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*Research Article*

# On Convective Dusty Flow Past a Vertical Stretching Sheet with Internal Heat Absorption

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The steady two-dimensional boundary layer flow of a viscous, incompressible, and electrically conducting dusty fluid past a vertical permeable stretching sheet under the influence of a transverse magnetic field with the viscous and Joule dissipation is investigated. The fluid particles are assumed to be heat absorbing and the temperature at the surface of the sheet is a result of convective heating. The governing nonlinear partial differential equations are transformed to a set of highly nonlinear coupled ordinary differential equations using a suitable similarity transformation and the resulting system is then solved numerically. It is found inter alia that the contributions of viscous and Joule dissipation in the flow are to increase the thickness of the thermal boundary layer.

## 1. Introduction

The study of boundary layer flows through continuously stretching sheet has attracted many researchers due to its bearing in many fluid engineering processes such as extrusion processes, melt spinning, hot rolling, wire drawing, glass-fiber production, manufacture of plastics, polymer and rubber sheets, performance of lubricants and paints, and movement of biological fluids Crane [1] first considered the steady two-dimensional boundary layer flow of a Newtonian fluid driven by a stretching elastic sheet moving in its own plane with a velocity varying linearly with the distance from a fixed point. Later, this work was extended by many researchers to investigate different aspects of the flow and heat transfer in a fluid of infinite extent surrounding a stretching sheet [2–7].

Magnetohydrodynamic flow through stretching sheets in the presence of free convective heat transfer has been investigated by a number of researchers due to its applications in metallurgical industry, such as the cooling of continuous strips and filaments drawn through a quiescent fluid. It is known that the properties of the final product depend significantly on the rate of cooling during the manufacturing processes. The rate of cooling can be controlled by drawing the strips in an electrically conducting fluid subject to a magnetic field, so that a final product of desired characteristics can be obtained [8, 9]. The free convection effect on MHD heat and mass transfer of a continuously moving permeable vertical surface was studied numerically by Yih [10]. He found that the Nusselt number and the Sherwood number increase with the increase in suction through the permeable wall. Ishak et al. [11] investigated the mixed convection boundary layer in the stagnation-point flow towards a stretching vertical sheet. Ishak et al. [12] also made an analysis for the steady two-dimensional magnetohydrodynamic flow of an incompressible viscous and electrically conducting fluid over a stretching sheet in its own plane. In this study, the stretching velocity, the surface temperature, and the transverse magnetic field were assumed to vary in a power law with the distance from the origin. Pal and Mondal [13] investigated the hydromagnetic non-Darcy flow and heat transfer characteristics over a stretching sheet taking into account the effect of Ohmic dissipation and thermal radiation. The internal heat absorption/generation exerts significant influence on the rate of heat transfer from a heated surface in several practical situations [14–17]. The effect of internal heat absorption/generation plays important role in the heat transfer of fluids undergoing exothermic or endothermic chemical reactions [14, 15]. Abo-Eldahab and El Aziz [15] studied the problem of hydromagnetic heat transfer over a continuously stretching surface in the presence of internal heat generation/absorption. The problem of magnetohydrodynamic mixed convection flow and heat transfer of a power-law fluid past a stretching surface in the presence of heat generation/absorption and thermal radiation is investigated by Chen [9].

The presence of dust particles in the flow of a viscous fluid has significant effects. The dust particles tend to retard the flow and to decrease the fluid temperature. Such flows are encountered in a wide variety of engineering problems such as nuclear reactor cooling, rain erosion, paint spraying, transport, waste water treatment, and combustion, The presence of solid particles such as ash or soot in combustion energy generators and their effect on performance of such devices led to studies of particulate suspension in electrically conducting fluid in the presence of magnetic field. Saffman [18] initiated the study of dusty fluids and discussed the stability of the laminar flow of a dusty gas in which the dust particles are uniformly distributed. Chamkha [19] considered unsteady laminar hydromagnetic fluid particle flow and heat transfer in channels and circular pipes considering two-phase continuum models. Attia [20] investigated effects of Hall current on Couette flow with heat transfer of a dusty conducting fluid in the presence of uniform suction/injection. S. Ghosh and A. K. Ghosh [21] studied hydromagnetic rotating flow of a dusty fluid near a pulsating plate when the flow is generated in the fluid particle system due to velocity tooth pulses subjected on the plate in the presence of a transverse magnetic field. Makinde and Chinyoka [22] studied unsteady fluid flow and heat transfer of a dusty fluid between two parallel plates with variable viscosity and thermal conductivity when the fluid is driven by a constant pressure gradient and subjected to a uniform external magnetic field applied perpendicular to the plates with a slip boundary condition. In all the above investigations, it has consistently been assumed that the temperature at the plate surface is constant. However, there exist several problems of physical interest which may require non-uniform conditions. Gireesha et al. [23] investigated the boundary flow and heat transfer of a dusty fluid flow over a stretching sheet with nonuniform heat source/sink. They considered two types of heating processes namely, (i) prescribed surface temperature and (ii) prescribed surface heat flux. Ramesh et al. [24] analyzed the steady two-dimensional MHD flow of a dusty fluid near the stagnation point over a permeable stretching sheet with the effect of non-uniform source/sink. Recently, the effects of time-dependent surface temperature on the flow and heat transfer of a viscous, incompressible, and electrically conducting dusty fluid are studied by Nandkeolyar et al. [25] and Nandkeolyar and Das [26]. They assumed that the temperature of the surface increases to a specific time and then remains constant. They also compared the flow of dusty fluids through a wall having time-dependent temperature with that of flow past an isothermal wall.

