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Research Article

A Berry-Esseen Type Bound in Kernel Density Estimation for Negatively Associated Censored Data

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We discuss the kernel estimation of a density function based on censored data when the survival and the censoring times form the stationary negatively associated (NA) sequences. Under certain regularity conditions, the Berry-Esseen type bounds are derived for the kernel density estimator and the Kaplan-Meier kernel density estimator at a fixed point x.

1. Introduction

Let $\{T_i; i \geq 1\}$ be a sequence of the true survival times. The random variables (r.v.s.) are not assumed to be mutually independent; it is assumed, however, that they have a common unknown continuous marginal distribution function (d.f.) $F(x) = P(T_i \leq x)$ and density function $f(x)$. Let the r.v.s. T_i be censored on the right by the censoring r.v.s. Y_i, so that one observes only (Z_i, δ_i), where

$$\begin{aligned} Z_i &= \min(T_i, Y_i) := T_i \wedge Y_i, \\ \delta_i &= I(T_i \leq Y_i), \quad i = 1, \ldots, n, \end{aligned} \tag{1}$$

here and in the sequel, and $I(A)$ is the indicator random variable of the event A. In this random censorship model, the censoring times Y_i, $i = 1, \ldots, n$, are assumed to have the common d.f. $G(y)$; they are also assumed to be independent of the r.v.s. T_i's. Following the convention in the survival literature, we assume that both X_i and Y_i are nonnegative random variables. In contrast to statistics for complete data, we observe only the pairs (Z_i, δ_i), $i = 1, \ldots, n$, and the estimators are based on these pairs.

The following nonparametric estimation of the distribution functions F and G due to Kaplan and Meier [1] is widely used to estimate F and G on the basis of the data (Z_i, δ_i):

$$\begin{aligned} \widehat{F}_n(x) &= 1 - \prod_{k=1}^{n} \left(1 - \frac{\delta_{(k)}}{n-k+1}\right)^{I(Z_{(k)} \leq x)} \\ &= 1 - \prod_{k=1}^{n} \left(\frac{n-k}{n-k+1}\right)^{I(\delta_{(k)}=1,\, Z_{(k)} \leq x)}, \\ \widehat{G}_n(x) &= 1 - \prod_{k=1}^{n} \left(\frac{n-k}{n-k+1}\right)^{I(\delta_{(k)}=0,\, Z_{(k)} \leq x)}, \end{aligned} \tag{2}$$

where $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ denote the order statistics of $Z_1, Z_2, \ldots, Z_n$ and $\delta_{(i)}$ is the concomitant of $Z_{(i)}$.

We introduce the kernel density estimator

$$f_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{Z_i - x}{h_n}\right) \frac{\delta_i}{1 - G(Z_i)}, \tag{3}$$

where $0 < h_n \to 0$ are bandwidths and K is some kernel function. When G is known, (3) can be used to estimate the common density of the lifetimes. However, in most practical cases G is unknown and must be replaced by the Kaplan-Meier estimator $\widehat{G}_n$, so the Kaplan-Meier kernel density estimator of the f is defined by

$$\widehat{f}_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{Z_i - x}{h_n}\right)\frac{\delta_i}{1-\widehat{G}_n(Z_i)}. \quad (4)$$

There is an extensive literature on the Kaplan-Meier estimator for censored independent observations. We refer to papers by Földes and Rejtő [2], Gu and Lai [3], Gill [4], and Sun and Zhu [5]. Sun and Zhu obtained the following Berry-Esseen bound for i.i.d. censored sequences.

Theorem A. *Let K be a bounded probability kernel function with compact support $[-1, 1]$ satisfying for integer $r \geq 2$,*

$$\frac{1}{j!}\int u^j K(u)\,\mathrm{d}u = \begin{cases} 1, & j = 0, \\ 0, & j = 1, 2, \ldots, r-1, \\ c_r \neq 0, & j = r. \end{cases} \quad (5)$$

Let f be r-order continuously differentiable and let G be continuously differentiable in a neighborhood of x with $f(x) > 0$ for $x < \tau_L$. Then

$$rl\sup_{y\in\mathbb{R}}\left|P\left((nh_n)^{1/2}\left[\widehat{f}_n(x) - f(x)\right] \leq y\sigma(x)\right) - \Phi(y)\right| = O(b_n), \quad (6)$$

where $\Phi(\cdot)$ denotes the standard normal distribution function, $b_n = (nh_n)^{-1/2} + n^{1/2}h_n^{r+1/2} + h_n^{1/4}$ and $\sigma^2(x) = (f(x)/(1-G(x)))\int K^2(t)\mathrm{d}t$.

However, the censored dependent data appear in a number of applications. For example, repeated measurements in survival analysis follow this pattern; see Kang and Koehler [6]. In the context of censored time series analysis, Shumway et al. [7] considered (hourly or daily) measurements of the concentration of a given substance subject to some detection limits, thus being potentially censored from the right. Lecoutre and Ould-Said [8], Cai [9], and Liang and Uña-Álvarez [10] studied the convergence for the stationary α-mixing data. However, the convergence for the NA data has not been reported.

The main purpose of this paper is to study the kernel density estimator and the Kaplan-Meier kernel estimator of a density function based on censored data when the survival and the censoring times form the stationary NA (see the following definition) sequences. Under certain regularity conditions, the Berry-Esseen type bounds are derived for the kernel density estimator and the Kaplan-Meier kernel estimator at a fixed point x.

Definition 1. Random variables $X_1, X_2, \ldots, X_n$, $n \geq 2$ are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{cov}\left(f_1(X_i; i \in A_1), f_2(X_j; j \in A_2)\right) \leq 0, \quad (7)$$

where f_1 and f_2 are increasing for every variable (or decreasing for every variable) such that this covariance exists. A sequence of random variables $\{X_i; i \geq 1\}$ is said to be NA if every finite subfamily is NA.

