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*Research Article*

# Bounds for the Arithmetic Mean in Terms of the Neuman-Sándor and Other Bivariate Means

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We present the largest values $\alpha_1$, $\alpha_2$, and $\alpha_3$ and the smallest values $\beta_1$, $\beta_2$, and $\beta_3$ such that the double inequalities $\alpha_1 M(a,b) + (1-\alpha_1)H(a,b) < A(a,b) < \beta_1 M(a,b) + (1-\beta_1)H(a,b)$, $\alpha_2 M(a,b) + (1-\alpha_2)\overline{H}(a,b) < A(a,b) < \beta_2 M(a,b) + (1-\beta_2)\overline{H}(a,b)$, and $\alpha_3 M(a,b) + (1-\alpha_3)He(a,b) < A(a,b) < \beta_3 M(a,b) + (1-\beta_3)He(a,b)$ hold for all $a,b>0$ with $a \neq b$, where $M(a,b)$, $A(a,b)$, $He(a,b)$, $H(a,b)$ and $\overline{H}(a,b)$ denote the Neuman-Sándor, arithmetic, Heronian, harmonic, and harmonic root-square means of $a$ and $b$, respectively.

## 1. Introduction

For $a,b>0$ with $a\neq b$ the Neuman-Sándor mean $M(a,b)$ [1] is defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}((a-b)/(a+b))}, \quad (1)$$

where $\sinh^{-1}(x) = \log(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the object intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a,b)$ can be found in the literature [1–10].

Let $\overline{H}(a,b) = \sqrt{2}ab/\sqrt{a^2+b^2}$, $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $He(a,b) = (a+\sqrt{ab}+b)/3$, $L(a,b) = (b-a)/(\log b - \log a)$, $P(a,b) = (a-b)/[4\arctan\sqrt{a/b} - \pi]$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic root-square, harmonic, geometric, Heronian, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic, and contraharmonic means of $a$ and $b$, respectively. Then it is known that the inequalities

$$\begin{aligned}\overline{H}(a,b) &< H(a,b) < G(a,b) < L(a,b)\\ &< He(a,b) < P(a,b) < A(a,b)\\ &< M(a,b) < T(a,b) < Q(a,b) < C(a,b)\end{aligned} \quad (2)$$

hold for all $a,b>0$ with $a\neq b$.

Neuman and Sándor [1, 2] proved that the inequalities

$$\frac{\pi}{4\log(1+\sqrt{2})}T(a,b) < M(a,b) < \frac{A(a,b)}{\log(1+\sqrt{2})},$$

$$\sqrt{2T^2(a,b) - Q^2(a,b)} < M(a,b) < \frac{T^2(a,b)}{Q(a,b)},$$

$$H(T(a,b), A(a,b)) < M(a,b) < L(A(a,b), Q(a,b)),$$

$$H(M(a,b), Q(a,b)) < T(a,b), \qquad M(a,b) < \frac{A^2(a,b)}{P(a,b)},$$

$$A^{2/3}(a,b)\,Q^{1/3}(a,b) < M(a,b) < \frac{2A(a,b)+Q(a,b)}{3},$$
$$\sqrt{A(a,b)\,T(a,b)} < M(a,b) < \sqrt{A^2(a,b)+T^2(a,b)},$$
$$\frac{G(x,y)}{G(1-x,1-y)} < \frac{L(x,y)}{L(1-x,1-y)} < \frac{P(x,y)}{P(1-x,1-y)} < \frac{A(x,y)}{A(1-x,1-y)} < \frac{M(x,y)}{M(1-x,1-y)} < \frac{T(x,y)}{T(1-x,1-y)},$$
$$\frac{1}{A(1-x,1-y)} - \frac{1}{A(x,y)} < \frac{1}{M(1-x,1-y)} - \frac{1}{M(x,y)} < \frac{1}{T(1-x,1-y)} - \frac{1}{T(x,y)},$$
$$A(x,y)\,A(1-x,1-y) < M(x,y)\,M(1-x,1-y) < T(x,y)\,T(1-x,1-y) \tag{3}$$