The aim of the present work is to investigate the steady two-dimensional boundary layer flow of a viscous, incompressible, and electrically conducting dusty fluid past a vertical permeable stretching sheet in the presence of a transverse magnetic field considering the effects of viscous and Joule dissipation and internal heat absorption. The governing nonlinear partial differential equations are subject to suitable similarity transformation and transformed to a set of nonlinear ordinary differential equations and solved using `bvp4c` routine of Matlab. The effects of several important parameters affecting the flow and heat transfer are studied with the help of suitable graphs and tables.

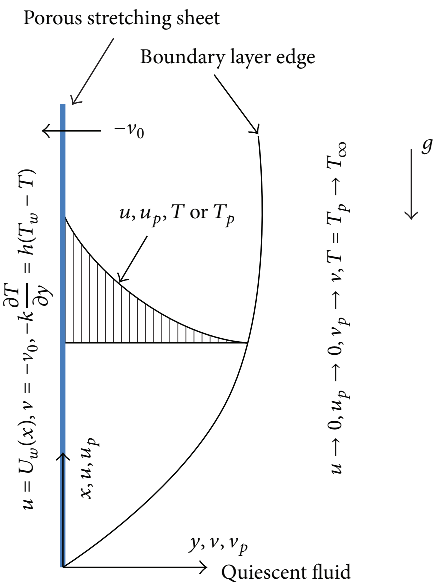



FIGURE 1: Geometry of the problem.

## 2. Mathematical Formulation of the Problem

We consider the steady two-dimensional boundary layer flow of a viscous, incompressible, electrically conducting, and heat absorbing dusty fluid past a vertical permeable stretching sheet under the influence of a transverse magnetic field. A cartesian coordinate system is used with the $x$-axis along the sheet and the $y$-axis normal to the sheet. The geometry of the problem is depicted in Figure 1. Two equal but opposite forces are applied along the sheet so that the wall is stretched, keeping the position of the origin unaltered. The flow is induced due to the stretching of the sheet in its own plane with the surface velocity $U_w(x)$. The transverse magnetic field of strength $B_0$ is acting normally to the stretching sheet. The temperature of the sheet is the result of a convective heating process via conduction which is characterized by a temperature $T_w$ and a heat transfer coefficient $h$. The fluid and the dust particle are assumed to be at rest at the beginning. It is also assumed that the dust particles are spherical in shape and uniform in size, and the number density of the dust particles is constant throughout the flow. Further, it is assumed that the induced magnetic field produced by the fluid motion is negligible in comparison to the applied one. This assumption is valid for low magnetic Reynolds number fluids [27].

Under the above assumptions, the equations governing the flow and heat transfer for the flow of a dusty fluid including the viscous and Joule dissipation effects are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial\left(\rho_p u_p\right)}{\partial x}+\frac{\partial\left(\rho_p v_p\right)}{\partial y}=0, \tag{2}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}+\frac{KN}{\rho}\left(u_p-u\right)-\frac{\sigma B_0^2}{\rho}u+g\beta^*\left(T-T_\infty\right), \tag{3}$$

$$u_p\frac{\partial u_p}{\partial x}+v_p\frac{\partial u_p}{\partial y}=\frac{K}{m}\left(u-u_p\right), \tag{4}$$

$$u_p\frac{\partial v_p}{\partial x}+v_p\frac{\partial v_p}{\partial y}=\frac{K}{m}\left(v-v_p\right), \tag{5}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{\sigma B_0^2}{\rho c_p}u^2+\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{N}{\rho\tau_T}\left(T_p-T\right)+\frac{N}{\rho c_p\tau_v}\left(u_p-u\right)^2-\frac{Q_0}{\rho c_p}\left(T-T_\infty\right), \tag{6}$$

$$u_p\frac{\partial T_p}{\partial x}+v_p\frac{\partial T_p}{\partial y}=\frac{c_p}{c_m\tau_T}\left(T-T_p\right), \tag{7}$$

where $(u, v)$ and $(u_p, v_p)$ are the velocity components of the fluid and dust particles along the $x$ and $y$ directions, respectively, $\rho$ and $\rho_p$ are density of the fluid and particle phase, respectively, $\mu$, $B_0$, $K$, $g$, and $\beta^*$ are the coefficient of viscosity of the fluid, applied magnetic field, Stokes' resistance coefficient, acceleration due to gravity, and volumetric coefficient of thermal expansion, respectively. $\tau = m/K$ is the relaxation time of particle phase, $m$ and $N$ are the mass concentration and number density of the particle phase, and $T$, $T_p$, and $T_\infty$ are the fluid temperature, particle temperature, and the fluid temperature in the free stream, respectively. $k$ is the thermal conductivity of the fluid. $\tau_T$ is the thermal equilibrium time and is the time required by a dust phase to adjust its temperature to that of fluid and $\tau_v$ is the relaxation time of the dust particle, that is, the time required by the dust phase to adjust its velocity relative to fluid. $c_p$, $c_m$ are the specific heat of fluid and dust particles. $Q_0$ is the internal heat absorption coefficient.