Obviously, if $\{X_i; i \geq 1\}$ is a sequence of NA random variables and $\{f_i; i \geq 1\}$ is a sequence of nondecreasing (or nonincreasing) functions, then $\{f_i(X_i); i \geq 1\}$ is also a sequence of NA random variables.

This definition was introduced by Joag-Dev and Proschan [11]. Statistical test depends greatly on sampling. The random sampling without replacement from a finite population is NA but is not independent. NA sampling has wide applications such as those in multivariate statistical analysis and reliability theory. Because of the wide applications of NA sampling, the limit behavior of NA random variables has received more and more attention recently. One can refer to Joag-Dev and Proschan [11] for fundamental properties, Matuła [12] for the three-series theorem, and Wu and Jiang [13, 14] for the strong convergence.

2. Main Results

In what follows, let L be the d.f. of the Z_i's, $\overline{L} := 1 - L$. Since the sequences $\{T_n; n \geq 1\}$ and $\{Y_n; n \geq 1\}$ are independent, it follows that $L = 1 - \overline{F}\,\overline{G} := 1 - (1-F)(1-G)$.

Define (possibly infinite) times τ_F, τ_G, and τ_L by

$$\tau_F = \inf\{y; F(y) = 1\}, \qquad \tau_G = \inf\{y; G(y) = 1\}, \qquad \tau_L = \inf\{y; L(y) = 1\}. \quad (8)$$

Then, $\tau_L = \tau_F \wedge \tau_G$.

We give the following four lemmas, which are helpful in proving our theorems.

Lemma 2 (Chang and Rao, [15]). *Let X and Y be random variables, then for any $a > 0$*

$$\sup_{y\in\mathbb{R}}|P(X+Y \leq y) - \Phi(y)| \leq \sup_{y\in\mathbb{R}}|P(X \leq y) - \Phi(y)| + \frac{a}{\sqrt{2\pi}} + P(|Y| > a), \quad (9)$$

here and in the sequel, where $\Phi(\cdot)$ denotes the standard normal distribution function.

Lemma 3 (Su et al. [16, Theorem 1]). *Let $\{X_i; i \geq 1\}$ be a sequence of NA r.v.s. with zero means and $\mathbb{E}|X_i|^p < \infty$, $i = 1, 2, \ldots$ and $p \geq 2$. Then for $S_n = \sum_{i=1}^n X_i$,*

$$\mathbb{E}|S_n|^p \leq c_p\left(\sum_{i=1}^n \mathbb{E}|X_i|^p + \left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{p/2}\right), \quad (10)$$

where $c_p > 0$ depends only on p.

Lemma 4. *Let $\{X_i; i \geq 1\}$ be a sequence of NA r.v.s. with continuous d.f. F, and let $F_n(x) := (1/n)\sum_{i=1}^n I(X_i < x)$ be the empirical d.f. based on the segments $X_1, \ldots, X_n$. Then*

$$\sup_{x\in\mathbb{R}}|F_n(x) - F(x)| = O\left(n^{-1/2}\ln^{1/2} n\right) \text{ a.s.} \quad (11)$$

Proof. Similar to the proof of Lemma 4 in Yang [17], we can prove Lemma 4. □

Lemma 5 (Wu and Chen [18, Theorem 1.3]). *Let $\{T_n;\ n \geq 1\}$ and $\{Y_n;\ n \geq 1\}$ be two sequences of NA r.v.s. Suppose that the sequences $\{T_n;\ n \geq 1\}$ and $\{Y_n;\ n \geq 1\}$ are independent. Then for any $0 < \tau < \tau_L$,*

$$\sup_{0\leq t\leq \tau}\left|\widehat{F}_n(t) - F(t)\right| = O\left(n^{-1/2}\ln^{1/2} n\right) \text{ a.s.} \tag{12}$$

In order to formulate our main results, we now list some assumptions.

(A_1) $\{Y_i;\ i \geq 1\}$ and $\{T_i;\ i \geq 1\}$ are two sequences of stationary NA random variables, and $\{Y_i\}$ and $\{T_i\}$ are independent.

(A_2) Suppose that $x < \tau_L$, $f(x) > 0$, and f and G have bounded derivative in a neighborhood of x.

(A_3) For all integers $j \geq 1$, the conditional distribution T_{j+1}, given $T_1 = x_1$, has a density $f_j(\cdot|x_1)$, and for all $x \in \mathbb{R}$, $f_j(x_2|x_1) \leq M_0$ for $x_1, x_2 \in U(x)$ and some $M_0 > 0$, where $U(x)$ represents a neighborhood of x.

(A_4) The kernel K is a bounded derivative function with $K(u) = 0$ for $|u| > 1$ and $\int_{-1}^{1} K(u)du = 1$.

(A_5) Let p_n, q_n, and $k_n = [n/(p_n + q_n)]$ be positive integers with

$$\lim_{n\to\infty} k_n = \infty, \qquad \lim_{n\to\infty}\frac{p_n k_n}{n} = 1, \qquad \lim_{n\to\infty} p_n h_n = 0,$$
$$\lim_{n\to\infty} nh_n = \infty, \qquad \lim_{n\to\infty} h_n^{-3} u(q_n) = 0, \tag{13}$$

where $u(n) := \sum_{j=n}^{\infty} |\mathrm{Cov}(T_1, T_{j+1})|$.

Remark 6. (A_5) Implies $\lim_{n\to\infty}(k_n q_n/n) = 0$ and $\lim_{n\to\infty}(p_n/n) = 0$.

Let $\sigma_n^2(x) = nh_n \operatorname{Var}(f_n(x))$, $\sigma^2(x) = (f(x)/(1 - G(x)))\int_{-1}^{1} K^2(t)dt$.