hold for all $a,b>0$ and $x,y\in(0,1/2]$ with $a\neq b$ and $x\neq y$. All the results stated above are in fact particular cases of more general and stronger results for the Schwab-Borchardt means [1, 2]. Some of them are based on the sequential method of Sándor [11]. In particular, Neuman and Sándor [1] also found that the inequality

$$M(a,b) < L(a_n,b_n) \tag{4}$$

holds for all $n\geq 0$ and $a,b>0$ with $a\neq b$, where $a_0=Q(a,b)$, $b_0=A(a,b)$, $a_{n+1}=(a_n+b_n)/2$, and $b_{n+1}=\sqrt{a_nb_n}$.

Li et al. [3] proved that the double inequality $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ holds for all $a,b>0$ with $a\neq b$, where $L_p(a,b)=[(b^{p+1}-a^{p+1})/((p+1)(b-a))]^{1/p}$ $(p\neq -1,0)$, $L_0(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, and $L_{-1}(a,b)=(b-a)/(\log b-\log a)$ is the $p$th generalized logarithmic mean of $a$ and $b$; $p_0=1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p}=2\log(1+\sqrt{2})$.

In [4], Neuman proved that the double inequalities

$$\alpha Q(a,b)+(1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b)+(1-\beta)A(a,b),$$
$$\lambda C(a,b)+(1-\lambda)A(a,b) < M(a,b) < \mu C(a,b)+(1-\mu)A(a,b) \tag{5}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq[1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})]=0.3249\cdots$, $\beta\geq 1/3$, $\lambda\leq[1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})=0.1345\cdots$, and $\mu\geq 1/6$.

The main purpose of this paper is to find the largest values $\alpha_1$, $\alpha_2$, and $\alpha_3$ and the smallest values $\beta_1$, $\beta_2$, and $\beta_3$ such that the double inequalities

$$\alpha_1 M(a,b)+(1-\alpha_1)H(a,b) < A(a,b) < \beta_1 M(a,b)+(1-\beta_1)H(a,b),$$
$$\alpha_2 M(a,b)+(1-\alpha_2)\overline{H}(a,b) < A(a,b) < \beta_2 M(a,b)+(1-\beta_2)\overline{H}(a,b),$$
$$\alpha_3 M(a,b)+(1-\alpha_3)He(a,b) < A(a,b) < \beta_3 M(a,b)+(1-\beta_3)He(a,b) \tag{6}$$

hold for all $a,b>0$ with $a\neq b$. All numerical computations are carried out using Mathematical software.

## 2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

**Lemma 1** (see [12, Lemma 1.1]). *Suppose that the power series $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $g(x)=\sum_{n=0}^{\infty}b_nx^n$ have the radius of convergence $r>0$ and $a_n,b_n>0$ for all $n\in\{0,1,2,\ldots\}$. Let $h(x)=f(x)/g(x)$; then the following hold.*

(1) *If the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0,r)$.*

(2) *If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for $0<n\leq n_0$ and (strictly) decreasing (increasing) for $n>n_0$, then there exists $x_0\in(0,r)$ such that $h(x)$ is (strictly) increasing (decreasing) on $(0,x_0)$ and (strictly) decreasing (increasing) on $(x_0,r)$.*

**Lemma 2.** *The function*

$$g(t)=\frac{t[\cosh(2t)+2\cosh(t)-3]}{\sinh(2t)+t\cosh(2t)-3t} \tag{7}$$

*is strictly decreasing on $(0,\log(1+\sqrt{2}))$, where $\sinh(t)=(e^t-e^{-t})/2$ and $\cosh(t)=(e^t+e^{-t})/2$ denote the hyperbolic sine and hyperbolic cosine functions, respectively.*

