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Research Article

Optimal Convergence Rates of Moving Finite Element Methods for Space-Time Fractional Differential Equations

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This paper studies the moving finite element methods for the space-time fractional differential equations. An optimal convergence rate of the moving finite element method is proved for the space-time fractional differential equations.

1. Introduction

Consider a time-dependent space fractional differential equation of the following form

$$\begin{gathered} {}_0D_t^{\alpha}u - p\,{}_aD_x^{\beta}u - q\,{}_xD_b^{\beta}u = f, \\ x \in \Omega := (a,b), \quad t \in I := (0,T], \end{gathered} \tag{1}$$

$$u(a,t) = 0, \quad u(b,t) = 0, \quad t \in I, \tag{2}$$

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{3}$$

where $0 < \alpha < 1$ and $1 < \beta < 2$, f and φ are given functions, ${}_0D_t^{\alpha}$ and ${}_aD_x^{\beta}$ represent left Caputo fractional differential operators for time and space, respectively, and ${}_xD_b^{\beta}$ denotes right Caputo fractional differential operator, p and q are two nonnegative constants satisfying $p + q = 1$.

For some nonlinear reaction terms $f = f(x,t,u)$, the above equation has finite-time blowup solution which means that the solution tends to infinity as time approaching to a finite time (see e.g., [1]). Moving mesh methods have great advantages in solving blowup problems (see, e.g., [2–9]). Therefore it is important to develop moving mesh methods for solving the fractional differential equations.

Although there are many references for developing and analyzing numerical methods on fixed mesh for solving fractional differential equations, the development of moving mesh methods for fractional differential equations is still in the early stage. Ma and Jiang [6] develop moving mesh collocation methods to solve nonlinear time fractional partial differential equations with blowup solutions. Jiang and Ma [10] analyze moving mesh finite element methods for time fractional partial differential equations and simulate the blow-up solutions. More recently, Ma et al. [11] provide a convergence analysis of moving finite element methods for space fractional differential equations with integer derivatives in time.

The convergence rates of moving finite element methods for integer partial differential equations are established by Bank et al. [12–14]. However, fractional derivatives in time will raise much challenge in the convergence analysis of moving finite element methods. The technique using interpolation in the paper [10] is not possible to derive the optimal convergence rates. In this paper, by introducing a fractional Ritz-projection operator, we obtain the optimal convergence rate which is consistent with the numerical predictions in the paper [10]. Moreover, we study the space-time fractional differential equations which are more complex than the time-fractional differential equations.

Throughout the paper, we use notation $A \lesssim B$ and $A \gtrsim B$ to denote $A \leq cB$ and $A \geq cB$, respectively, where C is a generic positive constant independent of any functions and numerical discretization parameters.

2. Preliminaries

Define left Riemann-Liouville fractional integral as

$$ {}_aD_x^{-\sigma}u(x) = \frac{1}{\Gamma(\sigma)}\int_a^x (x-\xi)^{\sigma-1}u(\xi)\,d\xi, \quad x>a,\ \sigma>0, \tag{4}$$

where $a \in \mathbb{R}$ or $a=-\infty$, and right Riemann-Liouville fractional integral as

$$ {}_xD_b^{-\sigma}u(x) = \frac{1}{\Gamma(\sigma)}\int_x^b (x-\xi)^{\sigma-1}u(\xi)\,d\xi, \quad x<b,\ \sigma>0, \tag{5}$$

where $b\in\mathbb{R}$ or $b=+\infty$. The Caputo left and right fractional derivatives are defined by, respectively,

$$\begin{aligned} {}_aD_x^{\mu}u(x) &= {}_aD_x^{-\sigma}D^n u(x), \quad \sigma = n-\mu,\ n-1\le\mu<n, \\ {}_xD_b^{\mu}u(x) &= {}_xD_b^{-\sigma}D^n u(x), \quad \sigma = n-\mu,\ n-1\le\mu<n. \end{aligned} \tag{6}$$

Define a functional space $H_0^\mu(\Omega)$, $\mu>0$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{H^\mu(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + \left\||\omega|^\mu \mathscr{F}(\tilde{u})\right\|_{L^2(\mathbb{R})}^2\right)^{1/2}, \tag{7}$$

where $\mathscr{F}(\tilde{u})$ denotes the Fourier transform of $\tilde{u}$, and $\tilde{u}$ is the extension of u by zero outside of Ω.

