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Research Article

Convergence of Viscosity Iteration Process for a Finite Family of Generalized Asymptotically Quasi-Nonexpansive Mappings

Zhiming Cheng

College of Mathematics and Computer Science, Yangtze Normal University, Chongqing 408100, China

Correspondence should be addressed to Zhiming Cheng; fsxczm@yznu.cn

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We introduce a general iteration method for a finite family of generalized asymptotically quasi-nonexpansive mappings. The results presented in the paper extend and improve some recent results in the works by Shahzad and Udomene (2006); L. Qihou (2001); Khan et al. (2008).

1. Introduction and Preliminaries

Let C be a nonempty subset of a real Banach space E and T a self-mapping of C. The set of fixed points of T is denoted by $F(T)$ and we assume that $F(T) \neq \emptyset$. The mapping T is said to be

(i) contractive mapping if there exists a constant α in $(0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, for all $x, y \in C$;

(ii) asymptotically nonexpansive mapping if there exists a sequence $\{u_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(iii) asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - p\| \leq (1 + u_n)\|x - p\|$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(iv) generalized asymptotically quasi-nonexpansive [1] if there exist two sequences $\{u_n\}$, $\{h_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty} h_n = 0$ such that

$$\|T^n x - p\| \leq (1 + u_n)\|x - p\| + h_n, \quad \forall x \in C, \; p \in F(T), \tag{1}$$

where $n = 1, 2, 3, \ldots$;

(v) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(vi) $(L-\gamma)$ uniform L-Lipschitz if there are constants $L > 0$ and $\gamma > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|^{\gamma}$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(vii) semicompact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

In (1), if $h_n = 0$ for all $n \geq 1$, then T becomes an asymptotically quasi-nonexpansive mapping; if $u_n = 0$ and $h_n = 0$ for all $n \geq 1$, then T becomes a quasi-nonexpansive mapping. It is known that an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive and a uniformly L-Lipschitzian mapping is $(L-1)$ uniform L-Lipschitz.

The mapping $T : C \to E$ is said to be demiclosed at 0 if for each sequence $\{x_n\} \subset C$ converging weakly to x_0 and $\{Tx_n\}$ converging strongly to 0, we have $Tx_0 = 0$.

A Banach space E is said to satisfy Opial's property if for each $x \in E$ and each sequence $\{x_n\}$ weakly convergent to x, the following condition holds for all $x \neq y$:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|. \tag{2}$$

Let C be a nonempty closed convex subset of a real Banach space E and $\{T_i : i = 1, 2, \ldots k\}$ a finite family of asymptotically nonexpansive mappings of C into itself. Suppose that

$\alpha_{in} \in [0,1]$, $n = 1, 2, 3, \ldots$, and $i = 1, 2, \ldots k$. Then we consider the following mapping of C into itself:

$$\begin{aligned} U_{1n} &= (1-\alpha_{1n}) I + \alpha_{1n} T_1^n U_{0n}, \\ U_{2n} &= (1-\alpha_{2n}) I + \alpha_{2n} T_2^n U_{1n}, \\ &\vdots \\ U_{(k-1)n} &= (1-\alpha_{(k-1)n}) I + \alpha_{(k-1)n} T_{k-1}^n U_{(k-2)n}, \\ W_n = U_{kn} &= (1-\alpha_{kn}) I + \alpha_{kn} T_k^n U_{(k-1)n}, \end{aligned} \tag{3}$$

where $U_{0n} = I$ (identity mapping). Such a mapping W_n is called the modified W-mapping generated by $T_1, T_2, \ldots, T_k$ and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$ (see [2, 3]).

In the sequel, we assume that $F = \bigcap_{i=1}^k F(T_i)$.

In 2008, Khan et al. [4] introduced the following iteration process for a family of asymptotically quasi-nonexpansive mappings, for an arbitrary $x_1 \in C$:

$$\begin{aligned} y_{1n} &= (1-\alpha_{1n}) x_n + \alpha_{1n} T_1^n y_{0n}, \\ y_{2n} &= (1-\alpha_{2n}) x_n + \alpha_{2n} T_2^n y_{1n}, \\ &\vdots \\ y_{(k-1)n} &= (1-\alpha_{(k-1)n}) x_n + \alpha_{(k-1)n} T_{k-1}^n y_{(k-2)n}, \\ x_{n+1} &= (1-\alpha_{kn}) x_n + \alpha_{kn} T_k^n y_{(k-1)n}, \end{aligned} \tag{4}$$