*Proof.* Making use of power series $\sinh(t)=\sum_{n=0}^{\infty}t^{2n+1}/(2n+1)!$ and $\cosh(t)=\sum_{n=0}^{\infty}t^{2n}/(2n)!$, the function $g(t)$ can be written as

$$g(t)=\frac{\sum_{n=0}^{\infty}\left[\left(2^{2n+2}+2\right)/(2n+2)!\right]t^{2n}}{\sum_{n=0}^{\infty}\left[(2n+5)\,2^{2n+2}/(2n+3)!\right]t^{2n}}. \tag{8}$$

Let

$$a_n=\frac{2^{2n+2}+2}{(2n+2)!},\qquad b_n=\frac{(2n+5)\,2^{2n+2}}{(2n+3)!}. \tag{9}$$

Then simple computation leads to

$$\frac{a_{n+1}}{b_{n+1}}-\frac{a_n}{b_n}=\frac{c_n}{(2n+5)(2n+7)\,2^{2n+3}}, \tag{10}$$

where

$$c_n = 2^{2n+5} - \left(12n^2 + 60n + 59\right). \tag{11}$$

It follows from (11) that

$$c_0 = -27, \qquad c_1 = -3, \tag{12}$$

$$c_n \geq 128n^2 - \left(12n^2 + 60n + 59\right) = 116n^2 - 60n - 59 > 0 \tag{13}$$

for all $n \geq 2$.

Equations (10) and (12) together with inequality (13) lead to the conclusion that the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is strictly decreasing for $0 \leq n \leq 1$ and strictly increasing for $n \geq 2$. Then from Lemma 1(2) and (8) together with (9) we clearly see that there exists $t_0 \in (0, \infty)$ such that $g(t)$ is strictly decreasing on $(0, t_0)$ and strictly increasing on $(t_0, \infty)$.

Let $t^* = \log(1 + \sqrt{2})$. Then simple computations lead to

$$\begin{gathered} \sinh\left(t^*\right) = 1, \qquad \cosh\left(t^*\right) = \sqrt{2}, \\ \sinh\left(2t^*\right) = 2\sqrt{2}, \qquad \cosh\left(2t^*\right) = 3. \end{gathered} \tag{14}$$

It is not difficult to verify that

$$g'\left(t^*\right) = -2t^{*2} + \left(2 - \sqrt{2}\right)t^* + 1 = -0.03734\cdots < 0. \tag{15}$$

From the piecewise monotonicity of $g(t)$ and inequality (15) we clearly see that $t^* = \log(1 + \sqrt{2}) < t_0$, which implies that $g(t)$ is strictly decreasing on $(0, \log(1 + \sqrt{2}))$. □

**Lemma 3.** *The inequality*

$$\sqrt{1 - t^2} > 1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} - \frac{5t^8}{128} - \frac{35t^{10}}{128} \tag{16}$$

*holds for all* $t \in (0, 1)$.

*Proof.* Simple computations lead to

$$\begin{aligned} &\left(1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} - \frac{5t^8}{128} - \frac{35t^{10}}{128}\right)^2 \\ &= 1 - t^2 - \frac{t^{10}}{16384} \\ &\quad \times \left(8064 - 4704t^2 - 1200t^4 - 585t^6 - 350t^8 - 1225t^{10}\right) \\ &< 1 - t^2 - \frac{t^{10}}{16384}(8064 - 4704 - 1200 - 585 - 350 - 1225) \\ &< 1 - t^2 \end{aligned} \tag{17}$$

for all $t \in (0, 1)$. □

**Lemma 4.** *The function* $f(t) = [t - \sinh^{-1}(t)]/[(1 - \sqrt{1 - t^2})\sinh^{-1}(t)]$ *is strictly decreasing in* $(0, 1)$.