Let $\gamma := \beta/2$. Then, the variational form of problem (1) with boundary conditions (2) and initial condition (3) is given by the following (see [15] for the derivation). Find $u\in H_0^\gamma(\Omega)$ such that

$$({}_0D_t^\alpha u, v) + B(u,v) = F(v), \quad \forall v\in H_0^\gamma(\Omega), \tag{8}$$

$$(u(x,0),v) = (\varphi(x),v), \quad \forall v\in H_0^\gamma(\Omega), \tag{9}$$

where

$$\begin{aligned} B(u,v) &:= p\left\langle {}_aD_x^\gamma u, {}_xD_b^\gamma v\right\rangle + q\left\langle {}_xD_b^\gamma u, {}_aD_x^\gamma v\right\rangle, \\ F(v) &:= \langle f, v\rangle, \end{aligned} \tag{10}$$

where (u,v) denotes L_2 inner product, $\langle\cdot,\cdot\rangle$ denotes the duality pairing of $H^{-\mu}(\Omega)$, and $H_0^\mu(\Omega)$, $\mu\ge 0$.

The properties of the bilinear form $B(\cdot,\cdot)$ are given by the following Lemma 1 whose proof can be found in [15].

Lemma 1. *The bilinear form $B(\cdot,\cdot)$ satisfies the following coercive and continuous properties over space $H_0^\gamma(\Omega)$:*

$$B(u,u) \gtrsim \|u\|_{H^\gamma(\Omega)}^2, \quad \forall u\in H_0^\gamma(\Omega), \tag{11}$$

$$|B(u,v)| \lesssim \|u\|_{H^\gamma(\Omega)}\|v\|_{H^\gamma(\Omega)}, \quad \forall u,v\in H_0^\gamma(\Omega). \tag{12}$$

3. Convergence Analysis of Moving Finite Element Method

Define a temporal mesh

$$\begin{aligned} &0\equiv t_0 < t_1 < \cdots < t_M \equiv T, \\ &\Delta t_n := t_n - t_{n-1}, \quad n=1,\ldots,M,\ \Delta t = \max_{1\le n\le M}\Delta t_n. \end{aligned} \tag{13}$$

Define spatial mesh (moving mesh) at time t_n,

$$\begin{aligned} &a\equiv x_0^n < x_1^n < \cdots < x_N^n \equiv b, \quad n=0,1,\ldots,M, \\ &h_k^n := x_k^n - x_{k-1}^n, \quad k=1,\ldots,N, \\ &h^n := \max_{1\le k\le N} h_k^n, \qquad h_n := \max_{0\le\ell\le n} h^\ell. \end{aligned} \tag{14}$$

Define a finite element space $\mathscr{V}^n \subset H_0^\gamma(\Omega)$ on the above moving mesh as

$$\mathscr{V}^n := \left\{v\in H_0^\gamma(\Omega)\cap C^0(\Omega) : v|_{[x_{k-1}^n, x_k^n]} \in P_{m-1}\right\}, \tag{15}$$

where P_{m-1} denotes the space of polynomials of degree less than or equal to $m-1$.

Then, the moving finite element method for the proposed problems is defined as follows: Find $U^n\in\mathscr{V}^n\subset H_0^\gamma(\Omega)$, for $n=1,\ldots,M$, such that

$$\begin{aligned} &\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^n b_k^n\left(U^k(x)-U^{k-1}(x),v\right) + B\left(U^n(x),v\right) \\ &\quad = F^n(v), \quad \forall v\in\mathscr{V}^n, \end{aligned} \tag{16}$$

$$\left(U^0(x),v\right) = (\varphi(x),v), \quad \forall v\in\mathscr{V}^0, \tag{17}$$

where $F^n(v) := \langle f(\cdot,t_n),v\rangle$ and

$$b_k^n = \frac{1}{\Delta t_k}\int_{t_{k-1}}^{t_k}\frac{ds}{(t_n-s)^\alpha}, \quad 1\le k\le n. \tag{18}$$

