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Research Article

On the Kronecker Products and Their Applications

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This paper studies the properties of the Kronecker product related to the mixed matrix products, the vector operator, and the vec-permutation matrix and gives several theorems and their proofs. In addition, we establish the relations between the singular values of two matrices and their Kronecker product and the relations between the determinant, the trace, the rank, and the polynomial matrix of the Kronecker products.

1. Introduction

The Kronecker product, named after German mathematician Leopold Kronecker (December 7, 1823–December 29, 1891), is very important in the areas of linear algebra and signal processing. In fact, the Kronecker product should be called the Zehfuss product because Johann Georg Zehfuss published a paper in 1858 which contained the well-known determinant conclusion $|\mathbf{A} \otimes \mathbf{B}| = |\mathbf{A}|^n |\mathbf{B}|^m$, for square matrices $\mathbf{A}$ and $\mathbf{B}$ with order m and n [1].

The Kronecker product has wide applications in system theory [2–5], matrix calculus [6–9], matrix equations [10, 11], system identification [12–15], and other special fields [16–19]. Steeba and Wilhelm extended the exponential functions formulas and the trace formulas of the exponential functions of the Kronecker products [20]. For estimating the upper and lower dimensions of the ranges of the two well-known linear transformations $\mathscr{T}_1(\mathbf{X}) = \mathbf{X} - \mathbf{AXB}$ and $\mathscr{T}_2(\mathbf{X}) = \mathbf{AX} - \mathbf{XB}$, Chuai and Tian established some rank equalities and inequalities for the Kronecker products [21]. Corresponding to two different kinds of matrix partition, Koning, Neudecker, and Wansbeek developed two generalizations of the Kronecker product and two related generalizations of the vector operator [22]. The Kronecker product has an important role in the linear matrix equation theory. The solution of the Sylvester and the Sylvester-like equations is a hotspot research area. Recently, the innovational and computationally efficient numerical algorithms based on the hierarchical identification principle for the generalized Sylvester matrix equations [23–25] and coupled matrix equations [10, 26] were proposed by Ding and Chen. On the other hand, the iterative algorithms for the extended Sylvester-conjugate matrix equations were discussed in [27–29]. Other related work is included in [30–32].

This paper establishes a new result about the singular value of the Kronecker product and gives a definition of the vec-permutation matrix. In addition, we prove the mixed products theorem and the conclusions on the vector operator in a different method.

This paper is organized as follows. Section 2 gives the definition of the Kronecker product. Section 3 lists some properties based on the the mixed products theorem. Section 4 presents some interesting results about the vector operator and the vec-permutation matrices. Section 5 discusses the determinant, trace, and rank properties and the properties of polynomial matrices.

2. The Definition and the Basic Properties of the Kronecker Product

Let $\mathbb{F}$ be a field, such as $\mathbb{R}$ or $\mathbb{C}$. For any matrices $\mathbf{A} = [a_{ij}] \in \mathbb{F}^{m\times n}$ and $\mathbf{B} \in \mathbb{F}^{p\times q}$, their Kronecker product (i.e., the direct product or tensor product), denoted as $\mathbf{A} \otimes \mathbf{B}$, is defined by

$$\mathbf{A} \otimes \mathbf{B} = \left[a_{ij}\mathbf{B}\right] = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \in \mathbb{F}^{(mp)\times(nq)}. \tag{1}$$

It is clear that the Kronecker product of two diagonal matrices is a diagonal matrix and the Kronecker product of two upper (lower) triangular matrices is an upper (lower) triangular matrix. Let $\mathbf{A}^T$ and $\mathbf{A}^H$ denote the transpose and the Hermitian transpose of matrix $\mathbf{A}$, respectively. $\mathbf{I}_m$ is an identity matrix with order $m \times m$. The following basic properties are obvious.

Basic properties as follows:

(1) $\mathbf{I}_m \otimes \mathbf{A} = \operatorname{diag}[\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}]$,

(2) if $\boldsymbol{\alpha} = [a_1, a_2, \ldots, a_m]^T$ and $\boldsymbol{\beta} = [b_1, b_2, \ldots, b_n]^T$, then, $\boldsymbol{\alpha}\boldsymbol{\beta}^T = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}^T = \boldsymbol{\beta}^T \otimes \boldsymbol{\alpha} \in \mathbb{F}^{m\times n}$,

(3) if $\mathbf{A} = [\mathbf{A}_{ij}]$ is a block matrix, then for any matrix $\mathbf{B}$, $\mathbf{A} \otimes \mathbf{B} = [\mathbf{A}_{ij} \otimes \mathbf{B}]$.

(4) $(\mu\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (\mu\mathbf{B}) = \mu(\mathbf{A} \otimes \mathbf{B})$,

(5) $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$,

(6) $\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$,

(7) $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}$,

(8) $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$,

(9) $(\mathbf{A} \otimes \mathbf{B})^H = \mathbf{A}^H \otimes \mathbf{B}^H$.

Property 2 indicates that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}^T$ are commutative. Property 7 shows that $\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}$ is unambiguous.

3. The Properties of the Mixed Products

This section discusses the properties based on the mixed products theorem [6, 33, 34].

