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*Research Article*

# Spherical Symmetric Solutions for the Motion of Relativistic Membranes in the Schwarzschild-Anti de Sitter Space-Time

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This paper concerns motion of relativistic membranes in the Schwarzschild-anti de Sitter space-time. We derive a nonlinear equation for relativistic membranes moving in the Schwarzschild-anti de Sitter space-time, discuss spherical symmetric solutions for the motion equations, and obtain some interesting physical results.

## 1. Introduction

This paper concerns the motion of relativistic strings in the Schwarzschild-anti de Sitter space-time. The metric reads

$$\begin{aligned} ds^2 = & -\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)dt^2+\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-1} \\ & \times dr^2 + r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \end{aligned} \tag{1}$$

which is a spherically symmetric solution of the Einstein field equations. The Schwarzschild-anti de Sitter space-time is a fundamental physical space-time; it plays an important role in general relativity, the theory of black holes, and modern cosmology.

As is well known, the theory of minimal surfaces/submanifolds has a long history originating from the papers of Lagrange in 1760 and the famous Plateau problem. And this theory plays a significant role in general relativity, the theory of black holes, particle physics, and so on. A good deal of attention has been paid to the theory of minimal surfaces in the Euclidean space $\mathbb{R}^n$ and Riemannian manifolds in recent years. On this topic, we refer to two classical books [1, 2]. For the theory of extremal surfaces/submanifolds in the Minkowski space-time, many important results have been derived ([3–6]). Particularly, the theory of extremal surfaces/submanifolds is very important in elementary particle physics. It is a relativistic string model that deals with a one-dimensional relativistic object, whose world surface is an extremal surface in the Minkowski space-time [7]. For the relativistic string theory, we refer to an excellent book by Barbashov and Nesterenko [8].

The authors in [9] simplified the description of relativistic membrane in the Minkowski space-time by the light-cone gauge. By variables transformations, a relationship was established between the dynamics of relativistic membrane and two-dimensional fluid dynamics. Moreover, in [10] they obtained a vector-valued equation of first order for relativistic membrane by introducing the orthonormal (1 + 3)-gauge, and then they deduced a second-order equation for minimal graph $z = z(t, x_1, x_2)$ in the Minkowski space-time $\mathbb{R}^{1+3}$ by the hodograph technique. Furthermore, using reduction of membrane equation in light-cone gauge, they obtained a second-order partial differential equation for the velocity potential. In the paper [11], a lot of simplifications of the equations of motion for the relativistic membrane were exhibited such as the orthonormal light-cone gauge, minimal graph method, and level set method. According to these reformulations, Hoppe found some classical solutions for the equations governing the motion of relativistic membrane in the Minkowski space-time.

Here we want to mention a result in [12]: the authors investigated the basic equations for the motion of relativistic membranes in the Schwarzschild space-time and got a nonlinear wave equation, and then they studied a spherical symmetric solution for the motion of relativistic membranes, giving many new physical results. By variational and geometrical methods, Huang and Kong in [13] obtained a new kind of equation describing the motion of relativistic membranes in the Minkowski space-time $\mathbb{R}^{1+n}$ ($n \geqslant 3$), gave some interesting properties, and showed that all plane-wave solutions of these equations were light-like extremal submanifolds and vice versa except for a type of special solution.

Kong and Zhang studied the motion of relativistic closed strings in the Minkowski space $\mathbb{R}^{1+n}$ in [14]; particularly, the authors obtained a general solution formula for this system of nonlinear equations. Based on the solution formula, they showed that the motion of closed strings was always time-periodic and extended the solution formula to finite relativistic strings. Moreover, in [15], Kong et al. investigated the dynamics of relativistic strings in the Minkowski space-time $\mathbb{R}^{1+n}$ ($n \geq 2$). They first obtained a system with $n$ nonlinear wave equations of Born-Infeld type describing the motion of the string, and then they showed that this system enjoyed some interesting geometric properties; in the end, they gave a sufficient and necessary condition for the global existence of extremal surfaces without a space-like point in $\mathbb{R}^{1+n}$. Furthermore, they made a lot of numerical analyses demonstrating that various topological singularities developed in finite time in the motion of the string.

This paper mainly focuses on the equations and spherical symmetric solutions for the motion of relativistic membranes in the Schwarzschild-anti de Sitter space-time. Concretely, we derive an interesting nonlinear wave equation for relativistic membranes and study systematically the spherical symmetric solutions for the motion of membranes.

