Research Article

Global Stability of a HIV-1 Model with General Nonlinear Incidence and Delays

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A HIV-1 model with two distributed intracellular delays and general incidence function is studied. Conditions are given under which the system exhibits the threshold behavior: the disease-free equilibrium E_0 is globally asymptotically stable if $R_0 \leq 1$; if $R_0 > 1$, then the unique endemic equilibrium E_1 globally asymptotically stable. Finally, it is shown that the given conditions are satisfied by several common forms of the incidence functions.

1. Introduction

The global stability is analyzed for a general mathematical model of HIV-1 pathogenesis proposed by Nelson and Perelson [1]. The general model includes two distributed intracellular delays and a combination therapy with a reverse transcriptase inhibitor and a protease inhibitor. All incidence functions in those papers are the bilinear functions. However, there are some pieces of evidence showing that a bilinear infection rate might not be an effective assumption when the number of target cells is large enough (see [2–4] for the review of evidence and models). The aim of this paper is to establish global stability for a delay integrodifferential equation with a general incidence term $f(x, v)$, the conditions given here are similar to those given in [4] for the ODE case.

In this paper, we consider the following HIV-1 model with a side class of nonlinear incidence rates and distributed delays:

$$
\begin{aligned}
\frac{dx(t)}{dt} &= s - d_T x(t) - (1 - n_{rt}) f(x(t), v(t)), \\
\frac{dy(t)}{dt} &= (1 - n_{rt}) \int_0^\infty G_1(\xi) f(x(t-\xi), \\
&\qquad\qquad v(t-\xi)) d\xi - \delta y(t), \\
\frac{dv(t)}{dt} &= (1 - n_p) N\delta \int_0^\infty G_2(\xi) y(t-\xi) d_\xi - cv(t),
\end{aligned}
\tag{1}
$$

where x, y, and v are the concentrations of uninfected target cells (T cells), productively infected cells, and infectious virus, respectively.

The form of the incidence function f is of fundamental importance. In this paper, we want to work with a function as general as possible but still possesses the properties necessary for conclusions to be made through mathematical analysis. Because of this, we will introduce conditions on f with which it may appear technical. However, as shown in Section 5, many commonly used incidence functions satisfy these conditions. For now, we assume only the following.

(A1) $f(x, v)$ is a nonnegative differentiable function on the nonnegative quadrant. Furthermore, $f(x, v)$ is positive if and only if both arguments are positive. The partial derivatives of f on x and v are denoted by f_1 and f_2, respectively. In Sections 3 and 4, it will be shown how the extra conditions on f imply in the local and global stability of an endemic equilibrium.

For the purpose of convenience, we rewrite (1) as

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \lambda - mx(t) - \beta f(x(t), v(t)), \\
\frac{dy(t)}{dt} &= \beta_1 \int_0^\infty g_1(\xi) f(x(t-\xi), v(t-\xi)) d\xi - ay(t), \\
\frac{dv(t)}{dt} &= k_1 \int_0^\infty g_2(\xi) y(t-\xi) d\xi - \mu v(t),
\end{aligned}
\tag{2}
$$

where $\lambda = s, m = d_T, \beta = 1 - n_{rt}, a = \delta, \mu = c, \beta_1 = (1 - n_{rt})\alpha_1,$ $k_1 = (1 - n_p)N\delta\alpha_2, g_i(\xi) = G_i(\xi)/\alpha_i, \alpha_i = \int_0^\infty G_i(\xi)d\xi, i = 1, 2.$

The paper is organized as follows. In Section 2, the basic reproduction number R_0 is determined and the equilibrium is found. The local stability of the equilibrium is studied in Section 3. The global dynamics are resolved in Section 4. In Section 5, examples are given of incidence functions which satisfy the assumptions that are used throughout the paper.

