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Research Article

Wave-Breaking Criterion for the Generalized Weakly Dissipative Periodic Two-Component Hunter-Saxton System

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This paper studies the wave-breaking criterion for the generalized weakly dissipative two-component Hunter-Saxton system in the periodic setting. We get local well-posedness for the generalized weakly dissipative two-component Hunter-Saxton system. We study a wave-breaking criterion for solutions and results of wave-breaking solutions with certain initial profiles.

1. Introduction

In recent years, the Hunter-Saxton equation [1]

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0 \tag{1}$$

models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal. In Hunter and Saxton [1], x is the space variable in a reference frame moving with the linearized wave velocity, t is a slow-time variable, and $u(t,x)$ is a measure of the average orientation of the medium locally around x at time t. In order to be more precise, the orientation of the molecules is described by the field of unit vectors $(\cos u(t,x), \sin u(t,x))$ [2]. The Hunter-Saxton equation also arises in a different physical context as the high-frequency limit [3, 4] of the Camassa-Holm equation for shallow water waves [5, 6] and a reexpression of the geodesic flow on the diffeomorphism group of the circle [7] with a bi-Hamiltonian structure [1, 8] which is completely integrable [4, 9]. Hunter and Saxton [1] explored the initial value problem for the Hunter and Saxton equation on the line (nonperiodic case) and on the unit circle $S = R/Z$ by using the method of characteristics, while Yin [2] studied it by using the Kato semigroup method. In addition, the two classes of admissible weak solutions, dissipative and conservative solutions, and their stability were investigated in [10–12]. Lenells [13] confirmed that the Hunter-Saxton equation also describes the geodesic flows on the quotient space of the infinite-dimensional group $D^s(S)$ modulo the subgroup of rotations $\mathrm{Rot}(S)$.

The Camassa-Holm equation admits many integrable multicomponent generalizations. So many authors studied the two-component Camassa-Holm system [14, 15]. Inspired by this, recently, the researchers have made a study of the global existence of solutions to a two-component generalized Hunter-Saxton system in the periodic setting as follows:

$$\begin{aligned} &u_{txx} + 2\sigma u_x u_{xx} + \sigma u u_{xxx} - \rho\rho_x + A u_x = 0, \\ &\qquad\qquad t > 0,\ x \in R, \\ &\rho_t + (\rho u)_x = 0, \quad t > 0,\ x \in R, \\ &u(t, x+1) = u(t,x), \quad \rho(t,x+1) = \rho(t,x), \\ &\qquad\qquad t \geq 0,\ x \in R, \\ &u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \quad x \in R. \end{aligned} \tag{2}$$

The authors of [16] have explored the particular choice of the parameter $\sigma = 1$. The authors of [17] have further studied the wave breaking and global existence for the system for the parameter $\sigma \in \mathrm{R}$ to determine a wave-breaking criterion for strong solutions by using the localization analysis in the transport equation theory.

In general, avoiding energy dissipation mechanisms in a real world is not so easy. Wu and Yin [18, 19] have investigated the blow-up phenomena and the blow-up rate of the strong solutions of the weakly dissipative CH equation and DP equation. Inspired by the results mentioned above, we are going to discuss the initial value problem associated with the generalized weakly dissipative periodic two-component Hunter-Saxton system

$$\begin{aligned}
&u_{txx} + 2\sigma u_x u_{xx} + \sigma u u_{xxx} - \rho\rho_x + A u_x \\
&\quad + \lambda\left(u - u_{xx}\right) = 0, \quad t > 0,\ x \in R, \\
&\rho_t + \left(\rho u\right)_x = 0, \quad t > 0,\ x \in R, \\
&u\left(t, x+1\right) = u\left(t, x\right), \quad \rho\left(t, x+1\right) = \rho\left(t, x\right), \\
&\qquad t \geq 0,\ x \in R, \\
&u\left(0, x\right) = u_0\left(x\right), \quad \rho\left(0, x\right) = \rho_0\left(x\right), \quad x \in R,
\end{aligned} \tag{3}$$

where $\sigma \in R$ is the new free parameter and $A \geq 0, \lambda < 0$.

Our major results of this paper are Theorems 11 and 12 (wave-breaking criterion). The remainder of the paper is organized as follows. Section 2 establishes the local well-posedness for (3) with the initial data in $H^s \times H^{s-1}$, $s \geq 2$. Section 3 deals with the wave breaking of this new system. Theorem 11, using transport equation theory, states a wave-breaking criterion which says that the wave breaking only depends on the slope of u, not the slope of ρ. Theorem 12 improves the blow-up criterion with a more precise condition.

Notation. Throughout this paper, $S = R/Z$ will denote the unit circle. By H^s, $s \geq 0$, we will represent the Sobolev spaces of equivalence classes of functions defined on the unit circle S which have square-integrable distributional derivatives up to order s. The H^s-norm will be designated by $\|\cdot\|_{H^s}$, and the norm of a vector $v \in H^s \times H^{s-1}$ will be written as $\|v\|_{H^s \times H^{s-1}}$. Also, the Lebesgue spaces of order $p \in [1, \infty]$ will be denoted by $L^p(S)$, and the norm of their elements will be denoted by $\|f\|_{L^p(S)}$. Finally, if $p = 2$, we agree on the convention $\|\cdot\|_{L^2(S)} = \|\cdot\|$.

2. Preliminaries

In this part, we will establish the local well-posedness for the Cauchy problem of system (3) by using Kato's theory. To pursue our goal, we give the results we wanted in brief.