The boundary conditions for the flow problem are

$$u=U_w(x)=cx,\qquad v=-v_0,$$
$$-k\frac{\partial T}{\partial y}=h\left(T_w-T\right)\quad\text{at } y=0,$$

$$u\longrightarrow 0,\qquad u_p\longrightarrow 0,\qquad v_p\longrightarrow v,$$
$$T\longrightarrow T_\infty,\qquad T_p\longrightarrow T_\infty,\qquad \rho_p\longrightarrow E\rho$$
$$\text{as } y\longrightarrow\infty, \tag{8}$$

where $c > 0$ is the stretching rate of the sheet, $E$ is the density ratio, $v_0 > 0$ is the suction velocity, and $h$ is the heat transfer coefficient.

Equation (1) is automatically satisfied through introducing the stream function $\psi(x, y) = \sqrt{c\nu}xf(\eta)$, such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. We further introduce the following variables:

$$u=cxf'(\eta),\qquad v=-\sqrt{c\nu}f(\eta),\qquad \eta=\sqrt{\frac{c}{\nu}}y,$$
$$u_p=cxF(\eta),\qquad v_p=\sqrt{c\nu}G(\eta),\qquad \rho_r=\frac{\rho_p}{\rho}=H(\eta),$$
$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\qquad \theta_p(\eta)=\frac{T_p-T_\infty}{T_w-T_\infty}, \tag{9}$$

where $\rho_r$ is the relative density. Substituting (9) in (2)–(7), we obtain

$$HF+HG'+H'G=0,$$
$$f'''+ff''-f'^2+l\alpha H\left(F-f'\right)-Mf'+\lambda\theta=0,$$
$$F^2+GF'-\alpha\left(f'-F\right)=0,$$
$$GG'-\alpha\left(G-f\right)=0, \tag{10}$$
$$\theta''+\Pr\left[M\mathrm{Ec}f'^2+\mathrm{Ec}f''^2+c_1N\left(\theta_p-\theta\right)+c_2N\mathrm{Ec}\left(F-f'\right)^2+f\theta'-\beta_h\theta\right]=0,$$
$$G\theta_p'-c_3\left(\theta-\theta_p\right)=0.$$

Table 1: Comparison of $-f''(0)$ values for various values of $M$ when $N = \mathrm{Gr} = S = 0$.

| | $M = 0$ | $M = 0.5$ | $M = 1$ | $M = 1.5$ | $M = 2$ |
|---|---|---|---|---|---|
| Yih [10] | 1.0000 | 1.2247 | 1.4142 | 1.5811 | 1.7321 |
| Present results | 1.000000 | 1.224745 | 1.414214 | 1.581139 | 1.732051 |

Table 2: Comparison of $-\theta'(0)$ values for various values of Pr when $M = \mathrm{Gr} = N = S = 0$.

| | Grubka and Bobba [5] | Ishak et al. [12] | Present results |
|---|---|---|---|
| Pr = 1 | 0.5820 | 0.5820 | 0.581977 |
| Pr = 3 | 1.1652 | 1.1652 | 1.165246 |
| Pr = 10 | 2.3080 | 2.3080 | 2.308003 |

Boundary conditions (8) are transformed to

$$\begin{gathered} f' = 1, \quad f = S, \quad \theta' = -\mathrm{Bi}\,(1-\theta) \quad \text{at } \eta = 0, \\ f' \longrightarrow 0, \quad F \longrightarrow 0, \quad G \longrightarrow -f, \\ \theta \longrightarrow 0, \quad \theta_p \longrightarrow 0, \quad H \longrightarrow E \\ \text{as } \eta \longrightarrow \infty, \end{gathered} \tag{11}$$

where primes denote differentiation with respect to $\eta$ and

$$\begin{gathered} M = \frac{\sigma B_0^2}{\rho c}, \quad l = \frac{mN}{\rho_p}, \quad \alpha = \frac{1}{\tau c}, \quad \lambda = \frac{\mathrm{Gr}_x}{\mathrm{Re}_x^2}, \\ \mathrm{Gr}_x = \frac{g\beta^* (T_w - T_\infty) x^3}{\nu^2}, \quad \mathrm{Re}_x = \frac{U_w x}{\nu}, \quad \mathrm{Pr} = \frac{\rho \nu c_p}{k}, \\ \mathrm{Ec} = \frac{U_w^2}{c_p (T_w - T_\infty)}, \quad S = \frac{v_0}{\sqrt{c\nu}}, \quad \mathrm{Bi} = \frac{h}{k}\sqrt{\frac{\nu}{c}}, \\ \beta_h = \frac{Q_0}{\rho c c_p}, \quad c_1 = \frac{1}{\rho c \tau_T}, \quad c_2 = \frac{1}{\rho c \tau_\nu}, \quad c_3 = \frac{c_p}{c_m c \tau_T}. \end{gathered} \tag{12}$$

The nondimensional parameters appearing in (10)-(11) and defined in (12) are the magnetic parameter $M$, the mass concentration of dust particles $l$, the fluid particle interaction parameter $\alpha$, the local thermal buoyancy parameter $\lambda$, the local Grashof number $\mathrm{Gr}_x$, the local Reynolds number $\mathrm{Re}_x$, the Prandtl number Pr, the Eckert number Ec, the suction parameter $S$, the Biot number Bi, the heat absorption parameter $\beta_h$, the local fluid particle interaction parameters for heat transfer $c_1$ and $c_3$, and the local fluid particle interaction parameter for velocity $c_2$. The value of $\lambda > 0$ corresponds to the buoyancy assisting flow while the value of $\lambda < 0$ corresponds to the buoyancy opposing flows and $\lambda = 0$ corresponds to the case of pure forced convection flow.