*Proof.* Differentiating $f(t)$ gives

$$\begin{aligned} &f'(t) \\ &= \Big(t\left(1 - t^2 - \sqrt{1 - t^2}\right) \\ &\quad + \sqrt{1 + t^2}\left[t\sinh^{-1}(t) + \sqrt{1 - t^2} - 1\right]\sinh^{-1}(t)\Big) \\ &\quad \times \left(\left(1 - \sqrt{1 - t^2}\right)^2\sqrt{1 - t^4}\left[\sinh^{-1}(t)\right]^2\right)^{-1}. \end{aligned} \tag{18}$$

Making use of the power series

$$\begin{aligned} \sqrt{1 + t^2} &= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n)!}{(2n - 1)4^n(n!)^2}t^{2n} \\ &= 1 + \frac{t^2}{2} - \frac{t^4}{8} + \frac{t^6}{16} - \frac{5t^8}{128} + \frac{7t^{10}}{256} - \cdots, \end{aligned} \tag{19}$$

$$\begin{aligned} \sqrt{1 - t^2} &= \sum_{n=0}^{\infty} \frac{(2n)!}{(1 - 2n)4^n(n!)^2}t^{2n} \\ &= 1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} - \frac{5t^8}{128} - \frac{7t^{10}}{256} - \cdots, \end{aligned} \tag{20}$$

$$\begin{aligned} \sinh^{-1}(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{(2n + 1)4^n(n!)^2}t^{2n+1} \\ &= t - \frac{t^3}{6} + \frac{3t^5}{40} - \frac{5t^7}{112} + \cdots, \end{aligned} \tag{21}$$

we get

$$\begin{aligned} \sqrt{1 + t^2}\sinh^{-1}(t) &= t + \frac{t^3}{3} - \frac{2t^5}{15} + \frac{8t^7}{105} - \frac{1091t^9}{13440} + \cdots \\ &< t + \frac{t^3}{3} - \frac{2t^5}{15} + \frac{8t^7}{105}, \\ t\sinh^{-1}(t) + \sqrt{1 - t^2} - 1 & \\ &= \frac{t^2}{2} - \frac{7t^4}{24} + \frac{t^6}{80} - \frac{75t^8}{896} + \cdots < \frac{t^2}{2} - \frac{7t^4}{34} + \frac{t^6}{80}, \end{aligned} \tag{22}$$

for $t \in (0, 1)$.

Let

$$\begin{aligned} g(t) &= t\left(1 - t^2 - \sqrt{1 - t^2}\right) \\ &\quad + \sqrt{1 + t^2}\left[t\sinh^{-1}(t) + \sqrt{1 - t^2} - 1\right]\sinh^{-1}(t) \\ &= -\left(1 - \sqrt{1 - t^2}\right)\left[t\sqrt{1 - t^2} + \sqrt{1 + t^2}\sinh^{-1}(t)\right] \\ &\quad + t\sqrt{1 + t^2}\left[\sinh^{-1}(t)\right]^2. \end{aligned} \tag{23}$$

We divide the proof into two cases.

*Case 1 ($t \in (0, \sqrt{10}/5)$).* Then from Lemma 3, (22), and (23) we have

$$\begin{aligned} g(t) &< t\left[1-t^2-\left(1-\frac{t^2}{2}-\frac{t^4}{8}-\frac{t^6}{16}-\frac{5t^8}{128}-\frac{35t^{10}}{128}\right)\right] \\ &\quad +\left(t+\frac{t^3}{3}-\frac{2t^5}{15}+\frac{8t^7}{105}\right)\times\left(\frac{t^2}{2}-\frac{7t^4}{24}+\frac{t^6}{80}\right) \\ &= \frac{t^2}{201600}\left(24235t^2+50309t^4+192t^6-17920\right) \\ &< \frac{t^2}{201600}\left[24235\left(\frac{\sqrt{10}}{5}\right)^2+50309\left(\frac{\sqrt{10}}{5}\right)^4 \right. \\ &\quad \left. +192\left(\frac{\sqrt{10}}{5}\right)^6-17920\right]<0. \end{aligned} \tag{24}$$