Theorem 7. *Suppose that (A_1)–(A_5) are satisfied; then*

$$\left|\sigma_n^2(x) - \sigma^2(x)\right| = O(a_n), \tag{14}$$

where $a_{1n} = (q_n k_n/n) + (q_n k_n u(p_n)/nh_n^3)$, $a_{2n} = p_n/n$, $a_{3n} = u(q_n)/h_n^3$, $a_n = p_n h_n + a_{1n}^{1/2} + a_{2n}^{1/2} + a_{3n} \to 0$.

Consider the following:

$$\sup_{y\in\mathbb{R}}\left|P\left(\frac{f_n(x) - \mathbb{E} f_n(x)}{\sqrt{\operatorname{Var} f_n(x)}} \leq y\right) - \Phi(y)\right| = O(b_n), \tag{15}$$

where $b_n = 1/(nh_n)^{1/2} + a_{1n}^{1/3} + a_{2n}^{1/3} + a_{3n}^{1/3} \to 0$.

Furthermore, if

$$\lim_{n\to\infty} n^{1/2} h_n^{3/2} = 0, \tag{16}$$

then

$$\sup_{y\in\mathbb{R}}\left|P\left(\frac{f_n(x) - f(x)}{\sqrt{\operatorname{Var} f_n(x)}} \leq y\right) - \Phi(y)\right| = O\left(b_n + n^{1/2} h_n^{3/2}\right). \tag{17}$$

Theorem 8. *Assume that the conditions of Theorem 7 hold. Then*

$$\begin{aligned}\sup_{y\in\mathbb{R}}&\left|P\left(\frac{\widehat{f}_n(x) - \mathbb{E}\widehat{f}_n(x)}{\sqrt{\operatorname{Var} f_n(x)}} \leq y\right) - \Phi(y)\right| \\ &= O\left(b_n + (h_n \ln n)^{1/4}\right).\end{aligned} \tag{18}$$

Furthermore, if (16) holds, then

$$\begin{aligned}\sup_{y\in\mathbb{R}}&\left|P\left(\frac{\widehat{f}_n(x) - f(x)}{\sqrt{\operatorname{Var} f_n(x)}} \leq y\right) - \Phi(y)\right| \\ &= O\left(b_n + (h_n \ln n)^{1/4} + n^{1/2} h_n^{3/2}\right).\end{aligned} \tag{19}$$

3. Proofs

Proof of Theorem 7. We observe that, by (3),

$$\begin{aligned}(nh_n)^{1/2} f_n(x) &= \sum_{i=1}^{n} \frac{1}{(nh_n)^{1/2}} K\left(\frac{Z_i - x}{h_n}\right)\frac{\delta_i}{1 - G(Z_i)} \\ &:= \sum_{i=1}^{n} Z_{ni} := S_n.\end{aligned} \tag{20}$$

Let $k_m = (m-1)(p_n+q_n)+1$, $l_m = (m-1)(p_n+q_n)+p_n+1$, $m = 1, 2, \ldots, k_n$, where

$$\begin{gathered}U_{nm} = \sum_{i=k_m}^{k_m+p_n-1} Z_{ni}, \qquad U'_{nm} = \sum_{i=l_m}^{l_m+q_n-1} Z_{ni}, \\ U'_{n,k_n+1} = \sum_{i=k_n(p_n+q_n)+1}^{n} Z_{ni}, \qquad S'_n = \sum_{m=1}^{k_n} U_{nm}, \\ S''_n = \sum_{m=1}^{k_n} U'_{nm}, \qquad S'''_n = U'_{n,k_n+1},\end{gathered} \tag{21}$$

and then

$$S_n = S'_n + S''_n + S'''_n. \tag{22}$$

By (20),

$$\begin{aligned}\sigma_n^2(x) &= \operatorname{Var} S_n = \operatorname{Var}\left(S'_n + S''_n + S'''_n\right) \\ &= \operatorname{Var} S'_n + \operatorname{Var} S''_n + \operatorname{Var} S'''_n + 2\operatorname{Cov}\left(S'_n, S''_n\right) \\ &\quad + 2\operatorname{Cov}\left(S'_n, S'''_n\right) + 2\operatorname{Cov}\left(S''_n, S'''_n\right).\end{aligned} \tag{23}$$

We first estimate $\operatorname{Var} S_n'$, $\operatorname{Var} S_n''$, and $\operatorname{Var} S_n'''$. Obviously, (A_1) implies that $\{U_{nm}\}$ and $\{Z_{ni}\}$ are stationary; thus,

$$\begin{aligned}
\operatorname{Var} S_n' &= \operatorname{Var}\left(\sum_{m=1}^{k_n} U_{nm}\right) \\
&= \sum_{m=1}^{k_n} \operatorname{Var} U_{nm} + 2\sum_{1\le i<j\le k_n} \operatorname{Cov}\left(U_{ni}, U_{nj}\right) \\
&= k_n \operatorname{Var} U_{n1} + 2\sum_{1\le i<j\le k_n} \operatorname{Cov}\left(U_{ni}, U_{nj}\right) \\
&= k_n p_n \operatorname{Var} Z_{n1} + 2k_n \sum_{1\le i<j\le p_n} \operatorname{Cov}\left(Z_{ni}, Z_{nj}\right) \\
&\quad + 2\sum_{1\le i<j\le k_n} \operatorname{Cov}\left(U_{ni}, U_{nj}\right) \\
&:= I_{n1} + I_{n2} + I_{n3}.
\end{aligned} \tag{24}$$

