned} & -2z^T(k) F^T(\xi) \begin{pmatrix} 0.5x(k) \\ -S_1(\xi) x(k+1) + S_1(\xi) A(\xi) x(k) + S_1(\xi) D(\xi) x(k-h(k)) + S_1(\xi) \sigma \omega(k) \\ -S_2(\xi) x(k+1) + S_2(\xi) A(\xi) x(k) + S_2(\xi) D(\xi) x(k-h(k)) + S_2(\xi) \sigma \omega(k) \\ -\sigma^T x(k+1) + \sigma^T A(\xi) x(k) + \sigma^T D(\xi) x(k-h(k)) + \sigma^T \sigma \omega(k) \end{pmatrix} z(k) \\ &= -z^T(k) F^T(\xi) \begin{pmatrix} 0.5I & 0 & 0 \\ S_1(\xi) A(\xi) & -S_1(\xi) & S_1(\xi) D(\xi) & S_1(\xi) \sigma \\ S_2(\xi) A(\xi) & -S_2(\xi) & S_2(\xi) D(\xi) & S_2(\xi) \sigma \\ \sigma^T A(\xi) & -\sigma^T & \sigma^T D(\xi) & \sigma^T \sigma \end{pmatrix} z(k) \\ &= -z^T(k) \begin{pmatrix} 0.5I & 0 & 0 \\ S_1(\xi) A(\xi) & -S_1(\xi) & S_1(\xi) D(\xi) & S_1(\xi) \sigma \\ S_2(\xi) A(\xi) & -S_2(\xi) & S_2(\xi) D(\xi) & S_2(\xi) \sigma \\ \sigma^T A(\xi) & -\sigma^T & \sigma^T D(\xi) & \sigma^T \sigma \end{pmatrix}^T F(\xi) z(k). \end{aligned} \quad (17)$$

Therefore, from (14), it follows that

$$\begin{aligned} E[\Delta V_1(k)] &= E \left[x^T(k) \left[-P(\xi) - S_1(\xi) A(\xi) - A(\xi)^T S_1^T(\xi) \right] x(k) \right. \\ &\quad + 2x^T(k) \left[S_1(\xi) - S_1(\xi) A(\xi) \right] x(k+1) \\ &\quad + 2x^T(k) \left[-S_1(\xi) D(\xi) \right. \\ &\quad \quad \left. - S_2(\xi) A(\xi) \right] x(k-h(k)) \\ &\quad + 2x^T(k) \left[-S_1(\xi) \sigma - \sigma^T A(\xi) \right] \omega(k) \\ &\quad + x(k+1) \left[P(\xi) + S_1(\xi) + S_1^T(\xi) \right] \\ &\quad \times x(k+1) \\ &\quad + 2x(k+1) \left[S_2(\xi) - S_1(\xi) D(\xi) \right] \\ &\quad \times x(k-h(k)) \\ &\quad + 2x(k+1) \left[\sigma^T - S_1(\xi) \sigma \right] \omega(k) \\ &\quad + x^T(k-h(k)) \left[-S_2(\xi) D(\xi) - D^T(\xi) S_2^T(\xi) \right] \\ &\quad \times x(k-h(k)) \\ &\quad + x^T(k-h(k)) \left[-S_2(\xi) \sigma - \sigma^T D(\xi) \right] \omega(k) \\ &\quad \left. + \omega^T(k) \left[-2\sigma^T \sigma \right] \omega(k) \right]. \end{aligned} \quad (18)$$