2. Equilibria and R_0

Assume the kernel functions G_1 and G_2 satisfy

$$G_i(\xi) \geq 0, \quad \text{for } \xi \geq 0,$$
$$0 < \int_0^\infty G_i(\xi)\,d\xi < 1, \quad i = 1, 2. \tag{3}$$

For any given initial condition,

$$\phi(\theta) = (x(\theta), y(\theta), v(\theta)) \in UC_g\left((-\infty, 0], \mathbf{R}_+^3\right), \tag{4}$$

system (2) satisfies the hypotheses that are sufficient to ensure the existence, uniqueness, and continuity of solutions [5, Theorems 2.1–2.3]; for the notation of UC_g, see [6, page 46]. Recall that $\alpha_i = \int_0^\infty G_i(\xi)d\xi$ and $g_i(\xi) = G_i(\xi)/\alpha_i$; thus, $\int_0^\infty g_i(\xi)d\xi = 1$. Then, the basic reproduction number R_0 defined in [7] can be rewritten as

$$R_0 = \frac{k_1\beta_1}{a\mu}f_2(E_0), \tag{5}$$

and for any value of the parameters, the disease-free equilibrium of (2) is given by

$$E_0 = (x_0, y_0, v_0) = \left(\frac{\lambda}{m}, 0, 0\right). \tag{6}$$

It is straightforward to show that if $R_0 \leq 1$, then (2) has only one nonnegative equilibrium, which is called the infection-free equilibrium E_0. The presence and number of endemic equilibrium depend on the form of the nonlinearity f, as well as the values of the parameters. In searching for equilibrium, we note that the equilibria of (2) are the same as the equilibria of the corresponding ordinary differential equation system. Here we give the following result.

Theorem 1. Let $(x(t), y(t), v(t))^T$ be the unique solution to system (2) with $x(0) > 0$, $y(0) > 0$, and $v(0) > 0$. Then $x(t), y(t),$ and $v(t)$ are positive for all $t > 0$. Moreover, the solution is bounded and thus exists globally.

Proof. Using the variation-of-constants formula, we obtain the positivity of $y(t)$ and $v(t)$.

For $x(t)$, we claim that if $x(0) > 0$, then $x(t) > 0$ for all $t > 0$. Otherwise, there exists $t' > 0$ such that $x(t') = 0$. Let t_0 be the first one that satisfies $x(t_0) = 0$; then $t_0 > 0$ and

$$\left.\frac{dx(t)}{dt}\right|_{t=t_0} = \lambda > 0, \tag{7}$$

which means that $x(t)$ is increasing at $t = t_0$ and it is a contradiction.

Next we will show that every solution is also bounded. It follows from the first equation of (2) that $dx(t)/dt < \lambda - mx(t)$. This implies that $\overline{\lim}_{t\to\infty}x(t) \leq \lambda/m$.

Let $V(t) = (\beta_1/\beta)\int_0^\infty g_1(\xi)x(t-\xi)d\xi + y(t)$; then

$$\left.\frac{dV(t)}{dt}\right|_{(2)}$$
$$= \frac{\beta_1\lambda}{\beta} - \frac{\beta_1(a-m)}{\beta}\int_0^\infty g_1(\xi)x(t-\xi)\,d\xi - av(t). \tag{8}$$

Note that $\int_0^\infty g_1(\xi)x(t-\xi)d\xi$ is ultimately bounded; then there exist positive constants C and T_0 such that

$$\left.\frac{dV(t)}{dt}\right|_{(2)} \leq \frac{\beta_1\lambda}{\beta} + C - aV(t), \quad \forall\, t \geq T_0. \tag{9}$$

This yields that $V(t)$ is eventually bounded and so is $y(t)$. By a similar argument, we can show the boundedness of $v(t)$. Therefore, the system (2) is point dissipative (see [1]) and hence the solution of (2) exists globally. □

Theorem 2. If $R_0 > 1$, then there exists an endemic equilibrium $E_1 = (x^*, y^*, v^*)$.