Apart from the velocity and temperature of the fluid and dust phases, the other physical quantities of practical interest are the skin friction coefficient $C_f$ and the local Nusselt number $\mathrm{Nu}_x$.

The skin friction coefficient $C_f$ is defined as

$$C_f = \frac{\tau_w}{\rho U_w^2}, \tag{13}$$

where

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}. \tag{14}$$

Using (14) in (13), we obtain

$$C_f \sqrt{\mathrm{Re}_x} = f''(0), \tag{15}$$

and the local Nusselt number is defined as

$$\mathrm{Nu}_x = \frac{x q_w}{k (T_w - T_\infty)}, \tag{16}$$

where

$$q_w = -k \left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{17}$$

Using (17) in (16), we obtain

$$\frac{\mathrm{Nu}_x}{\sqrt{\mathrm{Re}_x}} = -\theta'(0). \tag{18}$$

## 3. Numerical Solution and Validation of Results

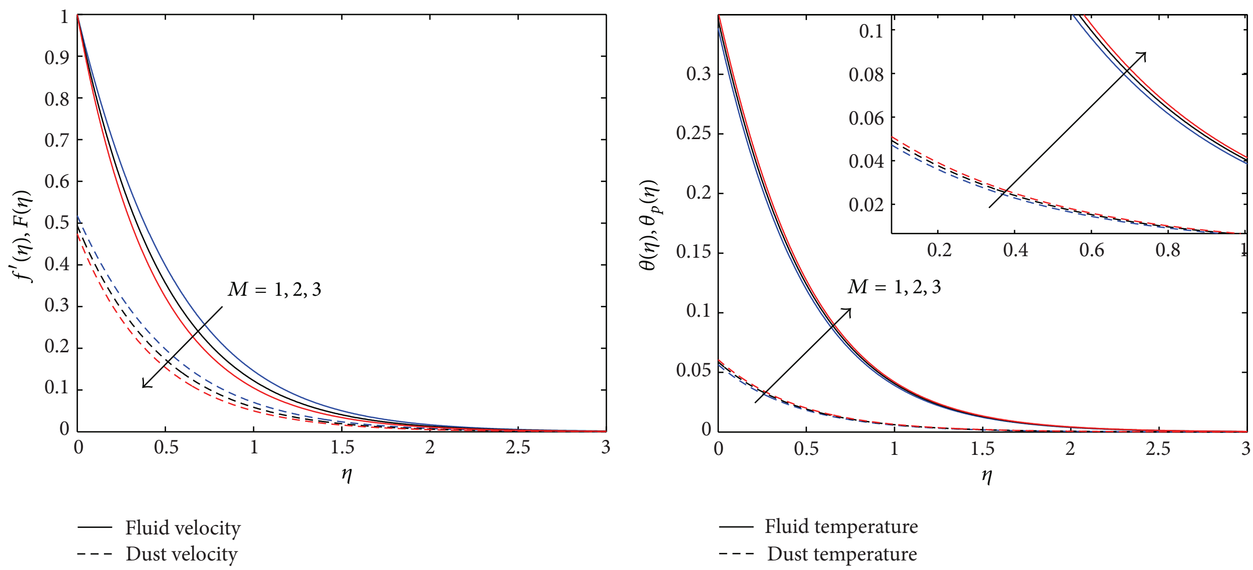
Equations (10) are highly nonlinear coupled ordinary differential equations, which are solved by the `bvp4c` routine of Matlab. In order to solve these equations they are first reduced to nine simultaneous ordinary differential equations as follows:

$$\begin{gathered} y_1' = y_2, \\ y_2' = y_3, \\ y_3' = -y_1 y_3 + y_2^2 - l\alpha y_9 (y_4 - y_2) + M y_2 - \lambda y_6, \\ y_4' = \frac{\alpha (y_2 - y_4) - y_4^2}{y_5}, \\ y_5' = -\frac{\alpha (y_5 + y_1)}{y_5}, \\ y_6' = y_7, \\ y_7' = -\mathrm{Pr}\left[ M\mathrm{Ec} y_2^2 + \mathrm{Ec} y_3^2 + c_1 N (y_8 - y_6) \right. \\ \left. + c_2 \mathrm{Ec} N (y_4 - y_2)^2 + y_1 y_7 - \beta_h y_6 \right], \\ y_8' = \frac{c_3 (y_6 - y_8)}{y_5}, \\ y_9' = -\frac{y_9 y_4}{y_5} + \frac{\alpha y_9 (y_5 + y_1)}{y_5^2}, \end{gathered} \tag{19}$$

where $y_1 = f$, $y_2 = f'$, $y_3 = f''$, $y_4 = F$, $y_5 = G$, $y_6 = \theta$, $y_7 = \theta'$, $y_8 = \theta_p$, and $y_9 = H$. The primes denote differentiation with respect to $\eta$.