By assumption (3), we have

$$\begin{aligned} E[\Delta V_1(k)] &= E \left[x^T(k) \left[-P(\xi) - S_1(\xi) A(\xi) - A(\xi)^T S_1^T(\xi) \right] x(k) \right. \\ &\quad + 2x^T(k) \left[S_1(\xi) - S_1(\xi) A(\xi) \right] x(k+1) \\ &\quad + 2x^T(k) \left[-S_1(\xi) D(\xi) - S_2(\xi) A(\xi) \right] \\ &\quad \times x(k-h(k)) \\ &\quad + x(k+1) \left[P(\xi) + S_1(\xi) + S_1^T(\xi) \right] x(k+1) \\ &\quad + 2x(k+1) \left[S_2(\xi) - S_1(\xi) D(\xi) \right] x(k-h(k)) \\ &\quad + x^T(k-h(k)) \left[-S_2(\xi) D(\xi) - D^T(\xi) S_2^T(\xi) \right] \\ &\quad \times x(k-h(k)) \\ &\quad \left. + \omega^T(k) \left[-2\sigma^T \sigma \right] \omega(k) \right]. \end{aligned} \quad (19)$$

Applying assumption (4), the following estimations holds:

$$\begin{aligned} & -\sigma^T(x(k), x(k-h(k)), k) \sigma_i(x(k), x(k-h(k)), k) \\ & \leq \rho_1 x^T(k) x(k) + \rho_2 x^T(k-h(k)) x(k-h(k)). \end{aligned} \quad (20)$$

Therefore, we have

$$\begin{aligned}
 E [\Delta V_1 (k)] &= E \left[x^T (k) \left[-P (\xi) - S_1 (\xi) A (\xi) - A (\xi)^T S_1^T (\xi) \right. \right. \\
 &\quad \left. \left. + 2\rho_1 I \right] x (k) \right. \\
 &\quad \left. + 2x^T (k) \left[S_1 (\xi) - S_1 (\xi) A (\xi) \right] x (k+1) \right. \\
 &\quad \left. + 2x^T (k) \left[-S_1 (\xi) D (\xi) - S_2 (\xi) A (\xi) \right] \right. \\
 &\quad \left. \times x (k-h(k)) \right. \\
 &\quad \left. + x (k+1) \left[P (\xi) + S_1 (\xi) + S_1^T (\xi) \right] \right. \\
 &\quad \left. \times x (k+1) \right. \\
 &\quad \left. + 2x (k+1) \left[S_2 (\xi) - S_1 (\xi) D (\xi) \right] \right. \\
 &\quad \left. \times x (k-h(k)) \right. \\
 &\quad \left. + x^T (k-h(k)) \left[-S_2 (\xi) D (\xi) - D^T (\xi) S_2^T (\xi) \right. \right. \\
 &\quad \left. \left. + 2\rho_2 I \right] x (k-h(k)) \right]. \tag{21}
 \end{aligned}$$

The expectation of the difference of $V_2(k)$ is given by

$$\begin{aligned}
 E [\Delta V_2 (k)] &= E \left[\sum_{i=k+1-h(k+1)}^k x^T (i) Q (\xi) x (i) \right. \\
 &\quad \left. - \sum_{i=k-h(k)}^{k-1} x^T (i) Q (\xi) x (i) \right] \\
 &= E \left[\sum_{i=k+1-h(k+1)}^{k-h_1} x^T (i) Q (\xi) x (i) \right. \\
 &\quad \left. + x^T (k) Q (\xi) x (k) \right. \\
 &\quad \left. - x^T (k-h(k)) Q (\xi) x (k-h(k)) \right. \\
 &\quad \left. + \sum_{i=k+1-h_1}^{k-1} x^T (i) Q (\xi) x (i) \right. \\
 &\quad \left. - \sum_{i=k+1-h(k)}^{k-1} x^T (i) Q (\xi) x (i) \right]. \tag{22}
 \end{aligned}$$

Since $h(k) \geq h_1$, we have

$$\sum_{i=k+1-h_1}^{k-1} x^T (i) Q (\xi) x (i) - \sum_{i=k+1-h(k)}^{k-1} x^T (i) Q (\xi) x (i) \leq 0, \tag{23}$$

and, hence, from (22), we have

$$\begin{aligned}
 E [\Delta V_2 (k)] &\leq E \left[\sum_{i=k+1-h(k+1)}^{k-h_1} x^T (i) Q (\xi) x (i) \right. \\
 &\quad \left. + x^T (k) Q (\xi) x (k) \right. \\
 &\quad \left. - x^T (k-h(k)) Q (\xi) x (k-h(k)) \right]. \tag{24}
 \end{aligned}$$