*Case 2 ($t \in [\sqrt{10}/5, 1)$).* Then from Lemma 3, (19), (20), and (23) together with $\sinh^{-1}(\sqrt{10}/5) = 0.5964\cdots$ we get

$$\begin{aligned} g(t) &< -\left[1-\left(1-\frac{t^2}{2}-\frac{t^4}{8}-\frac{t^6}{16}\right)\right] \\ &\quad \times\left[t\left(1-\frac{t^2}{2}-\frac{t^4}{8}-\frac{t^6}{16}-\frac{5t^8}{128}-\frac{35t^{10}}{128}\right)\right. \\ &\quad \left. +0.596\left(1+\frac{t^2}{2}-\frac{t^4}{8}\right)\right] \\ &\quad +(0.596)^2\left(1+\frac{t^2}{2}-\frac{t^4}{8}+\frac{t^6}{16}\right):= \\ g_1(t) &= 0.356409t-0.298t^2-0.3217955t^3-0.2235t^4 \\ &\quad +0.08448875t^5-0.03725t^6+0.0847755625t^7 \\ &\quad -0.0093125t^8+0.078125t^9+0.00465625t^{10} \\ &\quad +0.03515625t^{11}+0.1455078125t^{13} \\ &\quad +0.03662109375t^{15}+0.01708984375t^{17}. \end{aligned} \tag{25}$$

Numerical computations show that

$$g_1\left(\frac{\sqrt{10}}{5}\right) = -0.0000435, \qquad g_1(1) = -0.0510683\cdots, \tag{26}$$

$$g_1'\left(\frac{\sqrt{10}}{5}\right) = -0.52459\cdots, \qquad g_1'(1) = 2.4665\cdots, \tag{27}$$

$$g_1''\left(\frac{\sqrt{10}}{5}\right) = -1.86198\cdots, \qquad g_1''(1) = 43.2711\cdots, \tag{28}$$

$$g_1'''\left(\frac{\sqrt{10}}{5}\right) = 4.63578\cdots, \tag{29}$$

$$\begin{aligned} g_1^{(4)}(t) &> (-5.364+9.653865t) \\ &\quad +t^2\left(-13.41+71.2114725t-15.645t^2\right) \\ &> \left[-5.364+9.653865\times\left(\frac{\sqrt{10}}{5}\right)\right] \\ &\quad +t^2\left[\left(-13.41+71.2114725\times\left(\frac{\sqrt{10}}{5}\right)-15.645\right)\right] \\ &= 0.7416\cdots+t^2\times 15.9830\cdots>0. \end{aligned} \tag{30}$$

It follows from (29) and (30) that $g_1''(t)$ is strictly increasing in $[\sqrt{10}/5, 1)$. Then (28) leads to the conclusion that there exists $t_0 \in (\sqrt{10}/5, 1)$ such that $g_1'(t)$ is strictly decreasing in $[\sqrt{10}/5, t_0]$ and strictly increasing in $[t_0, 1)$.

From (27) and the piecewise monotonicity of $g_1'(t)$ we clearly see that there exists $t_1 \in (t_0, 1)$ such that $g_1(t)$ is strictly decreasing in $[\sqrt{10}/5, t_1]$ and strictly increasing in $[t_1, 1)$. Therefore,

$$g(t) < g_1(t) < 0 \tag{31}$$

for $t \in [\sqrt{10}/5, 1)$ follows from (25) and (26) together with the piecewise monotonicity of $g_1(t)$. □

**Lemma 5.** *Let $p \in (0,1)$, $\lambda_0 = \log(1+\sqrt{2})/[3-2\log(1+\sqrt{2})] = 0.7123\cdots$, and*

$$\varphi_p(t) = \frac{3p\sqrt{1-t^2}}{(p-1)t+2p-1} - \sinh^{-1}\left(\sqrt{1-t^2}\right). \tag{32}$$

