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Research Article

Stability and Hopf Bifurcation Analysis for a Stage-Structured Predator-Prey Model with Discrete and Distributed Delays

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We propose a three-dimensional stage-structured predatory-prey model with discrete and distributed delays. By use of a new variable, the original three-dimensional system transforms into an equivalent four-dimensional system. Firstly, we study the existence and local stability of positive equilibrium of the new system. And, by choosing the time delay τ as a bifurcation parameter, we show that Hopf bifurcation may occur as the time delay τ passes through some critical values. Secondly, by use of normal form theory and central manifold argument, we establish the direction and stability of Hopf bifurcation. At last, some simple discussion is presented.

1. Introduction

Since the pioneering theoretical works by Lotka [1] and Volterra [2], there were a lot of authors who studied all kinds of predator-prey models modeled by ordinary differential equations (ODEs) [3–5]. To reflect that the dynamical behavior of the models depends on the past history of the system, it is often necessary to incorporate time delays into the models. Therefore, a more realistic predator-prey model should be described by delayed differential equations (DDEs) [6–26]. Some of them investigated discrete delays [6–20]; others were about distributed delays [21–24]; and both discrete and distributed delays were studied in [25]. In general, delay differential equations exhibit more complicated dynamics on stability, periodic structure, bifurcation, and so on [26].

In the natural world, many individuals have a life story that takes them through two stages, immature and mature. The predator only catches the mature prey, as the immature preys are protected by their eggshells or refuge. Some predator-prey models with stage structure were investigated in [27–33]. Motivated by [25, 27, 31] and the references cited therein, in the present paper, we will consider the following stage-structured predator-prey model with discrete and distributed delay:

$$\begin{aligned} x_1'(t) &= rx_2(t) - d_1x_1(t) - re^{-d_1\tau}x_2(t-\tau), \\ x_2'(t) &= re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - b_1x_2(t)y(t), \\ y'(t) &= y(t)\left[-d_3 + b_2\int_{-\infty}^{t} F(t-s)x_2(s)\,\mathrm{d}s - \alpha y(t)\right], \end{aligned} \tag{1}$$

where $x_1(t)$, $x_2(t)$, and $y(t)$ can be interpreted as the population densities of the immature prey, mature prey, and predator at time t, respectively. r denotes the birth rate of the prey population; d_1, d_2, and d_3 denote the death rate of the immature prey, mature prey, and the predator; α is the density-depended death rate of the predator; b_1 denotes the per capita per unit time predation rate of the predator; the term $b_2\int_{-\infty}^{t} F(t-s)x_2(s)\mathrm{d}s$ is the conversion rate from prey to predator, and the distributed delay may interpret as digest delay. The function $F(s)$ is called the delayed kernel that is a nonnegative bounded function defined on $[0,\infty)$. Following the ideas of Cushing et al. [34], we define $F(t)$ as the following weak kernel function:

$$F(t) = \beta e^{-\beta t}, \quad \beta > 0. \tag{2}$$

Next, we define a new variable:

$$u(t) = \int_{-\infty}^{t} \beta e^{-\beta(t-s)} x_2(s)\, ds. \tag{3}$$

Then by use of linear chain trick technique, system (1) can be transformed into the following equivalent system:

$$\begin{aligned} x_1'(t) &= r x_2(t) - d_1 x_1(t) - r e^{-d_1\tau} x_2(t-\tau), \\ x_2'(t) &= r e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - b_1 x_2(t) y(t), \\ u'(t) &= \beta x_2(t) - \beta u(t), \\ y'(t) &= y(t)\left[-d_3 + b_2 u(t) - \alpha y(t)\right]. \end{aligned} \tag{4}$$

The organization of this paper is as follows: In Section 2, we will get the conditions for the existence and stability of positive equilibrium of system (4). The occurring condition for Hopf bifurcation is also obtained. In Section 3, by use of normal form theory and central manifold argument, we illustrate the direction and stability of Hopf bifurcation. In Section 4, we give some brief discussion.

2. Stability of Positive Equilibrium and Existence of Hopf Bifurcation

In this section, we will firstly investigate the existence and stability of positive equilibrium of system (4) then study the effect of time delay on the system (4); that is, we will choose τ as bifurcating parameter to analyze Hopf bifurcation.

Theorem 1. *There exists a unique positive equilibrium E^* for system* (4), *if assumption*

(H1) $re^{-d_2\tau} - d_2 > 0$ *holds. And* $E^* = (x_1^*, x_2^*, u^*, y^*)$, *with*

$$\begin{aligned} x_1^* &= \frac{r\left(1 - e^{-d_1\tau}\right)\left(\alpha r e^{-d_1\tau} - \alpha d_2 + b_1 b_3\right)}{b_1 b_2 d_1}, \\ x_2^* = u^* &= \frac{\alpha\left(r e^{-d_1\tau} - d_2\right) + b_1 b_3}{b_1 b_2}, \\ y^* &= \frac{r e^{-d_1\tau} - d_2}{b_1}. \end{aligned} \tag{5}$$

Linearizing system (4) at E^*, we get

$$\begin{aligned} x_1'(t) &= -d_1 x_1(t) + r x_2(t) - r e^{-d_1\tau} x_2(t-\tau), \\ x_2'(t) &= \left(-d_2 - b_1 y^*\right) x_2(t) + r e^{-d_1\tau} x_2(t-\tau) - b_1 x_2^* y(t), \\ u'(t) &= \beta x_2(t) - \beta u(t), \\ y'(t) &= b_2 y^* u(t) - \alpha y^* y(t), \end{aligned} \tag{6}$$

and the characteristic equation for system (6) takes the form

$$\begin{aligned} &\lambda^4 + h_1\lambda^3 + h_2\lambda^2 + h_3\lambda + h_4 \\ &\quad + \left[h_5\lambda^3 + h_6\lambda^2 + h_7\lambda + h_8\right] e^{-\lambda\tau} = 0, \end{aligned} \tag{7}$$

where

$$\begin{aligned} h_1 &= \left(d_1 + d_2 + b_1 y^*\right) + \left(\beta + \alpha y^*\right), \\ h_2 &= d_1\left(d_2 + b_1 y^*\right) + \left(\beta + \alpha y^*\right)\left(d_1 + d_2 + b_1 y^*\right) + \alpha\beta y^*, \\ h_3 &= d_1\left(d_2 + b_1 y^*\right)\left(\beta + \alpha y^*\right) + \alpha\beta y^*\left(d_1 + d_2 + b_1 y^*\right), \\ h_4 &= \alpha\beta d_1 y^*\left(d_2 + b_1 y^*\right) + \beta b_1 b_2 x_2^* y^*, \\ h_5 &= -r e^{-d_1\tau}, \\ h_6 &= -\left(\beta + \alpha y^* + d_1\right) r e^{-d_1\tau}, \\ h_7 &= -\left(\alpha\beta y^* + d_1\beta + \alpha d_1 y^*\right) r e^{-d_1\tau}, \\ h_8 &= -\alpha\beta d_1 y^* r e^{-d_1\tau}. \end{aligned} \tag{8}$$