The difference of $V_3(k)$ is given by

$$\begin{aligned}
 E [\Delta V_3 (k)] &= E \left[\sum_{j=-h_2+2}^{-h_1+1} \left[x^T (k) Q (\xi) x (k) \right. \right. \\
 &\quad \left. \left. - x^T (k+j-1) Q (\xi) \right. \right. \\
 &\quad \left. \left. \times x (k+j-1) \right] \right] \\
 &= E \left[(h_2 - h_1) x^T (k) Q (\xi) x (k) \right. \\
 &\quad \left. - \sum_{l=k+1-h_2}^{k-h_1} x^T (l) Q (\xi) x (l) \right]. \tag{25}
 \end{aligned}$$

Since

$$\sum_{i=k+1-h(k+1)}^{k-h_1} x^T (i) Q (\xi) x (i) - \sum_{i=k+1-h_2}^{k-h_1} x^T (i) Q (\xi) x (i) \leq 0, \tag{26}$$

we obtain from (24) and (25) that

$$\begin{aligned}
 E [\Delta V_2 (k) + \Delta V_3 (k)] &\leq E \left[(h_2 - h_1 + 1) x^T (k) Q (\xi) x (k) \right. \\
 &\quad \left. - x^T (k-h(k)) Q (\xi) x (k-h(k)) \right]. \tag{27}
 \end{aligned}$$

Therefore, combining the inequalities (21), (27) gives

$$E [\Delta V (k)] \leq E \left[\psi^T (k) T (\xi) \psi (k) \right], \tag{28}$$

where

$$\psi(k) = [x(k) \ x(k+1) \ x(k-h(k))]^T,$$

$$T(\xi) = \begin{pmatrix} M(\xi) & S_1(\xi) - S_1(\xi)A(\xi) & -S_1(\xi)D(\xi) - S_2(\xi)A(\xi) \\ S_1^T(\xi) - A^T(\xi)S_1^T(\xi) & P(\xi) + S_1(\xi) + S_1^T(\xi) & S_2(\xi) - S_1(\xi)D(\xi) \\ -D^T(\xi)S_1^T(\xi) - A^T(\xi)S_2^T(\xi) & S_2^T(\xi) - D^T(\xi)S_1^T(\xi) & -Q(\xi) - S_2(\xi)D(\xi) - D^T(\xi)S_2^T(\xi) + 2\rho_2 I \end{pmatrix}, \quad (29)$$

$$M(\xi) = (h_2 - h_1 + 1)Q(\xi) - P(\xi) - S_1(\xi)A(\xi) - A(\xi)^T S_1(\xi)^T + 2\rho_1 I.$$

Let us denote that

$$\begin{aligned} M_{ij} &:= (h_2 - h_1 + 1)Q_i - P_i - S_{1i}A_j - A_j^T S_{1i}^T + 2\rho_1 I, \\ (S_1 A)_{ij} &:= S_{1j}A_i + S_{1i}A_j, & (S_2 A)_{ij} &:= S_{2j}A_i + S_{2i}A_j, \\ (S_1 D)_{ij} &:= S_{1j}D_i + S_{1i}D_j, & (S_2 D)_{ij} &:= S_{2j}D_i + S_{2i}D_j, \end{aligned}$$

$$\begin{aligned} P_{ij} &= P_i + P_j, & Q_{ij} &= Q_i + Q_j, \\ S_{1ij} &= S_{1i} + S_{1j}, & S_{2ij} &= S_{2i} + S_{2j}. \end{aligned} \quad (30)$$