The paper is organized as follows. In Section 2, we recall the basic equations for the motion of relativistic membranes in the Schwarzschild-anti de Sitter space-time. Section 3 is devoted to a systematical study on the spherical symmetric solutions of the equations for the motion of relativistic membranes; at the same time, some new and interesting physical phenomena are discovered and illustrated. Some discussions are given in Section 4.

## 2. Basic Equation

A four-dimensional Lorentzian manifold $(M, g_{\mu\nu})$ is called the Schwarzschild-anti de Sitter space-time if the metric of $M$ can be written as (2) with $\lambda < 0$ ([16]). Consider

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -\left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right) dt^2 \\ &\quad + \left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right)^{-1} \\ &\quad \times dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \end{aligned} \tag{2}$$

where $m$ is a positive constant representing the mass of the universe and $\lambda$ is the cosmological constant. Assume $X = (t, r, \theta, \varphi)$ is a position vector of a point in the Schwarzschild-anti de Sitter space-time. Moreover, let $r_+$ be the largest root of the equation $1 - 2m/r - (\lambda/3) r^2 = 0$. Obviously, it holds that $0 < r_+ < 2m$. Since we are only interested in the motion of membrane in the region $r > r_+$, we may suppose that $r_0 > r_+$.

Consider the motion of a relativistic membrane in the Schwarzschild-anti de Sitter space-time

$$(t, \theta, \varphi) \longrightarrow (t, r(t, \theta, \varphi), \theta, \varphi). \tag{3}$$

In the coordinates $(t, \theta, \varphi)$, the induced metric of the submanifold $\mathscr{M}$ is

$$ds^2 = (dt, d\theta, d\varphi)\, \mathscr{G} (dt, d\theta, d\varphi)^T, \tag{4}$$

where

$$\mathscr{G} = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix}, \tag{5}$$

in which

$$\begin{aligned} g_{00} &= -\left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right) + \left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right)^{-1} r_t^2, \\ g_{01} &= g_{10} = \left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right)^{-1} r_t r_\theta, \\ g_{02} &= g_{20} = \left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right)^{-1} r_t r_\varphi, \\ g_{11} &= r^2 + \left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right)^{-1} r_\theta^2, \\ g_{12} &= g_{21} = \left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right)^{-1} r_\theta r_\varphi, \\ g_{22} &= r^2 \sin^2\theta + \left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right)^{-1} r_\varphi^2. \end{aligned} \tag{6}$$

We assume that the submanifold $\mathscr{M}$ is $C^2$ and time-like; that is,

$$\begin{aligned} \Delta \triangleq \det \mathscr{G} = &-\left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right) r^4 \sin^2\theta - r^2 r_\varphi^2 \\ &- r^2 \sin^2\theta r_\theta^2 + \left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right)^{-1} \\ &\times r^4 \sin^2\theta r_t^2 < 0. \end{aligned} \tag{7}$$

The area element of $\mathscr{M}$ is

$$d\mathscr{A} = \sqrt{-\Delta} dt\, d\theta\, d\varphi. \tag{8}$$

And the submanifold $\mathscr{M}$ is called *extremal* if $r = r(t, \theta, \varphi)$ is a critical point of the area functional

$$\mathscr{I} = \iiint \sqrt{-\Delta} dt\, d\theta\, d\varphi. \tag{9}$$

By calculations, we obtain the corresponding Euler-Lagrange equation as follows:

$$
\begin{aligned}
&\left[\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-1}-\frac{1}{\Delta}\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-2}r^4\sin^2\theta r_t^2\right]\\
&\quad\times r^4\sin^2\theta r_{tt}-\left[r^2+\frac{1}{\Delta}r^4r_\varphi^2\right]r_{\varphi\varphi}\\
&\quad-\left[1+\frac{1}{\Delta}r^2\sin^2\theta r_\theta^2\right]r^2\sin^2\theta r_{\theta\theta}+\frac{2}{\Delta}\\
&\quad\times\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-1}r^6\sin^2\theta r_t r_\varphi r_{t\varphi}\\
&\quad-\frac{2}{\Delta}r^4\sin^2\theta r_\theta r_\varphi r_{\theta\varphi}\\
&\quad+\frac{2}{\Delta}\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-1}r^6\sin^4\theta r_t r_\theta r_{t\theta}\\
&\quad+r^3\sin^2\theta\left(2-\frac{3m}{r}-\lambda r^2\right)-rr_\varphi^2\\
&\quad-rr_\theta^2\sin^2\theta-r^2r_\theta\sin 2\theta\\
&\quad+r_t^2r^3\sin^2\theta\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-2}\\
&\quad\times\left(2-\frac{5m}{r}-\frac{\lambda}{3}r^2\right)+\frac{r^2r_\varphi}{2\Delta}\\
&\quad\times\Big[-2rr_\varphi r_\theta^2\sin^2\theta\\
&\qquad+\left(-4+\frac{6m}{r}+2\lambda r^2\right)\\
&\qquad\times r^3\sin^2\theta r_\varphi+\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-2}\\
&\qquad\times r_t^2r^3\sin^2\theta r_\varphi\left(4-\frac{10m}{r}-\frac{2\lambda}{3}r^2\right)\\
&\qquad-2rr_\varphi^3\Big]+\frac{r^2r_\theta\sin^2\theta}{2\Delta}\\
&\quad\times\Big[-\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)r^4\sin 2\theta\\
&\qquad-2rr_\theta r_\varphi^2-2rr_\theta^3\sin^2\theta-r^2r_\theta^2\sin 2\theta\\
&\qquad+\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-1}r_t^2r^4\sin 2\theta\\
&\qquad+r^3r_\theta\sin^2\theta r_t^2\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-2}\\
&\qquad\times\left(4-\frac{10m}{r}-\frac{2\lambda}{3}r^2\right)\\
&\qquad+r^3r_\theta\sin^2\theta\left(-4+\frac{6m}{r}+2\lambda r^2\right)\Big]\\
&\quad-\frac{r^4r_t\sin^2\theta}{2\Delta}\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-1}\\
&\quad\times\Big[-2rr_tr_\theta^2\sin^2\theta+r^3r_t\sin^2\theta\\
&\qquad\times\left(-4+\frac{6m}{r}+2\lambda r^2\right)+r^3r_t^3\sin^2\theta\\
&\qquad\times\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)^{-2}\\
&\qquad\times\left(4-\frac{10m}{r}-\frac{2\lambda}{3}r^2\right)-2rr_tr_\varphi^2\Big]=0.
\end{aligned}
\tag{10}
$$

## 3. Spherical Symmetric Solutions

This section is concentrated on spherical symmetric solutions for the motion of relativistic membranes. Now we study the spherical symmetric solutions $r = r(t)$ for the motion of relativistic membranes. Therefore, in the present situation, (10) can be simplified as the following ordinary differential equation:

$$
\begin{aligned}
r_{tt}=&\frac{2r-m-(5/3)\lambda r^3}{r\left(r-2m-(\lambda/3)r^3\right)}r_t^2\\
&-\frac{\left(r-2m-(\lambda/3)r^3\right)\left(2r-3m-\lambda r^3\right)}{r^3}.
\end{aligned}
\tag{11}
$$

Consider the initial value problem for (11) with the initial data

$$
r(0)=r_0,\qquad r_t(0)=r_1,
\tag{12}
$$

where $r_0$ and $r_1$ are two constants meaning the initial position and initial velocity of the membrane, respectively. Since we are only interested in the motion of the membrane in the region $r > r_+$, without loss of generality, we may assume that $r_0 > r_+$.

Let

$$
z=r_t.
\tag{13}
$$

Equation (11) becomes the following form:

$$
\begin{aligned}
\frac{dz}{dr}=&\frac{2r-m-(5/3)\lambda r^3}{r\left(r-2m-(\lambda/3)r^3\right)}z\\
&-\frac{\left(r-2m-(\lambda/3)r^3\right)\left(2r-3m-\lambda r^3\right)}{r^3}\frac{1}{z}.
\end{aligned}
\tag{14}
$$

Using (14), we derive

$$\begin{aligned}\frac{d}{dr}\left(\frac{z^2}{r(r-2m-(\lambda/3)r^3)^3}\right) &= \frac{d}{dr}\left(\frac{1}{r(r-2m-(\lambda/3)r^3)^3}\right)z^2 \\ &\quad + \frac{2z}{r(r-2m-(\lambda/3)r^3)^3}\frac{dz}{dr} \\ &= -\frac{2(2r-3m-\lambda r^3)}{r^4(r-2m-(\lambda/3)r^3)^2} \\ &= \frac{d}{dr}\left(\frac{1}{r^3(r-2m-(\lambda/3)r^3)}\right).\end{aligned} \tag{15}$$

Integrating (15), we have

$$\begin{aligned}z^2 &= \frac{(r-2m-(\lambda/3)r^3)^2}{r^2} \\ &\quad\times\left[1+F(r_0,r_1)r^3\left(r-2m-\frac{\lambda}{3}r^3\right)\right],\end{aligned} \tag{16}$$

where

$$\begin{aligned}F(r_0,r_1) &= \frac{r_1^2}{r_0(r_0-2m-(\lambda/3)r_0^3)^3} \\ &\quad - \frac{1}{r_0^3(r_0-2m-(\lambda/3)r_0^3)}.\end{aligned} \tag{17}$$