Note that when $\tau = 0$, (7) becomes

$$\lambda^4 + \left(h_1 + h_5\right)\lambda^3 + \left(h_2 + h_6\right)\lambda^2 + \left(h_3 + h_7\right)\lambda + h_4 + h_8 = 0. \tag{9}$$

It is easy to confirm that

$$\begin{gathered} h_1 + h_5 = d_1 + \beta + \alpha y^* > 0, \\ h_2 + h_6 = \left(\beta + \alpha y^*\right) d_1 + \alpha\beta y^* > 0, \\ h_3 + h_7 = \alpha\beta y^* d_1 > 0, \\ h_4 + h_8 = \alpha\beta b_1 b_2 x_2^* y^* > 0, \\ \left(h_1 + h_5\right)\left(h_2 + h_6\right) - \left(h_3 + h_7\right) > 0, \\ \left(h_1 + h_5\right)\left[\left(h_2 + h_6\right)\left(h_3 + h_7\right) - \left(h_1 + h_5\right)\left(h_4 + h_8\right)\right] \\ - \left(h_3 + h_7\right)^2 > 0. \end{gathered} \tag{10}$$

Thus, by the Routh-Hurwitz criterion we know that all the roots of (9) have negative real parts, which means that the positive equilibrium E^* is locally asymptotically stable for $\tau = 0$.

Next, we will consider the case for $\tau > 0$. Suppose that there is a pure imaginary root $\lambda = i\omega$, $\omega > 0$. Then we get

$$\begin{aligned} &\omega^4 - h_1 i\omega^3 - h_2\omega^2 + h_3 i\omega + h_4 \\ &\quad + \left(-h_5 i\omega^3 - h_6\omega^2 + h_7 i\omega + h_8\right)\left(\cos\omega\tau - i\sin\omega\tau\right) = 0. \end{aligned} \tag{11}$$

Separating the real and imaginary parts, we have

$$\begin{aligned} \left(h_6\omega^2 - h_8\right)\cos\omega\tau + \left(h_5\omega^3 - h_7\omega\right)\sin\omega\tau &= \omega^4 - h_2\omega^2 + h_4, \\ \left(h_6\omega^2 - h_8\right)\sin\omega\tau + \left(-h_5\omega^3 + h_7\omega\right)\cos\omega\tau &= h_1\omega^3 - h_3\omega. \end{aligned} \tag{12}$$

Incorporating $\sin^2\omega\tau + \cos^2\omega\tau = 1$, we have

$$\omega^8 + f_1\omega^6 + f_2\omega^4 + f_3\omega^2 + f_4 = 0, \tag{13}$$

where

$$\begin{aligned} f_1 &= h_1^2 - 2h_2 - h_5^2, \\ f_2 &= 2h_4 + h_2^2 - 2h_1h_3 - h_6^2 + 2h_5h_7, \\ f_3 &= h_3^2 - 2h_2h_4 + 2h_6h_8 - h_7^2, \\ f_4 &= h_4^2 - h_8^2. \end{aligned} \tag{14}$$

Denote $z = \omega^2$. Then (13) becomes

$$z^4 + f_1z^3 + f_2z^2 + f_3z + f_4 = 0. \tag{15}$$

Let

$$G(z) = z^4 + f_1z^3 + f_2z^2 + f_3z + f_4. \tag{16}$$

Then the following assumption holds true.

(H2) Equation (15) has at least one positive real root.

In fact, if all the parameters of system (4) are given, it is easy to calculate the root of (15). Since $\lim_{z\to\infty}G(z) = +\infty$, we conclude that if $f_4 < 0$, then (15) has at least one positive real root. Without loss of generality, we assume that (15) has four positive root, defined by z_1, z_2, z_3, z_4, respectively. Then (13) has four positive roots as

$$\begin{aligned} \omega_1 &= \sqrt{z_1}, \qquad \omega_2 = \sqrt{z_2}, \\ \omega_3 &= \sqrt{z_3}, \qquad \omega_4 = \sqrt{z_4}. \end{aligned} \tag{17}$$

From (12), we obtain

$$\begin{aligned} \sin\omega\tau = {} & \big(h_5\omega^7 + (h_1h_6 - h_7 - h_2h_5)\,\omega^5 \\ & + (h_2h_7 + h_4h_5 - h_1h_8 - h_3h_6)\,\omega^3 \\ & + (h_3h_8 - h_4h_7)\,\omega\big) \\ & \times \left(h_5^2\omega^6 + \left(h_6^2 - 2h_5h_7\right)\omega^4 \right. \\ & \left. + \left(h_7^2 - 2h_6h_8\right)\omega^2 + h_8^2\right)^{-1}, \end{aligned} \tag{18}$$

$$\begin{aligned} \cos\omega\tau = {} & \big((h_6 - h_1h_5)\,\omega^6 + (h_1h_7 + h_3h_5 - h_2h_6 - h_8)\,\omega^4 \\ & + (h_4h_6 + h_2h_8 - h_3h_7)\,\omega^2 + h_4h_8\big) \\ & \times \left(h_5^2\omega^6 + \left(h_6^2 - 2h_5h_7\right)\omega^4 \right. \\ & \left. + \left(h_7^2 - 2h_6h_8\right)\omega^2 + h_8^2\right)^{-1}. \end{aligned} \tag{19}$$