From the convex combination of the expression of $P(\xi)$, $Q(\xi)$, $S_1(\xi)$, $S_2(\xi)$, $A(\xi)$, $D(\xi)$, we have

$$\begin{aligned} T(\xi) &= \sum_{i=1}^p \xi_i^2 \begin{pmatrix} M_{ii} & S_{1i} - S_{1i}A_i & -S_{1i}D_i - S_{2i}A_i \\ S_{1i}^T - A_i^T S_{1i}^T & P_i + S_{1i} + S_{1i}^T & S_{2i} - S_{1i}D_i \\ -D_i^T S_{1i}^T - A_i^T S_{2i}^T & S_{2i}^T - D_i^T S_{1i}^T & -Q_i - S_{2i}D_i - D_i^T S_{2i}^T + 2\rho_2 I \end{pmatrix} \\ &+ \sum_{i=1}^{p-1} \sum_{j=i+1}^p \xi_i \xi_j \begin{pmatrix} M_{ij} + M_{ji} & S_{1ij} - (S_1 A)_{ij} & -(S_1 D)_{ij} - (S_2 A)_{ij} \\ S_{1ij}^T - (A^T S_1^T)_{ij} & P_{ij} + S_{1ij} + S_{1ij}^T & S_{2ij} - (S_1 D)_{ij} \\ -(D^T S_1^T)_{ij} - (A^T S_2^T)_{ij} & S_{2ij}^T - (D^T S_1^T)_{ij} & -Q_{ij} - (S_2 D)_{ij} - (D^T S_2^T)_{ij} + 2\rho_2 I \end{pmatrix} \\ &= \sum_{i=1}^p \xi_i^2 \mathcal{M}_{ii}(P, Q, S_1, S_2) + \sum_{i=1}^{p-1} \sum_{j=i+1}^p \xi_i \xi_j [\mathcal{M}_{ij}(P, Q, S_1, S_2) + \mathcal{M}_{ji}(P, Q, S_1, S_2)]. \end{aligned} \quad (31)$$

Then, the conditions (i) and (ii) give

$$T(\xi) < -\sum_{i=1}^p \xi_i^2 \mathcal{S} + \frac{2}{p-1} \sum_{i=1}^{p-1} \sum_{j=i+1}^p \xi_i \xi_j \mathcal{S} \leq 0, \quad (32)$$

because of Proposition 3 as

$$(p-1) \sum_{i=1}^p \xi_i^2 - 2 \sum_{i=1}^{p-1} \sum_{j=i+1}^p \xi_i \xi_j = \sum_{i=1}^{p-1} \sum_{j=i+1}^p (\xi_i - \xi_j)^2 \geq 0, \quad (33)$$

and, hence, we finally obtain from (28) that

$$E[\Delta V(k)] \leq E[\psi^T(k) T(\xi) \psi(k)] < 0, \quad \forall k = 0, 1, 2, \dots, \quad (34)$$

which together with (12) and Definition 2 implies that the system (1) is robustly stable in the mean square. This completes the proof of the theorem. \square

Remark 5. The stability conditions of Theorem 4 are more appropriate for most of real systems since it is usually impossible in practice to know exactly the delay but lower and upper bounds are always possible.

3.2. Robust Stabilization. This section deals with a stabilization problem considered in [15] for constructing a delayed feedback controller, which stabilizes the resulting closed-loop system. The robust stability condition obtained in previous section will be applied to design a time-delayed state feedback controller for the discrete-time control system described by

$$\begin{aligned} x(k+1) &= A(\xi)x(k) + B(\xi)u(k) \\ &+ \sigma(x(k), x(k-h(k)), k)\omega(k), \end{aligned} \quad (35)$$

$$k = 0, 1, 2, \dots,$$

where $u(k) \in R^n$ is the control input, and the system matrices are subjected to uncertainties and belong to the polytope Ω given by

$$\Omega = \left\{ [A, B](\xi) := \sum_{i=1}^p \xi_i [A_i, B_i], \sum_{i=1}^p \xi_i = 1, \xi_i \geq 0 \right\}, \quad (36)$$