Noting (16), we obtain from (11) that

$$\begin{aligned}r_{tt} = \frac{r-2m-(\lambda/3)r^3}{r^3}\Big[2m-\frac{2}{3}\lambda r^3 &+ F(r_0,r_1)r^3\left(r-2m-\frac{\lambda}{3}r^3\right) \\ &\times\left(2r-m-\frac{5}{3}\lambda r^3\right)\Big].\end{aligned} \tag{18}$$

Define two functions $f(r)$ and $g(r)$ as follows:

$$\begin{aligned}f(r) &= r^3\left(r-2m-\frac{\lambda}{3}r^3\right), \\ g(r) &= r^3\left(r-2m-\frac{\lambda}{3}r^3\right)\left(2r-m-\frac{5}{3}\lambda r^3\right).\end{aligned} \tag{19}$$

Obviously $f(r)$ and $g(r)$ are strictly increasing positive functions when $r > r_+$. Hence, using the notations $f(r)$ and $g(r)$, (16) and (18) are written as

$$z^2 = \frac{(r-2m-(\lambda/3)r^3)^2}{r^2}\left[1+F(r_0,r_1)f(r)\right], \tag{20}$$

$$r_{tt} = \frac{r-2m-(\lambda/3)r^3}{r^3}\left[2m-\frac{2}{3}\lambda r^3+F(r_0,r_1)g(r)\right], \tag{21}$$

respectively.

In order to solve the Cauchy problem (11) and (12), that is, (21) and (12), we distinguish several cases as follows.

*Case 1.* Consider

$$r_1 \geqslant \frac{r_0-2m-(\lambda/3)r_0^3}{r_0}. \tag{22}$$

In this case, noting $r_0 > r_+$, it holds that

$$F(r_0,r_1) \geqslant 0, \qquad r_t(0) = r_1 > 0, \qquad r_{tt}(0) > 0. \tag{23}$$

According to the properties of $f(r)$ and $g(r)$, it follows from (20) that

$$z = r_t > 0, \tag{24}$$

which implies that $r$ is a strictly increasing function of $t$. Therefore, it follows that $r(t) > r(0) = r_0 > r_+$ for $t > 0$. Thus, we obtain from (21) that

$$r_{tt} > 0. \tag{25}$$

Case 1(a) $r_1 = (r_0 - 2m - (\lambda/3)r_0^3)/r_0$.
In the present situation, we have $F(r_0, r_1) = 0$, and

$$\begin{aligned}r_t &= \frac{r-2m-(\lambda/3)r^3}{r}, \\ r_{tt} &= \frac{r-2m-(\lambda/3)r^3}{r^3}\left(2m-\frac{2}{3}\lambda r^3\right).\end{aligned} \tag{26}$$

Clearly, because $r_t > 0$ and $r_{tt} > 0$, $r(t)$ tends to the infinity. We next prove that $r(t)$ goes to the positive infinity in finite time. In fact, it follows from (26) that

$$t = \int_{r_0}^{r}\frac{r}{r-2m-(\lambda/3)r^3}dr. \tag{27}$$

Let

$$T_{\max} = \int_{r_0}^{\infty}\frac{r}{r-2m-(\lambda/3)r^3}dr. \tag{28}$$

It is easy to see that $t \leq T_{\max}$. Now we show that $T_{\max} < \infty$. For a certain large $\tilde{r}$,

$$\begin{aligned}T_{\max} &= \int_{r_0}^{\tilde{r}}\frac{r}{r-2m-(\lambda/3)r^3}dr + \int_{\tilde{r}}^{\infty}\frac{r}{r-2m-(\lambda/3)r^3}dr \\ &< \int_{r_0}^{\tilde{r}}\frac{r}{r-2m-(\lambda/3)r^3}dr + \int_{\tilde{r}}^{\infty}\frac{r}{-(\lambda/3)r^3}dr \\ &= \int_{r_0}^{\tilde{r}}\frac{r}{r-2m-(\lambda/3)r^3}dr - \frac{3}{\lambda\tilde{r}} \\ &< \infty,\end{aligned} \tag{29}$$

which implies

$$r(t) \longrightarrow +\infty, \quad \text{when } t \nearrow T_{\max}. \tag{30}$$