In the scheme (16), $B(U^n,v)$ is the discretization of $B(u,v)$, and

$$\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^n b_k^n\left(U^k(x)-U^{k-1}(x),v\right) \tag{19}$$

is the discretization of the time-fractional derivative $({}_0D_t^\alpha u, v)$ in (8). To do the convergence analysis, we introduce a fractional Ritz projection operator (an analog of the standard one in [16]), $R_n : H_0^\gamma(\Omega) \to \mathscr{V}^n$ defined via, for $u\in H_0^\gamma(\Omega)$,

$$B(u-R_n u, v) = 0, \quad \forall v\in\mathscr{V}^n. \tag{20}$$

For the fractional Ritz projection operator we have the following estimation—Lemma 2.

Lemma 2. *For the fractional Ritz projection operator defined by (20) and $u\in H_0^\gamma(\Omega)\cap H^r(\Omega)$ ($\gamma\le r\le m$), one has the following estimation:*

$$\|u-R_n u\|_{L^2(\Omega)} \lesssim (h^n)^r\|u\|_{H^r(\Omega)}. \tag{21}$$

Proof. The proof of this lemma can be obtained by simply modifying the proof for Theorem 4.4 in [15]. □

We will also need the following lemma (see [10]) for proving our main results.

Lemma 3. *Suppose that positive numbers* ε_n, $n = 0, 1, \ldots, M$, *satisfy*

$$b_n^n \varepsilon_n \leq \sum_{k=2}^{n} \left(b_k^n - b_{k-1}^n\right) \varepsilon_{k-1} + b_1^n \mu + \kappa, \quad n = 1, \ldots, M, \quad (22)$$

where b_k^n, $k = 1, \ldots, n$, *are given by* (18), *and* κ, μ *are positive numbers. Then we have*

$$\varepsilon_n \leq \mu + \frac{\kappa}{b_1^n}, \quad n = 1, \ldots, M. \quad (23)$$

Proof. The proof can be found in [10, Lemma 2.4]. □

Theorem 4. *Assume that the solution of* (1)–(3) *satisfies* $u \in H_0^{\gamma}(\Omega) \cap H^r(\Omega)$ $(\gamma \leq r \leq m)$. *Then, the convergence estimation for the moving finite element method* (16)-(17) *is given by, for* $n = 1, \ldots, M$,

$$\begin{aligned} \left\|u(\cdot, t_n) - U^n\right\|_{L^2(\Omega)} \lesssim{} & h_n^r \left[\max_{0\leq t\leq T} \|u(x,t)\|_{H^r(\Omega)} \right. \\ & \left. + \max_{0\leq t\leq T} \left\|u_t(x,t)\right\|_{H^r(\Omega)}\right] \\ & + (\Delta t)^{2-\alpha} \max_{0\leq t\leq T} \left\|u_{tt}(x,t)\right\|_{L^2(\Omega)}. \end{aligned} \quad (24)$$

Proof. Define the local truncation error as

$$\mathcal{T}^n(x) := \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} b_k^n \left(u(x, t_k) - u(x, t_{k-1})\right) - {}_0D_t^{\alpha} u, \quad (25)$$

where $u(x,t)$ is the exact solution. From [10], we can derive that

$$\left\|\mathcal{T}^n\right\|_{L^2(\Omega)} \lesssim (\Delta t)^{2-\alpha} \max_{0\leq t\leq T} \left\|u_{tt}(x,t)\right\|_{L^2(\Omega)}. \quad (26)$$

From (8) we have the identity

$$\begin{aligned} & \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} b_k^n \left(u(\cdot, t_k) - u(\cdot, t_{k-1}), v\right) + B\left(u(\cdot, t_n), v\right) \\ & \quad = \left(\mathcal{T}^n, v\right) + F^n(v), \quad \forall v \in \mathcal{V}^n. \end{aligned} \quad (27)$$