We now provide the framework in which we will reformulate (3). To do this, we observe that we can write the first equation of (3) in the following integrated form:

$$u_{tx} + \frac{\sigma}{2}u_x^2 + \sigma u u_{xx} - \frac{1}{2}\rho^2 + Au + \lambda\partial_x^{-1}u - \lambda u_x = g\left(t\right), \tag{4}$$

where $\partial_x^{-1} f(x) = \int_0^x f(y)dy$ and $g(t)$ is determined by the periodicity of u to be

$$g\left(t\right) = -\int_S \left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au\right)dx. \tag{5}$$

Integrating both sides of (4) with respect to variable x, we get

$$\begin{aligned}
u_t + \sigma u u_x = {} & \partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g + \lambda u_x - \lambda\partial_x^{-1}u\right) \\
& + h\left(t\right),
\end{aligned} \tag{6}$$

where $h(t) : [0, \infty) \to R$ is an arbitrary continuous function. Therefore, (3) can be written in the "transport" form as follows:

$$\begin{aligned}
&u_t + \sigma u u_x = \partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au - \lambda\partial_x^{-1}u + \lambda u_x + g\right) \\
&\qquad + h\left(t\right), \quad t > 0,\ x \in R, \\
&\rho_t + u\rho_x = -u_x\rho, \quad t > 0,\ x \in R, \\
&u\left(t, x+1\right) = u\left(t, x\right), \quad \rho\left(t, x+1\right) = \rho\left(t, x\right), \\
&\qquad t \geq 0,\ x \in R, \\
&u\left(0, x\right) = u_0\left(x\right), \quad \rho\left(0, x\right) = \rho_0\left(x\right), \quad x \in R,
\end{aligned} \tag{7}$$

where $h(t) : [0, \infty) \to R$ is an arbitrary continuous function.

Next, we apply Kato's theory to establish the local well-posedness for the system (3). Consider the abstract quasi-linear evolution equation

$$\frac{dv}{dt} + A\left(v\right)v = f\left(v\right), \quad t \geq 0,\ v\left(0\right) = v_0. \tag{8}$$

Proposition 1 (see [20]). *Given the evolution equation* (8), *assume that the Kato conditions hold. For a fixed $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to the abstract quasi-linear evolution equation* (8) *such that*

$$v = v\left(\cdot, v_0\right) \in C\left(\left[0, T\right); Y\right) \cap C^1\left(\left[0, T\right); X\right). \tag{9}$$

Moreover, the map $v_0 \to v(\cdot, v_0)$ is continuous from Y to

$$C\left(\left[0, T\right); Y\right) \cap C^1\left(\left[0, T\right); X\right). \tag{10}$$

One may follow the similar argument as in [17] to obtain the following local well-posedness for (3).

Theorem 2. *Given any $X_0 = \binom{u_0}{\rho_0} \in H^s \times H^{s-1}$, $s \geq 2$, there exist a maximal $T = T(\sigma, A; \|X_0\|_{H^s \times H^{s-1}}) > 0$ and a unique solution $X = \binom{u}{\rho}$ to* (3) *such that*

$$\begin{aligned}
X = X\left(\cdot, X_0\right) \in {} & C\left(\left[0, T\right); H^s\left(S\right) \times H^{s-1}\left(S\right)\right) \\
& \cap C^1\left(\left[0, T\right); H^{s-1}\left(S\right) \times H^{s-2}\left(S\right)\right).
\end{aligned} \tag{11}$$

Moreover, the solution depends continuously on the initial data, that is, the mapping $X_0 \to X(\cdot, X_0) : H^s \times H^{s-1} \to C([0, T); H^s(S) \times H^{s-1}(S)) \cap C^1([0, T);$ and $H^{s-1}(S) \times H^{s-2}(S))$ is continuous, and the maximal existence time T can be chosen independently of the Sobolev order s.

Now, discuss the initial value problem for the Lagrangian flow map as follows:

$$\frac{\partial\varphi}{\partial t} = u(t, \varphi(t,x)), \quad t \in [0,T),$$
$$\varphi(0,x) = x, \quad x \in R, \tag{12}$$

where u is the first component of the solution X to (3). *Using classical results from ordinary differential equations, one can acquire the following result on φ which is of vital importance in the proof of the blow-up scenarios.*

Lemma 3 (see [17]). *Let $u \in C([0,T); H^s) \cap C^1([0,T); H^{s-1})$, $s \geq 2$. Then, initial value problem* (12) *admits a unique solution $\varphi \in C^1([0,T) \times R, R)$. Moreover, $\{\varphi(t,\cdot)\}_{t\in[0,T)}$ is increasing diffeomorphism of R with*

$$\varphi_x(t,x) = e^{\int_0^T u_x(\tau,\varphi(\tau,x))d\tau} > 0, \quad (t,x) \in [0,T) \times R. \tag{13}$$

Remark 4. Since $\varphi(t,\cdot) : R \rightarrow R$ is a diffeomorphism of the linear for every $t \in [0,T)$, the L^∞-norm of any function $v(t,\cdot) \in L^\infty$, $t \in [0,T)$ is preserved under the family of diffeomorphisms $\varphi(t,\cdot)$ with $t \in [0,T)$, that is,

$$\|v(t,\cdot)\|_{L^\infty(S)} = \left\|v(t,\varphi(t,\cdot))\right\|_{L^\infty(S)}, \quad t \in [0,T). \tag{14}$$

Similarly, we have

$$\inf_{x\in S} v(t,x) = \inf_{x\in S} v(t,\varphi(t,x)), \quad t \in [0,T),$$
$$\sup_{x\in S} v(t,x) = \sup_{x\in S} v(t,\varphi(t,x)), \quad t \in [0,T). \tag{15}$$

Lemma 5. *Let $X_0 = \binom{u_0}{\rho_0} \in H^s \times H^{s-1}$, $s \geq 2$, and let T be the maximal existence time of the solution $X = \binom{u}{\rho}$ to* (3) *with initial data X_0. Then, for all $t \in [0,T)$, we have the following results:*

$$\int_S \rho(t,x)\,dx = \int_S \rho_0(t)\,dx, \tag{16}$$

$$\int_S u_x^2(t,x) + \rho^2(t,x)\,dx \leq \int_S u_{0,x}^2(x) + \rho_0^2(x)\,dx \triangleq E_0. \tag{17}$$

Proof. On the one hand, integrating the second equation in (3) by parts and using the periodicity of u and ρ, we acquire

$$\frac{d}{dt}\int_S \rho\,dx = -\int_S (u\rho)_x dx = 0. \tag{18}$$

On the other hand, multiplying (4) by u_x and integrating by parts, considering the periodicity of u, we obtain

$$\frac{d}{dt}\int_S u_x^2 dx = -2\int_S u\rho\rho_x dx + 2\lambda\int_S u^2 dx + 2\lambda\int_S u_x^2 dx. \tag{19}$$

Multiplying the second equation in (3) by ρ and integrating by parts, we have

$$\frac{d}{dt}\int_S \rho^2 dx = 2\int_S u\rho\rho_x dx. \tag{20}$$

Adding the above two equations, we get

$$\frac{d}{dt}\int_S u_x^2 + \rho^2 dx = 2\lambda\|u\|_{H^2}^2, \quad \lambda < 0. \tag{21}$$