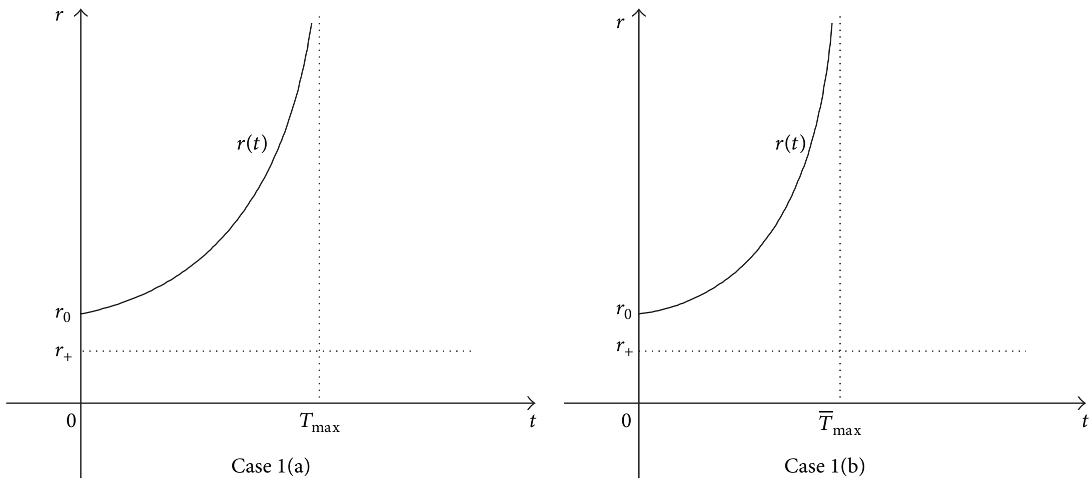


Figure 1: The graph of the solution $r = r(t)$ for Cases 1(a) and 2(b).

Equation (30) shows that the solution of the initial value problem (11) and (12) must blow up in finite time and the life span is $[0, T_{\max})$, where $T_{\max}$ is defined by (28). Different from the case that initial velocity is equal to critical velocity in [12], where the solution $r = r(t)$ of the Cauchy problem exists globally on the time $t \in [0, \infty)$, in this paper the solution of the initial value problem (11) and (12) blows up in finite time when initial velocity is equal to critical velocity. See Figure 1 for the graph of the solution $r = r(t)$ in the present situation.

Case 1(b) $r_1 > (r_0 - 2m - (\lambda/3)r_0^3)/r_0$.

In this case, we have $F(r_0, r_1) > 0$. Since $r_t > 0$ and $r_{tt} > 0$, $r(t)$ goes to the infinity. In what follows, we prove that $r(t)$ tends to the positive infinity in finite time.

Similar to the above case, it follows from (16) that

$$t = \int_{r_0}^{r} \frac{r}{r - 2m - (\lambda/3) r^3} \times \frac{1}{\sqrt{1 + F(r_0, r_1) r^3 (r - 2m - (\lambda/3) r^3)}} dr. \tag{31}$$

Denote

$$\overline{T}_{\max} = \int_{r_0}^{\infty} \frac{r}{r - 2m - (\lambda/3) r^3} \times \frac{1}{\sqrt{1 + F(r_0, r_1) r^3 (r - 2m - (\lambda/3) r^3)}} dr. \tag{32}$$

Obviously, we have $t \leqslant \overline{T}_{\max}$.

We next show that $\overline{T}_{\max} < \infty$. For a certain large $\bar{r}$,

$$\begin{aligned} \overline{T}_{\max} = & \int_{r_0}^{\bar{r}} \frac{r}{r - 2m - (\lambda/3) r^3} \times \frac{1}{\sqrt{1 + F(r_0, r_1) r^3 (r - 2m - (\lambda/3) r^3)}} dr \\ & + \int_{\bar{r}}^{\infty} \frac{r}{r - 2m - (\lambda/3) r^3} \times \frac{1}{\sqrt{1 + F(r_0, r_1) r^3 (r - 2m - (\lambda/3) r^3)}} dr \\ < & \int_{r_0}^{\bar{r}} \frac{r}{r - 2m - (\lambda/3) r^3} \times \frac{1}{\sqrt{1 + F(r_0, r_1) r^3 (r - 2m - (\lambda/3) r^3)}} dr \\ & + \int_{\bar{r}}^{\infty} \frac{r}{-(\lambda/3) r^6 \sqrt{-(\lambda/3) F(r_0, r_1)}} dr \\ = & \int_{r_0}^{\bar{r}} \frac{r}{r - 2m - (\lambda/3) r^3} \times \frac{1}{\sqrt{1 + F(r_0, r_1) r^3 (r - 2m - (\lambda/3) r^3)}} dr \\ & + \frac{1}{-(4/3)\lambda \sqrt{-(\lambda/3) F(r_0, r_1)} \bar{r}^4} < \infty. \end{aligned} \tag{33}$$