Denote

$$\begin{aligned} e_1 &= h_5^2, \qquad e_2 = h_6^2 - 2h_5h_7, \\ e_3 &= h_7^2 - 2h_6h_8, \qquad e_4 = h_8^2, \\ e_5 &= h_5, \qquad e_6 = h_1h_6 - h_7 - h_2h_5, \\ e_7 &= h_2h_7 + h_4h_5 - h_1h_8 - h_3h_6, \\ e_8 &= h_3h_8 - h_4h_7, \\ e_9 &= h_6 - h_1h_5, \\ e_{10} &= h_1h_7 + h_3h_5 - h_2h_6 - h_8, \\ e_{11} &= h_4h_6 + h_2h_8 - h_3h_7, \qquad e_{12} = -h_4h_8. \end{aligned} \tag{20}$$

Then $\cos\omega\tau$ can be written as

$$\cos\omega\tau = \frac{e_9\omega^6 + e_{10}\omega^4 + e_{11}\omega^2 + e_{12}}{e_1\omega^6 + e_2\omega^4 + e_3\omega^2 + e_4}, \tag{21}$$

from which we can get

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\arccos\left(\frac{e_9\omega_k^6 + e_{10}\omega_k^4 + e_{11}\omega_k^2 + e_{12}}{e_1\omega_k^6 + e_2\omega_k^4 + e_3\omega_k^2 + e_4}\right) + 2j\pi\right\}, \quad k = 1, 2, 3, 4;\ \ j = 0, 1, 2, \ldots. \tag{22}$$

Thus, $\pm i\omega_k$ is a pair of purely imaginary root of (7). Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3,4\}}\left\{\tau_k^{(0)}\right\}, \qquad \omega_0 = \omega, \tag{23}$$

In order to obtain the main result, it is necessary to make the following assumption:

(H3) $\mathrm{Re}(d\lambda/d\tau)|_{\tau=\tau_0} \neq 0$.

Taking the derivative of λ with respect to τ in (7), it is easy to obtain

$$\begin{aligned} & \left(4\lambda^3 + 3h_1\lambda^2 + 2h_2\lambda + h_3\right)\frac{d\lambda}{d\tau} \\ & \quad + \left(3h_5\lambda^2 + 2h_6\lambda + h_7\right)e^{-\lambda\tau}\frac{d\lambda}{d\tau} \\ & \quad - \left(h_5\lambda^3 + h_6\lambda^2 + h_7\lambda + h_8\right)e^{-\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right) = 0, \end{aligned} \tag{24}$$

and it may be rewritten as

$$\begin{aligned} \frac{d\lambda}{d\tau} = {} & \left(\left(h_5\lambda^4 + h_6\lambda^3 + h_7\lambda^2 + h_8\lambda\right)e^{-\lambda\tau}\right) \\ & \times \left(4\lambda^3 + 3h_1\lambda^2 + 2h_2\lambda + h_3 \right. \\ & \quad + \left(3h_5\lambda^2 + 2h_6\lambda + h_7\right)e^{-\lambda\tau} \\ & \quad \left. - \left(h_5\lambda^3 + h_6\lambda^2 + h_7\lambda + h_8\right)\tau e^{-\lambda\tau}\right)^{-1}, \end{aligned} \tag{25}$$

or equivalently, we have

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \left(4\lambda^3 + 3h_1\lambda^2 + 2h_2\lambda + h_3\right.\\ &\quad + \left(3h_5\lambda^2 + 2h_6\lambda + h_7\right)e^{-\lambda\tau}\\ &\quad \left.- \left(h_5\lambda^3 + h_6\lambda^2 + h_7\lambda + h_8\right)\tau e^{-\lambda\tau}\right)\\ &\quad \times\left(\left(h_5\lambda^4 + h_6\lambda^3 + h_7\lambda^2 + h_8\lambda\right)e^{-\lambda\tau}\right)^{-1}\\ &= \left(4\lambda^3 + 3h_1\lambda^2 + 2h_2\lambda + h_3\right.\\ &\quad \left.+ \left(3h_5\lambda^2 + 2h_6\lambda + h_7\right)e^{-\lambda\tau}\right)\\ &\quad \times\left(h_5\lambda^4 + h_6\lambda^3 + h_7\lambda^2 + h_8\lambda\right)^{-1} - \frac{\tau}{\lambda}.\end{aligned} \tag{26}$$

Taking $\lambda = i\omega$ into the above equation, we get

$$\begin{aligned}&\left(\frac{d\lambda}{d\tau}\right)^{-1}\\ &= \left(-4i\omega^3 - 3h_1\omega^2 + 2h_2 i\omega + h_3\right.\\ &\quad \left.+ \left(-3h_5\omega^2 + 2h_6 i\omega + h_7\right)(\cos\omega\tau - i\sin\omega\tau)\right)\\ &\quad \times\left(h_5\omega^4 - h_6 i\omega^3 - h_7\omega^2 + h_8 i\omega\right)^{-1} - \frac{\tau}{i\omega}\\ &= \left(-3h_1\omega^2 + h_3 - \left(3h_5\omega^2 - h_7\right)\cos\omega\tau + 2h_6\sin\omega\tau\right)\\ &\quad \times\left(h_5\omega^4 - h_7\omega^2 + i\left(h_8\omega - h_6\omega^3\right)\right)^{-1}\\ &\quad + \left(i\left[-4\omega^3 + 2h_2\omega + 2h_6\omega\cos\omega\tau\right.\right.\\ &\quad \left.\left.+ \left(3h_5\omega^2 - h_7\right)\sin\omega\tau\right]\right)\\ &\quad \times\left(h_5\omega^4 - h_7\omega^2 + i\left(h_8\omega - h_6\omega^3\right)\right)^{-1} - \frac{\tau}{i\omega}.\end{aligned} \tag{27}$$

Denote

$$Q = \left(h_5\omega^4 - h_7\omega^2\right)^2 + \left(h_8\omega - h_6\omega^3\right)^2 > 0. \tag{28}$$

Then we have

$$\begin{aligned}&Q\,\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\\ &= \left[-3h_1\omega^2 + h_3 - \left(3h_5\omega^2 - h_7\right)\cos\omega\tau + 2h_6\sin\omega\tau\right]\\ &\quad \times\left(h_5\omega^4 - h_7\omega^2\right)\\ &\quad + \left[-4\omega^3 + 2h_2\omega + 2h_6\omega\cos\omega\tau + \left(3h_5\omega^2 - h_7\right)\sin\omega\tau\right]\\ &\quad \times\left(h_8\omega - h_6\omega^3\right).\end{aligned} \tag{29}$$

Note that

$$\mathrm{Sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)|_{\tau=\tau_0}\right\} = \mathrm{Sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}|_{\tau=\tau_0}\right\}. \tag{30}$$

Now, we can employ a result from [35] to analyze (7).