where $y_{0n} = x_n$, $\alpha_{in} \in [0,1]$, $i = 1, 2, \ldots, k$, $n = 1, 2, \ldots$ and proved that the iterative sequence $\{x_n\}$ defined by (4) converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F}\|x - p\|$. With the help of (3), we write (4) as

$$x_{n+1} = W_n x_n. \tag{5}$$

Recently, Chang et al. [5] introduced the following iteration process of asymptotically nonexpansive mappings in Banach space:

$$\begin{aligned} x_{n+1} &= \lambda_n f(x_n) + (1-\lambda_n) T^n y_n, \\ y_n &= \beta_n x_n + (1-\beta_n) T^n x_n, \end{aligned} \tag{6}$$

where $\{\lambda_n\}, \{\beta_n\} \subset [0,1]$ and f is a fixed contractive mapping, and necessary and sufficient conditions are given for the iterative sequence $\{x_n\}$ to converge to the fixed points of T.

For a family of mappings, it is quite significant to devise a general iteration scheme which extends the iteration processes (4) and (6), simultaneously. Thereby, to achieve this goal, we introduce a new iteration process for a family of mappings as follows.

Let C be a nonempty closed convex subset of a real Banach space E, $\{T_i : C \to C, i = 1, 2, \ldots, k\}$ a family of generalized asymptotically quasi-nonexpansive mappings, and $f : C \to C$ a fixed contractive mapping with contractive coefficient $\alpha \in (0,1)$. For a given $x_1 \in C$, the iteration scheme is defined as follows:

$$x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n) W_n x_n, \tag{7}$$

where $\{\lambda_n\} \in [0,1]$ and W_n is the modified W-mapping generated by $T_1, T_2, \ldots, T_k$, and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$ for all positive integers n.

The purpose of this paper is to study the convergence problem of the iterative sequences $\{x_n\}$ defined by (7). The obtained results extend the corresponding results in [4–8], and Lemma 11 partly improves the method of proof of Lemma 3.1 in [4].

In what follows, we need the following useful known lemmas.

Lemma 1 (see [9]). *Let $\{a_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ be nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \le (1+\delta_n) a_n + \gamma_n, \tag{8}$$

where $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty \gamma_n < \infty$; then $\lim_{n\to\infty} a_n$ exists.

Moreover, if in addition, $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 (see [4]). *Let E be a uniformly convex Banach space, $0 < b \le t_n \le c < 1$ for all $n \ge 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E. Assume that $\limsup_{n\to\infty}\|x_n\| \le a$, $\limsup_{n\to\infty}\|y_n\| \le a$, and $\lim_{n\to\infty}\|t_n x_n + (1-t_n) y_n\| = a$ for some $a \ge 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

2. Main Results

Lemma 3. *Let C be a nonempty closed convex subset of a real Banach space E and T an asymptotically quasi-nonexpansive self-mapping of C with $\{u_n\} \subset [0,\infty)$ for all $n \ge 1$. Suppose $F(T) \ne \phi$. Then $F(T)$ is a closed subset in C.*

Proof. Let $\{z_n\}$ be an arbitrary sequence of $F(T)$ and $z_n \to z_0$ as $n \to \infty$. Since C is closed, we have $z_0 \in C$. For any $\epsilon > 0$, there exists a natural number N such that

$$\|z_n - z_0\| < \frac{\epsilon}{2 + u_1}, \quad \forall n \ge N. \tag{9}$$

Thus, we get

$$\begin{aligned} \|Tz_0 - z_0\| &\le \|Tz_0 - z_N\| + \|z_N - z_0\| \\ &\le (1+u_1)\|z_N - z_0\| + \|z_N - z_0\| \\ &= (2+u_1)\|z_N - z_0\| < \epsilon. \end{aligned} \tag{10}$$

Since ϵ is arbitrary, it follows that $\|Tz_0 - z_0\| = 0$; that is, $Tz_0 = z_0$. Hence $z_0 \in F(T)$ and $F(T)$ is closed. This completes the proof. □

Lemma 4. *Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_i : i = 1, 2, \ldots, k\}$ be k generalized asymptotically quasi-nonexpansive self-mappings of C with $\{u_{in}\}, \{h_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^\infty u_{in} < \infty$ and $\sum_{n=1}^\infty h_{in} < \infty$ for all $i \in \{1, 2, 3, \ldots, k\}$. Suppose $F \ne \emptyset$ and $\{\alpha_{in}\}_{n\ge1} \subset [0,1]$ for all $i \in \{1, 2, 3, \ldots, k\}$. Let W_n be the modified W-mapping generated by $T_1, T_2, \ldots, T_k$ and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$. Let the sequence $\{x_n\}$ be defined by (7) and assuming $\sum_{n=1}^\infty \lambda_n < \infty$, then*