We acquire

$$\int_S u_x^2(t,x) + \rho^2(t,x)\,dx \leq \int_S u_{0,x}^2(x) + \rho_0^2(x)\,dx \triangleq E_0. \tag{22}$$

This completes the proof of Lemma 5. □

Lemma 6. *Let $X_0 = \binom{u_0}{\rho_0} \in H^s \times H^{s-1}$, $s \geq 2$, and let T be the maximal existence time of the solution $X = \binom{u}{\rho}$ to* (3) *with initial data X_0. Then, for all $t \in [0,T)$, we have the following results:*

$$\int_S u^2(t,x)\,dx \leq e^{C_2 t}\left(\int_S u_0^2(x)\,dx + 1\right), \tag{23}$$

where $C_1 = \max(|\sigma| - \lambda, 1)E_0 + \sup_{t\in[0,\infty)}|h(t)| > 0$, $C_2 = C_1 + 4A - 2\lambda$.

Proof. By computing directly, we have

$$\begin{aligned}
&\left|\partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g + \lambda u_x - \lambda\partial_x^{-1}u\right) + h(t)\right| \\
&\quad\leq \int_0^x \left|\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g + \lambda u_x - \lambda\partial_x^{-1}u\right| dx + |h(t)| \\
&\quad\leq \frac{1}{2}\max(|\sigma| - \lambda, 1)E_0 + |g(t)| + |h(t)| \\
&\qquad + (|A| - \lambda)\int_0^1 |u|\,dx - \lambda \\
&\quad\leq \max(|\sigma| - \lambda, 1)E_0 + |h(t)| + (2|A| - \lambda)\int_S |u|\,dx - \lambda \\
&\quad:= C_1 + (2|A| - \lambda)\int_S |u|\,dx - \lambda,
\end{aligned} \tag{24}$$

where $C_1 = \max(|\sigma| - \lambda, 1)E_0 + \sup_{t\in[0,\infty)}|h(t)| > 0$ and

$$\begin{aligned}
|g(t)| &= \left|-\int_S\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au\right)dx\right| \\
&\leq \frac{1}{2}\max(|\sigma|, 1)E_0 + A\int_S |u|\,dx.
\end{aligned} \tag{25}$$

Multiplying (6) by u and integrating with respect to x, using the periodicity of u and (24), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_S u^2(t,x)\,dx \\
&= \int_S u u_t dx \\
&= -\sigma\int_S u_x u^2 dx \\
&\quad + \int_S u\Big[\partial_x^{-1}\Big(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + \lambda u_x \\
&\qquad - \lambda\partial_x^{-1}u + g\Big) + h(t)\Big]dx \\
&= \int_S u\Big[\partial_x^{-1}\Big(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + \lambda u_x \\
&\qquad -\lambda\partial_x^{-1}u + g\Big) + h(t)\Big]dx \\
&\le \Big(C_1 + (2|A| - \lambda)\int_S |u|\,dx - \lambda\Big)\int_S |u|\,dx \\
&\le C_1\int_S |u|\,dx + (2|A|-\lambda)\Big(\int_S |u|\,dx\Big)^2 - \lambda\int_S u^2 dx \\
&\le \Big(\frac{C_1}{2} + (2|A| - 2\lambda)\Big)\int_S u^2 dx + \frac{C_1-\lambda}{2} \\
&= \frac{C_2}{2}\int_S u^2 dx + \frac{C_1-\lambda}{2},
\end{aligned} \tag{26}$$

where $C_2 = C_1 + 4A - 2\lambda$; note that $C_2 > C_1$.

By Gronwall's inequality, we get

$$\begin{aligned}
\int_S u^2(t,x)\,dx &\le e^{C_2 t}\Big(\int_S u_0^2(x)\,dx + \frac{C_1-\lambda}{C_2}\Big) - \frac{C_1-\lambda}{C_2} \\
&\le e^{C_2 t}\Big(\int_S u_0^2(x)\,dx + 1\Big).
\end{aligned} \tag{27}$$

This completes the proof of Lemma 6. □

Lemma 7. *Assume that $u_0 \in H^s(S)$, $s \ge 2$, $u_0 \ne 0$, and that the corresponding solution $u(t,x)$ of (3) has a zero point for any time $t \ge 0$. Then, for all $t \in [0,T)$ we have*

$$\int_S u^2(t,x)\,dx \le \int_S u_x^2(t,x)\,dx \le E_0. \tag{28}$$

Proof. By assumption, there is $x_0 \in [0,1]$ such that $u(t,x_0) = 0$ for each $t \in [0,T)$.

Then, for $x \in S$, by holder equality, we have

$$\begin{aligned}
u^2(t,x) = \Big(\int_{x_0}^x u_x dx\Big)^2 &\le (x - x_0)\int_{x_0}^x u_x^2 dx, \\
&x \in \Big[x_0, x_0 + \frac{1}{2}\Big].
\end{aligned} \tag{29}$$

This implies $\sup_{x\in S} u^2(t,x) \le (1/2)\int_S u_x^2 dx$

$$\begin{aligned}
\int_S u^2(t,x)\,dx &\le \sup_{x\in S} u^2(t,x) \le \frac{1}{2}\int_S u_x^2 dx \\
&\le \int_S u_x^2 + \rho^2 dx \le E_0.
\end{aligned} \tag{30}$$ □

3. Wave-Breaking Criteria

In this section, by using transport equation theory, we obtain the wave-breaking criteria for solutions to (3). We first recall the following propositions.

Proposition 8 (1D Moser-type estimates). *The following estimates hold:*

(a) *For $s \ge 0$,*

$$\|fg\|_{H^s(R)} \le C\big(\|f\|_{L^\infty(R)}\|g\|_{H^s(R)} + \|f\|_{H^s(R)}\|g\|_{L^\infty(R)}\big). \tag{31}$$

(b) *For $s > 0$,*

$$\|f\partial_x g\|_{H^s(R)} \le C\big(\|f\|_{L^\infty(R)}\|\partial_x g\|_{H^s(R)} + \|f\|_{H^{s+1}(R)}\|g\|_{L^\infty(R)}\big). \tag{32}$$

(c) *For $s_1 \le 1/2$, $s_2 > 1/2$, $s_1 + s_2 > 0$,*

$$\|fg\|_{H^{s_1}(R)} \le C\|f\|_{H^{s_1}(R)}\|g\|_{H^{s_2}(R)}, \tag{33}$$

where $C's$ are constants that are independent of f and g.