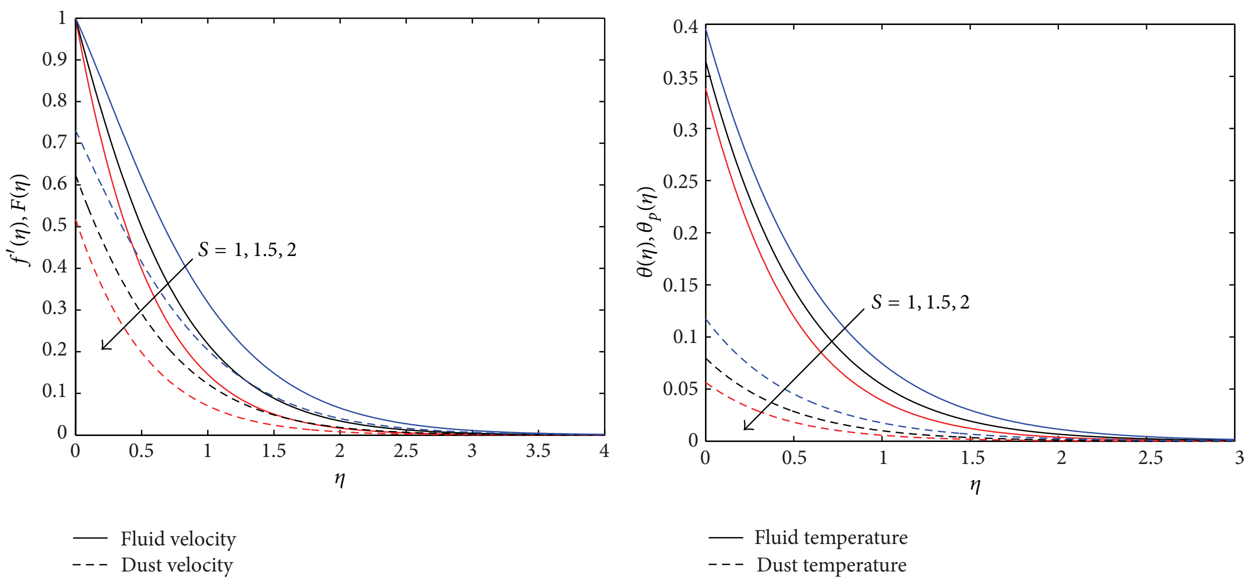
The boundary conditions for the above simultaneous ordinary differential equations are

$$\begin{gathered} y_1 = S, \quad y_2 = 1, \quad y_3 = s_1, \\ y_4 = s_2, \quad y_5 = s_3, \quad y_6 = s_4, \\ y_7 = -\mathrm{Bi}(1 - s_4), \quad y_8 = s_5, \quad y_9 = s_6, \\ \text{at } \eta = 0, \\ y_2 = 0, \quad y_4 = 0, \quad y_5 = -y_1, \\ y_6 = 0, \quad y_8 = 0, \quad y_9 = E, \\ \text{as } \eta \longrightarrow \infty, \end{gathered} \tag{20}$$




(a) (b)

Figure 2: Effect of $M$ on (a) $f'$, $F$ and (b) $\theta, \theta_p$ when $S = 2$, $\beta_h = 1$, $N = 1$, Bi $= 1$, and Ec $= 0.1$.




(a) (b)

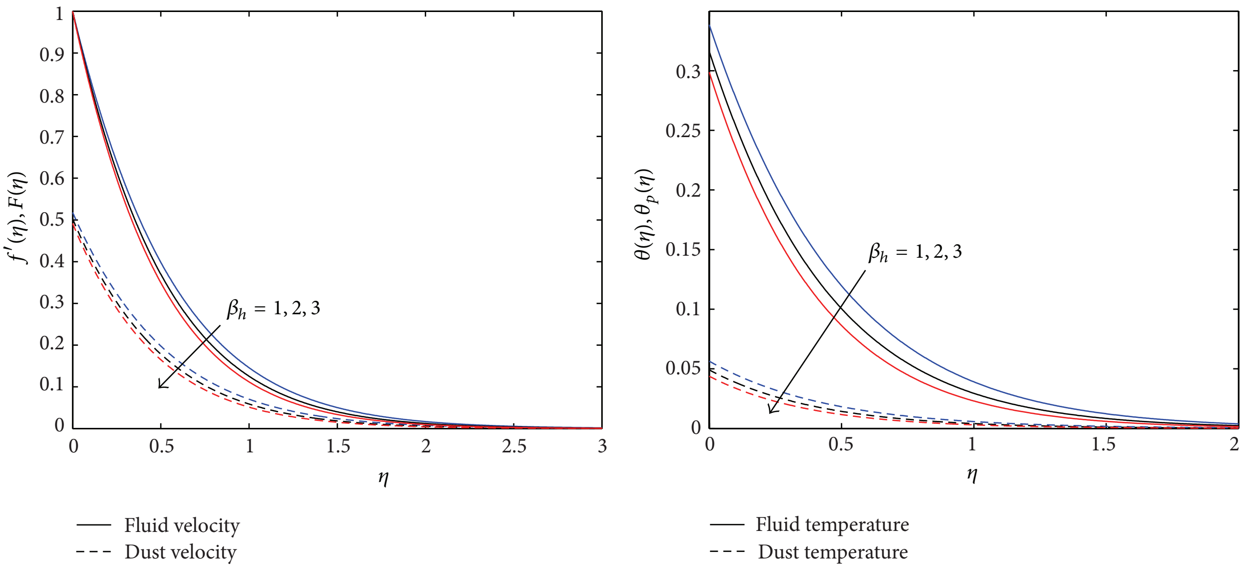
Figure 3: Effect of $S$ on (a) $f'$, $F$ and (b) $\theta, \theta_p$ when $M = 1$, $\beta_h = 1$, $N = 1$, Bi $= 1$, and Ec $= 0.1$.

where $s_i$ $(i = 1, 2, 3, 4, 5, 6)$ are the initial guesses for $y_3$, $y_4$, $y_5$, $y_6$, $y_8$, and $y_9$, respectively.