Theorem 1. *Let* $\mathbf{A} \in \mathbb{F}^{m\times n}$ *and* $\mathbf{B} \in \mathbb{F}^{p\times q}$*; then*

$$\mathbf{A} \otimes \mathbf{B} = \left(\mathbf{A} \otimes \mathbf{I}_p\right)\left(\mathbf{I}_n \otimes \mathbf{B}\right) = \left(\mathbf{I}_m \otimes \mathbf{B}\right)\left(\mathbf{A} \otimes \mathbf{I}_q\right). \tag{2}$$

Proof. According to the definition of the Kronecker product and the matrix multiplication, we have

$$\begin{aligned}
\mathbf{A} \otimes \mathbf{B} &= \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}\mathbf{I}_p & a_{12}\mathbf{I}_p & \cdots & a_{1n}\mathbf{I}_p \\ a_{21}\mathbf{I}_p & a_{22}\mathbf{I}_p & \cdots & a_{2n}\mathbf{I}_p \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{I}_p & a_{m2}\mathbf{I}_p & \cdots & a_{mn}\mathbf{I}_p \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B} \end{bmatrix} \\
&= \left(\mathbf{A} \otimes \mathbf{I}_p\right)\left(\mathbf{I}_n \otimes \mathbf{B}\right), \\
\mathbf{A} \otimes \mathbf{B} &= \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B} \end{bmatrix} \begin{bmatrix} a_{11}\mathbf{I}_q & a_{12}\mathbf{I}_q & \cdots & a_{1n}\mathbf{I}_q \\ a_{21}\mathbf{I}_q & a_{22}\mathbf{I}_q & \cdots & a_{2n}\mathbf{I}_q \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{I}_q & a_{m2}\mathbf{I}_q & \cdots & a_{mn}\mathbf{I}_q \end{bmatrix} \\
&= \left(\mathbf{I}_m \otimes \mathbf{B}\right)\left(\mathbf{A} \otimes \mathbf{I}_q\right).
\end{aligned} \tag{3}$$

□

From Theorem 1, we have the following corollary.

Corollary 2. *Let* $\mathbf{A} \in \mathbb{F}^{m\times m}$ *and* $\mathbf{B} \in \mathbb{F}^{n\times n}$*. Then*

$$\mathbf{A} \otimes \mathbf{B} = \left(\mathbf{A} \otimes \mathbf{I}_n\right)\left(\mathbf{I}_m \otimes \mathbf{B}\right) = \left(\mathbf{I}_m \otimes \mathbf{B}\right)\left(\mathbf{A} \otimes \mathbf{I}_n\right). \tag{4}$$

This mean that $\mathbf{I}_m \otimes \mathbf{B}$ *and* $\mathbf{A} \otimes \mathbf{I}_n$ *are commutative for square matrices* $\mathbf{A}$ *and* $\mathbf{B}$*.*

Using Theorem 1, we can prove the following mixed products theorem.

Theorem 3. *Let* $\mathbf{A} = [a_{ij}] \in \mathbb{F}^{m\times n}$, $\mathbf{C} = [c_{ij}] \in \mathbb{F}^{n\times p}$, $\mathbf{B} \in \mathbb{F}^{q\times r}$, *and* $\mathbf{D} \in \mathbb{F}^{r\times s}$*. Then*

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}). \tag{5}$$

Proof. According to Theorem 1, we have

$$\begin{aligned}
&(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) \\
&\quad = \left(\mathbf{A} \otimes \mathbf{I}_q\right)\left(\mathbf{I}_n \otimes \mathbf{B}\right)\left(\mathbf{C} \otimes \mathbf{I}_r\right)\left(\mathbf{I}_p \otimes \mathbf{D}\right) \\
&\quad = \left(\mathbf{A} \otimes \mathbf{I}_q\right)\left[\left(\mathbf{I}_n \otimes \mathbf{B}\right)\left(\mathbf{C} \otimes \mathbf{I}_r\right)\right]\left(\mathbf{I}_p \otimes \mathbf{D}\right) \\
&\quad = \left(\mathbf{A} \otimes \mathbf{I}_q\right)(\mathbf{C} \otimes \mathbf{B})\left(\mathbf{I}_p \otimes \mathbf{D}\right) \\
&\quad = \left(\mathbf{A} \otimes \mathbf{I}_q\right)\left[\left(\mathbf{C} \otimes \mathbf{I}_q\right)\left(\mathbf{I}_p \otimes \mathbf{B}\right)\right]\left(\mathbf{I}_p \otimes \mathbf{D}\right)
\end{aligned}$$

$$= \left[\left(\mathbf{A} \otimes \mathbf{I}_q\right)\left(\mathbf{C} \otimes \mathbf{I}_q\right)\right]\left[\left(\mathbf{I}_p \otimes \mathbf{B}\right)\left(\mathbf{I}_p \otimes \mathbf{D}\right)\right]$$
$$= \left[(\mathbf{AC}) \otimes \mathbf{I}_q)\right]\left[\mathbf{I}_p \otimes (\mathbf{BD})\right]$$
$$= (\mathbf{AC}) \otimes (\mathbf{BD}). \tag{6}$$

□

Let $\mathbf{A}^{[1]} := \mathbf{A}$ and define the Kronecker power by

$$\mathbf{A}^{[k+1]} := \mathbf{A}^{[k]} \otimes \mathbf{A} = \mathbf{A} \otimes \mathbf{A}^{[k]}, \quad k = 1, 2, \ldots. \tag{7}$$

From Theorem 3, we have the following corollary [7].

Corollary 4. *If the following matrix products exist, then one has*

(1) $(\mathbf{A}_1 \otimes \mathbf{B}_1)(\mathbf{A}_2 \otimes \mathbf{B}_2)\cdots(\mathbf{A}_p \otimes \mathbf{B}_p) = (\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_p) \otimes (\mathbf{B}_1\mathbf{B}_2\cdots\mathbf{B}_p)$,

(2) $(\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \cdots \otimes \mathbf{A}_p)(\mathbf{B}_1 \otimes \mathbf{B}_2 \otimes \cdots \otimes \mathbf{B}_p) = (\mathbf{A}_1\mathbf{B}_1) \otimes (\mathbf{A}_2\mathbf{B}_2) \otimes \cdots \otimes (\mathbf{A}_p\mathbf{B}_p)$,

(3) $[\mathbf{AB}]^{[k]} = \mathbf{A}^{[k]}\mathbf{B}^{[k]}$.