Proof. We look for solution (x^*, y^*, v^*) of the equations $dx/dt = 0, dy/dt = 0,$ and $dv/dt = 0$. We first note that $(dx/dt) + (\beta/\beta_1)(dy/dt) = 0$ implies $\lambda - mx^* - \beta f(x^*, v^*) + (\beta/\beta_1)ay^* = 0$ and $k_1y^* - \mu v^* = 0$, so $x^* = (\lambda - (a\beta/\beta_1)y^*)/m$, $v^* = k_1/\mu$. Let $H(y^*) = \beta_1 f((\lambda - (a\beta/\beta_1)y^*)/m, (k_1/\mu)y^*) - ay^*$. Then, $dy(t)/dt = 0$ whenever $H(y^*) = 0$. Thus, any zero of H in the interval $(0, \lambda\beta_1/a\beta)$ corresponds to an equilibrium with $x^*, y^*, v^* > 0$, that is, an endemic equilibrium.

Since $f(0, v) = f(x, 0) = 0$, it follows that $H(0) = 0$ and $H(\lambda\beta_1/a\beta) < 0$. The function H is continuous and so a sufficient condition for H to have a zero in $(0, \lambda\beta_1/a\beta)$ is that H is increasing at 0. Thus, there is an endemic equilibrium if

$$0 < \frac{dH}{dy}(0) = \beta_1\left(-\frac{a\beta}{m\beta_1}\right)f_1(E_0) + \frac{k_1\beta_1}{\mu}f_2(E_0) - a. \tag{10}$$

Since $f(x, 0) = 0$ for all x, it follows that $f_1(E_0) = 0$ and so (10) is equivalent to $R_0 = (k_1\beta_1/a\mu)f_2(E_0) > 1$. □

3. Local Stability of the Equilibria

Theorem 3. If $R_0 < 1$, then E_0 is locally asymptotically stable.

The proof is normal, so we omitted it. We now give conditions on f that are used here and after to show the locally and globally asymptotically stability of the endemic equilibrium. As a precondition, we assume that $R_0 > 1$ to guarantee the existence of endemic equilibrium $E_1 = (x^*, y^*, v^*)$. Consider the following.

(A2) $\text{sgn}(f(x, v^*) - f(x^*, v^*)) = \text{sgn}(x - x^*)$ for all $x \geq 0$.

(A3) For all $x, v > 0$, $f(x, v)/f(x, v^*)$ is in the closed interval with endpoints at 1 and v/v^*.

(A4) Either $f_1(x^*, v^*) > 0$ or $f_2(x^*, v^*) < f(x^*, v^*)/v^*$.

Theorem 4. *If $R_0 > 1$, and (A2), (A3), and (A4) hold, then any endemic equilibrium which exists is locally asymptotically stable.*

Proof. The characteristic equation of the linearization of (2) at endemic equilibrium $E_1 = (x^*, y^*, v^*)$ is

$$0 = (m + \lambda + \beta f_1 (E_1)) (a + \lambda) (\mu + \lambda)$$
$$- (m + \lambda) \beta_1 k_1 f_2 (E_1) \qquad (11)$$
$$\times \int_0^\infty g_1 (\xi) e^{-\lambda \xi} d\xi \int_0^\infty g_2 (\xi) e^{-\lambda \xi} d\xi.$$

We demonstrate that all zeros of it have negative real part. Since (A2) and (A3) hold, we get that $0 \leq \beta f_1(E_1)$ and $0 \leq f_2(E_1) \leq a\mu/\beta_1 k_1$. Suppose that λ has nonnegative real part, we deduce

$$\left| (m + \lambda) \beta_1 k_1 f_2 (E_1) \int_0^\infty g_1 (\xi) e^{-\lambda \xi} d\xi \int_0^\infty g_2 (\xi) e^{-\lambda \xi} d\xi \right|$$
$$\leq |m + \lambda| \beta_1 k_1 f_2 (E_1)$$
$$\leq \left| (m + \lambda + \beta f_1 (E_1)) (a + \lambda) (\mu + \lambda) \right|, \qquad (12)$$

and so the solutions λ with nonnegative real part if and only if all of the inequalities in (12) are in fact equal. The final inequality is strict unless $f_1(E_1) = 0$ (and $\lambda = 0$). The second last inequality is strict unless $f_2(E_1) = f(E_1)/v^*$. Assumption (A4) implies at least one is strict. Thus, the endemic equilibrium E_1 is locally asymptotically stable. □