This proves that

$$r(t) \longrightarrow +\infty, \quad \text{when } t \nearrow \overline{T}_{\max}. \tag{34}$$

Equation (34) implies that the solution of the Cauchy problem (11)-(12) must blow up in finite time and the life span is the interval $[0, \overline{T}_{\max})$, where $\overline{T}_{\max}$ is defined by (32). Compared to the case that initial velocity is larger than critical velocity in [12], in the present paper $\overline{T}_{\max}$ is controlled by the constant $\lambda$. See Figure 1 for the graph of the solution $r = r(t)$.

*Case 2.* Consider

$$\frac{r_0 - 2m - (\lambda/3) r_0^3}{r_0} \sqrt{\frac{2r_0 - 3m - \lambda r_0^3}{2r_0 - m - (5/3) \lambda r_0^3}} < r_1 < \frac{r_0 - 2m - (\lambda/3) r_0^3}{r_0}. \tag{35}$$

In this situation, we obtain from (17) and (21) that

$$-\frac{2m - (2/3) \lambda r_0^3}{g(r_0)} < F(r_0, r_1) < 0, \qquad r_t(0) = r_1 > 0, \qquad r_{tt}(0) > 0. \tag{36}$$

By the properties of $f(r)$ and $g(r)$, there exists a point $r^* \in (r_+, +\infty)$ such that

$$f(r^*) = -\frac{1}{F(r_0, r_1)}. \tag{37}$$

Thus, it follows from (20) that

$$z = r_t = 0, \quad \text{when } r = r^*. \tag{38}$$

On the one hand, noting (20) and the second inequality in (36), $r(t)$ will increase until a time $t^* > 0$. At the time $t^*$, it holds that

$$r_{tt} < 0, \quad \text{when } r = r^*. \tag{39}$$

By (36), we have $r_{tt} > 0$, when $r = r_0$. So there exists a point $r_\star \in (r_0, r^*)$, such that

$$r_{tt} = 0, \quad \text{when } r = r_\star. \tag{40}$$

On the other hand, when $r \to r_+ + 0$,

$$r_t \longrightarrow 0, \qquad r_{tt} \longrightarrow 0. \tag{41}$$

For $r \to r_+ + \delta$ ($\delta$ is sufficiently small and positive), both $r - 2m - (\lambda/3)r^3$ and $g(r)$ are small enough and positive; thus $r_{tt}$ is sufficiently small and $r_{tt} > 0$. Combined with (39), it holds that there exists a point $r_* \in (r_+ + \delta, r^*)$ such that

$$r_{tt} = 0, \quad \text{when } r = r_*. \tag{42}$$

Now we will discuss the developing with time of the membrane, that is, the properties enjoyed by the solution of the Cauchy problem (11) and (12). At the time $t^*$, it holds that

$$r = r^*, \qquad r_t = 0, \qquad r_{tt} < 0. \tag{43}$$

Therefore, $r_t(t) < 0$ for $t > t^*$. That is, when $t > t^*$, $r(t)$ is a strictly decreasing function; then there exists a time $t_*$ such that

$$r(t_*) = r_*, \qquad r_t(t_*) < 0, \qquad r_{tt}(t_*) = 0. \tag{44}$$

Hence, by (20) and (21), we obtain that for $t > t_*$

$$r(t) < r_*, \qquad r_t(t) < 0, \qquad r_{tt}(t) > 0. \tag{45}$$

Furthermore, from the fact that (42) holds when $r \to r_+ + 0$, we see that the solution of the Cauchy problem (11) and (12) exists globally on the time $t \in [0, \infty)$, and the solution satisfies the following decay properties:

$$r(t) \longrightarrow r_+, \qquad r_t(t) \longrightarrow 0, \qquad r_{tt}(t) \longrightarrow 0, \tag{46}$$

when $t \to \infty$. Figure 2 is the graph of the solution for Case 2.

*Case 3.* Consider

$$0 < r_1 \leqslant \frac{r_0 - 2m - (\lambda/3) r_0^3}{r_0} \times \sqrt{\frac{2r_0 - 3m - \lambda r_0^3}{2r_0 - m - (5/3) \lambda r_0^3}}. \tag{47}$$

In this situation, it follows from (17) and (21) that

$$-\frac{1}{f(r_0)} < F(r_0, r_1) \leqslant -\frac{2m - (2/3) \lambda r_0^3}{g(r_0)}, \qquad r_t(0) = r_1 > 0, \qquad r_{tt}(0) \leqslant 0. \tag{48}$$

Similar to Case 2, we can exactly analyze the properties enjoyed by the solution of the Cauchy problem (11) and (12). The graph of the solution is shown in Figure 2. Figure 2 shows that, in this case, the solution of the Cauchy problem (11) and (12) exists globally on the time $t \in [0, \infty)$, and the solution satisfies the decay condition (46).