Proposition 9 (see [21]). *Suppose that $s > -d/2$. Let v be a vector field such that ∇v belongs to $L^1([0,T];H^{s-1})$ if $s > 1 + d/2$ or to $L^1([0,T];H^{d/2}\cap L^\infty)$, otherwise. Suppose also that $f_0 \in H^s$, $F \in L^1([0,T];H^s)$ and that $f \in L^\infty([0,T];H^s)\cap C([0,T];S')$ solves the d-dimensional linear transport equations*

$$(T)\begin{cases}\partial_t f + v\cdot\nabla f = F, \\ f|_{t=0} = f_0.\end{cases} \tag{34}$$

Then $f \in C([0,T];H^s)$. More precisely, there exists a constant C depending only on s, p, and d such that the following statements hold:

(1) *If $s \ne 1 + d/2$,*

$$\|f\|_{H^s} \le \|f_0\|_{H^s} + \int_0^t \|F(\tau)\|_{H^s}d\tau + C\int_0^t V'(\tau)\,\|f(\tau)\|_{H^s}d\tau, \tag{35}$$

or

$$\|f\|_{H^s} \le e^{CV(t)}\Big(\|f_0\|_{H^s} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{H^s}d\tau\Big), \tag{36}$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{d/2}\cap L^\infty}d\tau$ if $s < 1 + d/2$ and $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{s-1}}d\tau$ else.

(2) *If $f = v$, then for all $s > 0$, estimates* (35) *and* (36) *hold with*

$$V(t) = \int_0^t \|\partial_x u(\tau)\|_{L^\infty} d\tau. \quad (37)$$

Proposition 10 (see [21]). *Let $0 < s < 1$. Suppose that $f_0 \in H^s$, $g \in L^1([0,T]; H^s)v$, $\partial_x v \in L^1([0,T]; L^\infty)$ and that $f \in L^\infty([0,T]; H^s) \cap C([0,T]; S')$ solves the 1-dimensional linear transport equation*

$$(T) \begin{cases} \partial_t f + v \cdot \nabla f = g, \\ f|_{t=0} = f_0. \end{cases} \quad (38)$$

Then $f \in C([0,T]; H^s)$. More precisely, there exists a constant C depending only on s such that the following statements hold:

$$\begin{aligned} \|f\|_{H^s} &\leq \|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s} d\tau \\ &\quad + C\int_0^t V'(\tau)\|f(\tau)\|_{H^s} d\tau \end{aligned} \quad (39)$$

or

$$\|f\|_{H^s} \leq e^{CV(t)}\left(\|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s} d\tau\right), \quad (40)$$

with $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{L^\infty}) d\tau$.

The above proposition was proved in [8] using Littlewood-Paley analysis for the transport equation and Moser-type estimates. Using this result and performing the same argument, as in [17], we can obtain the following blow-up criterion.

Theorem 11. *Let $X_0 = \binom{u_0}{\rho_0} \in H^s \times H^{s-1}$ with $s \geq 2$, and $X = \binom{u}{\rho}$ be the corresponding solution to* (3). *Assume that $T > 0$ is the maximal time of existence. Then*

$$T < \infty \Longrightarrow \int_0^T \|\partial_x u(\tau)\|_{L^\infty} d\tau = \infty. \quad (41)$$

Our next result describes the necessary and sufficient condition for the blow-up of solutions to (3).

Theorem 12. *Suppose that $\sigma \in R \setminus \{0\}$. Let $X_0 = \binom{u_0}{\rho_0} \in H^s \times H^{s-1}$, with $s \geq 2$, and let T be the maximal existence time of the solution $X = \binom{u}{\rho}$ to* (3) *with initial data X_0. Then, the solution blows up in finite time if and only if*

$$\liminf_{t \to T^-} \left\{\inf_{x \in S} \sigma u_x(t,x)\right\} = -\infty. \quad (42)$$

The approach one takes here is the method of characteristics. Applying the following lemma, we may carry out the estimates along the characteristics $\varphi(t,x)$ which captures $\sup_{x\in S} u_x(t,x)$ and $\inf_{x\in S} u_x(t,x)$.

Lemma 13 (see [22]). *Let $T > 0$ and let $v \in C^1([0,T]; H^2(R))$. Then, for every $t \in [0,T)$, there exists at least one point $\xi(t) \in R$ with*

$$m(t) := \inf_{x \in S} v_x(t,x) = v_x(t, \xi(t)), \quad (43)$$

and the function $m(t)$ is almost everywhere differentiable on $(0,T)$ with

$$\frac{dm(t)}{dt} = v_{tx}(t, \xi(t)), \quad \text{a.e. on } (0,T). \quad (44)$$

Lemma 14. *Let $X_0 = \binom{u_0}{\rho_0} \in H^s \times H^{s-1}$ with $s \geq 2$, and let T be the maximal existence time of the solution $X = \binom{u}{\rho}$ to* (3) *with initial data X_0. Then one has the following:*

(1) $\sigma \neq 0$

$$\begin{aligned} \sup_{x \in S} u_x(t,x) &\leq \|u_{0,x}\|_{L^\infty(S)} \\ &\quad + \sqrt{\frac{\|\rho_0\|^2_{L^\infty(S)} + K_1^2(T)}{\sigma} + \frac{\lambda^2}{\sigma^2}} + \frac{\lambda}{\sigma} \\ &\qquad (\sigma > 0) \end{aligned} \quad (45)$$

$$\begin{aligned} \inf_{x \in S} u_x(t,x) &\geq -\|u_{0,x}\|_{L^\infty(S)} \\ &\quad - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{K_2^2(T)}{\sigma}} + \frac{\lambda}{\sigma} \quad (\sigma < 0). \end{aligned} \quad (46)$$