A square matrix $\mathbf{A}$ is said to be a normal matrix if and only if $\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H$. A square matrix $\mathbf{A}$ is said to be a unitary matrix if and only if $\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H = \mathbf{I}$. Straightforward calculation gives the following conclusions [6, 7, 33, 34].

Theorem 5. *For any square matrices $\mathbf{A}$ and $\mathbf{B}$,*

(1) *if $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist, then $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$,*

(2) *if $\mathbf{A}$ and $\mathbf{B}$ are normal matrices, then $\mathbf{A} \otimes \mathbf{B}$ is a normal matrix,*

(3) *if $\mathbf{A}$ and $\mathbf{B}$ are unitary (orthogonal) matrices, then $\mathbf{A} \otimes \mathbf{B}$ is a unitary (orthogonal) matrix,*

Let $\lambda[\mathbf{A}] := \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ denote the eigenvalues of $\mathbf{A}$ and let $\sigma[\mathbf{A}] := \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ denote the nonzero singular values of $\mathbf{A}$. According to the definition of the eigenvalue and Theorem 3, we have the following conclusions [34].

Theorem 6. *Let $\mathbf{A} \in \mathbb{F}^{m\times m}$ and $\mathbf{B} \in \mathbb{F}^{n\times n}$; k and l are positive integers. Then $\lambda[\mathbf{A}^k \otimes \mathbf{B}^l] = \{\lambda_i^k\mu_j^l \mid i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n\} = \lambda[\mathbf{B}^l \otimes \mathbf{A}^k]$. Here, $\lambda[\mathbf{A}] = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ and $\lambda[\mathbf{B}] = \{\mu_1, \mu_2, \ldots, \mu_n\}$.*

According to the definition of the singular value and Theorem 3, for any matrices $\mathbf{A}$ and $\mathbf{B}$, we have the next theorem.

Theorem 7. *Let $\mathbf{A} \in \mathbb{F}^{m\times n}$ and $\mathbf{B} \in \mathbb{F}^{p\times q}$. If $\operatorname{rank}[\mathbf{A}] = r$, $\sigma[\mathbf{A}] = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, $\operatorname{rank}[\mathbf{B}] = s$, and $\sigma[\mathbf{B}] = \{\rho_1, \rho_2, \ldots, \rho_s\}$, then $\sigma[\mathbf{A} \otimes \mathbf{B}] = \{\sigma_i\rho_j \mid i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, s\} = \sigma[\mathbf{B} \otimes \mathbf{A}]$.*

Proof. According to the singular value decomposition theorem, there exist the unitary matrices $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$, $\mathbf{Q}$ which satisfy

$$\mathbf{A} = \mathbf{U}\begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{V}, \qquad \mathbf{B} = \begin{bmatrix}\boldsymbol{\Gamma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}, \tag{8}$$

where $\boldsymbol{\Sigma} = \operatorname{diag}[\sigma_1, \sigma_2, \ldots, \sigma_r]$ and $\boldsymbol{\Gamma} = \operatorname{diag}[\rho_1, \rho_2, \ldots, \rho_s]$. According to Corollary 4, we have

$$\begin{aligned}
\mathbf{A} \otimes \mathbf{B} &= \left\{\mathbf{U}\begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{V}\right\} \otimes \left\{\mathbf{W}\begin{bmatrix}\boldsymbol{\Gamma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}\right\}\\
&= (\mathbf{U} \otimes \mathbf{W})\left\{\begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \otimes \begin{bmatrix}\boldsymbol{\Gamma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right\}(\mathbf{V} \otimes \mathbf{Q})\\
&= (\mathbf{U} \otimes \mathbf{W})\begin{bmatrix}\boldsymbol{\Sigma} \otimes \begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}(\mathbf{V} \otimes \mathbf{Q})\\
&= (\mathbf{U} \otimes \mathbf{W})\begin{bmatrix}\boldsymbol{\Sigma} \otimes \Gamma & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}(\mathbf{V} \otimes \mathbf{Q}).
\end{aligned} \tag{9}$$

Since $\mathbf{U} \otimes \mathbf{W}$ and $\mathbf{V} \otimes \mathbf{Q}$ are unitary matrices and $\boldsymbol{\Sigma} \otimes \boldsymbol{\Gamma} = \operatorname{diag}[\sigma_1\rho_1, \sigma_1\rho_2, \ldots, \sigma_1\rho_s, \ldots, \sigma_r\rho_s]$, this proves the theorem. □

According to Theorem 7, we have the next corollary.

Corollary 8. *For any matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, one has $\sigma[\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}] = \sigma[\mathbf{C} \otimes \mathbf{B} \otimes \mathbf{A}]$.*

4. The Properties of the Vector Operator and the Vec-Permutation Matrix

In this section, we introduce a vector-valued operator and a vec-permutation matrix.

Let $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \in \mathbb{F}^{m\times n}$, where $\mathbf{a}_j \in \mathbb{F}^m$, $j = 1, 2, \ldots, n$; then the vector $\operatorname{col}[\mathbf{A}]$ is defined by

$$\operatorname{col}[\mathbf{A}] := \begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_n\end{bmatrix} \in \mathbb{F}^{mn}. \tag{10}$$

Theorem 9. *Let $\mathbf{A} \in \mathbb{F}^{m\times n}$, $\mathbf{B} \in \mathbb{F}^{n\times p}$, and $\mathbf{C} \in \mathbb{F}^{p\times n}$, Then*

(1) $(\mathbf{I}_p \otimes \mathbf{A})\operatorname{col}[\mathbf{B}] = \operatorname{col}[\mathbf{AB}]$,

(2) $(\mathbf{A} \otimes \mathbf{I}_p)\operatorname{col}[\mathbf{C}] = \operatorname{col}[\mathbf{C}\mathbf{A}^T]$.