Let $e^n = u(\cdot, t_n) - U^n$. Then subtracting (16) by (27) gives the error equation

$$\frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} b_k^n \left(e^k - e^{k-1}, v\right) + B\left(e^n, v\right) = \left(\mathcal{T}^n, v\right), \quad \forall v \in \mathcal{V}^n. \quad (28)$$

Define

$$\sigma^n := R_n u(\cdot, t_n) - U^n, \qquad \eta^n := R_n u(\cdot, t_n) - u(\cdot, t_n), \quad (29)$$

where R_n is the fractional Ritz projection operator defined by (20). Then,

$$e^n = \sigma^n - \eta^n. \quad (30)$$

Using (20), which tells us that $B(\eta^n, v) = 0$ for all $v \in \mathcal{V}^n$, we rewrite the error equation (28) as

$$\begin{aligned} & b_n^n \left(\sigma^n, v\right) + \Gamma(1-\alpha) B\left(\sigma^n, v\right) \\ & \quad = \sum_{k=2}^{n} \left(b_k^n - b_{k-1}^n\right) \left(\sigma^{k-1}, v\right) + b_1^n \left(\sigma^0, v\right) \\ & \qquad + \sum_{k=1}^{n} b_k^n \left(\eta^k - \eta^{k-1}, v\right) \\ & \qquad + \Gamma(1-\alpha) \left(\mathcal{T}^n, v\right), \quad \forall v \in \mathcal{V}^n. \end{aligned} \quad (31)$$

Choosing $v = \sigma^n$ in (31) and using Cauchy-Schwartz inequality with noting that $B(\sigma^n, \sigma^n) \geq 0$ (see (11)), we get

$$\begin{aligned} b_n^n \left\|\sigma^n\right\|_{L^2(\Omega)} \leq & \sum_{k=2}^{n} \left(b_k^n - b_{k-1}^n\right) \\ & \times \left\|\sigma^{k-1}\right\|_{L^2(\Omega)} + b_1^n \left\|\sigma^0\right\|_{L^2(\Omega)} \\ & + \sum_{k=1}^{n} b_k^n \left\|\eta^k - \eta^{k-1}\right\|_{L^2(\Omega)} \\ & + \Gamma(1-\alpha) \left\|\mathcal{T}^n\right\|_{L^2(\Omega)}. \end{aligned} \quad (32)$$

Using Lemma 3, we obtain that

$$\begin{aligned} \left\|\sigma^n\right\|_{L^2(\Omega)} \leq & \left\|\sigma^0\right\|_{L^2(\Omega)} \\ & + T \left[\sum_{k=1}^{n} b_k^n \left\|\eta^k - \eta^{k-1}\right\|_{L^2(\Omega)} \right. \\ & \left. + \Gamma(1-\alpha) \left\|\mathcal{T}^n\right\|_{L^2(\Omega)}\right]. \end{aligned} \quad (33)$$

Now we estimate the error

$$\begin{aligned} & \left\|\eta^k - \eta^{k-1}\right\|_{L^2(\Omega)} \\ & = \left\|\left(R_k u(\cdot, t_k) - u(\cdot, t_k)\right) - \left(R_{k-1} u(\cdot, t_{k-1}) - u(\cdot, t_{k-1})\right)\right\|_{L^2(\Omega)} \\ & = \left\|\left(R_k \left(u(\cdot, t_k) - u(\cdot, t_{k-1})\right)\right) - \left(u(\cdot, t_k) - u(\cdot, t_{k-1})\right) + \left(R_k u(\cdot, t_{k-1}) - R_{k-1} u(\cdot, t_{k-1})\right)\right\|_{L^2(\Omega)} \\ & \leq \left\|\left(R_k \left(u(\cdot, t_k) - u(\cdot, t_{k-1})\right)\right) - \left(u(\cdot, t_k) - u(\cdot, t_{k-1})\right)\right\|_{L^2(\Omega)} \\ & \quad + \left\|\left(R_k u(\cdot, t_{k-1}) - R_{k-1} u(\cdot, t_{k-1})\right)\right\|_{L^2(\Omega)}. \end{aligned} \quad (34)$$