The above simultaneous ordinary differential equations are solved subject to the boundary conditions using `bvp4c` routine. To validate the results of the present work, a comparison of values of $-f''(0)$ is presented in Table 1 and the comparison of values of $-\theta'(0)$ is presented in Table 2. The present results are found to be in excellent agreement with those of Yih [10], Grubka and Bobba [5], and Ishak et al. [12].

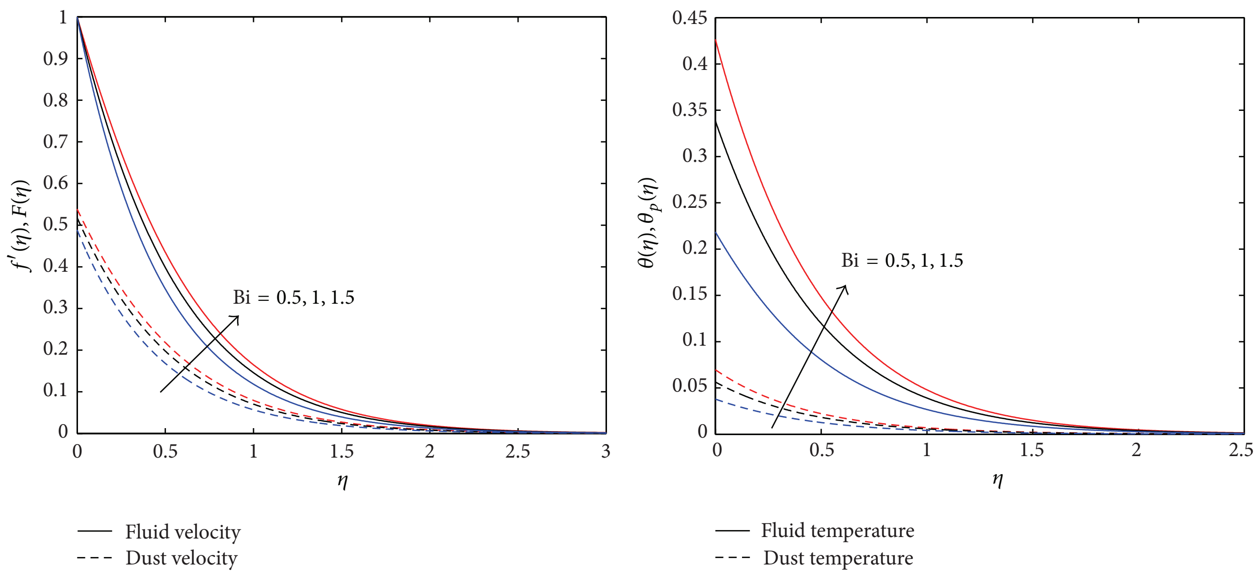
## 4. Results and Discussion

The effects of various flow parameters, namely, the magnetic parameter $M$, the suction parameter $S$, the heat absorption parameter $\beta_h$, the Biot number Bi, and the Eckert number Ec on the flow and heat transfer of the dusty fluid are investigated with the help of figures and tables. For the computation work, the default values of the parameters are taken as $\alpha = 5$, $c_1 = c_2 = c_3 = 1$, $E = 1$, $\lambda = 10$ (corresponding to buoyancy




(a) (b)

FIGURE 4: Effect of $S$ on (a) $f'$, $F$ and (b) $\theta$, $\theta_p$ when $M = 1$, $S = 2$, $N = 1$, Bi $= 1$, and Ec $= 0.1$.




(a) (b)

FIGURE 5: Effect of Bi on (a) $f'$, $F$ and (b) $\theta$, $\theta_p$ when $M = 1$, $S = 2$, $N = 1$, $\beta_h = 1$, and Ec $= 0.1$.

assisting flow), Pr $= 0.71$, and $l = 0.2$. Figure 2 shows the effect of magnetic parameter $M$ on the fluid and particle velocities and fluid and particle temperatures. The increase in the magnetic parameter signifies the increase in the strength of the applied magnetic field. It is observed that an increase in $M$ causes a decrease in the fluid and particle velocities but an increase in the fluid and particle temperatures. This effect on flow and heat transfer with respect to magnetic field is due to the resistive force which appears in the flow field due to the presence of magnetic field. The effect of Joule dissipation is important because it increases the temperature of the fluid and the dust phase with increases in the magnetic field. The thickness of momentum boundary layer decreases while the thermal boundary layer increases with an increase in the strength of the applied magnetic field.