Proof. Let $(\mathbf{B})_i$ denote the ith column of matrix $\mathbf{B}$; we have

$$\begin{aligned}
\left(\mathbf{I}_p \otimes \mathbf{A}\right)\operatorname{col}[\mathbf{B}] &= \begin{bmatrix}\mathbf{A} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{A} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}\end{bmatrix}\begin{bmatrix}(\mathbf{B})_1\\ (\mathbf{B})_2\\ \vdots\\ (\mathbf{B})_p\end{bmatrix}\\
&= \begin{bmatrix}\mathbf{A}(\mathbf{B})_1\\ \mathbf{A}(\mathbf{B})_2\\ \vdots\\ \mathbf{A}(\mathbf{B})_p\end{bmatrix} = \begin{bmatrix}(\mathbf{AB})_1\\ (\mathbf{AB})_2\\ \vdots\\ (\mathbf{AB})_p\end{bmatrix} = \operatorname{col}[\mathbf{AB}].
\end{aligned} \tag{11}$$

Similarly, we have

$$\begin{aligned}
&(\mathbf{A}\otimes\mathbf{I}_p)\,\mathrm{col}[\mathbf{C}]\\
&=\begin{bmatrix} a_{11}\mathbf{I}_p & a_{12}\mathbf{I}_p & \cdots & a_{1n}\mathbf{I}_p \\ a_{21}\mathbf{I}_p & a_{22}\mathbf{I}_p & \cdots & a_{2n}\mathbf{I}_p \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{I}_p & a_{m2}\mathbf{I}_p & \cdots & a_{mn}\mathbf{I}_p \end{bmatrix}\begin{bmatrix} (\mathbf{C})_1 \\ (\mathbf{C})_2 \\ \vdots \\ (\mathbf{C})_n \end{bmatrix}\\
&=\begin{bmatrix} a_{11}(\mathbf{C})_1 + a_{12}(\mathbf{C})_2 + \cdots + a_{1n}(\mathbf{C})_n \\ a_{21}(\mathbf{C})_1 + a_{22}(\mathbf{C})_2 + \cdots + a_{2n}(\mathbf{C})_n \\ \vdots \\ a_{m1}(\mathbf{C})_1 + a_{m2}(\mathbf{C})_2 + \cdots + a_{mn}(\mathbf{C})_n \end{bmatrix}\\
&=\begin{bmatrix} \mathbf{C}(\mathbf{A}^T)_1 \\ \mathbf{C}(\mathbf{A}^T)_2 \\ \vdots \\ \mathbf{C}(\mathbf{A}^T)_m \end{bmatrix} = \begin{bmatrix} (\mathbf{C}\mathbf{A}^T)_1 \\ (\mathbf{C}\mathbf{A}^T)_2 \\ \vdots \\ (\mathbf{C}\mathbf{A}^T)_m \end{bmatrix} = \mathrm{col}\left[\mathbf{C}\mathbf{A}^T\right].
\end{aligned} \tag{12}$$

□

Theorem 10. *Let* $\mathbf{A}\in\mathbb{F}^{m\times n}$, $\mathbf{B}\in\mathbb{F}^{n\times p}$, *and* $\mathbf{C}\in\mathbb{F}^{p\times q}$. *Then*

$$\mathrm{col}[\mathbf{ABC}] = (\mathbf{C}^T\otimes\mathbf{A})\,\mathrm{col}[\mathbf{B}]. \tag{13}$$

Proof. According to Theorems 9 and 1, we have

$$\begin{aligned}
\mathrm{col}[\mathbf{ABC}] &= \mathrm{col}[(\mathbf{AB})\mathbf{C}]\\
&= (\mathbf{C}^T\otimes\mathbf{I}_m)\,\mathrm{col}[\mathbf{AB}]\\
&= (\mathbf{C}^T\otimes\mathbf{I}_m)(\mathbf{I}_p\otimes\mathbf{A})\,\mathrm{col}[\mathbf{B}]\\
&= \left[(\mathbf{C}^T\otimes\mathbf{I}_m)(\mathbf{I}_p\otimes\mathbf{A})\right]\mathrm{col}[\mathbf{B}]\\
&= (\mathbf{C}^T\otimes\mathbf{A})\,\mathrm{col}[\mathbf{B}].
\end{aligned} \tag{14}$$

□

Theorem 10 plays an important role in solving the matrix equations [25, 35–37], system identification [38–54], and control theory [55–58].

Let $\mathbf{e}_{in}$ denote an n-dimensional column vector which has 1 in the ith position and 0's elsewhere; that is,

$$\mathbf{e}_{in} := [0,0,\ldots,0,1,0,\ldots,0]^T. \tag{15}$$

Define the vec-permutation matrix

$$\mathbf{P}_{mn} := \begin{bmatrix} \mathbf{I}_m\otimes\mathbf{e}_{1n}^T \\ \mathbf{I}_m\otimes\mathbf{e}_{2n}^T \\ \vdots \\ \mathbf{I}_m\otimes\mathbf{e}_{nn}^T \end{bmatrix} \in\mathbb{R}^{mn\times mn}, \tag{16}$$

which can be expressed as [6, 7, 33, 37]

$$\sum_{j=1}^{m}\sum_{k=1}^{n}(\mathbf{e}_{kn}\otimes\mathbf{e}_{jm})(\mathbf{e}_{jm}\otimes\mathbf{e}_{kn})^T. \tag{17}$$