(2) $\sigma = 0$

$$\begin{aligned} \sup_{x \in S} u_x(t,x) &\leq \sup_{x \in S} u_{0,x}(x) \\ &\quad + \frac{1}{2}\left(\sup_{x \in S} \rho_0^2(x) + K_1^2(T)\right)\frac{e^{\lambda t} - 1}{\lambda} \end{aligned} \quad (47)$$

$$\begin{aligned} \inf_{x \in S} u_x(t,x) &\geq \inf_{x \in S} u_{0,x}(x) \\ &\quad + \frac{1}{2}\left(\inf_{x \in S} \rho_0^2(x) - K_2^2(T)\right)\frac{e^{\lambda t} - 1}{\lambda}. \end{aligned} \quad (48)$$

The constants above are defined as follows:

$$K_1(T) = \sqrt{2A - \lambda + \frac{A}{2}E_0 + \frac{3A - 2\lambda}{2}\left[e^{C_2 T}\left(\|u_0\|^2_{L^2(S)} + 1\right)\right]}, \quad (49)$$

$$\begin{aligned} &K_2(T) \\ &= \sqrt{2A - \lambda + \frac{A+2}{2}E_0 + \frac{3A - 2\lambda}{2}\left[e^{C_2 T}\left(\|u_0\|^2_{L^2(S)} + 1\right)\right]}. \end{aligned} \quad (50)$$

Proof of Lemma 14. By Theorem 2 and a simple density argument, we show that the desired results are valid when $s \geq 3$, so we take $s = 3$ in the proof.

Let $\sigma > 0$. Using Lemma 13 and the fact that

$$\sup_{x\in S}\left[v_x(t,x)\right] = -\inf_{x\in S}\left[-v_x(t,x)\right]. \tag{51}$$

We can consider $M(t)$ and $\gamma(t)$ as follows:

$$M(t) := u_x(t,\xi(t)) = \sup_{x\in S}\left[u_x(t,x)\right], \quad t\in[0,T). \tag{52}$$

Hence,

$$u_{xx}(t,\xi(t)) = 0, \quad \text{a.e. on } t\in[0,T). \tag{53}$$

Take the trajectory $\varphi(t,x)$ defined in (12). Then we know that $\varphi(t,\cdot): R \to R$ is a diffeomorphism for every $t\in[0,T)$. Therefore, there exists $x_0(t)\in R$ such that

$$\varphi(t,x_0(t)) = \xi(t), \quad t\in[0,T). \tag{54}$$

Now, let

$$\gamma(t) = \rho(t,\varphi(t,x_0)), \quad t\in[0,T). \tag{55}$$

Therefore, along the trajectory $\varphi(t,x_0)$, (4) and the second equation of (3) become

$$\begin{gathered}M'(t) = -\frac{\sigma}{2}M^2(t) + \lambda M(t) + \frac{1}{2}\gamma^2(t) + f(t,\varphi(t,x_0))\\ \gamma'(t) = -\gamma M, \quad \text{a.e. } t\in[0,T),\end{gathered} \tag{56}$$

where the notation denotes the derivative with respect to t and f represents the function

$$\begin{aligned}f &= -Au - \lambda\partial_x^{-1}u + g(t)\\ &= -Au - \lambda\partial_x^{-1}u - \int_S\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au\right)dx.\end{aligned} \tag{57}$$

We first compute the upper and lower bounds for f for later use in getting the blow-up result as follows:

$$\begin{aligned}f &= -Au - \lambda\int_0^x u\,dx - \frac{\sigma}{2}\int_S u_x^2dx - \frac{1}{2}\int_S\rho^2dx + A\int_S u\,dx\\ &\le -Au - \lambda\int_0^x u\,dx + A\int_S u\,dx \le \frac{A}{2}\left(1+u^2\right)\\ &\quad + \frac{A-\lambda}{2}\left(1+\int_S u^2dx\right).\end{aligned} \tag{58}$$

Since $u^2 \le (1/2)\int_S(u^2+u_x^2)dx$, (17), we obtain the upper bound for f

$$\begin{aligned}f &\le \frac{A}{2}\left(1+\frac{1}{2}\int_S\left(u^2+u_x^2\right)dx\right) + \frac{A-\lambda}{2}\left(1+\int_S u^2dx\right)\\ &\le A - \frac{\lambda}{2} + \frac{A}{4}\int_S\left(\rho^2+u_x^2\right)dx + \frac{3A-2\lambda}{4}\int_S u^2dx\\ &\le A - \frac{\lambda}{2} + \frac{A}{4}E_0 + \frac{3A-2\lambda}{4}\left[e^{C_2t}\left(\int_S u_0^2(x)\,dx + 1\right)\right]\\ &\le A - \frac{\lambda}{2} + \frac{A}{4}E_0 + \frac{3A-2\lambda}{4}\\ &\quad\times\left[e^{C_2T}\left(\left\|u_0\right\|_{L^2(S)}^2+1\right)\right] := \frac{1}{2}K_1^2(T).\end{aligned} \tag{59}$$

Now we turn to the lower bound of f. Using previous arguments, we get

$$\begin{aligned}-f &= Au + \lambda\int_0^x u\,dx + \frac{\sigma}{2}\int_S u_x^2dx + \frac{1}{2}\int_S\rho^2dx - A\int_S u\,dx\\ &\le A\frac{1+u^2}{2} + \frac{\max(|\sigma|,1)}{2}\int_S\left(\rho^2+u_x^2\right)dx\\ &\quad + \frac{A-\lambda}{2}\left(1+\int_S u^2dx\right)\\ &\le A - \frac{\lambda}{2} + \frac{A+2\max(|\sigma|,1)}{4}\int_S\left(\rho^2+u_x^2\right)dx\\ &\quad + \frac{3A-2\lambda}{4}\int_S u^2dx\\ &\le A - \frac{\lambda}{2} + \frac{A+2\max(|\sigma|,1)}{4}E_0\\ &\quad + \frac{3A-2\lambda}{4}\left[e^{C_2T}\left(\left\|u_0\right\|_{L^2(S)}^2+1\right)\right].\end{aligned} \tag{60}$$

When $\sigma < 0$, we have a finer estimate

$$\begin{aligned}-f &\le A - \frac{\lambda}{2} + \frac{A+2}{4}\int_S\left(\rho^2+u_x^2\right)dx + \frac{3A-2\lambda}{4}\int_S u^2dx\\ &\le A - \frac{\lambda}{2} + \frac{A+2}{4}E_0 + \frac{3A-2\lambda}{4}\\ &\quad\times\left[e^{C_2T}\left(\left\|u_0\right\|_{L^2(S)}^2+1\right)\right] := \frac{1}{2}K_2^2(T).\end{aligned} \tag{61}$$

