*Research Article*

# Approximate Controllability of Fractional Integrodifferential Evolution Equations

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This paper addresses the issue of approximate controllability for a class of control system which is represented by nonlinear fractional integrodifferential equations with nonlocal conditions. By using semigroup theory, $p$-mean continuity and fractional calculations, a set of sufficient conditions, are formulated and proved for the nonlinear fractional control systems. More precisely, the results are established under the assumption that the corresponding linear system is approximately controllable and functions satisfy non-Lipschitz conditions. The results generalize and improve some known results.

## 1. Introduction

In recent years, use theory of fractional calculus and fractional differential equations has gained importance and popularity due to its applications in various fields of science and engineering. Various physical phenomena in science and engineering can be successfully modeled by using fractional calculus theory. Due to its tremendous scope and applications, several papers have been devoted to study the existence of mild solutions of fractional differential equations (see [1–4] and references therein). On the other hand, controllability is an important property of a control system which plays an important role in the analysis and design of control systems [5–8]. Most literatures in this direction so far have been concerned with controllability of nonlinear differential equations in infinite-dimensional spaces without fractional derivatives (see [9] and references therein). Using generalized open mapping theorem, a set of sufficient conditions for constrained local relative controllability near the origin are formulated and proved for the semilinear systems with delayed controls in [10, 11].

Recently, only few papers deal with the controllability of fractional dynamical systems [12–14]. Klamka [15, 16] derived a set of sufficient conditions for the local controllability of

finite-dimensional fractional discrete-time semilinear systems. However, the problem of controllability for fractional systems has not been fully investigated, and there is still room open for further research in this area [17]. Moreover, the approximate controllable systems are more prevalent, and very often, approximate controllability is completely adequate in applications (see [18–21] and references therein). Therefore, it is important, in fact necessary, to study the weaker concept of controllability, namely, approximate controllability for nonlinear fractional integrodifferential systems. Motivated by this fact, in this paper, we consider the approximate controllability of the fractional nonlinear integrodifferential evolution equations with nonlocal initial condition in the following form:

$$
\begin{aligned}
{}^{C}D^{q}x(t) &= -Ax(t) + Bu(t) + t^{n}f(t, x(t), (Hx)(t)), \\
& \quad t \in J = [0, b], \quad n \in Z^{+}, \quad q \in (0, 1), \quad (1) \\
x(0) &= g(x) + x_{0} \in X_{\gamma},
\end{aligned}
$$

where the state $x(\cdot)$ takes the values in a Hilbert space $X$, $^{C}D^{q}$ denotes Caputo derivative, $-A : D(A) \to X$ is the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$ on $X$; the control function $u(\cdot)$ is given in $L^{2}(J, U)$; $U$ is a

Hilbert space; $B$ is a bounded linear operator from $U$ into $X$; the operator $H$ is defined by $(Hx)(t) = \int_0^b h(t, s, x(s))ds$; the nonlinear term $f : J \times X_\gamma \times X_\gamma \rightarrow X$(or $X_\gamma$) is a given function, where here $J = [0, b]$; and $X_\gamma = D(A^\gamma)$ $(0 < \gamma < 1)$ is a Hilbert space with the norm $\|x\|_\gamma = \|A^\gamma x\|$ for $x \in X_\gamma$. The functions $f$, $h$, and $g$ will be specified later. In fact, our results in this paper are motivated by the recent work of [20], and the fractional integrodifferential equations are studied in [4]. The main objective of this paper is to derive conditions for the approximate controllability of (1) with non-Lipschitz coefficients, and the associated linear system is approximately controllable.

## 2. Preliminaries

In this section, we provide definitions, lemmas, and notations necessary to establish our main results [4]. Let $\|f\|_{L^p(J,R^+)}$ denote the $L^p(J, R^+)$ norm of $f$ whenever $f \in L^p(J, R^+)$ for some $p$ with $1 < p < \infty$. Let $\mathscr{C}_\gamma$ denote the Banach space $C(J, X_\gamma)$ endowed with sup norm given by $\|x\|_\infty \equiv \sup_{t \in J} \|x\|_\gamma$, for $x \in \mathscr{C}_\gamma$, $\gamma \in (0, 1)$.

Let us recall the following known results.