These two definitions of the vec-permutation matrix are equivalent; that is,

$$\sum_{j=1}^{m}\sum_{k=1}^{n}(\mathbf{e}_{kn}\otimes\mathbf{e}_{jm})(\mathbf{e}_{jm}\otimes\mathbf{e}_{kn})^T = \mathbf{P}_{mn}. \tag{18}$$

In fact, according to Theorem 3 and the basic properties of the Kronecker product, we have

$$\begin{aligned}
&\sum_{j=1}^{m}\sum_{k=1}^{n}(\mathbf{e}_{kn}\otimes\mathbf{e}_{jm})(\mathbf{e}_{jm}\otimes\mathbf{e}_{kn})^T\\
&\quad=\sum_{j=1}^{m}\sum_{k=1}^{n}(\mathbf{e}_{kn}\otimes\mathbf{e}_{jm})(\mathbf{e}_{jm}^T\otimes\mathbf{e}_{kn}^T)\\
&\quad=\sum_{j=1}^{m}\sum_{k=1}^{n}(\mathbf{e}_{kn}\mathbf{e}_{jm}^T)\otimes(\mathbf{e}_{jm}\mathbf{e}_{kn}^T)\\
&\quad=\sum_{j=1}^{m}\sum_{k=1}^{n}(\mathbf{e}_{kn}\otimes\mathbf{e}_{jm}^T)\otimes(\mathbf{e}_{jm}\otimes\mathbf{e}_{kn}^T)\\
&\quad=\sum_{j=1}^{m}\sum_{k=1}^{n}\left[\mathbf{e}_{kn}\otimes(\mathbf{e}_{jm}^T\otimes\mathbf{e}_{jm})\otimes\mathbf{e}_{kn}^T\right]\\
&\quad=\sum_{k=1}^{n}\left[\mathbf{e}_{kn}\otimes\sum_{j=1}^{m}(\mathbf{e}_{jm}^T\otimes\mathbf{e}_{jm})\otimes\mathbf{e}_{kn}^T\right]\\
&\quad=\sum_{k=1}^{n}\left[\mathbf{e}_{kn}\otimes\mathbf{I}_m\otimes\mathbf{e}_{kn}^T\right]\\
&\quad=\begin{bmatrix} \mathbf{I}_m\otimes\mathbf{e}_{1n}^T \\ \mathbf{I}_m\otimes\mathbf{e}_{2n}^T \\ \vdots \\ \mathbf{I}_m\otimes\mathbf{e}_{nn}^T \end{bmatrix}\\
&\quad=\mathbf{P}_{mn}.
\end{aligned} \tag{19}$$

Based on the definition of the vec-permutation matrix, we have the following conclusions.

Theorem 11. *According to the definition of* $\mathbf{P}_{mn}$, *one has*

(1) $\mathbf{P}_{mn}^T = \mathbf{P}_{nm}$,

(2) $\mathbf{P}_{mn}^T\mathbf{P}_{mn} = \mathbf{P}_{mn}\mathbf{P}_{mn}^T = \mathbf{I}_{mn}$.

That is, $\mathbf{P}_{mn}$ *is an* $(mn)\times(mn)$ *permutation matrix.*

Proof. According to the definition of $\mathbf{P}_{mn}$, Theorem 3, and the basic properties of the Kronecker product, we have