Figure 3 exhibits the effect of suction parameter $S$ on the flow and heat transfer. An increase in $S$, which marks the increase in the suction velocity through the sheet, decreases the velocity and temperature for both the fluid and dust phases. The cause of the decreasing effect on the fluid velocity is acceleration of the velocity towards the plate due to the flow through the pores of the plate. This has importance

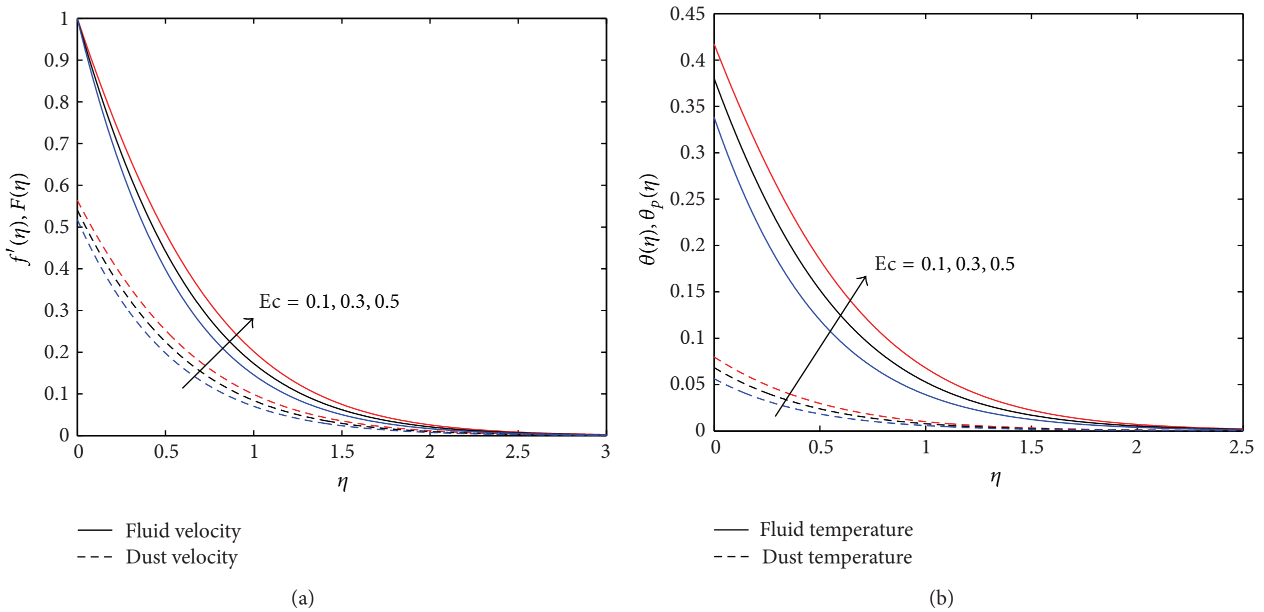



FIGURE 6: Effect of Ec on (a) $f'$, $F$ and (b) $\theta, \theta_p$ when $M = 1$, $S = 2$, $N = 1$, $\beta_h = 1$, and Bi = 1.

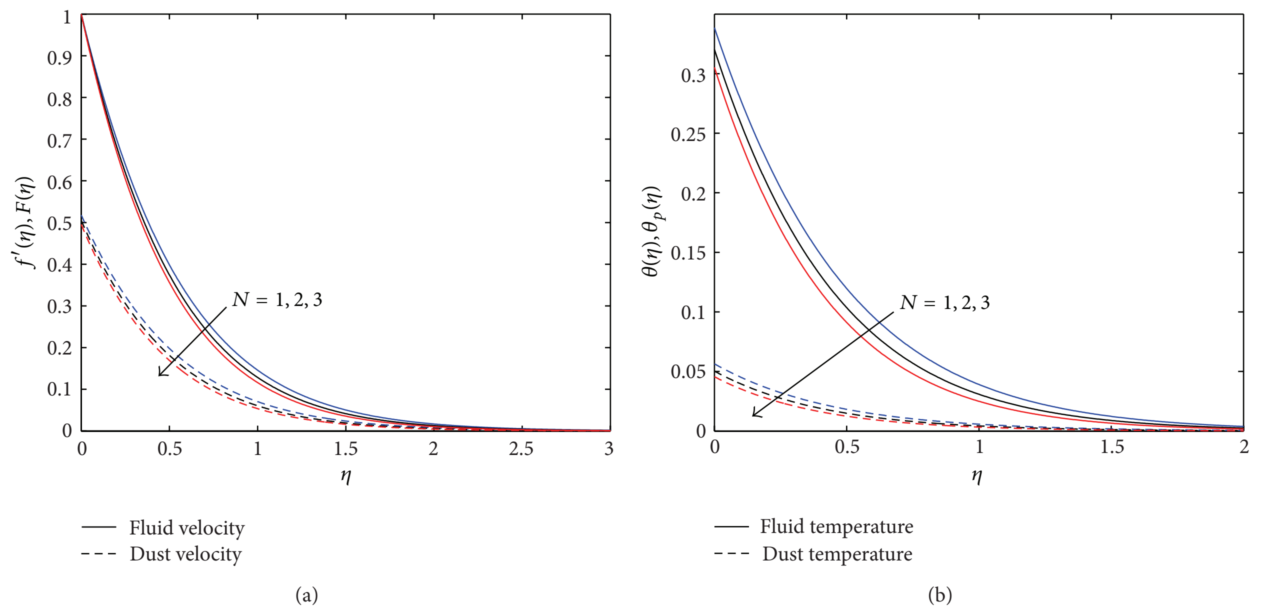



FIGURE 7: Effect of $N$ on (a) $f'$, $F$ and (b) $\theta, \theta_p$ when $M = 1$, $S = 2$, Ec = 0.1, $\beta_h = 1$, and Bi = 1.

in delaying the boundary layer formation in the flow field. The fluid temperature decreases as a result of the heat removal with the fluid flowing through the pores. Both the momentum and thermal boundary layer thicknesses are decreased with the increase in the suction through the pores of the sheet.