Using Taylor theorem and Lemma 2, we have

$$\begin{aligned} & \left\|\left(R_k \left(u(\cdot, t_k) - u(\cdot, t_{k-1})\right) - \left(u(\cdot, t_k) - u(\cdot, t_{k-1})\right)\right)\right\|_{L^2(\Omega)} \\ & \quad \lesssim \Delta t_k \left(h^k\right)^r \left\|u_t(\cdot, t_k)\right\|_{H^r(\Omega)}. \end{aligned} \quad (35)$$

If the fixed spatial mesh is taken, then $R_k \equiv R_{k-1}$ and thereby

$$\left\|\left(R_k u\left(\cdot, t_{k-1}\right) - R_{k-1} u\left(\cdot, t_{k-1}\right)\right)\right\|_{L^2(\Omega)} = 0. \quad (36)$$

Therefore, for moving spatial mesh, it is reasonable to assume that

$$\begin{aligned} &\left\|\left(R_k u\left(\cdot, t_{k-1}\right) - R_{k-1} u\left(\cdot, t_{k-1}\right)\right)\right\|_{L^2(\Omega)} \\ &\quad \lesssim \Delta t_k \left(h^k\right)^r \left\|u\left(\cdot, t_{k-1}\right)\right\|_{H^r(\Omega)}, \end{aligned} \quad (37)$$

which is verified numerically by examples in the next section. In addition subtracting (17) by (9) gives that

$$\left(e^0, v\right) = 0, \quad \forall v \in \mathscr{V}^0. \quad (38)$$

Taking $v = \sigma^0$ into (38) we derive that

$$\left\|\sigma^0\right\|^2_{L^2(\Omega)} = \left(\eta^0, \sigma^0\right) \leq \left\|\eta^0\right\|_{L^2(\Omega)} \left\|\sigma^0\right\|_{L^2(\Omega)}. \quad (39)$$

Consequently, it follows from Lemma 2 that

$$\left\|\sigma^0\right\|_{L^2(\Omega)} \leq \| \eta^0 \|_{L^2(\Omega)} \lesssim \left(h^0\right)^r \|u(x,0)\|_{H^r(\Omega)}. \quad (40)$$

Combining (26), (34), (35), (37), and (40) into (33) gives that

$$\begin{aligned} \left\|\sigma^n\right\|_{L^2(\Omega)} &\lesssim h_n^r \left[\max_{0\leq t\leq T} \|u(x,t)\|_{H^r(\Omega)} + \max_{0\leq t\leq T} \left\|u_t(x,t)\right\|_{H^r(\Omega)}\right] \\ &\quad + (\Delta t)^{2-\alpha} \max_{0\leq t\leq T} \left\|u_{tt}(x,t)\right\|_{L^2(\Omega)}. \end{aligned} \quad (41)$$

Finally, by applying Lemma 2 and (41) to

$$\left\|e^n\right\|_{L^2(\Omega)} \leq \left\|\eta^n\right\|_{L^2(\Omega)} + \left\|\sigma^n\right\|_{L^2(\Omega)}, \quad (42)$$

which is led by the triangle inequality, we complete the proof of this theorem. □

Remark 5. In the above proof, we use assumption (37). Now we give comments on the assumption. For fixed meshed, the finite element spaces for time level t_{k-1} and t_k are equal. Therefore, the Ritz-projections of u on the finite element spaces remain unchanged and, thus, the left-hand side of (37) is zero, that is,

$$\left\|\left(R_k u\left(\cdot, t_{k-1}\right) - R_{k-1} u\left(\cdot, t_{k-1}\right)\right)\right\|_{L^2(\Omega)} = 0. \quad (43)$$

For moving spatial mesh, the finite element spaces for time level t_{k-1} and t_k are not the same and the different structure highly depends on the mesh movement. However, the difference between the adjacent finite element spaces will not be significant unless the mesh movement is too fast. Therefore, it is reasonable to assume the inequality (37) holds.

For integer partial differential equations, assumptions on the mesh movement are generally required for proving the optimal convergence rates for moving finite element methods (see, e.g., [12–14]) and moving finite difference methods (see, e.g., [17]). Not surprisingly, conditions on the mesh movement are needed to prove the optimal convergence rates for moving mesh methods for the fractional differential equations. Numerical examples in the next section show that if the mesh satisfies the condition (44), which is normally used in the papers addressing the convergence analysis of moving mesh methods (see, e.g., [17, 18]), then (37) is verified.