Lemma 2 (see [35]). *Consider the exponential polynomial*

$$\begin{aligned}&P\left(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_m}\right)\\ &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)}\\ &\quad + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1}\\ &\quad + \cdots + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m},\end{aligned} \tag{31}$$

where $\tau_i \geq 0$ $(i = 1, 2, \dots, m)$ and $p_j^{(i)}$ $(i = 0, 1, \dots, m; j = 1, 2, \dots, n)$ are constants. As $(\tau_1, \tau_2, \dots, \tau_m)$ vary, the sum of the order of the zeros of P $(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$ on the open right half-plane can change only if a zero appears on or crosses the imaginary axis.

From Lemma 2, it is easy to obtain the following theorem.

Theorem 3. *Suppose that (H1), (H2), and (H3) hold. Then the following results hold true.*

(i) *The positive equilibrium E^* of system (4) (x_1^*, x_2^*, u^*, y^*) is asymptotically stable for $\tau \in [0, \tau_0)$.*

(ii) *The positive equilibrium E^* of system (4) undergoes a Hopf bifurcation when $\tau = \tau_0$. That is system (4) has a periodic solution bifurcating from the positive equilibrium E^* near $\tau = \tau_0$.*

3. Direction and Stability of the Hopf Bifurcation

In this section, we will derive the explicit formulae for determining the properties of the Hopf bifurcation at critical value of τ_0 by using the normal form and the center manifold theory [35]. Throughout this section, we always assume that system (4) undergoes Hopf bifurcation at the positive equilibrium E^* for $\tau = \tau_0$, and then $\pm\omega_0$ is the corresponding purely imaginary roots of the characteristic equation at the positive equilibrium E^*.

Let $u_1 = x_1 - x_1^*$, $u_2 = x_2 - x_2^*$, $u_3 = u - u^*$, $u_4 = y - y^*$, $\overline{u_i}(t) = u_i(\tau t)$, and $\tau = \tau_0 + \mu$, and dropping the bars for simplification of notations, then system (4) is transformed into FDE defined in $C = C([-1, 0], R^4)$ as

$$u'(t) = L_\mu(u_t) + g(\mu, x_t), \tag{32}$$

where $u_t = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in R^4$, $L_\mu : C \to R$, $g : R \times C \to R$, and

$$\begin{aligned} L_\mu(\phi) &= (\tau_0+\mu)\begin{pmatrix} -d_1 & r & 0 & 0 \\ 0 & -d_2-b_1y^* & 0 & -b_1x_2^* \\ 0 & \beta & -\beta & 0 \\ 0 & 0 & b_2y^* & -\alpha y^* \end{pmatrix} \\ &\quad\times \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \\ \phi_4(0) \end{pmatrix} \\ &\quad + (\tau_0+\mu)\begin{pmatrix} 0 & -re^{-d_1\tau} & 0 & 0 \\ 0 & re^{-d_1\tau} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \\ \phi_4(-1) \end{pmatrix}, \end{aligned} \tag{33}$$

$$g(\mu,\phi) = (\tau_0+\mu)\begin{pmatrix} 0 \\ -b_1\phi_2(0)\phi_4(0) \\ 0 \\ b_2\phi_4(0)\phi_3(0) - \alpha\phi_4(0)^2 \end{pmatrix}, \tag{34}$$

where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta))^T \in C$. By the Riesz representation theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \quad \text{for } \phi \in C. \tag{35}$$

In fact, we can choose

$$\begin{aligned} \eta(\theta,\mu) &= (\tau_0+\mu) \\ &\quad\times \begin{pmatrix} -d_1 & r & 0 & 0 \\ 0 & -d_2-b_1y^* & 0 & -b_1x_2^* \\ 0 & \beta & -\beta & 0 \\ 0 & 0 & b_2y^* & -\alpha y^* \end{pmatrix}\delta(\theta) \\ &\quad + (\tau_0+\mu)\begin{pmatrix} 0 & -re^{-d_1(\tau_0+\mu)} & 0 & 0 \\ 0 & re^{-d_1(\tau_0+\mu)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\delta(\theta+1), \end{aligned} \tag{36}$$

where δ is the Dirac delta function. For $\phi \in C^1([-1, 0], R^4)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0, \end{cases} \tag{37}$$

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ g(\mu,\phi), & \theta = 0. \end{cases}$$

Then system (32) is equivalent to

$$u'(t) = A(\mu)u_t + R(\mu)u_t, \tag{38}$$

where $u_t = u(t+\theta)$, $\theta \in [-1, 0]$. For $\psi \in C^1([0, 1], \nu(R^4)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0, 1], \\ \displaystyle\int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0, \end{cases} \tag{39}$$

and a bilinear inner product

$$\begin{aligned} \langle\psi(s), \phi(\theta)\rangle &= \overline{\psi}(0)\phi(0) \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)\, d\eta(\theta)\phi(\xi)\, d\xi, \end{aligned} \tag{40}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in Section 2, we know that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively.