4. Global Stability of the Equilibria

Theorem 5. *If $R_0 \leq 1$, then E_0 is globally stable; that is,*

$$\lim_{t \to \infty} (x(t), y(t), v(t)) = E_0. \qquad (13)$$

Proof. Let $K(t) = K_1(t) + K_2(t) + K_3(t)$ with

$$K_1 (t) = \frac{1}{2} \left(x(t) - \frac{\lambda}{m} \right)^2 + \frac{\lambda \beta}{m \beta_1} y(t) + \frac{a \lambda \beta}{k_1 m \beta_1} v(t),$$

$$K_2 (t) = \frac{\lambda \beta}{m} \int_0^\infty H_1 (\xi) \left(f (x(t - \xi), v(t - \xi)) \right.$$
$$\left. + f_2 (x_0, 0) v(t - \xi) \right) d\xi, \qquad (14)$$

$$K_3 (t) = \frac{a \lambda \beta}{m \beta_1} \int_0^\infty H_2 (\xi) y(t - \xi) d\xi,$$

where $H_i(t) = \int_t^\infty g_i(\xi) d\xi$, $i = 1, 2$.

It is clear that $K(t) \geq 0$ and $K(t) = 0$ if and only if $x(t) = \lambda/m$, $y(t) = v(t) = 0$. The derivative of K_1 along the solution of (2) is

$$\left. \frac{dK_1 (t)}{dt} \right|_{(2)} = \left(x(t) - \frac{\lambda}{m} \right) (\lambda - mx(t) - \beta f (x(t), v(t)))$$
$$- \frac{a \lambda \beta}{m \beta_1} y(t) - \frac{a \lambda \beta \mu}{k_1 m \beta_1} v(t)$$
$$+ \frac{\lambda \beta}{m} \int_0^\infty g_1 (\xi) f (x(t - \xi), v(t - \xi)) d\xi$$
$$+ \frac{a \lambda \beta}{m \beta_1} \int_0^\infty g_2 (\xi) y(t - \xi) d\xi. \qquad (15)$$

Noticing that $H_1(0) = 1$, $H_1(\infty) = 0$, and $dH_1(t) = -g_1(t)dt$, and using integration by parts, we calculate the derivatives of K_2 and K_3

$$\left. \frac{dK_2 (t)}{dt} \right|_{(2)} = \frac{\lambda \beta}{m} f (x, v) + \frac{\lambda \beta}{m} f_2 (x_0, 0)$$
$$- \frac{\lambda \beta}{m} \int_0^\infty g_1 (\xi) f (x(t - \xi), v(t - \xi)) d\xi$$
$$- \frac{\lambda \beta}{m} f_2 (x_0, 0) \int_0^\infty g_1 (\xi) v(t - \xi) d\xi.$$

$$\left. \frac{dK_3 (t)}{dt} \right|_{(2)} = \frac{a \lambda \beta}{m \beta_1} \left(y(t) - \int_0^\infty g_2 (\xi) y(t - \xi) d\xi \right). \qquad (16)$$

Thus,

$$\left. \frac{dK (t)}{dt} \right|_{(2)} = -m \left(x(t) - \frac{\lambda}{m} \right)^2$$
$$- \beta x(t) f (x(t), v(t)) - \frac{a \lambda \beta \mu}{m \beta_1 k_1} (1 - R_0) v(t)$$
$$- \frac{\lambda \beta}{m} f_2 (x_0, 0) \int_0^\infty g_1 (\xi) v(t - \xi) d\xi \leq 0. \qquad (17)$$

Recalling that f is nonnegative and is positive if and only if both arguments are positive, we must have that $\{E_0\}$ is the largest invariant subset of $\{dK/dt = 0\}$; then the global stability of E_0 follows from the classical Lyapunov-LaSalle invariance principle (see [6], Theorem 2.7.1). □

Theorem 6. *If $R_0 > 1$, and (A2) and (A3) hold, then E_1 is globally stable in $D = R_+^3 \setminus E_0$.*

Proof. Let $W(t) = W_1(x(t), y(t), v(t)) + W_2(x(t), y(t), v(t)) + W_3(x(t), y(t), v(t))$.