(1) *there exist two sequences* $\{\nu_n\}$ *and* $\{\xi_n\}$ *in* $[0,\infty)$ *with* $\sum_{n=1}^{\infty}\nu_n<\infty$, $\sum_{n=1}^{\infty}\xi_n<\infty$ *such that*

$$\|x_{n+1}-p\|\le(1+\nu_n)^k\|x_n-p\|+\xi_n,\quad\forall p\in F,\ n\ge 1;\tag{11}$$

(2) *there exists a constant* $M_1>0$, *such that*

$$\|x_{n+m}-p\|\le M_1\|x_n-p\|+M_1\sum_{i=n}^{\infty}\xi_i,\quad\forall p\in F,\ n,m=1,2,3,\ldots,\tag{12}$$

where $\{\xi_n\}\in[0,\infty)$ *and* $\Sigma_{n=1}^{\infty}\xi_n<\infty$.

Proof. (1) Let $\nu_n=\max_{1\le i\le k}u_{in}$, for all n. Since $\sum_{n=1}^{\infty}u_{in}<\infty$ for each i, we can get $\sum_{n=1}^{\infty}\nu_n<\infty$. For all $p\in F$, it follows from (3) that

$$\begin{aligned}\|U_{1n}x_n-p\|&\le(1-\alpha_{1n})\|x_n-p\|+\alpha_{1n}\|T_1^nx_n-p\|\\&\le(1-\alpha_{1n})\|x_n-p\|\\&\quad+\alpha_{1n}[(1+u_{1n})\|x_n-p\|+h_{1n}]\\&\le(1+u_{1n})\|x_n-p\|+h_{1n}\\&\le(1+\nu_n)\|x_n-p\|+h_{1n}.\end{aligned}\tag{13}$$

Assume that $\|U_{jn}x_n-p\|\le(1+\nu_n)^j\|x_n-p\|+(1+\nu_n)^{j-1}\sum_{i=1}^{j}h_{in}$ for some $1\le j\le k-1$. Then

$$\begin{aligned}&\|U_{(j+1)n}x_n-p\|\\&\le(1-\alpha_{(j+1)n})\|x_n-p\|\\&\quad+\alpha_{(j+1)n}\|T_{j+1}^nU_{jn}x_n-p\|\\&\le(1-\alpha_{(j+1)n})\|x_n-p\|\\&\quad+\alpha_{(j+1)n}\left(\left(1+u_{(j+1)n}\right)\|U_{jn}x_n-p\|+h_{(j+1)n}\right)\\&\le(1-\alpha_{(j+1)n})\|x_n-p\|+\alpha_{(j+1)n}h_{(j+1)n}\\&\quad+\alpha_{(j+1)n}\left(1+u_{(j+1)n}\right)\\&\quad\times\left((1+\nu_n)^j\|x_n-p\|+(1+\nu_n)^{j-1}\sum_{i=1}^{j}h_{in}\right)\\&\le\left(\left(1-\alpha_{(j+1)n}\right)+\alpha_{(j+1)n}(1+\nu_n)^{j+1}\right)\|x_n-p\|\\&\quad+(1+\nu_n)^j\sum_{i=1}^{j}h_{in}+h_{(j+1)n}\\&\le\left(\left(1-\alpha_{(j+1)n}\right)(1+\nu_n)^{j+1}+\alpha_{(j+1)n}(1+\nu_n)^{j+1}\right)\\&\quad\times\|x_n-p\|+(1+\nu_n)^j\sum_{i=1}^{j+1}h_{in}\\&\le(1+\nu_n)^{j+1}\|x_n-p\|+(1+\nu_n)^j\sum_{i=1}^{j+1}h_{in}.\end{aligned}\tag{14}$$

Thus, by induction, we have

$$\|U_{jn}x_n-p\|\le(1+\nu_n)^j\|x_n-p\|+(1+\nu_n)^{j-1}\sum_{i=1}^{j}h_{in},\tag{15}$$

for all $j=1,2,\ldots,k$. Hence,

$$\begin{aligned}\|W_nx_n-p\|=\|U_{kn}x_n-p\|&\le(1+\nu_n)^k\|x_n-p\|\\&\quad+(1+\nu_n)^{k-1}\sum_{i=1}^{k}h_{in}.\end{aligned}\tag{16}$$