where $A_i, B_i, i = 1, 2, \dots, p$, are given constant matrices with appropriate dimensions. As in [8], we consider a parameter-dependent delayed feedback control law

$$u(k) = F(\xi)x(k-h(k)), \quad k = -h_2, \dots, 0, \quad (37)$$

where $h(k)$ is the time-varying delay function satisfying $0 < h_1 \leq h(k) \leq h_2$, and $F(\xi)$ is the controller gain to be determined. Applying the feedback controller (37) to the system (35), the closed-loop time-delay system is

$$x(k+1) = A(\xi)x(k) + B(\xi)F(\xi)x(k-h(k)) + \sigma(x(k), x(k-h(k)), k)\omega(k), \quad (38)$$

$$k = 0, 1, 2, \dots$$

Definition 6. The system (35) is robustly stabilizable in the mean square if there is a delayed feedback control (37) such that the closed-loop delay system (38) is robustly stable in the mean square.

Let

$$\mathcal{M}_{ij}(P, Q, S_1) = \begin{pmatrix} (h_2 - h_1 + 1)Q_i - P_i - S_{1i}A_j - A_j^T S_{1i}^T + 2\rho_1 I & S_{1i} - S_{1i}A_j & -P_i - S_{1i}A_j \\ S_{1i}^T - A_j^T S_{1i}^T & P_i + S_{1i} + S_{1i}^T & S_{1i} - P_i \\ -P_i - A_j^T S_{1i}^T & S_{1i}^T - P_i & -Q_i - P_i - P_i + 2\rho_2 I \end{pmatrix}, \quad (39)$$

$$\mathcal{S} = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The following theorem can be derived from Theorem 4.

Theorem 7. *The system (35) is robustly stabilizable in the mean square by the delayed feedback control (37), where*

$$F(\xi) = B^T(\xi) [B(\xi)B^T(\xi)]^{-1} S_1^T(\xi) [S_1(\xi)S_1^T(\xi)]^{-1} P(\xi), \quad (40)$$

if there exist symmetric matrices $P_i > 0, Q_i > 0, i = 1, 2, \dots, p$, and constant matrices $S_{1i}, i = 1, 2, \dots, p, S \geq 0$, satisfying the following LMIs:

- (i) $\mathcal{M}_{ii}(P, Q, S_1) + \mathcal{S} < 0, i = 1, 2, \dots, p;$
- (ii) $\mathcal{M}_{ij}(P, Q, S_1) + \mathcal{M}_{ji}(P, Q, S_1) - (2/(p-1))\mathcal{S} < 0, i = 1, 2, \dots, p-1; j = i+1, \dots, p.$

Proof. Taking $S_{1i} = S_{2i}$ and using the feedback control (37), the closed-loop system becomes system (Σ_ξ) , where $D(\xi) = B(\xi)F(\xi) = S_1^T(\xi)[S_1(\xi)S_1^T(\xi)]^{-1}P(\xi)$. Since $S_1(\xi)D(\xi) = P(\xi)$, the robust stability condition of the closed-loop system (38), by Theorem 4, is immediately derived. \square

Remark 8. The stabilization conditions of Theorem 7 are more appropriate for most of real systems since it is usually impossible in practice to know exactly the delay but lower and upper bounds are always possible.

4. Numerical Examples

To illustrate the effectiveness of the previous theoretical results, we consider the following numerical examples.