From (A_1), (A_2), and (A_4), we obtain

$$\begin{aligned}
\operatorname{Var} Z_{n1} &= \frac{1}{nh_n} \operatorname{Var} K\left(\frac{Z_1 - x}{h_n}\right)\frac{\delta_1}{1-G(Z_1)} \\
&= \frac{1}{nh_n}\left\{\mathbb{E}\left[K^2\left(\frac{Z_1 - x}{h_n}\right)\frac{\delta_1}{(1-G(Z_1))^2}\right] - \left[\mathbb{E}K\left(\frac{Z_1 - x}{h_n}\right)\frac{\delta_1}{1-G(Z_1)}\right]^2\right\} \\
&= \frac{1}{nh_n}\iint K^2\left(\frac{\min(u,v) - x}{h_n}\right) \times \frac{I(u<v)}{(1-G(\min(u,v)))^2}\,\mathrm{d}F(u)\,\mathrm{d}G(v) \\
&\quad - \frac{1}{nh_n}\left[\iint K\left(\frac{\min(u,v)-x}{h_n}\right) \times \frac{I(u<v)}{1-G(\min(u,v))}\,\mathrm{d}F(u)\,\mathrm{d}G(v)\right]^2 \\
&= \frac{1}{nh_n}\int\int_{u<v} K^2\left(\frac{u-x}{h_n}\right)\frac{1}{(1-G(u))^2}\,\mathrm{d}F(u)\,\mathrm{d}G(v) \\
&\quad - \frac{1}{nh_n}\left[\int\int_{u<v} K\left(\frac{u-x}{h_n}\right)\frac{1}{1-G(u)}\,\mathrm{d}F(u)\,\mathrm{d}G(v)\right]^2 \\
&= \frac{1}{nh_n}\left\{\int K^2\left(\frac{u-x}{h_n}\right)\frac{f(u)}{1-G(u)}\,\mathrm{d}u - \left[\int K\left(\frac{u-x}{h_n}\right)f(u)\,\mathrm{d}u\right]^2\right\} \\
&= \frac{1}{n}\left\{\int_{-1}^{1} K^2(u)\frac{f(x+uh_n)}{1-G(x+uh_n)}\,\mathrm{d}u - h_n\left[\int_{-1}^{1} K(u) f(x+uh_n)\,\mathrm{d}u\right]^2\right\}
\end{aligned} \tag{25}$$

$$= O\left(\frac{1}{n}\right). \tag{26}$$

Hence, by (A_5), $I_{n1} = O(1)$.

For $i<j$ and $x<\tau_L$, by $(A_1)-(A_4)$,

$$\begin{aligned}
&\left|\operatorname{Cov}\left(Z_{ni}, Z_{nj}\right)\right| \\
&\quad = \left|\operatorname{Cov}\left(Z_{n1}, Z_{n,j-i+1}\right)\right| \\
&\quad = \frac{1}{nh_n}\left|\operatorname{Cov}\left(K\left(\frac{Z_1-x}{h_n}\right)\frac{\delta_1}{1-G(Z_1)}, K\left(\frac{Z_{j-i+1}-x}{h_n}\right)\frac{\delta_{j-i+1}}{1-G(Z_{j-i+1})}\right)\right| \\
&\quad \le \frac{c}{nh_n}\left\{\mathbb{E}\left|K\left(\frac{T_1-x}{h_n}\right)K\left(\frac{T_{j-i+1}-x}{h_n}\right)\right| + \left(\mathbb{E}K\left(\frac{Z_1-x}{h_n}\right)\right)^2\right\} \\
&\quad \le \frac{c}{nh_n}\left\{\iint\left|K\left(\frac{u-x}{h_n}\right)K\left(\frac{v-x}{h_n}\right)\right| \times f(u) f_{j-i}(v\mid u)\,\mathrm{d}u\,\mathrm{d}v + \left(\int K\left(\frac{u-x}{h_n}\right)f(u)\,\mathrm{d}u\right)^2\right\} \\
&\quad \le \frac{ch_n}{n}\left\{\iint_{-1}^{1}|K(u)K(v)|\,\mathrm{d}u\,\mathrm{d}v + \left(\int_{-1}^{1}K(u) f(x+h_n u)\,\mathrm{d}u\right)^2\right\} \\
&\quad = O\left(\frac{h_n}{n}\right).
\end{aligned} \tag{27}$$

Therefore, by (A_5),

$$\begin{aligned}
\left|I_{n2}\right| &= 2k_n\left|\sum_{1\le i<j\le p_n}\operatorname{Cov}\left(Z_{ni}, Z_{nj}\right)\right| \\
&= O\left(\frac{k_n p_n^2 h_n}{n}\right) = O\left(p_n h_n\right) \longrightarrow 0.
\end{aligned} \tag{28}$$