By (7) and (16), we obtain

$$\begin{aligned}\|x_{n+1}-p\|&\le\lambda_n\|f(x_n)-p\|+(1-\lambda_n)\|W_nx_n-p\|\\&\le\lambda_n\|f(x_n)-f(p)\|+\lambda_n\|f(p)-p\|\\&\quad+(1-\lambda_n)\|W_nx_n-p\|\\&\le\lambda_n\alpha\|x_n-p\|+\lambda_n\|f(p)-p\|+(1-\lambda_n)\\&\quad\times\left[(1+\nu_n)^k\|x_n-p\|+(1+\nu_n)^{k-1}\sum_{i=1}^{k}h_{in}\right]\\&\le(1+\nu_n)^k\|x_n-p\|\\&\quad+(1-\lambda_n)(1+\nu_n)^{k-1}\sum_{i=1}^{k}h_{in}\\&\quad+\lambda_n\|f(p)-p\|.\end{aligned}\tag{17}$$

Since $\sum_{n=1}^{\infty}\nu_n<\infty$, $\{\nu_n\}_{n=1}^{\infty}$ is bounded. Setting $M=\max\{\sup_n(1+\nu_n)^{k-1},\|f(p)-p\|\}$, we get that

$$\|x_{n+1}-p\|\le(1+\nu_n)^k\|x_n-p\|+\xi_n,\quad\forall p\in F,\ n\ge 1,\tag{18}$$

where $\xi_n=M(\sum_{i=1}^{k}h_{in}+\lambda_n)$ and $\sum_{n=1}^{\infty}\xi_n<\infty$. This completes the proof of (1).

(2) If $t\ge 0$, then $1+t\le e^t$ and consequently, $(1+t)^k\le e^{kt}$, $k=1,2,\ldots$. Thus, from part (1), we get

$$\begin{aligned}\|x_{n+m}-p\|&\le(1+\nu_{n+m-1})^k\|x_{n+m-1}-p\|+\xi_{n+m-1}\\&\le\exp\{k\nu_{n+m-1}\}\|x_{n+m-1}-p\|+\xi_{n+m-1}\\&\le\exp\{k\nu_{n+m-1}\}\\&\quad\times(\exp\{k\nu_{n+m-2}\}\|x_{n+m-2}-p\|+\xi_{n+m-2})\\&\quad+\xi_{n+m-1}\end{aligned}$$

$$\leq \exp\{k(\nu_{n+m-1}+\nu_{n+m-2})\}\|x_{n+m-2}-p\| + \exp\{k\nu_{n+m-1}\}(\xi_{n+m-2}+\xi_{n+m-1})$$
$$\vdots$$
$$\leq \exp\left\{k\sum_{i=n}^{n+m-1}\nu_i\right\}\|x_n-p\| + \exp\left\{k\sum_{i=n+1}^{n+m-1}\nu_i\right\}\sum_{i=n}^{n+m-1}\xi_i$$
$$\leq M_1\|x_n-p\| + M_1\sum_{i=n}^{\infty}\xi_i, \tag{19}$$

for any positive integers m, n, where $M_1 = \exp\{k\sum_{i=1}^{\infty}\nu_i\}$, $\sum_{i=1}^{\infty}\xi_i < \infty$. This completes the proof of (2). □

Remark 5. Lemma 4 generalizes Lemma 2.1 in [4].

Theorem 6. *Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_i : i = 1,2,\ldots,k\}$ be k generalized asymptotically quasi-nonexpansive self-mappings of C with $\{u_{in}\},\{h_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}u_{in} < \infty$ and $\sum_{n=1}^{\infty}h_{in} < \infty$ for all $i \in \{1,2,3,\ldots,k\}$. Let $\{\alpha_{in}\}_{n\geq1} \subset [0,1]$ for all $i \in \{1,2,3,\ldots,k\}$ and let W_n be a modified W-mapping generated by $T_1,T_2,\ldots,T_k$ and $\alpha_{1n},\alpha_{2n},\ldots,\alpha_{kn}$. Suppose that $F\neq\emptyset$ is closed and $\sum_{n=1}^{\infty}\lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion* (7)*; then the sequence $\{x_n\}$ converges strongly to $p \in F$ if and only if* $\liminf_{n\to\infty}d(x_n,F) = 0$.