*Case 4.* Consider

$$-\frac{r_0 - 2m - (\lambda/3) r_0^3}{r_0} \times \sqrt{\frac{2r_0 - 3m - \lambda r_0^3}{2r_0 - m - (5/3) \lambda r_0^3}} < r_1 \leqslant 0. \tag{49}$$

In this case, it holds that

$$-\frac{1}{f(r_0)} \leq F(r_0, r_1) < -\frac{2m - (2/3) \lambda r_0^3}{g(r_0)}, \qquad r_t(0) = r_1 \leqslant 0, \qquad r_{tt}(0) < 0. \tag{50}$$

Similar to the discussion in Cases 2 and 3, we can exactly analyze the properties enjoyed by the solution of the Cauchy problem (11) and (12). See Case 4 in Figure 3 for the graph of the solution. In this case, the solution of the Cauchy problem (11) and (12) also exists globally on the time $t \in [0, \infty)$, and the solution $r = r(t)$ satisfies the decay condition (46).

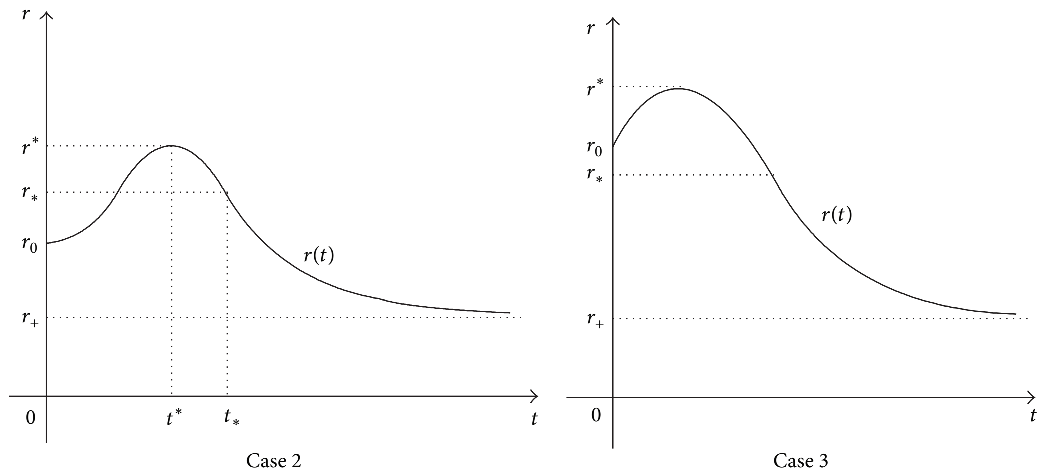


FIGURE 2: The graph of the solution $r = r(t)$ for Cases 2 and 3.

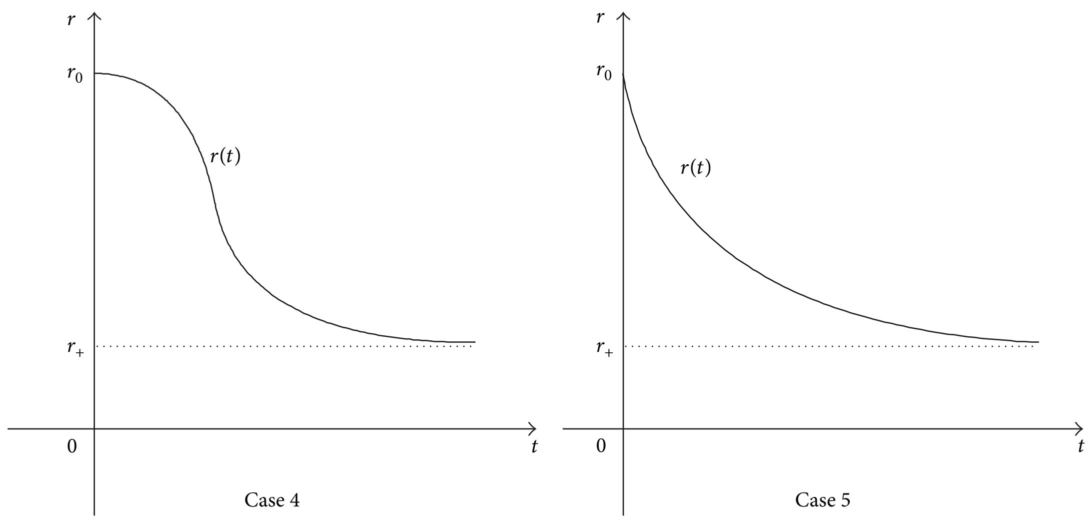


FIGURE 3: The graph of the solution $r = r(t)$ for Cases 4 and 5.