The fractional integral of order $\beta$ with the lower limit 0 for a function $f$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^{1-\beta}} ds, \quad t > 0, \ \beta > 0, \quad (2)$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Riemann-Liouville derivative of order $\beta$ with lower limit zero for a function $f : [0, \infty) \rightarrow R$ can be written as

$$^L D^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\beta+1-n}} ds, \quad (3)$$
$$t > 0, \quad n-1 < \beta < n.$$

The Caputo derivative of order $\beta$ for a function $f : [0, \infty) \rightarrow R$ can be written as

$$D^\beta f(t) = {}^L D^\beta \left( f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad (4)$$
$$t > 0, \quad n-1 < \beta < n.$$

Remark 1 (see [4]). (i) If $f(t) \in C^n[0, \infty)$, then

$$D^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^n(s)}{(t-s)^{\beta+1-n}} ds = I^{n-\beta} f^n(s), \quad (5)$$
$$t > 0, \quad 0 \le n-1 < \beta < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If $f$ is an abstract function with values in $X$, then integrals which appear in the above results are taken in Bochner's sense.

For additional details concerning the fractional derivative, we refer the reader to [3].

To define the mild solution for the control system (1), we associate problem (1) to the following integral equation [4]:

$$x(t) = \mathscr{T}(t) \left[ x_0 + g(x) \right] + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) Bu(s) ds$$
$$+ \int_0^t (t-s)^{q-1} s^n \mathscr{S}(t-s) f(s, x(s), (Hx)(s)) ds,$$
$$t \in J, \quad (6)$$

where $\mathscr{T}(t) = \int_0^\infty \xi_q(\theta) T(t^q \theta) d\theta$, $\mathscr{S}(t) = q \int_0^\infty \theta \xi_q(\theta) T(t^q \theta) d\theta$, $\xi_q(\theta) = (1/q) \theta^{-1-1/q} \overline{w}_q(\theta^{-1/q}) \ge 0$, $\overline{w}_q(\theta) = (1/\pi) \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} (\Gamma(nq+1)/n!) \sin(n\pi q)$, $\theta \in (0, \infty)$, and $\xi_q$ is a probability density function defined on $(0, \infty)$; that is $\xi_q(\theta) \ge 0$, $\theta \in (0, \infty)$, and $\int_0^\infty \xi_q(\theta) d\theta = 1$.

Definition 2. A function $x(\cdot; x_0, u) \in C(J, X_\gamma)$ is said to be a mild solution of (1) if for any $u(\cdot) \in L^2(J, U)$ the integral equation (6) is satisfied.

Definition 3. The system (1) is said to be approximately controllable on the interval $J$ if $\overline{\mathfrak{R}(b, x_0)} = X$, where

$$\mathfrak{R}(b, x_0) = \left\{ x_b(x_0; u)(0) : u(\cdot) \in L^2(J, U) \right\} \quad (7)$$

is called the reachable set of system (1) at terminal time $b$, and its closure in $X$ is denoted by $\overline{\mathfrak{R}(b, x_0)}$; let $x_b(x_0; u)$ be the state value of (1) at terminal time $b$ corresponding to the control $u$ and the initial value $x_0 \in X$.

Consider the following linear fractional differential control system

$$D_t^q x(t) = Ax(t) + Bu(t), \quad t \in J = [0, b], \ 0 < q < 1,$$
$$x(0) = x_0. \quad (8)$$

The approximate controllability for the system (8) is a natural generalization of approximate controllability for the linear first order control system (see [18]). It is convenient at this point to introduce the controllability operator associated with the linear system

$$\Gamma_0^b = \int_0^b (b-s)^{q-1} \mathscr{S}(b-\tau) BB^* \mathscr{S}^*(b-\tau) d\tau, \quad (9)$$

where $B^*$ denotes the adjoint of $B$ and $\mathscr{S}^*(t)$ is the adjoint of $\mathscr{S}(t)$. It is straightforward that the operator $\Gamma_0^b$ is a linear bounded operator. Let $R(\alpha, \Gamma_0^b) = (\alpha I + \Gamma_0^b)^{-1}$ for $\alpha > 0$.

**Lemma 4.** *The linear fractional control system* (8) *is approximately controllable on $J$ if and only if $\alpha R(\alpha, \Gamma_0^b) \rightarrow 0$ as $\alpha \rightarrow 0^+$ in the strong operator topology.*

The proof of this lemma is a straightforward adaptation of the proof of Theorem 2 of [18].

**Lemma 5** (see [4]). *For each $\psi \in L^p(J, X)$ with $1 \leq p < +\infty$, one has $\lim_{h\to 0}\int_0^b \|\psi(t+h) - \psi(t+h)\|^p dt = 0$, where $\psi(s) = 0$ for $s$ does not belong to $J$.*

**Lemma 6** (see [4]). *The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties.*

(i) *For fixed $t \geq 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are linear and bounded operators. For any $x \in X$,*

$$\|\mathscr{T}(t)x\| \leq M\|x\|, \qquad \|\mathscr{S}(t)x\| \leq \frac{Mq}{\Gamma(q+1)}\|x\|. \quad (10)$$