With

$$W_1(x(t), y(t), v(t)) = \int_0^\infty g_1(\xi) U(t, \xi) d\xi,$$

$$W_2(x(t), y(t), v(t))$$

$$= \frac{\beta}{\beta_1} y^* h\left(\frac{y}{y_*}\right) + \beta f(x^*, v^*) \int_0^\infty H_1(\xi) h\left(\frac{v(t-\xi)}{v_*}\right) d\xi,$$

$$W_3(x(t), y(t), v(t))$$

$$= \frac{a\beta}{k_1 \beta_1} F(v(t), v^*) + \frac{a\beta}{\beta_1} \int_0^\infty H_2(\xi) F(y(t-\xi), y^*) d\xi,$$

$$(18)$$

where $U(t, \xi) = x(t-\xi) - x^* - \int_{x^*}^{x(t-\xi)} (f(x^*, v^*)/f(\sigma, v^*)) d\sigma$, $h(t) = t - 1 - \ln t$, and $F(u, u^*) = u - u^* \ln u - (u^* - u^* \ln u^*)$; it is easy to see that $h(t)$ has the strict global minimum $h(1) = 0$ for $t > 0$, $F(u, u^*) \geq F(u^*, u^*) = 0$ for $u > 0$ and any positive constant u^*. By Theorem 1, all solutions are positive and bounded. Thus, $W(t)$ is well defined and $W(t) \geq 0$, in which the equality holds if and only if $y(t) = y^*, v(t) = v^*$ and $x(t-\xi) = x^*, y(t-\xi) = y^*, v(t-\xi) = v^*$ for almost all $\xi \in [0, \infty)$. For clarity, the derivatives of W_1, W_2, and W_3 will be calculated separately and then combined to obtain dW/dt.

First, $dW_1/dt = \int_0^\infty g_1(\xi)(dU(t, \xi)/dt) d\xi$, where

$$\frac{dU(t, \xi)}{dt} = \left(1 - \frac{f(x^*, v^*)}{f(x(t-\xi), v^*)}\right)$$

$$\times (\lambda - mx(t-\xi) - \beta f(x(t-\xi), v(t-\xi)))$$

$$= -m(x(t-\xi) - x^*)\left(1 - \frac{f(x^*, v^*)}{f(x(t-\xi), v^*)}\right)$$

$$+ \beta f(x^*, v^*)\left(1 - \frac{f(x(t-\xi), v(t-\xi))}{f(x^*, v^*)}\right)$$

$$- \frac{f(x^*, v^*)}{f(x(t-\xi), v^*)}$$

$$+ \frac{f(x(t-\xi), v(t-\xi))}{f(x(t-\xi), v^*)}\right).$$

$$(19)$$

Since $\lambda = mx^* + \beta f(x^*, v^*)$. That is,

$$\frac{dW_1}{dt} = -m \int_0^\infty g_1(\xi)(x(t-\xi) - x^*)$$

$$\times \left(1 - \frac{f(x^*, v^*)}{f(x(t-\xi), v^*)}\right) d\xi$$

$$+ \beta f(x^*, v^*)$$

$$\times \int_0^\infty g_1(\xi)\left(1 - \frac{f(x(t-\xi), v(t-\xi))}{f(x^*, v^*)}\right.$$