Suppose that $q(\theta) = (1, q_1, q_2, q_3)^T e^{i\theta\omega_0\tau_0}$ is the eigenvectors of $A(0)$ corresponding to $i\omega_0\tau_0$. Then $A(0)q(\theta) = i\omega_0\tau_0 q(\theta)$. It follows from the definition of $A(0)$ and $\eta(\theta,\mu)$ that

$$\begin{aligned} &\tau_0\begin{pmatrix} -d_1 & r & 0 & 0 \\ 0 & -d_2-b_1y^* & 0 & -b_1x_2^* \\ 0 & \beta & -\beta & 0 \\ 0 & 0 & b_2y^* & -\alpha y^* \end{pmatrix}q(0) \\ &\quad + \tau_0\begin{pmatrix} 0 & -re^{-d_1\tau_0} & 0 & 0 \\ 0 & re^{-d_1\tau_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}q(-1) = i\omega_0\tau_0 q(0). \end{aligned} \tag{41}$$

Because of $q(-1) = q(0)e^{-i\omega_0\tau_0}$, then we get

$$\begin{aligned} &\begin{pmatrix} i\omega_0+d_1 & -r+re^{-d_1\tau_0}e^{-i\omega_0\tau_0} & 0 & 0 \\ 0 & i\omega_0+d_2+b_1y^*-re^{-d_1\tau_0}e^{-i\omega_0\tau_0} & 0 & b_1x_2^* \\ 0 & -\beta & i\omega_0+\beta & 0 \\ 0 & 0 & -b_2y^* & i\omega_0+\alpha y^* \end{pmatrix} \\ &\quad\times\begin{pmatrix} 1 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{42}$$

from which we obtain

$$\begin{aligned} q_1 &= \frac{i\omega_0 - d_1}{r(1-e^{-d_1\tau_0}e^{-i\omega_0\tau_0})}, \\ q_2 &= \frac{\beta(d_1 - i\omega_0)}{r(i\omega_0+\beta)(1-e^{-d_1\tau_0}e^{-i\omega_0\tau_0})}, \\ q_3 &= \frac{b_2y^*\beta(i\omega_0+d_1)}{r(i\omega_0+\beta)(i\omega_0+\alpha y^*)(1-e^{-d_1\tau_0}e^{-i\omega_0\tau_0})}. \end{aligned} \tag{43}$$

Similarly, let $q^*(\theta) = D(1, q_1^*, q_2^*, q_3^*)e^{i\theta\omega_0\tau_0}$ be the eigenvectors of A^* corresponding to $-i\omega_0\tau_0$, and according to the definition of A^* we have

$$\tau_0 \begin{pmatrix} -d_1 & 0 & 0 & 0 \\ r & -d_2 - b_1 y^* & \beta & 0 \\ 0 & 0 & -\beta & b_2 y^* \\ 0 & -b_1 x_2^* & 0 & -\alpha y^* \end{pmatrix} q^*(0) + \tau_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ -re^{-d_1\tau_0} & re^{-d_1\tau_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} q^*(-1) = -i\omega_0\tau_0 q^*(0). \tag{44}$$

Note that $q^*(-1) = q^*(0)e^{i\omega_0\tau_0}$. Then we get

$$\begin{pmatrix} -i\omega_0 + d_1 & 0 & 0 & 0 \\ re^{-d_1\tau_0}e^{i\omega_0\tau_0} - r & d_2 - i\omega_0 + b_1 y^* - re^{-d_1\tau_0}e^{i\omega_0\tau_0} & -\beta & 0 \\ 0 & 0 & \beta - i\omega_0 & -b_2 y^* \\ 0 & b_1 x_2^* & 0 & \alpha y^* - i\omega_0 \end{pmatrix} \times \begin{pmatrix} 1 \\ q_1^* \\ q_2^* \\ q_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{45}$$

from which we can obtain

$$\begin{aligned} q_1^* &= \left((i\omega_0 - \alpha y^*)(i\omega_0 + \beta)\left(1 - e^{-d_1\tau_0}e^{i\omega_0\tau_0}\right)\right) \\ &\quad \times \left((i\omega_0 + \beta)(i\omega_0 - \alpha y^*)\right. \\ &\quad \times \left(-i\omega_0 + d_2 + b_1 y^* - re^{-d_1\tau_0}e^{-i\omega_0\tau_0}\right) \\ &\quad \left. - \beta b_1 x_2^* b_2 y^*\right)^{-1}, \\ q_2^* &= \left(rb_1 x_2^* b_2 y^* (i\omega_0 + \beta)\left(1 - e^{-d_1\tau_0}e^{i\omega_0\tau_0}\right)\right) \\ &\quad \times \left(\left(\omega_0^2 + \beta^2\right)(i\omega_0 - \alpha y^*)\right. \\ &\quad \times \left(-i\omega_0 + d_2 + b_1 y^* - re^{-d_1\tau_0}e^{-i\omega_0\tau_0}\right) \\ &\quad \left. - (\beta - i\omega_0)\beta b_1 x_2^* b_2 y^*\right)^{-1}, \\ q_3^* &= \left(rb_1 x_2^* (i\omega_0 + \beta)\left(1 - e^{-d_1\tau_0}e^{i\omega_0\tau_0}\right)\right) \\ &\quad \times \left((i\omega_0 + \beta)(i\omega_0 - \alpha y^*)\right. \\ &\quad \times \left(-i\omega_0 + d_2 + b_1 y^* - re^{-d_1\tau_0}e^{-i\omega_0\tau_0}\right) \\ &\quad \left. - \beta b_1 x_2^* b_2 y^*\right)^{-1}. \end{aligned} \tag{46}$$

By (40), we get

$$\begin{aligned} &\langle q^*(s), q(\theta)\rangle \\ &= \overline{D}\left(1, \overline{q}_1^*, \overline{q}_2^*, \overline{q}_3^*\right)(1, q_1, q_2, q_3)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{D}\left(1, \overline{q}_1^*, \overline{q}_2^*, \overline{q}_3^*\right) \\ &\quad \times e^{-i\omega_0\tau_0(\xi-\theta)} d\eta(\theta)(1, q_1, q_2, q_3)^T e^{i\omega_0\tau_0\xi} d\xi \\ &= \overline{D}\left\{1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* + q_3\overline{q}_3^* \right. \\ &\quad \left. - \int_{-1}^{0}\left(1, \overline{q}_1^*, \overline{q}_2^*, \overline{q}_3^*\right)\theta e^{i\omega_0\tau_0\theta} d\eta(\theta)(1, q_1, q_2, q_3)^T\right\} \\ &= \overline{D}\left[1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* + q_3\overline{q}_3^* \right. \\ &\quad \left. + \tau_0 r q_1\left(\overline{q}_1^* - 1\right)e^{-d_2\tau_0}e^{-i\omega_0\tau_0}\right]. \end{aligned} \tag{47}$$

Then we can choose $\overline{D}$ such that $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \overline{q}(\theta)\rangle = 0$.