The effect of heat absorption parameter $\beta_h$ on the velocity and temperature for both the fluid and dust phases is demonstrated in Figure 4. The heat absorption parameter $\beta_h$ measures the amount of heat flux absorbed by the fluid. It is noticed that an increase in $\beta_h$ causes a decrease in the velocity and temperature for both the fluid and dust phases. The behavior is as per expectations as the heat absorption by the fluid causes a decrease in the kinetic and thermal energy of the fluid and as a result the velocity and temperature for both the phases decrease. The thickness of both the momentum and thermal boundary layers decreases with an increase in the heat absorption by the fluid.

Figure 5 presents the effect of increase in the Biot number Bi on the flow and heat transfer. Biot number Bi gives the ratio of the heat transfer resistances inside and at the surface of a body which measures the convective heat transfer rate

Table 3: Values of $-f''(0)$ and $-\theta'(0)$ for different values of the parameters.

| $M$ | $S$ | $\beta_h$ | Bi | Ec | $N$ | $-f''(0)$ | $-\theta'(0)$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 1 | 0.1 | 1 | 1.73694589 | 0.66142424 |
| 2 | — | — | — | — | — | 2.03962304 | 0.65409733 |
| 3 | — | — | — | — | — | 2.30749884 | 0.64766402 |
| 1 | 1 | 1 | 1 | 0.1 | 1 | 0.68814355 | 0.60401190 |
| — | 1.5 | — | — | — | — | 1.20287314 | 0.63508597 |
| — | 2 | — | — | — | — | 1.73694589 | 0.66142424 |
| 1 | 2 | 1 | 1 | 0.1 | 1 | 1.73694589 | 0.66142424 |
| — | — | 2 | — | — | — | 1.88561032 | 0.68377933 |
| — | — | 3 | — | — | — | 1.99471106 | 0.70124323 |
| 1 | 2 | 1 | 0.5 | 0.1 | 1 | 2.11897411 | 0.39070844 |
| — | — | — | 1 | — | — | 1.73694589 | 0.66142424 |
| — | — | — | 1.5 | — | — | 1.46083806 | 0.85993205 |
| 1 | 2 | 1 | 1 | 0.1 | 1 | 1.73694589 | 0.66142424 |
| — | — | — | — | 0.3 | — | 1.49212499 | 0.61949317 |
| — | — | — | — | 0.5 | — | 1.25526077 | 0.58242785 |
| 1 | 2 | 1 | 1 | 0.1 | 1 | 1.73694589 | 0.66142424 |
| — | — | — | — | — | 2 | 1.86119164 | 0.67973786 |
| — | — | — | — | — | 3 | 1.95733253 | 0.69468425 |

between fluid and the surface of the sheet. It is shown that the increase in Bi has an increasing effect on the velocity and temperature for both the fluid and dust phases. An increase in the convective heat transfer rate contributes to the thickening of the momentum and thermal boundary layers.

The effect of Eckert number Ec, which signifies the viscous dissipation of the fluid, on the flow and heat transfer is exhibited in Figure 6. It is observed that an increase in the viscous dissipation of the fluid tends to increase the velocity and temperature for both the phases. The reason for this effect is that the viscosity of the fluid takes energy from the motion of the fluid and transforms it into the internal energy of the fluid which results in the heating up of the fluid, and an increase in the fluid temperature is encountered. The momentum and thermal boundary layers get thicker with the increase in the viscous dissipation.

The effect of number density of dust particles $N$ is depicted in Figure 7. Number density of dust particles measures the density of dust particles in the flow system so that the value $N = 0$ corresponds to the clean fluid. It is depicted in the figures that the increase in the number density of dust particles causes a decrease in the velocity and temperature for both the fluid and dust phases. The central reason for this is the presence of dust particles which causes retardation into the fluid flow. The dust particles tend to absorb the heat when they come in contact with the fluid and this causes a decrease in the fluid temperature. The presence of dust particles causes a decrease in both the momentum and thermal boundary layer thicknesses.

The effects of magnetic field, suction, heat absorption, convective heat transfer at the plate, viscous dissipation, and number density of dust particles on $-f''(0)$ and $-\theta'(0)$ which measures the coefficient of skin friction and local Nusselt number at the sheet, respectively, are presented in Table 3. We found that the skin friction increases with an increase in strength of magnetic field, suction, heat absorption, and the number density of dust particles while it is oppositely affected by convective heat transfer rate at the plate and viscous dissipation of the fluid. The Nusselt number at the plate is increased with an increase in suction, heat absorption, convective heat transfer, and number density of dust particles whereas it is reversely influenced by the magnetic field and viscous dissipation.

## 5. Conclusions

The combined effects of viscous and Joule dissipation on the steady two-dimensional boundary layer flow of a viscous, incompressible, and electrically conducting dusty fluid past a vertical permeable stretching sheet under the influence of a transverse magnetic field with internal heat absorption effects are investigated numerically. The important findings of practical interest are as follows.

(i) The momentum boundary layer thickness increases with an increase in the convective heat transfer rate from the plate and viscous dissipation whereas it decreases with an increase in magnetic field strength, suction, heat absorption, and number density of dust particles.

(ii) The contribution of viscous and Joule dissipation is to increase the thickness of the thermal boundary layer while its thickness decreases with the increase in suction, heat absorption, convective heat transfer, and the number density of dust particles.

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