Proof. We will only prove the sufficiency; the necessity is obvious. From Lemma 4(1), we have

$$\|x_{n+1}-p\| \leq (1+\nu_n)^k\|x_n-p\| + \xi_n, \tag{20}$$

for all $p \in F$ and all n. Therefore,

$$d(x_{n+1},F) \leq (1+\nu_n)^k d(x_n,F) + \xi_n = \left(1+\sum_{r=1}^{k}\frac{k(k-1)\cdots(k-r+1)}{r!}\nu_n^r\right) \times d(x_n,F) + \xi_n. \tag{21}$$

As $\sum_{n=1}^{\infty}\nu_n < \infty$, so $\sum_{r=1}^{k}(k(k-1)\cdots(k-r+1)/r!)\nu_n^r < \infty$. By Lemma 1 and $\liminf_{n\to\infty}d(x_n,F) = 0$, we get that $\lim_{n\to\infty}d(x_n,F) = 0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 4(2), we have

$$\|x_{n+m}-p\| \leq M_1\|x_n-p\| + M_1\sum_{i=n}^{\infty}\xi_i \quad \forall p \in F,\ n,m\geq 1. \tag{22}$$

Hence, for all integers $m \geq 1$ and all $p \in F$,

$$\|x_{n+m}-x_n\| \leq \|x_{n+m}-p\| + \|x_n-p\| \leq (M_1+1)\|x_n-p\| + M_1\sum_{j=n}^{\infty}\xi_j. \tag{23}$$

Taking infimum over $p \in F$ in (23) gives

$$\|x_{n+m}-x_n\| \leq (M_1+1)d(x_n,F) + M_1\sum_{j=n}^{\infty}\xi_j. \tag{24}$$

Now, since $\lim_{n\to\infty}d(x_n,F) = 0$ and $\sum_{j=1}^{\infty}\xi_j < \infty$, given $\epsilon > 0$, there exists an integer $N_1 > 0$ such that for all $n \geq N_1$, $d(x_n,F) < \epsilon/(2(M_1+2))$ and $\sum_{j=n}^{\infty}\xi_n < \epsilon/(2(M_1+1))$. So for all integers $n \geq N_1$, $m \geq 1$, we obtain from (24) that

$$\|x_{n+m}-x_n\| < \epsilon, \quad \forall n \geq N_1,\ m\geq 1. \tag{25}$$

Hence, $\{x_n\}$ is a Cauchy sequence in E. Since E is complete, there exists $q \in E$ such that $\lim_{n\to\infty}x_n = q$. We now show that $q \in F$. Since $d(x_n,F) \to 0$ and $x_n \to q$ as $n \to \infty$, for each $\overline{\epsilon} > 0$, there exists an integer $N_2 > 0$ such that, $d(x_n,F) = \inf_{p\in F}\|x_n-p\| < \overline{\epsilon}/3$ and $\|x_n-q\| < \overline{\epsilon}/2$ for all $n \geq N_2$. In particular, we have $d(x_{N_2},F) = \inf_{p\in F}\|x_{N_2}-p\| < \overline{\epsilon}/3$; that is, there exists a $\overline{p} \in F$ such that $\|x_{N_2}-\overline{p}\| < \overline{\epsilon}/2$; hence

$$\|q-\overline{p}\| \leq \|x_{N_2}-q\| + \|x_{N_2}-\overline{p}\| < \overline{\epsilon}. \tag{26}$$

Since F is a closed subset of E, we obtain $q \in F$. This completes the proof. □

Remark 7. Theorem 6 generalizes and extends Theorem 2.2 of Khan et al. [4], Theorem 3.1 of Ghosh and Debnath [8], Theorem 3.2 of Shahzad and Udomene [6], and Theorem 1 of Qihou [7] together with its Corollaries 1 and 2.

Asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings are all generalized asymptotically quasi-nonexpansive, by Theorem 6 and Lemma 3, so we have

Corollary 8. *Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_i : i = 1,2,\ldots,k\}$ be k asymptotically quasi-nonexpansive self-mappings of C with $\{u_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}u_{in} < \infty$ for all $i \in \{1,2,3,\ldots,k\}$. Let $\{\alpha_{in}\}_{n\geq1} \subset [0,1]$ for all $i \in \{1,2,3,\ldots,k\}$ and let W_n be a modified W-mapping generated by $T_1,T_2,\ldots,T_k$ and $\alpha_{1n},\alpha_{2n},\ldots,\alpha_{kn}$. Suppose $F\neq\emptyset$ and $\sum_{n=1}^{\infty}\lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion* (7)*. Then the sequence $\{x_n\}$ converges strongly to $p \in F$ if and only if* $\liminf_{n\to\infty}d(x_n,F) = 0$.