Nest, we will use the ideas in [35] to compute the coordinates describing center manifold C_0 at $\mu = 0$. Define

$$\begin{aligned} z(t) &= \langle q^*, u_t\rangle, \\ W(t, \theta) &= u_t(\theta) - 2\,\mathrm{Re}\{z(t), q(\theta)\}. \end{aligned} \tag{48}$$

On the center manifold C_0, we have $W(t, \theta) = W(z(t), \overline{z}(t), \theta)$, and

$$W(z(t), \overline{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \ldots, \tag{49}$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of $q^*(s)$ and $\overline{q}(\theta)$. Note that W is real if u_t is real, and we only consider real solutions. For the solution $u_t \in C_0$ of (38), since $\mu = 0$ and (38), we have

$$\begin{aligned} z'(t) &= i\omega_0\tau_0 z + \overline{q}^*(0)\, g(0, W(z, \overline{z}, 0)) + 2\,\mathrm{Re}\{zq(0)\} \\ &= i\omega_0\tau_0 z + \overline{q}^*(0)\, g_0(z, \overline{z}). \end{aligned} \tag{50}$$

Then, the above equation can be denoted as

$$z'(t) = i\omega_0\tau_0 z(t) + f(z, \overline{z}), \tag{51}$$

where

$$\begin{aligned} f(z, \overline{z}) = \overline{q}^*(0)\, g_0(z, \overline{z}) &= f_{20}\frac{z^2}{2} + f_{11}z\overline{z} \\ &\quad + f_{02}\frac{\overline{z}^2}{2} + f_{21}\frac{z^2\overline{z}}{2} + \cdots. \end{aligned} \tag{52}$$

From (48) and (49), we have

$$\begin{aligned} u_t &= (u_{1t}(\theta), u_{2t}(\theta), u_{3t}(\theta), u_{4t}(\theta)) \\ &= W(t, \theta) + 2\,\mathrm{Re}\{z(t), q(\theta)\} \\ &= W(t, \theta) + zq(\theta) + \overline{z}\overline{q(\theta)}, \end{aligned} \tag{53}$$

and then we can obtain

$$u_{2t}(0) = W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + q_1 z + \overline{q_1}\,\overline{z} + O\left(\left|z,\overline{z}^3\right|\right),$$
$$u_{3t}(0) = W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0) z\overline{z} + W_{02}^{(3)}(0)\frac{\overline{z}^2}{2} + q_2 z + \overline{q_2}\,\overline{z} + O\left(\left|z,\overline{z}\right|^3\right),$$
$$u_{4t}(0) = W_{20}^{(4)}(0)\frac{z^2}{2} + W_{11}^{(4)}(0) z\overline{z} + W_{02}^{(4)}(0)\frac{\overline{z}^2}{2} + q_3 z + \overline{q_3}\,\overline{z} + O\left(\left|(z,\overline{z})\right|^3\right). \tag{54}$$

From the definition of $g(\mu, u_t)$, we have

$$f(z,\overline{z}) = \overline{q}^*(0)\, g_0(z,\overline{z}) = \tau_0\overline{D}\left(1,\overline{q}_1^*,\overline{q}_2^*,\overline{q}_3^*\right) \times \begin{pmatrix} 0 \\ -2b_1 u_{2t}(0)\, u_{4t}(0) \\ 0 \\ b_2 u_{3t}(0)\, u_{4t}(0) - \alpha u_{4t}^2(0) \end{pmatrix} \tag{55}$$

$$\begin{aligned} &= \tau_0\overline{D}\Big\{z^2\left[-2b_1\overline{q}_1^* q_1 q_3 + b_2\overline{q}_3^* q_2 q_3 - \alpha\overline{q}_3^* q_3^2\right] \\ &\quad + 2z\overline{z}\left[-2b_1\overline{q}_1^*\operatorname{Re}\{q_1\overline{q_3}\} + b_2\overline{q}_3^*\operatorname{Re}\{q_2\overline{q_3}\} - \alpha\overline{q}_3^*\operatorname{Re}\{q_3\overline{q_3}\}\right] \\ &\quad + \overline{z^2}\left[-b_1\overline{q}_1^*\overline{q_1 q_3} + b_2\overline{q}_3^*\overline{q_2 q_3} - \alpha\overline{q}_3^*\overline{q_3 q_3}\right] \\ &\quad + \frac{1}{2}z^2\overline{z}\Big[-b_1 q_1^*\left(W_{20}^{(2)}(0)\overline{q_3} + 2W_{11}^{(2)}(0) q_3 + 2W_{11}^{(4)}(0) q_1 + W_{20}^{(4)}(0)\overline{q_1}\right) \\ &\quad + b_2 q_3^*\left(W_{20}^{(3)}(0)\overline{q_3} + 2W_{11}^{(3)}(0) q_3 + 2W_{11}^{(4)}(0) q_2 + W_{20}^{(4)}(0)\overline{q_2}\right) \\ &\quad - \alpha q_3^*\left(2W_{20}^{(4)}(0)\overline{q_3} + 4W_{11}^{(4)}(0) q_3\right)\Big] + \cdots\Big\}. \end{aligned} \tag{56}$$