$$\left. - \frac{f(x^*, v^*)}{f(x(t-\xi), v^*)}\right) d\xi$$

$$+ \beta f(x^*, v^*) \int_0^\infty g_1(\xi) \frac{f(x(t-\xi), v(t-\xi))}{f(x(t-\xi), v^*)} d\xi.$$

$$(20)$$

$\square$

Next, we calculate dW_2/dt. Consider

$$\frac{dW_2}{dt} = \beta f(x^*, v^*)$$

$$\times \int_0^\infty g_1(\xi) \frac{f(x(t-\xi), v(t-\xi))}{f(x^*, v^*)}$$

$$\times \left(1 - \frac{y^*}{y}\right) d\xi - \frac{\beta a}{\beta_1} y(t) + \frac{\beta}{\beta_1} a y^*$$

$$+ \beta f(x^*, v^*) \int_0^\infty g_1(\xi)$$

$$\times \left(h\left(\frac{v}{v^*}\right) - h\left(\frac{v(t-\xi)}{v^*}\right)\right) d\xi$$

$$= \beta f(x^*, v^*) \int_0^\infty g_1(\xi)$$

$$\times \left[\frac{f(x(t-\xi), v(t-\xi))}{f(x^*, v^*)}\right.$$

$$\left. - \frac{y^*}{y} \frac{f(x(t-\xi), v(t-\xi))}{f(x^*, v^*)} - \frac{y^*}{y}\right] d\xi$$

$$+ \beta f(x^*, v^*) \int_0^\infty g_1(\xi)$$

$$\times \left[1 + h\left(\frac{v}{v^*}\right) - h\left(\frac{v(t-\xi)}{v^*}\right)\right] d\xi.$$

$$(21)$$

Using the fact that $H_2(0) = 1$, $H_2(\infty) = 0$, and $dH_2 = -g_2(t) dt$, we get

$$\frac{dW_3}{dt} = \frac{a\beta}{k_1 \beta_1}\left(1 - \frac{v^*}{v}\right)\left(k_1 \int_0^\infty g_2(\xi) y(t-\xi) d\xi - \mu v(t)\right)$$

$$- \frac{a\beta}{\beta_1} \int_0^\infty H_2(\xi) \frac{dF(y(t-\xi), y^*)}{d\xi} d\xi$$

$$= -\frac{a\beta}{\beta_1} \int_0^\infty g_2(\xi) \frac{v^*}{v} y(t-\xi) d\xi - \frac{a\beta\mu}{k_1 \beta_1}(v - v^*)$$

$$+ \frac{a\beta}{\beta_1} y(t) + \frac{a\beta}{\beta_1} y^* \int_0^\infty g_2(\xi) \ln \frac{y(t-\xi)}{y} d\xi$$

$$= -\beta f(x^*, v^*) \int_0^\infty g_2(\xi) h\left(\frac{v^*}{v} \frac{y(t-\xi)}{y}\right) d\xi$$

$$+ \beta f(x^*, v^*) \int_0^\infty g_1(\xi)\left(h\left(\frac{y}{y^*}\right) - h\left(\frac{v}{v^*}\right)\right) d\xi.$$

$$(22)$$

The third quality holds since $y^* = (\beta_1/a)f(x^*, v^*)$, $v^* = (k_1/\mu)y^*$. We conclude that