*Then $\varphi_{1/2}(t) < 0$ and $\varphi_{\lambda_0}(t) > 0$ for all $t \in (0,1)$.*

*Proof.* We first prove that $\varphi_{1/2}(t) < 0$ for $t \in (0,1)$. From (32) one has

$$\varphi_{1/2}(1) = 0, \tag{33}$$

$$\varphi_{1/2}'(t) = \frac{4t-2t^2+(1/4)t^3-(3t-3/4)\sqrt{2-t^2}}{\sqrt{1-t^2}\sqrt{2-t^2}(2-(1/2)t)^2}. \tag{34}$$

Let

$$f(t) = 4t-2t^2+\frac{1}{4}t^3-\left(3t-\frac{3}{4}\right)\sqrt{2-t^2}. \tag{35}$$

Then

$$\begin{aligned} f(1) &= 0, \\ f'(t) &= \frac{24t^2-3t-24+\left(16-16t+3t^2\right)\sqrt{2-t^2}}{4\sqrt{2-t^2}}. \end{aligned} \tag{36}$$

We divide the proof into two cases.

*Case 1 ($t \in (0, 0.7)$).* Then we clearly see that

$$\begin{aligned}
&\frac{24t^2 - 3t - 24 + \left(16 - 16t + 3t^2\right)\sqrt{2 - t^2}}{\sqrt{2 - t^2}} \\
&\quad < 24t^2 - \frac{3t + 24}{\sqrt{2}} + \left(16 - 16t + 3t^2\right) \\
&\quad = 27t^2 - \left(\frac{3}{2}\sqrt{2} + 16\right)t + 16 - 12\sqrt{2} \\
&\quad = 27\left(t - \frac{3\sqrt{2} + 32 + \sqrt{5376\sqrt{2} - 5870}}{108}\right) \\
&\qquad \times \left(t - \frac{3\sqrt{2} + 32 - \sqrt{5376\sqrt{2} - 5870}}{108}\right) \\
&\quad = 27\left(t - 0.72105\cdots\right)\left(t + 0.04985\cdots\right) < 0.
\end{aligned} \tag{37}$$

*Case 2 ($t \in [0.7, 1)$).* Then we get

$$\begin{aligned}
&\left[24t^2 - 3t - 24 + \left(16 - 16t + 3t^2\right)\sqrt{2 - t^2}\right]\sqrt{2 - t^2} \\
&\quad = \left(24t^2 - 3t - 24\right)\sqrt{2 - t^2} + 32 - 32t - 10t^2 + 16t^3 - 3t^4 \\
&\quad < 24t^2 - 3t - 24 + 32 - 32t - 10t^2 + 16t^3 - 3t^4 \\
&\quad = -\left(1 - t\right)\left(27t + 13t^2 - 3t^3 - 8\right) < 0.
\end{aligned} \tag{38}$$

It follows from (32) and (33) together with Cases 1 and 2 that $f(t) < 0$ for $t \in (0, 1)$. Then from (34) and (35) we know that $\varphi_{1/2}(t)$ is strictly decreasing in $(0, 1)$.

Therefore, $\varphi_{1/2}(t) < 0$ for $t \in (0, 1)$ follows from (33) and the monotonicity of $\varphi_{1/2}(t)$.

Next, we prove that $\varphi_{\lambda_0}(t) > 0$ for $t \in (0, 1)$. From (32) we clearly see that we only have to prove that

$$\begin{aligned}
\varphi_{\lambda_0}\left(\sqrt{1 - t^2}\right) &= \frac{3\lambda_0 t}{\left(\lambda_0 - 1\right)\sqrt{1 - t^2} + 2\lambda_0 + 1} - \sinh^{-1}(t) \\
&> 0
\end{aligned} \tag{39}$$

for all $t \in (0, 1)$.