Combining (59) and (60), we obtain

$$\begin{aligned}|f| &\le A - \frac{\lambda}{2} + \frac{A+2\max(|\sigma|,1)}{4}E_0\\ &\quad + \frac{3A-2\lambda}{4}\left[e^{C_2T}\left(\left\|u_0\right\|_{L^2(S)}^2+1\right)\right] := \frac{1}{2}K_3^2(T).\end{aligned} \tag{62}$$

Since $s\ge 3$, we have $u\in C_0^1(S)$. Therefore,

$$\sup_{x\in S} u_x(t,x) \ge 0, \qquad \inf_{x\in S} u_x(t,x) \le 0, \quad t\in[0,T). \tag{63}$$

Hence, $M(t) > 0$ for $t \in [0, T)$. From the second equation of (55), we obtain

$$\gamma(t) = \gamma(0) e^{-\int_0^t M(\tau) d\tau}. \tag{64}$$

Hence,

$$|\rho(t, \varphi(t, x_0))| = |\gamma(t)| \le |\gamma(0)| \le \|\rho_0\|_{L^\infty(S)}. \tag{65}$$

For any given $x \in S$, define

$$\begin{aligned} p_1(t) = {} & M(t) - \|u_{0,x}\|_{L^\infty(S)} \\ & - \sqrt{\frac{\|\rho_0\|^2_{L^\infty(S)} + K_1^2(T)}{\sigma} + \frac{\lambda^2}{\sigma^2}} - \frac{\lambda}{\sigma} \quad (\sigma > 0). \end{aligned} \tag{66}$$

Observing that $p_1(t)$ is a C^1-differentiable function on $[0, T)$ and satisfies

$$\begin{aligned} p_1(0) = {} & M(0) - \|u_{0,x}\|_{L^\infty(S)} \\ & - \sqrt{\frac{\|\rho_0\|^2_{L^\infty(S)} + K_1^2(T)}{\sigma} + \frac{\lambda^2}{\sigma^2}} - \frac{\lambda}{\sigma} \\ \le {} & M(0) - \|u_{0,x}\|_{L^\infty(S)}. \end{aligned} \tag{67}$$

We now claim that $p_1(t) \le 0$ $t \in [0, T)$.

Assume the contrary that there is $t_0 \in [0, T)$ such that $p_1(t_0) > 0$.

Let $t_1 = \max\{t < t_0 : p_1(t) = 0\}$. Then $p_1(t_1) = 0$ and $p_1'(t_1) \ge 0$, or equivalently,

$$M(t_1) = \|u_{0,x}\|_{L^\infty(S)} + \sqrt{\frac{\|\rho_0\|^2_{L^\infty(S)} + K_1^2(T)}{\sigma} + \frac{\lambda^2}{\sigma^2}} + \frac{\lambda}{\sigma} \tag{68}$$

and $M'(t_1) \ge 0$ a.e. $t \in [0, T)$. On the other hand, we have

$$\begin{aligned} M'(t_1) = {} & -\frac{\sigma}{2} M^2(t_1) + \lambda M(t_1) + \frac{1}{2}\gamma^2(t_1) \\ & + f(t_1, \varphi(t_1, x_0)) \quad \text{a.e. } t \in [0, T) \\ \le {} & -\frac{\sigma}{2}\left(M(t_1) - \frac{\lambda}{\sigma}\right)^2 + \frac{1}{2}\|\rho_0\|^2_{L^\infty(S)} + \frac{\lambda^2}{2\sigma} \\ & + \frac{K_1^2(T)}{2} < 0, \end{aligned} \tag{69}$$

which is a contradiction. Therefore, $p_1(t) \le 0$ for all $t \in [0, T)$. Since x is arbitrarily chosen, we obtain (45).

To derive (46) in the case of $\sigma < 0$, we consider $\widetilde{M}(t)$ and $\widetilde{\gamma}(t)$ as in Lemma 13:

$$\widetilde{M}(t) := u_x(t, \zeta(t)) = \inf_{x \in S} [u_x(t, x)] \quad t \in [0, T). \tag{70}$$

Hence,

$$u_{xx}(t, \zeta(t)) = 0 \quad \text{a.e. } t \in [0, T). \tag{71}$$

Using previous arguments, we take the characteristic $\varphi(t, x)$ defined in (13) and choose $x_1(t) \in R$ such that

$$\varphi(t, x_1(t)) = \zeta(t). \tag{72}$$

Let

$$\widetilde{\gamma}(t) = \rho(t, \varphi(t, x)), \quad t \in [0, T). \tag{73}$$

Hence, along the trajectory $\widetilde{M}'(t) = \lambda \widetilde{M}(t) + (1/2)\gamma^2(t) + f(t, \varphi(t, x_0)) \ge \lambda \widetilde{M}(0) + (1/2)\gamma^2(0) + (1/2)K_2^2(T)$, (4) and the second equation of (3) become

$$\begin{aligned} & \widetilde{M}'(t) = -\frac{\sigma}{2}\widetilde{M}^2(t) + \lambda \widetilde{M}(t) + \frac{1}{2}\gamma^2(t) + f(t, \varphi(t, x_0)) \\ & \widetilde{\gamma}'(t) = -\widetilde{\gamma}\widetilde{M}, \quad \text{a.e. } t \in [0, T). \end{aligned} \tag{74}$$

Define

$$\begin{aligned} p_2(t) = {} & \widetilde{M}(t) + \|u_{0,x}\|_{L^\infty(S)} \\ & + \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{K_2^2(T)}{\sigma}} - \frac{\lambda}{\sigma} \quad (\sigma < 0). \end{aligned} \tag{75}$$

For any given $x \in S$, Note that $p_2(t)$ is also C^1-differentiable function on $[0, T)$ and satisfies

$$\begin{aligned} p_2(0) = {} & \widetilde{M}(0) + \|u_{0,x}\|_{L^\infty(S)} \\ & + \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{K_2^2(T)}{\sigma}} - \frac{\lambda}{\sigma} \ge \widetilde{M}(0) + \|u_{0,x}\|_{L^\infty(S)} \ge 0. \end{aligned} \tag{76}$$

We now claim that $p_2(t) \ge 0$, for any $t \in [0, T)$.