$$
\begin{aligned}
\mathbf{P}_{mn}^T &= \begin{bmatrix} \mathbf{I}_m \otimes \mathbf{e}_{1n}^T \\ \mathbf{I}_m \otimes \mathbf{e}_{2n}^T \\ \vdots \\ \mathbf{I}_m \otimes \mathbf{e}_{nn}^T \end{bmatrix}^T \\
&= \left[\mathbf{I}_m^T \otimes \mathbf{e}_{1n}, \mathbf{I}_m^T \otimes \mathbf{e}_{2n}, \dots, \mathbf{I}_m^T \otimes \mathbf{e}_{nn}\right] \\
&= \begin{bmatrix} \mathbf{e}_{1m}^T \otimes \mathbf{e}_{1n} & \mathbf{e}_{1m}^T \otimes \mathbf{e}_{2n} & \cdots & \mathbf{e}_{1m}^T \otimes \mathbf{e}_{nn} \\ \mathbf{e}_{2m}^T \otimes \mathbf{e}_{1n} & \mathbf{e}_{2m}^T \otimes \mathbf{e}_{2n} & \cdots & \mathbf{e}_{2m}^T \otimes \mathbf{e}_{nn} \\ \vdots & \vdots & & \vdots \\ \mathbf{e}_{mm}^T \otimes \mathbf{e}_{1n} & \mathbf{e}_{mm}^T \otimes \mathbf{e}_{2n} & \cdots & \mathbf{e}_{mm}^T \otimes \mathbf{e}_{nn} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{e}_{1n} \otimes \mathbf{e}_{1m}^T & \mathbf{e}_{2n} \otimes \mathbf{e}_{1m}^T & \cdots & \mathbf{e}_{nn} \otimes \mathbf{e}_{1m}^T \\ \mathbf{e}_{1n} \otimes \mathbf{e}_{2m}^T & \mathbf{e}_{2n} \otimes \mathbf{e}_{2m}^T & \cdots & \mathbf{e}_{nn} \otimes \mathbf{e}_{2m}^T \\ \vdots & \vdots & & \vdots \\ \mathbf{e}_{1n} \otimes \mathbf{e}_{mm}^T & \mathbf{e}_{2n} \otimes \mathbf{e}_{mm}^T & \cdots & \mathbf{e}_{nn} \otimes \mathbf{e}_{mm}^T \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{I}_n \otimes \mathbf{e}_{1m}^T \\ \mathbf{I}_n \otimes \mathbf{e}_{2m}^T \\ \vdots \\ \mathbf{I}_n \otimes \mathbf{e}_{mm}^T \end{bmatrix} \\
&= \mathbf{P}_{nm},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\mathbf{P}_{mn}\mathbf{P}_{mn} \\
&\quad= \begin{bmatrix} \mathbf{I}_m \otimes \mathbf{e}_{1n}^T \\ \mathbf{I}_m \otimes \mathbf{e}_{2n}^T \\ \vdots \\ \mathbf{I}_m \otimes \mathbf{e}_{nn}^T \end{bmatrix} \left[\mathbf{I}_m \otimes \mathbf{e}_{1n}, \mathbf{I}_m \otimes \mathbf{e}_{2n}, \dots, \mathbf{I}_m \otimes \mathbf{e}_{nn}\right] \\
&\quad= \begin{bmatrix} \mathbf{I}_m \otimes \left(\mathbf{e}_{1n}^T\mathbf{e}_{1n}\right) & \mathbf{I}_m \otimes \left(\mathbf{e}_{1n}^T\mathbf{e}_{2n}\right) & \cdots & \mathbf{I}_m \otimes \left(\mathbf{e}_{1n}^T\mathbf{e}_{nn}\right) \\ \mathbf{I}_m \otimes \left(\mathbf{e}_{2n}^T\mathbf{e}_{1n}\right) & \mathbf{I}_m \otimes \left(\mathbf{e}_{2n}^T\mathbf{e}_{2n}\right) & \cdots & \mathbf{I}_m \otimes \left(\mathbf{e}_{2n}^T\mathbf{e}_{nn}\right) \\ \vdots & \vdots & & \vdots \\ \mathbf{I}_m \otimes \left(\mathbf{e}_{nn}^T\mathbf{e}_{1n}\right) & \mathbf{I}_m \otimes \left(\mathbf{e}_{nn}^T\mathbf{e}_{2n}\right) & \cdots & \mathbf{I}_m \otimes \left(\mathbf{e}_{nn}^T\mathbf{e}_{nn}\right) \end{bmatrix} \\
&\quad= \begin{bmatrix} \mathbf{I}_m & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_m \end{bmatrix} \\
&\quad= \mathbf{I}_{mn},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\mathbf{P}_{mn}^T\mathbf{P}_{mn} &= \left[\mathbf{I}_m \otimes \mathbf{e}_{1n}, \mathbf{I}_m \otimes \mathbf{e}_{2n}, \dots, \mathbf{I}_m \otimes \mathbf{e}_{nn}\right] \begin{bmatrix} \mathbf{I}_m \otimes \mathbf{e}_{1n}^T \\ \mathbf{I}_m \otimes \mathbf{e}_{2n}^T \\ \vdots \\ \mathbf{I}_m \otimes \mathbf{e}_{nn}^T \end{bmatrix} \\
&= \mathbf{I}_m \otimes \left[\sum_{i=1}^{n} \mathbf{e}_{in}\mathbf{e}_{in}^T\right] \\
&= \mathbf{I}_m \otimes \mathbf{I}_n \\
&= \mathbf{I}_{mn}.
\end{aligned}
\tag{22}
$$

For any matrix $\mathbf{A} \in \mathbb{F}^{m\times n}$, we have $\operatorname{col}[\mathbf{A}] = \mathbf{P}_{mn}\operatorname{col}[\mathbf{A}^T]$. □

Theorem 12. *If* $\mathbf{A} \in \mathbb{F}^{m\times n}$ *and* $\mathbf{B} \in \mathbb{F}^{p\times q}$, *then one has* $\mathbf{P}_{mp}(\mathbf{A}\otimes \mathbf{B})\mathbf{P}_{nq}^T = \mathbf{B} \otimes \mathbf{A}$.

Proof. Let $\mathbf{B} := [b_{ij}] = \begin{bmatrix} \mathbf{B}^1 \\ \mathbf{B}^2 \\ \vdots \\ \mathbf{B}^p \end{bmatrix}$, where $\mathbf{B}^i \in \mathbb{F}^{1\times q}$ and $i = 1, 2, \dots, p$, and $j = 1, 2, \dots, q$. According to the definition of $\mathbf{P}_{mn}$ and the Kronecker product, we have

$$
\begin{aligned}
&\mathbf{P}_{mp}(\mathbf{A}\otimes\mathbf{B})\mathbf{P}_{nq}^T \\
&\quad= \begin{bmatrix} \mathbf{I}_m \otimes \mathbf{e}_{1p}^T \\ \mathbf{I}_m \otimes \mathbf{e}_{2p}^T \\ \vdots \\ \mathbf{I}_m \otimes \mathbf{e}_{pp}^T \end{bmatrix} \left[(\mathbf{A})_1 \otimes \mathbf{B}, (\mathbf{A})_2 \otimes \mathbf{B}, \dots, (\mathbf{A})_n \otimes \mathbf{B}\right] \mathbf{P}_{nq}^T \\
&\quad= \begin{bmatrix} (\mathbf{A})_1 \otimes \mathbf{B}^1 & (\mathbf{A})_2 \otimes \mathbf{B}^1 & \cdots & (\mathbf{A})_n \otimes \mathbf{B}^1 \\ (\mathbf{A})_1 \otimes \mathbf{B}^2 & (\mathbf{A})_2 \otimes \mathbf{B}^2 & \cdots & (\mathbf{A})_n \otimes \mathbf{B}^2 \\ \vdots & \vdots & & \vdots \\ (\mathbf{A})_1 \otimes \mathbf{B}^p & (\mathbf{A})_2 \otimes \mathbf{B}^p & \cdots & (\mathbf{A})_n \otimes \mathbf{B}^p \end{bmatrix} \mathbf{P}_{nq}^T \\
&\quad= \begin{bmatrix} \mathbf{A} \otimes \mathbf{B}^1 \\ \mathbf{A} \otimes \mathbf{B}^2 \\ \vdots \\ \mathbf{A} \otimes \mathbf{B}^p \end{bmatrix} \left[\mathbf{I}_n \otimes \mathbf{e}_{1q}, \mathbf{I}_n \otimes \mathbf{e}_{2q}, \dots, \mathbf{I}_n \otimes \mathbf{e}_{qq}\right] \\
&\quad= \begin{bmatrix} \mathbf{A}b_{11} & \mathbf{A}b_{12} & \cdots & \mathbf{A}b_{1q} \\ \mathbf{A}b_{21} & \mathbf{A}b_{22} & \cdots & \mathbf{A}b_{2q} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}b_{p1} & \mathbf{A}b_{p2} & \cdots & \mathbf{A}b_{pq} \end{bmatrix} \\
&\quad= \mathbf{B} \otimes \mathbf{A}.
\end{aligned}
\tag{23}
$$