Inequality (39) can be rewritten as

$$\lambda_0 > \frac{\left(1 - \sqrt{1 - t^2}\right)\sinh^{-1}(t)}{3t - \left(2 + \sqrt{1 - t^2}\right)\sinh^{-1}(t)}. \tag{40}$$

Let

$$\begin{aligned}
h(t) &= \frac{\left(1 - \sqrt{1 - t^2}\right)\sinh^{-1}(t)}{3t - \left(2 + \sqrt{1 - t^2}\right)\sinh^{-1}(t)} \\
&= \frac{1}{1 + 3\left[t - \sinh^{-1}(t)\right] / \left[\left(1 - \sqrt{1 - t^2}\right)\sinh^{-1}(t)\right]}.
\end{aligned} \tag{41}$$

Then inequality (40) follows from Lemma 4 and (41) together with $h(1) = \lambda_0$. □

## 3. Main Results

**Theorem 6.** *The double inequality*

$$\begin{aligned}
&\alpha_1 M(a, b) + \left(1 - \alpha_1\right) H(a, b) \\
&\quad < A(a, b) < \beta_1 M(a, b) + \left(1 - \beta_1\right) H(a, b)
\end{aligned} \tag{42}$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 6/7 = 0.8571\cdots$ and $\beta_1 \geq \log(1 + \sqrt{2}) = 0.8813\cdots$.*

*Proof.* Since $H(a, b)$, $M(a, b)$, and $A(a, b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $x = (a - b)/(a + b)$ and $t = \sinh^{-1}(x)$. Then $x \in (0, 1)$, $t \in (0, \log(1 + \sqrt{2}))$, and

$$\begin{aligned}
\frac{A(a, b) - H(a, b)}{M(a, b) - H(a, b)} &= \frac{x^2}{x/\sinh^{-1}(x) - \left(1 - x^2\right)} \\
&= \frac{t\cosh(2t) - t}{2\sinh(t) + t\cosh(2t) - 3t}.
\end{aligned} \tag{43}$$

Let

$$f(t) = \frac{t\cosh(2t) - t}{2\sinh(t) + t\cosh(2t) - 3t}. \tag{44}$$

Then $f(t)$ can be rewritten as

$$\begin{aligned}
&f(t) \\
&= \frac{t\sum_{n=0}^{\infty}\left(2^{2n}/(2n)!\right)t^{2n} - t}{2\sum_{n=0}^{\infty}\left(t^{2n+1}/(2n+1)!\right) + \sum_{n=0}^{\infty}\left(2^{2n}/(2n)!\right)t^{2n+1} - 3t} \\
&= \frac{\sum_{n=1}^{\infty}\left(2^{2n}/(2n)!\right)t^{2n}}{\sum_{n=1}^{\infty}\left[\left(2 + (2n+1)2^{2n}\right)/(2n+1)!\right]t^{2n+1}} \\
&= \frac{\sum_{n=0}^{\infty}\left(2^{2n+2}/(2n+2)!\right)t^{2n}}{\sum_{n=0}^{\infty}\left[\left(2 + (2n+3)2^{2n+2}\right)/(2n+3)!\right]t^{2n}} \\
&:= \frac{\sum_{n=0}^{\infty}a_n t^{2n}}{\sum_{n=0}^{\infty}b_n t^{2n}}, \\
&\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{(12n + 34)2^{2n+2}}{\left[(2n+3)2^{2n+2} + 2\right]\left[(2n+5)2^{2n+4} + 2\right]} > 0
\end{aligned} \tag{45}$$

for all $n \geq 0$.