Comparing the coefficients with those of (52), we obtain

$$f_{20} = 2\tau_0\overline{D}\left[-b_1\overline{q}_1^* q_1 q_3 + b_2\overline{q}_3^* q_2 q_3 - \alpha\overline{q}_3^* q_3^2\right],$$
$$f_{11} = 2\tau_0\overline{D}\left[-b_1\overline{q}_1^*\operatorname{Re}\{q_1\overline{q_3}\} + b_2\overline{q}_3^*\operatorname{Re}\{q_2\overline{q_3}\} - \alpha\overline{q}_3^*\operatorname{Re}\{q_3\overline{q_3}\}\right],$$
$$f_{02} = 2\tau_0\overline{D}\left[-b_1\overline{q}_1^*\overline{q_1}\,\overline{q_3} + b_2\overline{q}_3^*\overline{q_2}\,\overline{q_3} - \alpha\overline{q}_3^*\overline{q_3^2}\right],$$
$$\begin{aligned} f_{21} = \tau_0\overline{D}\Big[&-b_1 q_1^*\left(W_{20}^{(2)}(0)\overline{q_3} + 2W_{11}^{(2)}(0) q_3 + 2W_{11}^{(4)}(0) q_1 + W_{20}^{(4)}(0)\overline{q_1}\right) \\ &+ b_2 q_3^*\left(W_{20}^{(3)}(0)\overline{q_3} + 2W_{11}^{(3)}(0) q_3 + 2W_{11}^{(4)}(0) q_2 + W_{20}^{(4)}(0)\overline{q_2}\right) \\ &- \alpha q_3^*\left(2W_{20}^{(4)}(0)\overline{q_3} + 4W_{11}^{(4)}(0) q_3\right)\Big]. \end{aligned} \tag{57}$$

In order to determine f_{21} we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (38) and (48), we have

$$\begin{aligned} \dot{W} &= \dot{u}_t - \dot{z}q - \dot{\overline{z}}\overline{q} \\ &= \begin{cases} A(0)W - 2\operatorname{Re}\{\overline{q}^*(0)\, g_0 q(\theta)\}, & \theta \in [-1,0), \\ A(0)W - 2\operatorname{Re}\{\overline{q}^*(0)\, g_0 q(\theta)\} + g_0, & \theta = 0, \end{cases} \\ &= A(0)W - H(z,\overline{z},\theta), \end{aligned} \tag{58}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\overline{z} + H_{02}\frac{\overline{z^2}}{2} + \cdots. \tag{59}$$

Note that on the center manifold C_0 near the origin

$$\dot{W} = W_z\dot{z} + W_{\overline{z}}\dot{\overline{z}}, \tag{60}$$

and thus we obtain

$$\left(A(0) - 2i\omega_0\tau_0\right) W_{20}(\theta) = -H_{20}(\theta),$$
$$A(0)\, W_{11}(\theta) = -H_{11}(\theta). \tag{61}$$

By (58), we know that, for $\theta \in [-1,0)$,

$$H(z,\overline{z},\theta) = -\overline{q}^*(0)\, g_0 q(\theta) - q_0^*\overline{g_0 q}(\theta) = -f q(\theta) - \overline{f}\,\overline{q(\theta)}. \tag{62}$$

Comparing the coefficients with those in (59), we get

$$H_{20}(\theta) = -f_{20} q(\theta) - \overline{f}_{02}\overline{q}(\theta),$$
$$H_{11}(\theta) = -f_{11} q(\theta) - \overline{f}_{11}\overline{q}(\theta). \tag{63}$$

From (61), (63), and the definition of A, we have

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + f_{20} q(\theta) + \overline{f}_{02}\overline{q}(\theta). \tag{64}$$

Noting that $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$, we get

$$W_{20}(\theta) = \frac{i f_{20}}{\omega_0\tau_0} q(0)\, e^{i\omega_0\tau_0\theta} + \frac{i\overline{f}_{02}}{3\omega_0\tau_0}\overline{q}(0)\, e^{-i\omega_0\tau_0\theta} + E_1 e^{2i\omega_0\tau_0\theta}, \tag{65}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_1^{(4)}) \in R^4$ is a constant vector.

Similarly, from (61) and (62), we can get

$$W_{11}(\theta)=\frac{i\overline{f}_{11}}{\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta}+E_2, \tag{66}$$

where $E_2=(E_2^{(1)},E_2^{(2)},E_2^{(3)},E_2^{(4)})\in R^4$ is a constant vector.

Next, we will find out E_1 and E_2. In fact, from the definition of A and (61), we can obtain

$$\int_{-1}^{0}\mathrm{d}\eta(\theta)W_{20}(\theta)=2i\omega_0\tau_0W_{20}(0)-H_{20}(0), \tag{67}$$

$$\int_{-1}^{0}\mathrm{d}\eta(\theta)W_{11}(\theta)=-H_{11}(0), \tag{68}$$

where $\eta(\theta)=\eta(0,\theta)$. From (59) and (61), when $\theta=0$ we have

$$\begin{aligned}H(z,\overline{z},0)&=-2\,\mathrm{Re}\{\overline{q}^*(0)g_0q(0)\}+g_0\\&=-\overline{q}^*(0)g_0q(0)-q^*(0)\overline{g_0}\,\overline{q}(0)+g_0\\&=-g(z,\overline{z})q(0)-\overline{g}(z,\overline{z})\overline{q}(0)+g_0.\end{aligned} \tag{69}$$

That is,

$$\begin{aligned}&H_{20}(0)\frac{z^2}{2}+H_{11}(0)z\overline{z}+H_{02}(0)\frac{\overline{z^2}}{2}\\&=-q(0)\left(f_{20}\frac{z^2}{2}+f_{11}z\overline{z}+f_{02}\frac{\overline{z^2}}{2}+\cdots\right)\\&\quad-\overline{q}(0)\left(\overline{f}_{20}\frac{\overline{z}^2}{2}+\overline{f}_{11}z\overline{z}+\overline{f_{02}}\frac{z^2}{2}+\cdots\right)+g_0.\end{aligned} \tag{70}$$

By (34), we have

$$g_0=\tau_0\begin{pmatrix}0\\-b_1u_{2t}(0)u_{4t}(0)\\0\\b_2u_{3t}(0)u_{4t}(0)-\alpha u_{4t}^2(0)\end{pmatrix}. \tag{71}$$

By (48) and (49), we have

$$\begin{aligned}u_t(\theta)&=W(t,\theta)+2\,\mathrm{Re}\{z(t)q(\theta)\}\\&=W(t,\theta)+z(t)q(\theta)+\overline{z}(t)\overline{q}(\theta)\\&=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+z(t)q(\theta)\\&\quad+\overline{z}(t)\overline{q}(\theta)+\cdots.\end{aligned} \tag{72}$$