□

From Theorem 12, we have the following corollaries.

Corollary 13. *If* $\mathbf{A} \in \mathbb{F}^{m\times n}$, *then* $\mathbf{P}_{mr}(\mathbf{A}\otimes \mathbf{I}_r)\mathbf{P}_{nr}^T = \mathbf{I}_r \otimes \mathbf{A}$.

Corollary 14. *If* $\mathbf{A} \in \mathbb{F}^{m\times n}$ *and* $\mathbf{B} \in \mathbb{F}^{n\times m}$, *then*

$$\mathbf{B} \otimes \mathbf{A} = \mathbf{P}_{mn} (\mathbf{A} \otimes \mathbf{B}) \mathbf{P}_{nm}^{T} = \mathbf{P}_{mn} \left[(\mathbf{A} \otimes \mathbf{B}) \mathbf{P}_{mn}^{2}\right] \mathbf{P}_{nm}^{T}. \quad (24)$$

That is, $\lambda[\mathbf{B} \otimes \mathbf{A}] = \lambda[(\mathbf{A} \otimes \mathbf{B})\mathbf{P}_{mn}^{2}]$. *When* $\mathbf{A} \in \mathbb{F}^{n\times n}$ *and* $\mathbf{B} \in \mathbb{F}^{t\times t}$, *one has* $\mathbf{B} \otimes \mathbf{A} = \mathbf{P}_{nt}(\mathbf{A} \otimes \mathbf{B})\mathbf{P}_{nt}^{T}$. *That is, if* $\mathbf{A}$ *and* $\mathbf{B}$ *are square matrices, then* $\mathbf{A} \otimes \mathbf{B}$ *is similar to* $\mathbf{B} \otimes \mathbf{A}$.

5. The Scalar Properties and the Polynomials Matrix of the Kronecker Product

In this section, we discuss the properties [6, 7, 34] of the determinant, the trace, the rank, and the polynomial matrix of the Kronecker product.

For $\mathbf{A} \in \mathbb{F}^{m\times m}$ and $\mathbf{B} \in \mathbb{F}^{n\times n}$, we have $|\mathbf{A}\otimes\mathbf{B}| = |\mathbf{A}|^n|\mathbf{B}|^m = |\mathbf{B} \otimes \mathbf{A}|$. If $\mathbf{A}$ and $\mathbf{B}$ are two square matrices, then we have $\mathrm{tr}[\mathbf{A} \otimes \mathbf{B}] = \mathrm{tr}[\mathbf{A}]\,\mathrm{tr}[\mathbf{B}] = \mathrm{tr}[\mathbf{B} \otimes \mathbf{A}]$. For any matrices $\mathbf{A}$ and $\mathbf{B}$, we have $\mathrm{rank}[\mathbf{A} \otimes \mathbf{B}] = \mathrm{rank}[\mathbf{A}]\,\mathrm{rank}[\mathbf{B}] = \mathrm{rank}[\mathbf{B} \otimes \mathbf{A}]$. According to these scalar properties, we have the following theorems.

Theorem 15. (1) *Let* $\mathbf{A}, \mathbf{C} \in \mathbb{F}^{m\times m}$ *and* $\mathbf{B}, \mathbf{D} \in \mathbb{F}^{n\times n}$. *Then*

$$\begin{aligned} |(\mathbf{A} \otimes \mathbf{B}) (\mathbf{C} \otimes \mathbf{D})| &= |(\mathbf{A} \otimes \mathbf{B})| \, |(\mathbf{C} \otimes \mathbf{D})| \\ &= (|\mathbf{A}| \, |\mathbf{C}|)^n (|\mathbf{B}||\mathbf{D}|)^m \\ &= |\mathbf{AC}|^n |\mathbf{BD}|^m. \end{aligned} \quad (25)$$

(2) *If* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, *and* $\mathbf{D}$ *are square matrices, then*

$$\begin{aligned} \mathrm{tr}\,[(\mathbf{A} \otimes \mathbf{B}) (\mathbf{C} \otimes \mathbf{D})] &= \mathrm{tr}\,[(\mathbf{AC}) \otimes (\mathbf{BD})] \\ &= \mathrm{tr}\,[\mathbf{AC}]\,\mathrm{tr}\,[\mathbf{BD}] \\ &= \mathrm{tr}\,[\mathbf{CA}]\,\mathrm{tr}\,[\mathbf{DB}]. \end{aligned} \quad (26)$$