Example 9 (robust stability). Consider system Σ_ξ for $p = 2$, where the delay function $h(k)$ is given by

$$h(k) = 1 + 28 \sin^2 \frac{k\pi}{2}, \quad k = 0, 1, 2, \dots,$$

$$A_1 = \begin{pmatrix} -30.5 & 1 \\ 2 & -3.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -35.5 & 1 \\ 3 & -4.5 \end{pmatrix}, \quad (41)$$

$$D_1 = \begin{pmatrix} -1.5 & 0.1 \\ 0.4 & -2.15 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -2.5 & 0.2 \\ 0.3 & -1.85 \end{pmatrix}.$$

By using the LMI Toolbox in MATLAB, the LMIs (i) and (ii) of Theorem 4 are feasible with $h_1 = 1, h_2 = 29, \rho_1 = 0.011, \rho_2 = 0.015$, and we use the condition in the Theorem 4 for this example. The solutions of LMI verify as follow of the form

$$P_1 = \begin{pmatrix} 4.6120 & 0.2565 \\ 0.2565 & 3.3703 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2.9556 & 0.0381 \\ 0.0381 & 3.5256 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0.1402 & 0.0109 \\ 0.0109 & 0.0145 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0.3550 & 0.0101 \\ 0.0101 & 0.2101 \end{pmatrix},$$

$$S_{11} = \begin{pmatrix} -0.0596 & -0.0430 \\ 0.0031 & 0.0453 \end{pmatrix},$$

$$S_{12} = \begin{pmatrix} -0.0197 & -0.0095 \\ 0.0045 & 0.0375 \end{pmatrix},$$

$$S_{21} = \begin{pmatrix} 0.0006 & 0.0250 \\ -0.0029 & -0.1615 \end{pmatrix},$$

$$S_{22} = \begin{pmatrix} -0.0002 & 0.0133 \\ -0.0030 & -0.1228 \end{pmatrix},$$

$$S = \begin{pmatrix} 2.0759 & 0.0459 \\ 0.0459 & 1.3271 \end{pmatrix}. \quad (42)$$

Therefore, the system is robustly stable.

Example 10 (robust stabilization). Consider system (35) for $p = 2$, where the delay function $h(k)$ is given by

$$h(k) = 1 + 34 \sin^2 \frac{k\pi}{2}, \quad k = 0, 1, 2, \dots,$$

$$A_1 = \begin{pmatrix} -30.5 & 1 \\ 2 & -3.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -35.5 & 1 \\ 3 & -4.5 \end{pmatrix}, \quad (43)$$

$$B_1 = \begin{pmatrix} -1.5 & 0.1 \\ 0.4 & -2.15 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -2.5 & 0.2 \\ 0.3 & -1.85 \end{pmatrix}.$$

By using the LMI Toolbox in MATLAB, the LMIs (i) and (ii) of Theorem 7 are feasible with $h_1 = 1, h_2 = 35, \rho_1 = 0.011, \rho_2 = 0.015$, and we use the condition in the Theorem 7 for this example. The solutions of LMI verify as follow of the form

$$P_1 = \begin{pmatrix} 1.3886 & -0.0760 \\ -0.0760 & 1.3559 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1.6286 & 0.0649 \\ 0.0649 & 1.5243 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0.0097 & -0.0048 \\ -0.0048 & 0.0057 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0.0728 & -0.0159 \\ -0.0159 & 0.0621 \end{pmatrix},$$

$$S_{11} = \begin{pmatrix} -0.0274 & 0.0827 \\ -0.0133 & -0.2222 \end{pmatrix},$$

$$S_{12} = \begin{pmatrix} -0.0209 & 0.0619 \\ -0.0226 & -0.1942 \end{pmatrix},$$

$$S = \begin{pmatrix} 0.5954 & -0.0672 \\ -0.0672 & 0.5469 \end{pmatrix}. \tag{44}$$