By (A_1), (A_2), (A_4), and Lemma 2.3 of Zhang [19], for $l \geq 1$,

$$\begin{aligned}
&\left|\operatorname{Cov}(U_{n1}, U_{n,l+1})\right| \\
&= \left|\sum_{i=1}^{p_n} \sum_{j=l(p_n+q_n)+1}^{l(p_n+q_n)+p_n} \operatorname{Cov}(Z_{ni}, Z_{nj})\right| \\
&= \left|\sum_{i=1}^{p_n} \left\{\sum_{j=1}^{i-1} \operatorname{Cov}(Z_{n,i-j+1}, Z_{n,l(p_n+q_n)+1}) + \sum_{j=i}^{p_n} \operatorname{Cov}(Z_{n1}, Z_{n,l(p_n+q_n)+j-i+1})\right\}\right| \\
&= \left|\sum_{r=1}^{p_n} (p_n - r + 1) \operatorname{Cov}(Z_{n1}, Z_{n,l(p_n+q_n)+r}) + \sum_{r=1}^{p_n-1} (p_n - r) \operatorname{Cov}(Z_{n,r+1}, Z_{n,l(p_n+q_n)+1})\right| \\
&\leq p_n \sum_{r=l(p_n+q_n)}^{l(p_n+q_n)+p_n-1} \left|\operatorname{Cov}(Z_{n1}, Z_{n,r+1})\right| + p_n \sum_{r=l(p_n+q_n)-(p_n-1)}^{l(p_n+q_n)-1} \left|\operatorname{Cov}(Z_{n1}, Z_{n,r+1})\right| \\
&= p_n \sum_{r=l(p_n+q_n)-(p_n-1)}^{l(p_n+q_n)+p_n-1} \left|\operatorname{Cov}(Z_{n1}, Z_{n,r+1})\right| \\
&= \frac{p_n}{nh_n} \sum_{r=l(p_n+q_n)-(p_n-1)}^{l(p_n+q_n)+p_n-1} \left|\operatorname{Cov}\left(K\left(\frac{Z_1 - x}{h_n}\right) \times \frac{\delta_1}{1 - G(Z_1)}, K\left(\frac{Z_{r+1} - x}{h_n}\right) \times \frac{\delta_{r+1}}{1 - G(Z_{r+1})}\right)\right| \\
&\leq c \frac{p_n}{nh_n^3} \sum_{r=l(p_n+q_n)-(p_n-1)}^{l(p_n+q_n)+p_n-1} \left|\operatorname{Cov}(T_1, T_{r+1})\right|.
\end{aligned} \tag{29}$$

Thus, by (A_1) and (A_5),

$$\begin{aligned}
|I_{n3}| &= \left|2 \sum_{1\leq i<j\leq k_n} \operatorname{Cov}(U_{ni}, U_{nj})\right| \\
&= \left|2 \sum_{l=1}^{k_n-1} (k_n - l) \operatorname{Cov}(U_{n1}, U_{n,l+1})\right| \\
&\leq \frac{cp_n k_n}{nh_n^3} \sum_{l=1}^{k_n-1} \sum_{r=l(p_n+q_n)-(p_n-1)}^{l(p_n+q_n)+p_n-1} \left|\operatorname{Cov}(T_1, T_{r+1})\right| \\
&\leq \frac{cp_n k_n}{n} \frac{1}{h_n^3} \sum_{r=q_n}^{\infty} \left|\operatorname{Cov}(T_1, T_{r+1})\right| \\
&= O\left(\frac{u(q_n)}{h_n^3}\right) = O(a_{3n}) \longrightarrow 0.
\end{aligned} \tag{30}$$

Therefore, by the combination of (A_5), (24), (26), (28), and (30),

$$\operatorname{Var} S_n' = O(1) + O(p_n h_n) + O(a_{3n}) = O(1). \tag{31}$$

Similarly,

$$\begin{aligned}
\operatorname{Var} S_n'' &= k_n \operatorname{Var} U_{n1}' + 2 \sum_{1\leq i<j\leq k_n} \operatorname{Cov}(U_{ni}', U_{nj}') \\
&= k_n q_n \operatorname{Var} Z_{n1} + 2k_n \sum_{1\leq i<j\leq q_n} \operatorname{Cov}(Z_{ni}, Z_{nj}) + 2 \sum_{1\leq i<j\leq k_n} \operatorname{Cov}(U_{ni}', U_{nj}') \\
&= O\left(\frac{q_n k_n}{n} + \frac{q_n^2 k_n h_n}{n} + \frac{q_n k_n u(p_n)}{nh_n^3}\right) \\
&= O\left(\frac{q_n k_n}{n} + \frac{q_n k_n u(p_n)}{nh_n^3}\right) = O(a_{1n}).
\end{aligned} \tag{32}$$

By (26), (27), (A_1), (A_5), and $n - k_n(p_n + q_n) \leq p_n + q_n \leq 2p_n$,

$$\begin{aligned}
\operatorname{Var} S_n''' &= \operatorname{Var}\left(\sum_{i=k_n(p_n+q_n)+1}^{n} Z_{ni}\right) \\
&= (n - k_n(p_n + q_n)) \operatorname{Var}(Z_{n1}) + 2 \sum_{k_n(p_n+q_n)+1\leq i<j\leq n} \operatorname{Cov}(Z_{ni}, Z_{nj}) \\
&= O\left(\frac{p_n}{n} + \frac{p_n^2 h_n}{n}\right) = O\left(\frac{p_n}{n}\right) \\
&= O(a_{2n}).
\end{aligned} \tag{33}$$

By (25), (A_2), and (A_4),

$$\begin{aligned}\left|n\operatorname{Var}Z_{n1}-\sigma^2(x)\right| &= \left|\int_{-1}^{1}K^2(u)\left(\frac{f(x+uh_n)}{1-G(x+uh_n)}-\frac{f(x)}{1-G(x)}\right)\mathrm{d}u-h_n\left[\int_{-1}^{1}K(u)f(x+uh_n)\,\mathrm{d}u\right]^2\right|\\ &\le \left|\int_{-1}^{1}K^2(u)\frac{(1-G(x))(f(x+uh_n)-f(x))+f(x)(G(x+uh_n)-G(x))}{(1-G(x+uh_n))(1-G(x))}\mathrm{d}u\right|\\ &\quad + h_n\left[\int_{-1}^{1}K(u)f(x+uh_n)\,\mathrm{d}u\right]^2\\ &= O(h_n).\end{aligned} \tag{34}$$

Note that $|\operatorname{Cov}(X,Y)| \le (\operatorname{Var}X\operatorname{Var}Y)^{1/2}$ for any random variables X and Y; from (31)–(33),

$$\begin{aligned}\left|\operatorname{Cov}(S_n',S_n'')\right| &= O(a_{1n}^{1/2}),\\ \left|\operatorname{Cov}(S_n',S_n''')\right| &= O(a_{2n}^{1/2}),\\ \left|\operatorname{Cov}(S_n'',S_n''')\right| &= O(a_{1n}^{1/2}a_{2n}^{1/2}).\end{aligned} \tag{35}$$

Therefore, from the combination of (23) and (31)–(34), it follows that

$$\begin{aligned}&\left|\sigma_n^2(x)-\sigma^2(x)\right|\\ &\quad= \left|n\operatorname{Var}Z_{n1}-\sigma^2(x)+O\left(p_nh_n+a_{3n}+a_{1n}^{1/2}+a_{2n}^{1/2}\right)\right|\\ &\quad= O\left(p_nh_n+a_{3n}+a_{1n}^{1/2}+a_{2n}^{1/2}\right)\\ &\quad= O(a_n).\end{aligned} \tag{36}$$

Thus, (14) holds.