$$
\begin{aligned}
\frac{dW}{dt} = &-m \int_0^\infty g_1(\xi)\left(x(t-\xi) - x^*\right) \\
&\times \left(1 - \frac{f(x^*, v^*)}{f(x(t-\xi), v^*)}\right) d\xi \\
&- \beta f(x^*, v^*) \int_0^\infty g_2(\xi) h\left(\frac{v^*}{v} \frac{y(t-\xi)}{y}\right) d\xi \\
&+ \beta f(x^*, v^*) \int_0^\infty g_1(\xi) C(\xi) d\xi,
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
C(\xi) = &\, 2 - \frac{f(x^*, v^*)}{f(x(t-\xi), v^*)} + \frac{f(x(t-\xi), v(t-\xi))}{f(x(t-\xi), v^*)} \\
&- \frac{y^*}{y} \frac{f(x(t-\xi), v(t-\xi))}{f(x^*, v^*)} - \frac{y}{y^*} \\
&+ h\left(\frac{y}{y^*}\right) - h\left(\frac{v(t-\xi)}{v^*}\right) \\
= &-h\left(\frac{f(x^*, v^*)}{f(x(t-\xi), v^*)}\right) \\
&- h\left(\frac{y^*}{y} \frac{f(x(t-\xi), v(t-\xi))}{f(x^*, v^*)}\right) \\
&+ h\left(\frac{f(x(t-\xi), v(t-\xi))}{f(x(t-\xi), v^*)}\right) - h\left(\frac{v(t-\xi)}{v^*}\right).
\end{aligned}
\tag{24}
$$

Since the function $h(t)$ is monotone on each side of 1 and is minimized at $t = 1$, (A3) implies that $h((f(x(t-\xi), v(t-\xi)))/(f(x(t-\xi), v^*))) \le h(v(t-\xi)/v^*)$. Therefor, noting that $h \ge 0$ and (A2), we have $dW/dt \le 0$. So the solutions must tend to M, the largest invariant subset of $\{dW/dt = 0\}$.

To have dW/dt equal zero, it is necessary to have $(x(t-\xi) - x^*)(1 - (f(x^*, v^*)/f(x(t-\xi), v^*))) = 0$ for almost every $\xi \in [0, \infty)$, which holds if and only if $x(t) = x^*$; $v^* y(t-\xi) = v(t)y(t)$ and $y^* f(x(t-\xi), v(t-\xi)) = y(t)f(x^*, v^*)$ for almost every $\xi \in [0, \infty)$.

Since M is invariant with respect to (2), we get

$$
\begin{aligned}
\frac{dy(t)}{dt} &= \beta_1 \int_0^\infty g_1(\xi) f(x(t-\xi), v(t-\xi)) d\xi - ay(t) \\
&= \beta_1 \int_0^\infty g_1(\xi) \frac{y(t)f(x^*, v^*)}{y^*} d\xi - ay(t) = 0.
\end{aligned}
\tag{25}
$$

So this determines $y(t)$ to be a constant. Using $v^* y(t-\xi) = v(t)y(t)$, we obtain $v(t) = v^*$ and this yields that $y(t) = y^*$. Thus, each element of M satisfies $x(t) = x^*$, $y(t) = y^*$, and $v(t) = v^*$ for all $t > 0$. This shows that

$$
\lim_{t \to \infty} (x(t), y(t), v(t)) = (x^*, y^*, v^*) = E_1. \tag{26}
$$

This completes the proof.

5. Examples

We now give some examples of incidence functions for which the hypotheses are satisfied.

Example 1 (mass action). Let $f(x, v) = xv$. The hypotheses (A1)–(A4) are satisfied and so global dynamics are determined by the magnitude of R_0. The global behavior of this model was previously studied in [1, 8] and was fully resolved in [9].

Example 2 (saturating incidence). Let $f(x, v) = \beta x(t)v(t)/(1 + \alpha x(t))$ for some constant $\beta > 0$, $\alpha > 0$. The hypotheses (A1)–(A4) are satisfied and so global dynamics are determined by the magnitude of R_0. The global properties of this model were studied in [10].

Example 3 (Holling type II incidence). Let $f(x, v) = \beta x(t)v(t)/(1 + v(t))$ for some constant $\beta > 0$. The hypotheses (A1)–(A4) are satisfied and so global dynamics are determined by the magnitude of R_0. The global properties of this model were studied in [11].

Example 4 (Beddington-DeAngelis incidence). Let $f(x, v) = x(t)v(t)/(1 + ax(t) + bv(t))$ for some constants $a > 0$ and $b > 0$. The hypotheses (A1)–(A4) are satisfied and so global dynamics are determined by the magnitude of R_0. The global properties were studied in [12, 13].

Acknowledgment

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