4. Numerical Studies of Fractional Ritz Projections

In this section, we verify assumption (37) via numerical examples. To this end, we calculate the fractional Ritz projection (defined by (20)) for a given function $g(x)$.

Example 6. Let $g(x) = x^{\mu_1} - x^{\mu_2}$, μ_1, $\mu_2 > 0$, the moving meshes $\{x_k^n\}_{k=0}^N$ be generated by de Boor' algorithm [19] based on equidistribution principle and satisfy

$$\left|h_k^{n+1} - h_k^n\right| \lesssim \Delta t_{n+1} \min\left(h_k^{n+1}, h_k^n\right), \quad (44)$$

which is often used in the analysis (see, e.g., [17, 18]). The bilinear form is given by

$$B(u,v) := \frac{1}{2}\left\langle {}_0D_x^\gamma u, {}_xD_1^\gamma v\right\rangle + \frac{1}{2}\left\langle {}_xD_1^\gamma u, {}_0D_x^\gamma v\right\rangle. \quad (45)$$

We calculate the fractional Ritz projection $R_n g$ on the 1st-order FEM spaces $\mathscr{V}^n$ and verify the error estimation (37).

On meshes $\{x_k^n\}_{k=0}^N$, we construct piecewise linear finite element spaces

$$\mathscr{V}^n = \text{span}\left\{\phi_k^n(x), k = 1, \ldots, N-1\right\}, \quad (46)$$

where $\phi_k^n(x)$, $k = 1, \ldots, N-1$ are hat functions. So the fractional Ritz projection of function g on the finite element spaces $\mathscr{V}^n$ can be written as

$$R_n g(x) = \sum_{k=1}^{N-1} g_k^n \phi_k^n(x). \quad (47)$$

Inserting (47) into (20) gives that

$$B\left(g - \sum_{k=1}^{N-1} g_k^n \phi_k^n(x), v\right) = 0, \quad \forall v \in \mathscr{V}^n. \quad (48)$$

Thus, we may obtain a system of algebraic equations by taking $v = \phi_i^n(x)$, $i = 1, \ldots, N-1$:

$$\mathbf{A}\mathbf{g}^n = \mathbf{b}, \quad (49)$$

for unknown vector

$$\mathbf{g}^n := \left(g_1^n, \ldots, g_{N-1}^n\right)^T, \quad (50)$$

with matrix $\mathbf{A}$ and $\mathbf{b}$ given by

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} B\left(\phi_1^n, \phi_1^n\right) & \cdots & B\left(\phi_{N-1}^n, \phi_1^n\right) \\ B\left(\phi_1^n, \phi_2^n\right) & \cdots & B\left(\phi_{N-1}^n, \phi_2^n\right) \\ \vdots & \vdots & \vdots \\ B\left(\phi_1^n, \phi_{N-1}^n\right) & \cdots & B\left(\phi_{N-1}^n, \phi_{N-1}^n\right) \end{pmatrix}, \\ \mathbf{b} &= \left(B\left(g, \phi_1^n\right), B\left(g, \phi_2^n\right), \ldots, B\left(g, \phi_{N-1}^n\right)\right)^T. \end{aligned} \quad (51)$$

We check the rate (37) in the following way: For fixed time meshsize $\Delta t_n \equiv 1/M$ (M fixed), calculate the space rate for varying N,

$$\text{Rate for space} := \left|\frac{\log(\text{Error}(N)/\text{Error}(N-1))}{\log((N-1)/N)}\right|, \quad (52)$$

Table 1: Rate for space for $g(x) = x^{\mu_1} - x^{\mu_2}$ with $\mu_1 = 7/2$, $\mu_2 = 7/3$ for Example 6 using 1st-order FEM.