It follows from (45) together with Lemma 1(1) that $f(t)$ is strictly increasing in $(0, \log(1+\sqrt{2}))$. Note that

$$\lim_{t\to 0} f(t) = \frac{a_0}{b_0} = \frac{6}{7}, \qquad f\left(\log\left(1+\sqrt{2}\right)\right) = \log\left(1+\sqrt{2}\right). \tag{46}$$

Therefore, Theorem 6 follows from (43) and (44) together with (46) and the monotonicity of $f(t)$. □

**Theorem 7.** *The double inequality*

$$\alpha_2 M(a,b) + (1-\alpha_2)\overline{H}(a,b) < A(a,b) < \beta_2 M(a,b) + (1-\beta_2)\overline{H}(a,b) \tag{47}$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq \log(1+\sqrt{2}) = 0.8813\cdots$ and $\beta_2 \geq 9/10$.*

*Proof.* Since $\overline{H}(a,b)$, $M(a,b)$, and $A(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$ and $t = \sinh^{-1}(x)$. Then $x \in (0,1)$, $t \in (0, \log(1+\sqrt{2}))$, and

$$\frac{A(a,b) - \overline{H}(a,b)}{M(a,b) - \overline{H}(a,b)} = \frac{\left(\sqrt{1+x^2} + x^2 - 1\right)\sinh^{-1}(x)}{x\sqrt{1+x^2} - (1-x^2)\sinh^{-1}(x)} = \frac{t\left[\cosh(2t) + 2\cosh(t) - 3\right]}{\sinh(2t) + t\cosh(2t) - 3t}. \tag{48}$$

Simple computations lead to

$$\lim_{t\to 0} \frac{t\left[\cosh(2t) + 2\cosh(t) - 3\right]}{\sinh(2t) + t\cosh(2t) - 3t} = \frac{9}{10}, \qquad \lim_{t\to\log(1+\sqrt{2})} \frac{t\left[\cosh(2t) + 2\cosh(t) - 3\right]}{\sinh(2t) + t\cosh(2t) - 3t} = \log\left(1+\sqrt{2}\right). \tag{49}$$

Therefore, Theorem 7 follows from Lemma 2, (48), and (49). □

**Theorem 8.** *The double inequality*

$$\alpha_3 M(a,b) + (1-\alpha_3) He(a,b) < A(a,b) < \beta_3 M(a,b) + (1-\beta_3) He(a,b) \tag{50}$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 1/2$ and $\beta_3 \geq \log(1+\sqrt{2})/[3 - 2\log(1+\sqrt{2})] = 0.7123\cdots$.*

*Proof.* Since $H(a,b)$, $M(a,b)$, and $A(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b) \in (0,1)$, $0 < p < 1$, and $\lambda_0 = \log(1+\sqrt{2})/[3 - 2\log(1+\sqrt{2})]$; then

$$\frac{A(a,b) - He(a,b)}{M(a,b) - He(a,b)} = \frac{\left(1 - \sqrt{1-x^2}\right)\sinh^{-1}(x)}{3x - \left(2 + \sqrt{1-x^2}\right)\sinh^{-1}(x)}, \tag{51}$$

$$\begin{aligned} &pM(a,b) + (1-p) He(a,b) - A(a,b) \\ &= \frac{A(a,b)}{3}\left[\frac{3px}{\sinh^{-1}(x)} - (p-1)\sqrt{1-x^2} - (1+2p)\right] \\ &= \frac{A(a,b)\left[(1+2p) - (1-p)\sqrt{1-x^2}\right]}{3\sinh^{-1}(x)}\varphi_p\left(\sqrt{1-x^2}\right), \end{aligned} \tag{52}$$

where $\varphi_p(t)$ is defined as in Lemma 5.

Note that

$$\lim_{x\to 0} \frac{\left(1 - \sqrt{1-x^2}\right)\sinh^{-1}(x)}{3x - \left(2 + \sqrt{1-x^2}\right)\sinh^{-1}(x)} = \frac{1}{2}, \qquad \lim_{x\to 1} \frac{\left(1 - \sqrt{1-x^2}\right)\sinh^{-1}(x)}{3x - \left(2 + \sqrt{1-x^2}\right)\sinh^{-1}(x)} = \lambda_0. \tag{53}$$

Therefore, Theorem 8 follows from (51)–(53) together with Lemma 5. □

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