*Case 5.* Consider

$$r_1 \leqslant -\frac{r_0 - 2m - (\lambda/3)\, r_0^3}{r_0} \times \sqrt{\frac{2r_0 - 3m - \lambda r_0^3}{2r_0 - m - (5/3)\,\lambda r_0^3}}. \tag{51}$$

In this case, we have

$$F(r_0, r_1) \geqslant -\frac{2m - (2/3)\,\lambda r_0^3}{g(r_0)}, \qquad r_t(0) = r_1 < 0, \qquad r_{tt}(0) \geqslant 0. \tag{52}$$

In the present situation we only consider the case that $F(r_0, r_1) \geqslant 0$; then it follows from (21) that

$$r_{tt} > 0, \tag{53}$$

provided that $r > r_+$.

Therefore, noting (53) and using (42), we know that the solution of the Cauchy problem (11) and (12) exists globally on the time $t \in [0, \infty)$, and the solution $r = r(t)$ satisfies the decay conditions (46). The graph and detailed information of the solution in the present situation are shown in Figure 3.

## 4. Discussions

The Schwarzschild-anti de Sitter space-time is a fundamental physical space-time and plays an important role in general relativity and the theory of black holes. The notion of extremal surfaces/submanifolds is used to formulate the Wilson criterion of quark confinement in gauge models of strong interactions. The theory of extremal surfaces/submanifolds is important in the elementary particle physics. The world sheets of relativistic membranes moving in physical space-times are nothing but extremal surfaces/submanifolds in these space-times. Some beautiful and deep results have been obtained, but unfortunately only a few results have been known for relativistic membranes moving in the Schwarzschild-anti de Sitter space-time.

In the present paper, we study the equations and spherical symmetric solutions for relativistic membranes moving in the Schwarzschild-anti de Sitter space-time. First, a nonlinear wave equation for the motion of relativistic membranes in the Schwarzschild-anti de Sitter space-time is derived, and then spherical symmetric solutions for the motion of relativistic membranes are studied systematically. We can summarize the main results as follows. In this paper we consider a spherical membrane centered at the black hole $r = 0$ and assume that $r_0$ and $r_1$ represent the initial position and initial velocity of the membrane, respectively. For a given initial position $r_0 > r_+$, we find a critical initial velocity $v_c = (r_0 - 2m - (\lambda/3)r_0^3)/r_0$ such that the motion with a different initial velocity has essentially different behavior. Exactly, (i) when $r_1 > v_c$, the solution for motion of the spherical membrane must blow up in finite time and the life span of the solution is $[0, \overline{T}_{\max})$, where $\overline{T}_{\max}$ is defined by (32) and $\overline{T}_{\max} < \infty$. (ii) When $r_1 = v_c$, the solution also blows up in finite time and the life span of the solution is $[0, T_{\max})$, where $T_{\max}$ is given by (28) and $T_{\max} < \infty$. (iii) When $r_1 < v_c$, the solution exists globally on the time $t \in [0, \infty)$, and the solution satisfies the following decay conditions:

$$r(t) \longrightarrow r_+, \qquad r_t(t) \longrightarrow 0, \qquad r_{tt}(t) \longrightarrow 0, \tag{54}$$

when $t \to \infty$.

In geometry, our results show that, for a given spherical membrane centered at the black hole $r = 0$, (i) if the initial velocity is larger than the critical velocity $v_c$, then singularities in the motion will develop in finite time, precisely, at the time $\overline{T}_{\max}$; that is, the spherical membrane disappears at the time $\overline{T}_{\max}$ in this case. (ii) If the initial velocity is equal to the critical velocity $v_c$, similar to the above case, singularities will develop in finite time, and the spherical membrane disappears at the time $T_{\max}$ in this situation. (iii) If the initial velocity is less than the critical velocity $v_c$, then motion of the spherical membrane will not stop at any finite time, and the spherical membrane converges to the event horizon $r = r_+$ when the time tends to infinity. From the graphs we see that motion of the spherical membrane may be different due to the different initial velocity. In this case, the moving velocity of the membrane goes to zero when the membrane tends to the event horizon.

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