Corollary 9. *Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_i : i = 1,2,\ldots,k\}$ be k asymptotically nonexpansive self-mappings of C with $\{u_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}u_{in} < \infty$ for all $i \in \{1,2,3,\ldots,k\}$. Let $\{\alpha_{in}\}_{n\geq1} \subset [0,1]$ for all $i \in \{1,2,3,\ldots,k\}$ and let W_n be a modified W-mapping generated by $T_1,T_2,\ldots,T_k$ and $\alpha_{1n},\alpha_{2n},\ldots,\alpha_{kn}$.*

Suppose $F \neq \emptyset$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7). Then the sequence $\{x_n\}$ converges strongly to $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Corollary 10. *Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_i : i = 1, 2, \ldots, k\}$ be k generalized asymptotically quasi-nonexpansive self-mappings of C with $\{u_{in}\}, \{h_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in \{1, 2, 3, \ldots, k\}$. Let $\{\alpha_{in}\}_{n\geq 1} \subset [0, 1]$ for all $i \in \{1, 2, 3, \ldots, k\}$ and let W_n be a modified W-mapping generated by $T_1, T_2, \ldots, T_k$ and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$. Suppose that $F \neq \emptyset$ is closed and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7). Then the sequence $\{x_n\}$ converges strongly to $p \in F$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.*

3. Results in Uniformly Convex Banach Spaces

Lemma 11. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $\{T_i : i = 1, 2, \ldots, k\}$ be k $(L - \gamma)$ uniform Lipschitz and generalized asymptotically quasi-nonexpansive self-mappings of C with $\{u_{in}\}, \{h_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in \{1, 2, 3, \ldots, k\}$. Let $\alpha_{in} \in [\delta, 1 - \delta]$ for some $\delta \in (0, 1/2)$ and let W_n be a modified W-mapping generated by $T_1, T_2, \ldots, T_k$ and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$. Suppose $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7). Then $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$ for each $j \in \{1, 2, 3, \ldots k\}$.*

Proof. Let $p \in F$ and $\nu_n = \max_{1\leq i\leq k} u_{in}$, for all n. By Lemma 1 and Lemma 4(1), it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$. Assume that

$$\lim_{n\to\infty} \|x_n - p\| = c. \tag{27}$$

From (2) and (27) we obtain that

$$\limsup_{n\to\infty} \left\|U_{jn} x_n - p\right\| \leq c, \quad \forall 1 \leq j \leq k. \tag{28}$$

From (7), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\lambda_n f(x_n) + (1 - \lambda_n) W_n x_n - p\| \\ &\leq \lambda_n \alpha \|x_n - p\| + \lambda_n \|f(p) - p\| \\ &\quad + (1 - \lambda_n) \|U_{kn} x_n - p\|; \end{aligned} \tag{29}$$

therefore,

$$\liminf_{n\to\infty} \|U_{kn} x_n - p\| \geq c. \tag{30}$$

From (28) and (30) we can obtain that

$$\lim_{n\to\infty} \|U_{kn} x_n - p\| = c. \tag{31}$$

Suppose that $\lim_{n\to\infty} \|U_{(j+1)n} x_n - p\| = c$ for some $1 \leq j \leq k - 1$. Since

$$\begin{aligned} \left\|U_{(j+1)n} x_n - p\right\| &\leq \left(1 - \alpha_{(j+1)n}\right) \|x_n - p\| \\ &\quad + \alpha_{(j+1)n} \left\|T_{j+1}^n U_{jn} x_n - p\right\| \\ &\leq \left(1 - \alpha_{(j+1)n}\right) \|x_n - p\| + \alpha_{(j+1)n} \\ &\quad \times \left[\left(1 + u_{(j+1)n}\right) \left\|U_{jn} x_n - p\right\| + h_{(j+1)n}\right], \end{aligned} \tag{32}$$