N	$\gamma = 0.4$		$\gamma = 0.7$	
	Error	Rate	Error	Rate
8	$2.2257e-3$	—	$2.3272e-3$	—
16	$4.9491e-4$	2.1	$5.4391e-4$	2.1
32	$1.3344e-4$	1.9	$1.4496e-4$	1.9
64	$3.0457e-5$	2.1	$3.3110e-5$	2.1
128	$7.4518e-6$	2.0	$7.9799e-6$	2.1
256	$1.8840e-6$	2.0	$1.9807e-6$	2.0

Table 2: Rate for time for $g(x) = x^{\mu_1} - x^{\mu_2}$ with $\mu_1 = 7/2$, $\mu_2 = 7/3$ for Example 6 using 1st-order FEM.

M	$\gamma = 0.4$		$\gamma = 0.7$	
	Error	Rate	Error	Rate
32	$7.8436e-5$	—	$7.8611e-5$	—
64	$4.2750e-5$	0.9	$4.2814e-5$	0.9
128	$2.2223e-5$	0.9	$2.2225e-5$	0.9
256	$1.1253e-5$	1.0	$1.1265e-5$	1.0
512	$5.6105e-6$	1.0	$5.6164e-6$	1.0
1024	$2.8238e-6$	1.0	$2.8266e-6$	1.0

for fixed N, calculate the time rate for varying time meshsize $\Delta t_n = 1/M$,

$$\text{Rate for time} := \left| \frac{\log(\text{Error}(M)/\text{Error}(M-1))}{\log((M-1)/M)} \right|. \quad (53)$$

From Tables 1 and 2, we can see that the convergence order for space is 2 and the convergence order for time is 1, which are consistent with (37) where $r = 2$ for the use of linear finite element methods.

Example 7. We calculate the fractional Ritz projection for function $g(x) = x^{\mu_1} - x^{\mu_2}$, $\mu_1, \mu_2 > 0$ on the following constructed 3rd-order FEM spaces $\mathcal{V}^n$. Also, like Example 6, we restrict the moving meshes $\{x_k^n\}_{k=0}^N$ to satisfy (44) and we use the bilinear form

$$B(u,v) := \frac{1}{2}\left\langle {}_0D_x^{\gamma} u, {}_x D_1^{\gamma} v \right\rangle + \frac{1}{2}\left\langle {}_x D_1^{\gamma} u, {}_0 D_x^{\gamma} v \right\rangle. \quad (54)$$

On meshes $\{x_k^n\}_{k=0}^N$, we construct finite element spaces (3rd-order piecewise polynomials)

$$\mathcal{V}^n = \operatorname{span}\left\{\phi_{k/3}^n(x), k = 1, \ldots, 3N-1\right\}, \quad (55)$$

where $\phi_{k/3}^n(x)$, $k = 1, \ldots, 3N-1$ are basis functions defined by

$$\phi_k^n(x) = \begin{cases} \ell_{k-1,3}^n(x), & x \in [x_{k-1}^n, x_k^n]; \\ \ell_{k,0}^n(x), & x \in [x_k^n, x_{k+1}^n]; \\ 0, & \text{otherwise}, \end{cases}$$
$$k = 1, \ldots, N-1, \quad (56)$$
$$\phi_{k+j/3}^n = \begin{cases} \ell_{k,j}^n(x), & x \in [x_k^n, x_{k+1}^n]; \\ 0, & \text{otherwise}, \end{cases}$$
$$j = 1, 2, \ k = 0, 1, \ldots, N-1,$$

where $\ell_{k,j}^n(x)$, $j = 0, 1, 2, 3$, are the cubic Lagrange basis functions with respect to local mesh points x_k^n, $x_{k+1/3}^n$, $x_{k+2/3}^n$, x_{k+1}^n, where

$$x_{k+1/3}^n =: x_k^n + \frac{(x_{k+1}^n - x_k^n)}{3},$$
$$x_{k+2/3}^n =: x_k^n + \frac{2(x_{k+1}^n - x_k^n)}{3}. \quad (57)$$