Thus, we obtain

$$H_{20}(0)=-f_{20}q(0)-\overline{f}_{02}\overline{q}(0)+2\tau_0\begin{pmatrix}0\\-b_1q_1q_3\\0\\b_2q_2q_3-\alpha q_3^2\end{pmatrix}, \tag{73}$$

$$\begin{aligned}H_{11}(0)&=-f_{11}q(0)-\overline{f}_{11}\overline{q}(0)+2\tau_0\\&\quad\times\begin{pmatrix}0\\-b_1\,\mathrm{Re}\{q_1\overline{q_3}\}\\0\\b_2\,\mathrm{Re}\{q_2\overline{q_3}\}-\alpha\overline{q}_3^*|q_3|^2\end{pmatrix}.\end{aligned} \tag{74}$$

Note that

$$\begin{aligned}&\left(i\omega_0\tau_0I-\int_{-1}^{0}e^{i\omega_0\tau_0\theta}d\eta(\theta)\right)q(0)=0,\\&\left(-i\omega_0\tau_0I-\int_{-1}^{0}e^{-i\omega_0\tau_0\theta}d\eta(\theta)\right)\overline{q}(0)=0.\end{aligned} \tag{75}$$

Then, by substituting (65) and (73) into (67), we will get

$$\left(2i\omega_0\tau_0I-\int_{-1}^{0}e^{2i\omega_0\tau_0\theta}d\eta(\theta)\right)E_1=2\tau_0\begin{pmatrix}0\\-b_1q_1q_3\\0\\b_2q_2q_3-\alpha q_3^2\end{pmatrix}, \tag{76}$$

or equivalently,

$$\begin{aligned}&\begin{pmatrix}2i\omega_0+d_1&-r+re^{-d_1\tau_0}e^{-2i\omega_0\tau_0}&0&0\\0&2i\omega_0+d_2+b_1y^*-re^{-d_1\tau_0}e^{-2i\omega_0\tau_0}&0&b_1x_2^*\\0&-\beta&2i\omega_0+\beta&0\\0&0&-b_2y^*&2i\omega_0-\alpha y^*\end{pmatrix}E_1\\&=2\begin{pmatrix}0\\-b_1q_1q_3\\0\\b_2q_2q_3-\alpha q_3^2\end{pmatrix},\end{aligned} \tag{77}$$

from which we can get

$$\begin{aligned}&E_1\\&=2\begin{pmatrix}2i\omega_0+d_1&-r+re^{-d_1\tau_0}e^{-2i\omega_0\tau_0}&0&0\\0&2i\omega_0+d_2+b_1y^*-re^{-d_1\tau_0}e^{-2i\omega_0\tau_0}&0&b_1x_2^*\\0&-\beta&2i\omega_0+\beta&0\\0&0&-b_2y^*&2i\omega_0-\alpha y^*\end{pmatrix}^{-1}\\&\quad\times\begin{pmatrix}0\\-b_1q_1q_3\\0\\b_2q_2q_3-\alpha q_3^2\end{pmatrix}.\end{aligned} \tag{78}$$

Similarly, by substituting (66) and (74) into (68), we will get

$$\begin{aligned}&\begin{pmatrix}d_1&-r+re^{-d_1\tau_0}&0&0\\0&d_2+b_1y^*-re^{-d_1\tau_0}&0&b_1x_2^*\\0&-\beta&\beta&0\\0&0&-b_2y^*&-\alpha y^*\end{pmatrix}E_2\\&=2\begin{pmatrix}0\\-b_1\,\mathrm{Re}\{q_1\overline{q_3}\}\\0\\b_2\,\mathrm{Re}\{q_2\overline{q_3}\}-\alpha\,\mathrm{Re}\{q_3\overline{q_3}\}\end{pmatrix},\\&E_2=2\begin{pmatrix}d_1&-r+re^{-d_1\tau_0}&0&0\\0&d_2+b_1y^*-re^{-d_1\tau_0}&0&b_1x_2^*\\0&-\beta&\beta&0\\0&0&-b_2y^*&-\alpha y^*\end{pmatrix}^{-1}\\&\quad\times\begin{pmatrix}0\\-b_1\,\mathrm{Re}\{q_1\overline{q_3}\}\\0\\b_2\,\mathrm{Re}\{q_2\overline{q_3}\}-\alpha\,\mathrm{Re}\{q_3\overline{q_3}\}\end{pmatrix}\end{aligned} \tag{79}$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (65) and (66). Furthermore, f_{21} can be expressed by the parameters and delay. Thus, we can compute the following values:

$$\begin{aligned}
c_1(0) &= \frac{i}{2\omega_0\tau_0}\left(f_{20}f_{11} - 2|f_{11}|^2 - \frac{|f_{02}|^2}{3}\right) + \frac{f_{21}}{2}, \mu_2 \\
&= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \beta_2 \\
&= 2\operatorname{Re}\{c_1(0)\}, T_2 \\
&= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_0)\}}{\omega\tau_0},
\end{aligned} \tag{80}$$

which determine the qualities of bifurcating periodic solution in the center manifold at critical value τ_0; that is, μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau > \tau_0$ ($\tau < \tau_0$); β_2 determines the stability of the bifurcating periodic solution: the bifurcating periodic solution is stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines period of the bifurcating periodic solution: the period increases (decreases) if $T_2 > 0$ (< 0).

4. Discussion

In this paper, we propose a three-dimensional stage-structured predatory-prey model with discrete and distributed delay. Then, by introducing a new variable, the original system is transformed into an equivalent four-dimensional system (4). In Section 2, we analyze the existence and local stability of the positive equilibrium of the system (4). In fact, by use of the Routh-Hurwitz criterion, we know that the positive equilibrium E^* is locally asymptotically stable for $\tau = 0$. Then, by some computation, we get a threshold $\tau = \tau_0$ in Theorem 3. The results indicate that if $\tau < \tau_0$, the positive equilibrium E^* of system (4) is asymptotically stable; if $\tau = \tau_0$, the positive equilibrium E^* of system (4) will undergo a Hopf bifurcation. In Section 3, by use of normal form theory and central manifold argument, we establish the formulae for the direction and the stability of the Hopf bifurcation. Our theoretical results show that stage structure and delay play an important role in the dynamics of the system, and delay may lead to complicated dynamic behaviors, such as Hopf bifurcation.

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