Suppose not, then there is $\widetilde{t} \in [0, T)$ such that $p_2(\widetilde{t_0}) < 0$. Define

$$t_2 = \max\{t < \widetilde{t_0} : p_2(t) = 0\}. \tag{77}$$

Then, $p_2(t_2) = 0$ and $p_2'(t_2) < 0$, or equivalently,

$$\widetilde{M}(t_2) = -\|u_{0,x}\|_{L^\infty(S)} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{K_2^2(T)}{\sigma}} + \frac{\lambda}{\sigma} \tag{78}$$

and $\widetilde{M}'(t_2) \le 0$ a.e. $t \in [0, T)$. However, we have

$$\begin{aligned} \widetilde{M}'(t_2) = {} & -\frac{\sigma}{2}\widetilde{M}^2(t_2) + \lambda \widetilde{M}(t_2) + \frac{1}{2}\gamma^2(t_2) \\ & + f(t_2, \varphi(t_2, x_0)) \quad \text{a.e. } t \in [0, T) \\ \ge {} & -\frac{\sigma}{2}\left(-\|u_{0,x}\|_{L^\infty(S)} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{K_2^2(T)}{\sigma}}\right)^2 \\ & + \frac{\lambda^2}{2\sigma} - \frac{1}{2}K_2^2(T) > 0. \end{aligned} \tag{79}$$

Therefore, $p_2(t) \geq 0$ for any $t \in [0,T)$. Since x is chosen arbitrarily, we obtain (46).

Let $\sigma = 0$. Using previous arguments, (56) becomes

$$M'(t) = -\frac{\sigma}{2}M^2(t) + \lambda M(t) + \frac{1}{2}\gamma^2(t) + f(t,\varphi(t,x_0))$$
$$\gamma'^{(t)} = -\gamma M, \quad \text{a.e. } t \in [0,\mathrm{T}), \tag{80}$$

where the notation denotes the derivative with respect to t and f represents the function

$$\begin{aligned} f &= -Au - \lambda\partial_x^{-1}u + g(t) \\ &= -Au - \lambda\partial_x^{-1}u - \int_S\left(\frac{1}{2}\rho^2 - Au\right)dx. \end{aligned} \tag{81}$$

We first compute the upper and lower bounds for f for later use in getting the blow-up result:

$$\begin{aligned} f &= -Au - \lambda\int_0^x u\,dx - \frac{1}{2}\int_S \rho^2 dx + A\int_S u\,dx \\ &\leq -Au - \lambda\int_0^x u\,dx + A\int_S u\,dx \leq \frac{A}{2}\left(1+u^2\right) \\ &\quad + \frac{A-\lambda}{2}\left(1+\int_S u^2 dx\right) \\ &\leq \frac{A}{2}\left(1+\frac{1}{2}\int_S\left(u^2+u_x^2\right)dx\right) + \frac{A-\lambda}{2}\left(1+\int_S u^2 dx\right) \\ &\leq A - \frac{\lambda}{2} + \frac{A}{4}\int_S\left(\rho^2+u_x^2\right)dx + \frac{3A-2\lambda}{4}\int_S u^2 dx \\ &\leq A - \frac{\lambda}{2} + \frac{A}{4}E_0 + \frac{3A-2\lambda}{4}\left[e^{C_2 t}\left(\int_S u_0^2(x)\,dx + 1\right)\right] \\ &\leq A - \frac{\lambda}{2} + \frac{A}{4}E_0 + \frac{3A-2\lambda}{4}\left[e^{C_2 T}\left(\left\|u_0\right\|_{L^2(S)}^2 + 1\right)\right]. \end{aligned} \tag{82}$$

Now, we turn to the lower bound of f:

$$\begin{aligned} -f &= Au + \lambda\int_0^x u\,dx + \frac{1}{2}\int_S \rho^2 dx - A\int_S u\,dx \\ &\leq A\frac{1+u^2}{2} + \frac{1}{2}\int_S \rho^2 dx + \frac{A-\lambda}{2}\left(1+\int_S u^2 dx\right) \\ &\leq A - \frac{\lambda}{2} + \frac{A+2}{4}\int_S\left(\rho^2+u_x^2\right)dx + \frac{3A-2\lambda}{4}\int_S u^2 dx \\ &\leq A - \frac{\lambda}{2} + \frac{A+2}{4}E_0 + \frac{3A-2\lambda}{4}\left[e^{C_2 T}\left(\left\|u_0\right\|_{L^2(S)}^2 + 1\right)\right]. \end{aligned} \tag{83}$$

Combining (82) and (83), we obtain

$$|f| \leq A - \frac{\lambda}{2} + \frac{A+2}{4}E_0 + \frac{3A-2\lambda}{4}\left[e^{C_2 T}\left(\left\|u_0\right\|_{L^2(S)}^2 + 1\right)\right]. \tag{84}$$

We know $M(t) > 0$ for $t \in [0,T)$. From the second equation of (81), we obtain that

$$\gamma(t) = \gamma(0)\,e^{-\int_0^t M(\tau)d\tau}. \tag{85}$$

Hence,

$$\left|\rho(t,\varphi(t,x_0))\right| = \left|\gamma(t)\right| \leq \left|\gamma(0)\right|. \tag{86}$$

Therefore, we have

$$\begin{aligned} M'(t) &= \lambda M(t) + \frac{1}{2}\gamma^2(t) + f(t,\varphi(t,x_0)) \\ &\leq \lambda M(0) + \frac{1}{2}\gamma^2(0) + \frac{1}{2}K_1^2(T), \end{aligned} \tag{87}$$

$$M'(t) - \lambda M(t) \leq \frac{1}{2}\left(\sup_{x\in S}\rho_0^2(x) + K_1^2(T)\right) \quad \text{a.e. } t \in [0,T). \tag{88}$$

Integrating (88) on $[0,t]$, we prove (47) as follows:

$$M(t) \leq \sup_{x\in S}u_{0,x}(x) + \frac{1}{2}\left(\sup_{x\in S}\rho_0^2(x) + K_1^2(T)\right)\frac{e^{\lambda t}-1}{\lambda}. \tag{89}$$

To obtain a lower bound for $\inf_{x\in S}u_x(t,x)$, we use the same argument.