so we obtain that

$$\liminf_{n\to\infty} \left\|U_{jn} x_n - p\right\| \geq c. \tag{33}$$

From (28) and (33), we have that

$$\lim_{n\to\infty} \left\|U_{jn} x_n - p\right\| = c. \tag{34}$$

Thus, by induction, we have

$$\lim_{n\to\infty} \left\|U_{jn} x_n - p\right\| = c, \tag{35}$$

for each $j = 1, 2, 3, \ldots, k$. That is,

$$\lim_{n\to\infty} \left\|\left(1 - \alpha_{jn}\right)(x_n - p) + \alpha_{jn}\left(T_j^n U_{(j-1)n} x_n - p\right)\right\| = c, \tag{36}$$

for each $j = 1, 2, 3, \ldots, k$. From (28), we obtain

$$\limsup_{n\to\infty} \left\|T_j^n U_{(j-1)n} x_n - p\right\| \leq c, \tag{37}$$

for each $j = 1, 2, 3, \ldots, k$. By Lemma 2, we get

$$\lim_{n\to\infty} \left\|T_j^n U_{(j-1)n} x_n - x_n\right\| = 0, \quad \forall 1 \leq j \leq k. \tag{38}$$

If $j = 1$, from (38), we have

$$\lim_{n\to\infty} \left\|T_1^n x_n - x_n\right\| = 0. \tag{39}$$

If $j = 2, 3, \ldots, k$, then we have

$$\begin{aligned} \left\|T_j^n x_n - x_n\right\| &\leq \left\|T_j^n x_n - T_j^n U_{(j-1)n} x_n\right\| + \left\|T_j^n U_{(j-1)n} x_n - x_n\right\| \\ &\leq L\left\|x_n - U_{(j-1)n} x_n\right\|^{\gamma} \\ &\quad + \left\|T_j^n U_{(j-1)n} x_n - x_n\right\| \\ &= L\Big(\alpha_{(j-1)n} \left\|x_n - T_{j-1}^n U_{(j-2)n} x_n\right\|\Big)^{\gamma} \\ &\quad + \left\|T_j^n U_{(j-1)n} x_n - x_n\right\|. \end{aligned} \tag{40}$$

Hence,

$$\lim_{n\to\infty} \left\|T_j^n x_n - x_n\right\| = 0, \quad \forall 1 \leq j \leq k. \tag{41}$$

Note that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\lambda_n f(x_n) + (1 - \lambda_n) W_n x_n - x_n\| \\ &\leq \lambda_n \left(\alpha \|x_n - p\| + \|f(p) - p\| + \|x_n - p\|\right) \end{aligned}$$

$$
\begin{aligned}
&+ (1-\lambda_n)\|W_n x_n - x_n\| \\
&= \lambda_n (\alpha \|x_n - p\| + \|f(p) - p\| + \|x_n - p\|) \\
&+ (1-\lambda_n)\alpha_{kn}\|T_k^n U_{(k-1)n} x_n - x_n\|;
\end{aligned} \tag{42}
$$

therefore, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{43}$$

Now, we observe that

$$
\begin{aligned}
\|x_n - T_j x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_j^{n+1} x_{n+1}\| \\
&+ \|T_j^{n+1} x_{n+1} - T_j^{n+1} x_n\| + \|T_j^{n+1} x_n - T_j x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_j^{n+1} x_{n+1}\| \\
&+ L\|x_{n+1} - x_n\|^{\gamma} + L\|T_j^n x_n - x_n\|^{\gamma}.
\end{aligned} \tag{44}
$$

By (41) and (43), we have

$$\lim_{n\to\infty}\|x_n - T_j x_n\| = 0, \tag{45}$$

for $j = 1, 2, 3, \ldots, k$. This completes the proof. □

Theorem 12. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $\{T_i : i = 1, 2, \ldots, k\}$ be k $(L-\gamma)$ uniform Lipschitz and generalized asymptotically quasi-nonexpansive self-mappings of C with $\{u_{in}\}, \{h_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in \{1, 2, 3, \ldots, k\}$. Let $\alpha_{in} \in [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$ and let W_n be a modified W-mapping generated by $T_1, T_2, \ldots, T_k$ and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$. Suppose $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and there exists one member in $\{T_i^m : i = 1, 2, \ldots, k\}$ which is semicompact for some positive integer m. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7). Then $\{x_n\}$ converges strongly to some common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$.*