Now, we prove (15). Let $\widetilde{S}_n=(S_n-\mathbb{E}S_n)/\sigma_n(x)$, $\widetilde{S}_n'=(S_n'-\mathbb{E}S_n')/\sigma_n(x)$, $\widetilde{S}_n''=(S_n''-\mathbb{E}S_n'')/\sigma_n(x)$, $\widetilde{S}_n'''=(S_n'''-\mathbb{E}S_n''')/\sigma_n(x)$. Then, $\widetilde{S}_n=\widetilde{S}_n'+\widetilde{S}_n''+\widetilde{S}_n'''$. According to Lemma 2, (14), (20), (32), and (33), we have

$$\begin{aligned}&\sup_{y\in\mathbb{R}}\left|P\left(\frac{f_n(x)-\mathbb{E}f_n(x)}{\sqrt{\operatorname{Var}f_n(x)}}\le y\right)-\Phi(y)\right|\\ &\quad=\sup_{y\in\mathbb{R}}\left|P\left(\widetilde{S}_n'+\widetilde{S}_n''+\widetilde{S}_n'''\le y\right)-\Phi(y)\right|\\ &\quad\le\sup_{y\in\mathbb{R}}\left|P\left(\widetilde{S}_n'\le y\right)-\Phi(y)\right|+\frac{a_{1n}^{1/3}}{\sqrt{2\pi}}\\ &\qquad+P\left(\widetilde{S}_n''>a_{1n}^{1/3}\right)+\frac{a_{2n}^{1/3}}{\sqrt{2\pi}}+P\left(\widetilde{S}_n'''>a_{2n}^{1/3}\right)\\ &\quad=\sup_{y\in\mathbb{R}}\left|P\left(\widetilde{S}_n'\le y\right)-\Phi(y)\right|+O\left(a_{1n}^{1/3}+a_{2n}^{1/3}\right).\end{aligned} \tag{37}$$

Let ξ_{nm}, $m=1,2,\ldots,k_n$ be independent random variables with the same distribution as $\widetilde{U}_{nm}:=(U_{nm}-\mathbb{E}U_{nm})/\sigma_n(x)$ for $m=1,2,\ldots,k_n$. Put $H_n=\sum_{m=1}^{k_n}\xi_{nm}$, $B_n^2=\sum_{m=1}^{k_n}\operatorname{Var}\widetilde{U}_{nm}=\sum_{m=1}^{k_n}\operatorname{Var}\xi_{nm}=\operatorname{Var}H_n$. Obviously,

$$\begin{aligned}&\sup_{y\in\mathbb{R}}\left|P\left(\widetilde{S}_n'\le y\right)-\Phi(y)\right|\\ &\quad\le\sup_{y\in\mathbb{R}}\left|P\left(\widetilde{S}_n'\le y\right)-P(H_n\le y)\right|\\ &\qquad+\sup_{y\in\mathbb{R}}\left|\Phi\left(\frac{y}{B_n}\right)-\Phi(y)\right|\\ &\qquad+\sup_{y\in\mathbb{R}}\left|P(H_n\le y)-\Phi\left(\frac{y}{B_n}\right)\right|\\ &\quad:=J_{1n}+J_{2n}+J_{3n}.\end{aligned} \tag{38}$$

Note that $\operatorname{Var}S_n=\sigma_n^2(x)$ and $B_n^2=(\operatorname{Var}S_n'-I_{n3})/\sigma_n^2(x)$ from (20) and (24). By (14), (30), (32), and (33),

$$\begin{aligned}J_{2n}&=\sup_{y\in\mathbb{R}}\left|\Phi\left(\frac{y}{B_n}\right)-\Phi(y)\right|\le\left|B_n^2-1\right|\\ &=\left|\frac{\operatorname{Var}S_n'-I_{n3}-\operatorname{Var}S_n}{\sigma_n^2(x)}\right|\\ &\le c\left|\operatorname{Var}S_n'-\operatorname{Var}S_n\right|+c\left|I_{n3}\right|\\ &\le c\operatorname{Var}\left(S_n''+S_n'''\right)\\ &\quad+2\left|\operatorname{Cov}\left(S_n',S_n''+S_n'''\right)\right|+O(a_{3n})\\ &=O\left(a_{1n}^{1/2}+a_{2n}^{1/2}+a_{3n}\right)\longrightarrow 0.\end{aligned} \tag{39}$$

Note that ξ_{nm}, $m=1,2,\ldots,k_n$, are independent random variables, and $B_n^2=\operatorname{Var}H_n$. Therefore, by $B_n\to 1$ (from (39)), (14), and Berry-Esseen inequality (cf. Petrov [20, page 154, Theorem 5.7]), there exists some constant $c > 0$ such that

$$
\begin{aligned}
J_{3n} &= \sup_{y\in\mathbb{R}} \left| P\left(\frac{H_n}{B_n} \le y\right) - \Phi(y) \right| \\
&\le c \frac{\sum_{m=1}^{k_n} \mathbb{E}|\xi_{nm}|^3}{B_n^3} \le c \sum_{m=1}^{k_n} \mathbb{E}|U_{nm}|^3 \\
&= \sum_{m=1}^{k_n} \mathbb{E} \left| \sum_{i=k_m}^{k_m+p_n-1} Z_{ni} \right|^3 .
\end{aligned}
\tag{40}
$$