Therefore, the system is robustly stabilizable with the feedback control

$$u(k) = B^T(\xi) [B(\xi) B^T(\xi)]^{-1} S_1^T(\xi) [S_1(\xi) S_1^T(\xi)]^{-1} P(\xi) x(k-h(k))$$

$$= (\xi_1 B_1 + \xi_2 B_2)^T [(\xi_1 B_1 + \xi_2 B_2) (\xi_1 B_1 + \xi_2 B_2)^T]^{-1}$$

$$\times (\xi_1 S_{11} + \xi_2 S_{12})^T [(\xi_1 S_{11} + \xi_2 S_{12}) (\xi_1 S_{11} + \xi_2 S_{12})^T]^{-1} (\xi_1 P_1 + \xi_2 P_2) (\xi) x(k-h(k))$$

$$= \begin{pmatrix} -1.5\xi_1 - 2.5\xi_2 & 0.4\xi_1 + 0.3\xi_2 \\ 0.1\xi_1 + 0.2\xi_2 & -2.15\xi_1 - 1.85\xi_2 \end{pmatrix} \times \begin{pmatrix} -1.5\xi_1 - 2.5\xi_2 & 0.4\xi_1 + 0.3\xi_2 \\ 0.1\xi_1 + 0.2\xi_2 & -2.15\xi_1 - 1.85\xi_2 \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} -1.5\xi_1 - 2.5\xi_2 & 0.1\xi_1 + 0.2\xi_2 \\ 0.4\xi_1 + 0.3\xi_2 & -2.15\xi_1 - 1.85\xi_2 \end{pmatrix}^{-1} \times \begin{pmatrix} -0.0274\xi_1 - 0.0209\xi_2 & -0.0133\xi_1 - 0.0226\xi_2 \\ 0.0827\xi_1 + 0.0619\xi_2 & -0.2222\xi_1 - 0.1942\xi_2 \end{pmatrix}$$

$$\times \begin{pmatrix} -0.0274\xi_1 - 0.0209\xi_2 & -0.0133\xi_1 - 0.0226\xi_2 \\ 0.0827\xi_1 + 0.0619\xi_2 & -0.2222\xi_1 - 0.1942\xi_2 \end{pmatrix}^{-1} \times \begin{pmatrix} -0.0274\xi_1 - 0.0209\xi_2 & 0.0827\xi_1 + 0.0619\xi_2 \\ -0.0133\xi_1 - 0.0226\xi_2 & -0.2222\xi_1 - 0.1942\xi_2 \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} 1.3886\xi_1 + 1.6286\xi_2 & -0.0760\xi_1 + 0.0649\xi_2 \\ -0.0760\xi_1 + 0.0649\xi_2 & 1.3559\xi_1 + 1.5243\xi_2 \end{pmatrix}$$

$$= \begin{pmatrix} -2.0829\xi_1^2 - 5.9144\xi_1\xi_2 - 4.0715\xi_2^2 & -0.0304\xi_1^2 + 0.0260\xi_1\xi_2 + 0.0195\xi_2^2 \\ -0.0076\xi_1^2 - 0.0087\xi_1\xi_2 + 0.0128\xi_2^2 & -2.9152\xi_1^2 - 5.7856\xi_1\xi_2 - 2.8200\xi_2^2 \end{pmatrix} x(k-h(k)). \tag{45}$$

Therefore, the feedback delayed controller is

$$u_1(k) = [-2.0829\xi_1^2 - 5.9144\xi_1\xi_2 - 4.0715\xi_2^2]$$

$$\times x_1(k-h(k))$$

$$+ [-0.0304\xi_1^2 + 0.0260\xi_1\xi_2 + 0.0195\xi_2^2]$$

$$\times x_2(k-h(k)),$$

$$u_2(k) = [-0.0076\xi_1^2 - 0.0087\xi_1\xi_2 + 0.0128\xi_2^2]$$

$$\times x_1(k-h(k))$$

$$+ [-2.9152\xi_1^2 - 5.7856\xi_1\xi_2 - 2.8200\xi_2^2]$$

$$\times x_2(k-h(k)). \tag{46}$$

5. Conclusion

In this paper, new delay-dependent mean square robust stability conditions for linear polytopic delay-difference stochastic equations with interval time-varying delays have been presented in terms of LMIs. An application to mean square robust stabilization of discrete stochastic control systems

with time-delayed feedback controllers has been studied. Numerical examples have been given to demonstrate the effectiveness of the proposed conditions.

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