Proof. By Lemma 11, we have

$$\lim_{n\to\infty}\|x_n - T_j x_n\| = 0, \tag{46}$$

for each $j = 1, 2, 3, \ldots k$. Without loss of generality, we may assume that T_1^m is semicompact for some $m \ge 1$; then we have

$$
\begin{aligned}
\|T_1^m x_n - x_n\| &\le \|T_1^m x_n - T_1^{m-1} x_n\| + \|T_1^{m-1} x_n - T_1^{m-2} x_n\| \\
&+ \cdots + \|T_1 x_n - x_n\| \\
&\le \|T_1 x_n - x_n\| + (m-1) L\|T_1 x_n - x_n\|^{\gamma} \longrightarrow 0.
\end{aligned} \tag{47}
$$

Since T_1^m is semicompact, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in C$. Hence, we have

$$\|q - T_i q\| = \lim_{j\to\infty}\|x_{n_j} - T_j x_{n_j}\| = 0, \tag{48}$$

for each $i = 1, 2, 3, \ldots, k$. This implies that $q \in F$. By Corollary 10, $\{x_n\}$ converges strongly to some common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$. □

Theorem 13. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $\{T_i : i = 1, 2, \ldots, k\}$ be k $(L-\gamma)$ uniform Lipschitz and generalized asymptotically quasi-nonexpansive self-mappings of C with $\{u_{in}\}, \{h_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in \{1, 2, 3, \ldots, k\}$. Let $\alpha_{in} \in [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$ and let W_n be a modified W-mapping generated by $T_1, T_2, \ldots, T_k$ and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$. Suppose $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and each $I - T_i$, $i = 1, 2, \ldots, k$, is demiclosed at 0. If E satisfies Opial's condition, then the sequence $\{x_n\}$ defined by (7) converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$.*

Proof. From the proof of Lemma 11, we know that $\{x_n\}$ is a bounded sequence in C. Since E is uniformly convex, it must be reflexive. Therefore, there exists a subsequence $\{x_{n_j}\}$ in $\{x_n\}$ converging weakly to $u \in C$. By Lemma 11, $\lim_{j\to\infty}\|x_{n_j} - T_i x_{n_j}\| = 0$ and $I - T_i$ is demiclosed at 0 for $i = 1, 2, \ldots k$, so we obtain $T_i u = u$. That is, $u \in F$. Suppose that there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to $v \in C$. As above, we can prove $v \in F$. By (27) we know that $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. Assume $v \neq u$. Then by the Opial's condition, we have

$$
\begin{aligned}
\lim_{n\to\infty}\|x_n - u\| &= \lim_{n_j\to\infty}\|x_{n_j} - u\| < \lim_{n_j\to\infty}\|x_{n_j} - v\| \\
&= \lim_{n\to\infty}\|x_n - v\| = \lim_{n_k\to\infty}\|x_{n_k} - v\| \\
&< \lim_{n_k\to\infty}\|x_{n_k} - u\| = \lim_{n\to\infty}\|x_n - u\|,
\end{aligned} \tag{49}
$$

which is a contradiction. Hence $u = v$. This implies that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$. □

Remark 14. Lemma 11, Theorem 12, and Theorem 13 extend Lemma 3.1, Theorem 3.3, and Theorem 3.2 of Khan et al. [4], respectively.

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] N. Shahzad and H. Zegeye, "Strong convergence of an implicit iteration process for a finite family of generalized asymptotically quasi-nonexpansive maps," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1058–1065, 2007.

[2] K. Nakajo, K. Shimoji, and W. Takahashi, "On strong convergence by the hybrid method for families of mappings in Hilbert spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 1-2, pp. 112–119, 2009.

[3] W. Takahashi and K. Shimoji, “Convergence theorems for nonexpansive mappings and feasibility problems,” *Mathematical and Computer Modelling*, vol. 32, no. 11-13, pp. 1463–1471, 2000.

[4] A. R. Khan, A.-A. Domlo, and H. Fukhar-ud-din, “Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 1–11, 2008.

[5] S. S. Chang, H. W. J. Lee, C. K. Chan, and J. K. Kim, “Approximating solutions of variational inequalities for asymptotically nonexpansive mappings,” *Applied Mathematics and Computation*, vol. 212, no. 1, pp. 51–59, 2009.

[6] N. Shahzad and A. Udomene, “Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces,” *Fixed Point Theory and Applications*, vol. 2006, Article ID 18909, 10 pages, 2006.

[7] L. Qihou, “Iterative sequences for asymptotically quasi-nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 259, no. 1, pp. 1–7, 2001.

[8] M. K. Ghosh and L. Debnath, “Convergence of Ishikawa iterates of quasi-nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 207, no. 1, pp. 96–103, 1997.

[9] H. K. Xu, “Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process,” *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 301–308, 1993.