Since $\sigma = 0$, (80) becomes

$$\begin{aligned} \widetilde{M}'(t) &= \lambda\widetilde{M}(t) + \frac{1}{2}\gamma^2(t) + f(t,\varphi(t,x_1)), \\ \widetilde{\gamma}'(t) &= -\widetilde{\gamma}\widetilde{M}, \quad \text{a.e. } t \in [0,T). \end{aligned} \tag{90}$$

Because of $\widetilde{M}(t) < 0$, we get from the second equation of (90) that

$$\widetilde{\gamma}(t) = \widetilde{\gamma}(0)\,e^{-\int_0^t M(\tau)d\tau}. \tag{91}$$

This means that

$$\left|\rho(t,\varphi(t,x_1))\right| = \left|\gamma(t)\right| \geq \left|\gamma(0)\right|. \tag{92}$$

Then,

$$\begin{aligned} \widetilde{M}'(t) &= \lambda\widetilde{M}(t) + \frac{1}{2}\gamma^2(t) + f(t,\varphi(t,x_0)) \\ &\geq \lambda\widetilde{M}(0) + \frac{1}{2}\gamma^2(0) + \frac{1}{2}K_2^2(T), \end{aligned} \tag{93}$$

$$\widetilde{M}'(t) - \lambda\widetilde{M}(t) \geq \frac{1}{2}\left(\inf_{x\in S}\rho_0^2(x) - K_2^2(T)\right) \quad \text{a.e. } t \in [0,T). \tag{94}$$

Integrating (94) on $[0,t]$, we prove (48). This completes the proof of Lemma 14. □

Lemma 15. *Suppose that $\sigma \in R \setminus \{0\}$. Suppose $X_0 = \binom{u_0}{\rho_0} \in H^s \times H^{s-1}$ with $s \geq 2$, and let T be the maximal existence time* *of the solution* $X = \binom{u}{\rho}$ *to* (3) *with initial data* X_0. *Then we have*

$$\rho(t, \varphi(t,x))\, \varphi_x(t,x) = \rho_0(x), \quad (t,x) \in [0,T] \times S. \tag{95}$$

Moreover, if there exists $M > 0$ *such that*

$$\inf_{(t,x)\in[0,T]\times S} \sigma u_x(t,x) \geq -M, \quad (t,x) \in [0,T] \times S. \tag{96}$$

Then

$$\begin{aligned} \|\rho(t,\cdot)\|_{L^\infty(S)} &= \|\rho(t,\varphi(t,\cdot))\|_{L^\infty(S)} \\ &\leq e^{MT/\sigma} \|\rho_0\|_{L^\infty(S)} \quad (\sigma > 0) \end{aligned} \tag{97}$$

$$\begin{aligned} \|\rho(t,\cdot)\|_{L^\infty(S)} &= \|\rho(t,\varphi(t,\cdot))\|_{L^\infty(S)} \\ &\leq e^{NT/\sigma} \|\rho_0\|_{L^\infty(S)} \quad (\sigma < 0), \end{aligned} \tag{98}$$

where $N = \|u_{0,x}\|_{L^\infty(S)} + \sqrt{\lambda^2/\sigma^2 - K_2^2(T)/\sigma} - \lambda/\sigma$ *and* $K_2(T)$ *is given in* (50).

Proof. Differentiating the left hand side of (95) with respect to t, in view of the relations (12) and (3), we obtain

$$\begin{aligned} &\frac{d}{dt}\{\rho(t,\varphi(t,x))\,\varphi_x(t,x)\} \\ &\quad = [\rho_t(t,\varphi) + \rho_x(t,\varphi)\,\varphi_t(t,x)] \\ &\qquad \times \varphi_x(t,x) + \rho(t,\varphi)\,\varphi_{xt}(t,x) \\ &\quad = [\rho_t(t,\varphi) + \rho_x(t,\varphi)\,\varphi_t(t,x)]\,\varphi_x(t,x) \\ &\qquad + \rho(t,\varphi)\,u_x(t,\varphi)\,\varphi_x(t,x) \\ &\quad = [\rho_t(t,\varphi) + \rho_x(t,\varphi)\,\varphi_t(t,x) + \rho(t,\varphi)\,u_x(t,\varphi)] \\ &\qquad \times \varphi_x(t,x) = 0. \end{aligned} \tag{99}$$

This completes the proof of (95). In view of the assumption (96) and $\sigma > 0$, we obtain $u(t,x) \geq -(M/\sigma)$ $(t,x) \in [0,T] \times S$.

By Lemma 3 and (95), we have

$$\begin{aligned} \|\rho(t,\cdot)\|_{L^\infty} &= \|\rho(t,\varphi(t,\cdot))\|_{L^\infty} = \left\| e^{-\int_0^t u_x(\tau,\cdot)d\tau} \rho_0(\cdot) \right\|_{L^\infty} \\ &\leq e^{MT/\sigma} \|\rho_0(\cdot)\|_{L^\infty}. \end{aligned} \tag{100}$$

To obtain (98), we use a similar argument as before. Using (13) and the lower bound for $u_x(t,x)$ in (46), it follows that

$$\begin{aligned} \|\rho(t,\cdot)\|_{L^\infty} &= \|\rho(t,\varphi(t,\cdot))\|_{L^\infty} = \left\| e^{-\int_0^t u_x(\tau,\cdot)d\tau} \rho_0(\cdot) \right\|_{L^\infty} \\ &\leq e^{NT/\sigma} \|\rho_0(\cdot)\|_{L^\infty}, \end{aligned} \tag{101}$$

which proves (98). This completes the proof of Lemma 15. □

Proof of Theorem 12. Suppose that $T < \infty$ and that (42) is not valid. Then, there is some positive number $M > 0$ such that

$$\sigma u_x(t,x) \geq -M, \quad (t,x) \in [0,T] \times S. \tag{102}$$

It now follows from Lemma 14 that $|u_x(t,x)| \leq C$, where $C = C(A, M, \sigma, E_0, \lambda, \|u_0\|, T)$. Therefore, Theorem 11 implies that the maximal existence time $T = \infty$, which contradicts with the assumption that $T < \infty$.

Conversely, the Sobolev embedding theorem $H^s(S) \to L^\infty(S)$ with $s > 1/2$ implies that if (70) holds, the corresponding solution blows up in finite time, which completes the proof of Theorem 12. □

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