Similar to (26), we can get $\mathbb{E}|Z_{n1}|^3 = O(1/n^{3/2}h_n^{1/2})$ and $\mathbb{E}Z_{n1}^2 = O(1/n)$. It is easy to see from Property P7 of Joag-Dev and Proschan [11] that $\{Z_n; n \ge 1\}$ is also sequence of NA r.v.s., so by using Lemma 3, we have

$$
\begin{aligned}
J_{3n} &\le c \sum_{m=1}^{k_n} \left[\sum_{i=k_m}^{k_m+p_n-1} \mathbb{E}|Z_{ni}|^3 + \left(\sum_{i=k_m}^{k_m+p_n-1} \mathbb{E}Z_{ni}^2 \right)^{3/2} \right] \\
&\le ck_n \left(p_n \frac{1}{n^{3/2}h_n^{1/2}} + \left(\frac{p_n}{n}\right)^{3/2} \right) \\
&= O\left(\frac{1}{(nh_n)^{1/2}} + a_{2n}^{1/2} \right).
\end{aligned}
\tag{41}
$$

Assume that $\varphi(t)$ and $\psi(t)$ are the characteristic functions of $\widetilde{S}_n'$ and H_n, respectively. By Esseen inequality (cf. Petrov [20, page 146, Theorem 5.3]), for any $T > 0$, there exists some constant $c > 0$ such that

$$
\begin{aligned}
J_{1n} &= \sup_{y\in\mathbb{R}} \left| P\left(\widetilde{S}_n' \le y\right) - P(H_n \le y) \right| \\
&\le \int_{-T}^{T} \left| \frac{\varphi(t) - \psi(t)}{t} \right| dt \\
&\quad + T \sup_{y\in\mathbb{R}} \int_{|u|\le(c/T)} \left| P(H_n \le u + y) - P(H_n \le y) \right| du \\
&:= J_{1n}' + J_{1n}''.
\end{aligned}
\tag{42}
$$

By Theorem 10 in Newman [21], (14), and (30),

$$
\begin{aligned}
|\varphi(t) - \psi(t)| &= \left| \mathbb{E}\exp\left(\mathrm{i}t \sum_{m=1}^{k_n} \widetilde{U}_{nm} \right) - \prod_{m=1}^{k_n} \mathbb{E}\exp\left(\mathrm{i}t\widetilde{U}_{nm}\right) \right| \\
&\le 2 \sum_{1\le i<j\le k_n} t^2 \left| \mathrm{Cov}\left(\widetilde{U}_{ni}, \widetilde{U}_{nj}\right) \right| \\
&\le c \sum_{1\le i<j\le k_n} t^2 \left| \mathrm{Cov}\left(U_{ni}, U_{nj}\right) \right| \\
&= t^2 O(a_{3n}).
\end{aligned}
\tag{43}
$$

Therefore,

$$
J_{1n}' = O\left(a_{3n}T^2\right). \tag{44}
$$

On applying (39)–(41), we have

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}} \left| P(H_n \le u + y) - P(H_n \le y) \right| \\
&\quad \le \sup_{y\in\mathbb{R}} \left[\left| P\left(\frac{H_n}{B_n} \le \frac{u+y}{B_n}\right) - \Phi\left(\frac{u+y}{B_n}\right) \right| \right. \\
&\qquad + \left| P\left(\frac{H_n}{B_n} \le \frac{y}{B_n}\right) - \Phi\left(\frac{y}{B_n}\right) \right| \\
&\qquad \left. + \left| \Phi\left(\frac{u+y}{B_n}\right) - \Phi\left(\frac{y}{B_n}\right) \right| \right] \\
&\quad = O\left(\frac{1}{(nh_n)^{1/2}} + a_{2n}^{1/2} + |u| \right).
\end{aligned}
\tag{45}
$$

Thus,

$$
J_{1n}'' = O\left(\frac{1}{(nh_n)^{1/2}} + a_{2n}^{1/2} + \frac{1}{T} \right). \tag{46}
$$

Choosing $T = a_{3n}^{-1/3}$, then by (42)–(46),

$$
J_{1n} = O\left(\frac{1}{(nh_n)^{1/2}} + a_{2n}^{1/2} + a_{3n}^{1/3} \right). \tag{47}
$$

Therefore, the combination of (37)–(39), (41), (47), and (15) holds.

Finally, we prove (17). By Lemma 2 and (15), for any $a > 0$,

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}} \left| P\left(\frac{f_n(x) - f(x)}{\sqrt{\mathrm{Var}\, f_n(x)}} \le y \right) - \Phi(y) \right| \\
&\quad = \sup_{y\in\mathbb{R}} \left| P\left(\frac{f_n(x) - \mathbb{E}f_n(x)}{\sqrt{\mathrm{Var}\, f_n(x)}} \right. \right. \\
&\qquad \left. \left. + \frac{\mathbb{E}f_n(x) - f(x)}{\sqrt{\mathrm{Var}\, f_n(x)}} \le y \right) - \Phi(y) \right| \\
&\quad \le \sup_{y\in\mathbb{R}} \left| P\left(\frac{f_n(x) - \mathbb{E}f_n(x)}{\sqrt{\mathrm{Var}\, f_n(x)}} \le y \right) - \Phi(y) \right| \\
&\qquad + \frac{a}{\sqrt{2\pi}} + P\left(\frac{\left|\mathbb{E}f_n(x) - f(x)\right|}{\sqrt{\mathrm{Var}\, f_n(x)}} > a \right) \\
&\quad = O(b_n) + \frac{a}{\sqrt{2\pi}} + P\left(\frac{\left|\mathbb{E}f_n(x) - f(x)\right|}{\sqrt{\mathrm{Var}\, f_n(x)}} > a \right).
\end{aligned}
\tag{48}
$$