So, the fractional Ritz projection of function g on the finite element spaces $\mathcal{V}^n$ can be written as

$$R_n g(x) = \sum_{k=1}^{3N-1} g_{k/3}^n \phi_{k/3}^n(x). \quad (58)$$

Inserting (58) into (20) gives that

$$B\left(g - \sum_{k=1}^{3N-1} g_{k/3}^n \phi_{k/3}^n(x), v\right) = 0, \quad \forall v \in \mathcal{V}^n. \quad (59)$$

Thus, we may obtain a system of algebraic equations by taking $v = \phi_{i/3}^n(x)$, $i = 1, \ldots, 3N-1$:

$$\mathbf{A}\mathbf{g}^n = \mathbf{b}, \quad (60)$$

for unknown vector

$$\mathbf{g}^n := \left(g_{1/3}^n, g_{2/3}^n, \ldots, g_{(3N-1)/3}^n\right)^T, \quad (61)$$

with matrix $\mathbf{A}$ and $\mathbf{b}$ given by

$$\mathbf{A} = \begin{pmatrix} B(\phi_{1/3}^n, \phi_{1/3}^n) & \cdots & B(\phi_{(3N-1)/3}^n, \phi_{1/3}^n) \\ B(\phi_{1/3}^n, \phi_{2/3}^n) & \cdots & B(\phi_{(3N-1)/3}^n, \phi_{2/3}^n) \\ \vdots & \vdots & \vdots \\ B(\phi_{1/3}^n, \phi_{(3N-1)/3}^n) & \cdots & B(\phi_{(3N-1)/3}^n, \phi_{(3N-1)/3}^n) \end{pmatrix},$$
$$\mathbf{b} = \left(B(g, \phi_{1/3}^n), B(g, \phi_{2/3}^n), \ldots, B(g, \phi_{(3N-1)/3}^n)\right)^T. \quad (62)$$

We check the convergence rate (37) in the same way as Example 6. From the numerics in Tables 3 and 4, we can see that the convergence order for space is 4 and the convergence order for time is 1, which are consistent with (37) where $r = 4$ for the use of 3rd-order finite element methods.

TABLE 3: Rate for space for $g(x) = x^{\mu_1} - x^{\mu_2}$ with $\mu_1 = 7/2$, $\mu_2 = 7/3$ for Example 7 using 3rd-order FEM.

N	$\gamma = 0.4$		$\gamma = 0.7$	
	Error	Rate	Error	Rate
4	$9.3105e-3$	—	$9.3105e-3$	—
8	$7.2803e-4$	3.7	$7.2803e-4$	3.8
16	$5.0060e-5$	3.8	$5.0059e-5$	3.9
32	$3.2670e-6$	3.9	$3.2670e-6$	3.9
64	$2.0783e-7$	4.0	$2.0783e-7$	4.0
128	$1.3076e-8$	4.0	$1.3083e-8$	4.0
256	$8.4731e-10$	3.9	$7.7842e-10$	4.0

TABLE 4: Rate for time for $g(x) = x^{\mu_1} - x^{\mu_2}$ with $\mu_1 = 7/2$, $\mu_2 = 7/3$ for Example 7 using 3rd-order FEM.

M	$\gamma = 0.4$		$\gamma = 0.7$	
	Error	Rate	Error	Rate
16	$5.9067e-4$	—	$5.9068e-4$	—
32	$1.1010e-4$	2.4	$1.1011e-4$	2.4
64	$3.2024e-5$	1.7	$3.2025e-5$	1.7
128	$1.3049e-5$	1.2	$1.3049e-5$	1.2
256	$6.4572e-6$	1.0	$6.4572e-6$	1.0
512	$3.1807e-6$	1.0	$3.1808e-6$	1.0
1024	$1.5882e-6$	1.0	$1.5882e-6$	1.0
2048	$7.9375e-7$	1.0	$7.9375e-7$	1.0

5. Conclusions

This paper studied the moving finite element methods for space-time fractional differential equations. The proof using interpolation (see [10]) was not possible to give the optimal convergence rates. However, using fractional Ritz projection operator proposed in this paper, the optimal convergence rates were obtained, although a natural assumption, which was numerically verified, was used. The proposed moving finite element methods can be readily implemented and applied to the nonlinear fractional differential equations with blowup solutions. These further studies on the applications will be carried out elsewhere.

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