(3) *Let* $\mathbf{A} \in \mathbb{F}^{m\times n}$, $\mathbf{C} \in \mathbb{F}^{n\times p}$, $\mathbf{B} \in \mathbb{F}^{q\times r}$, *and* $\mathbf{D} \in \mathbb{F}^{r\times s}$; *then*

$$\begin{aligned} \mathrm{rank}\,[(\mathbf{A} \otimes \mathbf{B}) (\mathbf{C} \otimes \mathbf{D})] &= \mathrm{rank}\,[(\mathbf{AC}) \otimes (\mathbf{BD})] \\ &= \mathrm{rank}\,[\mathbf{AC}]\,\mathrm{rank}\,[\mathbf{BD}]. \end{aligned} \quad (27)$$

Theorem 16. *If* $f(x, y) := x^r y^s$ *is a monomial and* $f(\mathbf{A}, \mathbf{B}) := \mathbf{A}^{[r]} \otimes \mathbf{B}^{[s]}$, *where* r, s *are positive integers, one has the following conclusions.*

(1) *Let* $\mathbf{A} \in \mathbb{F}^{m\times m}$ *and* $\mathbf{B} \in \mathbb{F}^{n\times n}$. *Then*

$$\left| f(\mathbf{A}, \mathbf{B}) \right| = |\mathbf{A}|^{rm^{r-1}n^s} |\mathbf{B}|^{sm^r n^{s-1}}. \quad (28)$$

(2) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are square matrices, then*

$$\mathrm{tr}\left[f(\mathbf{A}, \mathbf{B}) \right] = f(\mathrm{tr}\,[\mathbf{A}], \mathrm{tr}\,[\mathbf{B}]). \quad (29)$$

(3) *For any matrices* $\mathbf{A}$ *and* $\mathbf{B}$, *one has*

$$\mathrm{rank}\left[f(\mathbf{A}, \mathbf{B}) \right] = f(\mathrm{rank}\,[\mathbf{A}], \mathrm{rank}\,[\mathbf{B}]). \quad (30)$$

If $\lambda[\mathbf{A}] = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ and $f(x) = \sum_{i=1}^{k} c_i x^i$ is a polynomial, then the eigenvalues of

$$f(\mathbf{A}) = \sum_{i=1}^{k} c_i \mathbf{A}^i \quad (31)$$

are

$$f(\lambda_j) = \sum_{i=1}^{k} c_i \lambda_j^i, \quad j = 1, 2, \ldots, m. \quad (32)$$

Similarly, consider a polynomial $f(x, y)$ in two variables x and y:

$$f(x, y) = \sum_{i,j=1}^{k} c_{ij} x^i y^j, \quad c_{ij}, x, y \in \mathbb{F}, \quad (33)$$

where k is a positive integer. Define the polynomial matrix $f(\mathbf{A}, \mathbf{B})$ by the formula

$$f(\mathbf{A}, \mathbf{B}) = \sum_{i,j=1}^{k} c_{ij} \mathbf{A}^i \otimes \mathbf{B}^j. \quad (34)$$

According to Theorem 3, we have the following theorems [34].

Theorem 17. *Let* $\mathbf{A} \in \mathbb{F}^{m\times m}$ *and* $\mathbf{B} \in \mathbb{F}^{n\times n}$; *if* $\lambda[\mathbf{A}] = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ *and* $\lambda[\mathbf{B}] = \{\mu_1, \mu_2, \ldots, \mu_n\}$, *then the matrix* $f(\mathbf{A}, \mathbf{B})$ *has the eigenvalues*

$$f(\lambda_r, \mu_s) = \sum_{i,j=1}^{k} c_{ij} \lambda_r^i \mu_s^j, \quad r = 1, 2, \ldots, m, \; s = 1, 2, \ldots, n. \quad (35)$$

Theorem 18 (see [34]). *Let* $\mathbf{A} \in \mathbb{F}^{m\times m}$. *If* $f(z)$ *is an analytic function and* $f(\mathbf{A})$ *exists, then*

$$f(\mathbf{I}_n \otimes \mathbf{A}) = \mathbf{I}_n \otimes f(\mathbf{A}),$$
$$f(\mathbf{A} \otimes \mathbf{I}_n) = f(\mathbf{A}) \otimes \mathbf{I}_n.$$

Finally, we introduce some results about the Kronecker sum [7, 34]. The Kronecker sum of $\mathbf{A} \in \mathbb{F}^{m\times m}$ and $\mathbf{B} \in \mathbb{F}^{n\times n}$, denoted as $\mathbf{A} \oplus \mathbf{B}$, is defined by

$$\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_n + \mathbf{I}_m \otimes \mathbf{B}.$$

Theorem 19. *Let* $\mathbf{A} \in \mathbb{F}^{m\times m}$, *and* $\mathbf{B} \in \mathbb{F}^{n\times n}$. *Then*

$$\exp[\mathbf{A} \oplus \mathbf{B}] = \exp[\mathbf{A}] \otimes \exp[\mathbf{B}],$$
$$\sin(\mathbf{A} \oplus \mathbf{B}) = \sin(\mathbf{A}) \otimes \cos(\mathbf{B}) + \cos(\mathbf{A}) \otimes \sin(\mathbf{B}),$$
$$\cos(\mathbf{A} \oplus \mathbf{B}) = \cos(\mathbf{A}) \otimes \cos(\mathbf{B}) - \sin(\mathbf{A}) \otimes \sin(\mathbf{B}).$$

6. Conclusions

This paper establishes some conclusions on the Kronecker products and the vec-permutation matrix. A new presentation about the properties of the mixed products and the vector operator is given. All these obtained conclusions make the theory of the Kronecker product more complete.

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