Applying (14), (A_3), (A_4), and differential mean value theorem, there exists a constant $0 < \theta < 1$, such that

$$\begin{aligned}
&\frac{\left|\mathbb{E} f_n(x) - f(x)\right|}{\sqrt{\operatorname{Var} f_n(x)}} \\
&\quad \leq c\sqrt{nh_n}\left|\frac{1}{h_n}\mathbb{E}K\left(\frac{Z_1 - x}{h_n}\right)\frac{\delta_1}{1 - G(Z_1)} - f(x)\right| \\
&\quad = c\sqrt{nh_n}\left|\int_{-1}^{1} K(u)\left(f(x + uh_n) - f(x)\right)\mathrm{d}u\right| \\
&\quad = c\sqrt{nh_n}\int_{-1}^{1}\left|K(u)\, uh_n f'(x + \theta uh_n)\right|\mathrm{d}u \\
&\quad = O\left(n^{1/2}h_n^{3/2}\right).
\end{aligned} \tag{49}$$

Hence, there exists a constant M sufficiently large such that $|\mathbb{E} f_n(x) - f(x)|/\sqrt{\operatorname{Var} f_n(x)} < Mn^{1/2}h_n^{3/2}$. Let $a = Mn^{1/2}h_n^{3/2}$ in (48); then $P(|\mathbb{E} f_n(x) - f_n(x)|/\sqrt{\operatorname{Var} f_n(x)} > a) = P(\phi) = 0$. Therefore, by (48), (16) holds. □

Proof of Theorem 8. Using (15) and Lemma 2,

$$\begin{aligned}
&\sup_{y\in\mathbb{R}}\left|P\left(\frac{\widehat{f}_n(x) - \mathbb{E}\widehat{f}_n(x)}{\sqrt{\operatorname{Var} f_n(x)}} \leq y\right) - \Phi(y)\right| \\
&\quad = \sup_{y\in\mathbb{R}}\left|P\left(\frac{f_n(x) - \mathbb{E} f_n(x)}{\sqrt{\operatorname{Var} f_n(x)}} + \frac{\widehat{f}_n(x) - f_n(x) - \mathbb{E}\left(\widehat{f}_n(x) - f_n(x)\right)}{\sqrt{\operatorname{Var} f_n(x)}} \leq y\right) - \Phi(y)\right| \\
&\quad \leq \sup_{y\in\mathbb{R}}\left|P\left(\frac{f_n(x) - \mathbb{E} f_n(x)}{\sqrt{\operatorname{Var} f_n(x)}} \leq y\right) - \Phi(y)\right| + \frac{(h_n \ln n)^{1/4}}{\sqrt{2\pi}} \\
&\qquad + P\left(\frac{\left|\widehat{f}_n(x) - f_n(x) - \mathbb{E}\left(\widehat{f}_n(x) - f_n(x)\right)\right|}{\sqrt{\operatorname{Var} f_n(x)}} > (h_n \ln n)^{1/4}\right) \\
&\quad \leq O\left(b_n + (h_n \ln n)^{1/4}\right) + c\frac{\mathbb{E}\left((nh_n)^{1/2}\left|\widehat{f}_n(x) - f_n(x)\right|\right)}{(h_n \ln n)^{1/4}}.
\end{aligned} \tag{50}$$

Let $L_n(x) = n^{-1}\sum_{i=1}^{n} I(Z_i \leq x)$ be the empirical d.f. of L. Then, by (2),

$$L_n(x) = 1 - \left(1 - \widehat{F}_n(x)\right)\left(1 - \widehat{G}_n(x)\right). \tag{51}$$

Thus, by Lemmas 4 and 5, for $\tau < \tau_L$,

$$\begin{aligned}
&\sup_{0\leq x\leq\tau}\left|\widehat{G}_n(x) - G(x)\right| \\
&\quad \leq \sup_{0\leq x\leq\tau}\frac{1}{1 - \widehat{F}_n(x)} \times\left\{\left|L_n(x) - L(x)\right| + (1 - G(x))\left|\widehat{F}_n(x) - F(x)\right|\right\} \\
&\quad = O\left(\sqrt{\frac{\ln n}{n}}\right).
\end{aligned} \tag{52}$$

Using (14), we get

$$\begin{aligned}
&\mathbb{E}\left((nh_n)^{1/2}\left|\widehat{f}_n(x) - f_n(x)\right|\right) \\
&\quad \leq \frac{1}{\sqrt{nh_n}}\sum_{i=1}^{n}\mathbb{E}\left\{\left|K\left(\frac{Z_i - x}{h_n}\right)\right| \times\frac{\left|\widehat{G}_n(Z_i) - G(Z_i)\right|}{(1 - G(Z_i))\left(1 - \widehat{G}_n(Z_i)\right)}\delta_i\right\} \\
&\quad \leq c\sqrt{\frac{\ln n}{h_n}}\left|\mathbb{E}K\left(\frac{T_1 - x}{h_n}\right)\right| \\
&\quad = c\sqrt{\frac{\ln n}{h_n}}h_n\int_{-1}^{1}|K(t)|\, f(x + h_n t)\,\mathrm{d}t \\
&\quad = O\left(\sqrt{h_n \ln n}\right).
\end{aligned} \tag{53}$$

Therefore, (18) holds from (50) and (53).

Using (18), similar to the proof of (17), we can prove (19). This completes the proof of Theorem 8. □

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