(ii) *$\{\mathscr{T}(t), t \geq 0\}$ and $\{\mathscr{S}(t), \geq 0\}$ are strongly continuous.*

(iii) *For every $t > 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are also compact operators.*

(iv) *For any $x \in X$, $\beta \in (0,1)$, and $\gamma \in (0,1)$ one has*

$$A\mathscr{S}(t)x = A^{1-\beta}\mathscr{S}(t)A^\beta x, \quad t \in J,$$

$$\left\|A^\gamma \mathscr{S}(t)\right\| \leq \frac{M_\gamma q\Gamma(2-\gamma)}{\Gamma(1+q(1-\gamma))}t^{-\gamma q}, \quad 0 < t \leq b. \quad (11)$$

(v) *For fixed $t \geq 0$ and any $x \in X_\gamma$, one has*

$$\|\mathscr{T}(t)x\|_\gamma \leq M\|x\|_\gamma, \qquad \|\mathscr{S}(t)x\|_\gamma \leq \frac{Mq}{\Gamma(q+1)}\|x\|_\gamma. \quad (12)$$

(vi) *$\mathscr{T}_\gamma(t)$ and $\mathscr{S}_\gamma(t)$, $t > 0$ are uniformly continuous; that is, for each fixed $t > 0$ and $\epsilon > 0$, there exists $h > 0$ such that*

$$\left\|\mathscr{T}_\gamma(t+\epsilon) - \mathscr{T}_\gamma(t)\right\|_\gamma < \epsilon, \quad \text{for } t+\epsilon \geq 0, \ |\epsilon| < h,$$

$$\left\|\mathscr{S}_\gamma(t+\epsilon) - \mathscr{S}_\gamma(t)\right\|_\gamma < \epsilon, \quad \text{for } t+\epsilon \geq 0, \ |\epsilon| < h, \quad (13)$$

*where $\mathscr{T}_\gamma(t) = \int_0^\infty \xi_q(\theta)T_\gamma(t^q\theta)d\theta$, $\mathscr{S}_\gamma(t) = q\int_0^\infty \theta\xi_q(\theta)T_\gamma(t^q\theta)d\theta$.*

## 3. Main Result

In this section, we present our main result on approximate controllability of control system (1). We prove that under certain conditions, approximate controllability of the linear system (8) implies the approximate controllability of nonlinear fractional system (1). In order to establish the result, we require the following assumptions.

($H_1$) The function $f : J \times X_\gamma \times X_\gamma \to X_\gamma$ is Carathéodory, and there exists a positive function $\rho \in L^p(J, R^+)$ for some $p$ with $1 < p < \infty$ such that $\|f(t, x, y)\|_\gamma \leq \rho(t)$ for all $x, y \in X_\gamma$ and $t \in J$.

($H_2$) The function $g : \mathscr{C}_\gamma \to X_\gamma$ is completely continuous, and there exist $\beta_1, \beta_2 > 0$ such that

$$\|g(x)\|_\gamma \leq \beta_1\|x\|_\infty + \beta_2. \quad (14)$$

($H_3$) The function $h : D_h \times X_\gamma \to X_\gamma$ is continuous, and there exist $L_1, L_2 > 0$ such that $\|h(t, s, x) - h(t, s, y)\|_\gamma \leq L_1\|x - y\|_\gamma + L_2$ for each $(t, s) \in D_h = \{(t, s) \in R^2, 0 \leq s, t \leq b\}$ and $x, y \in X_\gamma$.

($H_4$) The linear fractional control system (8) is approximately controllable.

($H_5$) The function $f : J \times X_\gamma \times X_\gamma \to X_\gamma$ is continuous and uniformly bounded, and there exists $N > 0$ such that $\|f(t, x, y)\| \leq N$ for all $(t, x, y) \in J \times X_\gamma \times X_\gamma$.

($H_6$) The semigroup $T(t)$ is compact.

In order to prove the required result, for $\alpha > 0$, we define the operator $F_\alpha$ on $C(J, X_\gamma)$ as

$$(F_\alpha x)(t) = z(t), \quad (15)$$

where

$$z(t) = \mathscr{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)Bv(s)\,ds$$

$$+ \int_0^t (t-s)^{q-1}s^n\mathscr{S}(t-s)f(s, x(s), (Hx)(s))\,ds,$$

$$t \in J, \quad (16)$$

$$v(t) = B^*\mathscr{S}^*(b-t)R\left(\alpha, \Gamma_0^b\right)p(x(\cdot)),$$

$$p(x(\cdot)) = x_b - \mathscr{T}(b)[x_0 + g(x)]$$

$$- \int_0^b (b-s)^{q-1}s^n\mathscr{S}(b-s) \quad (17)$$

$$\times f(s, x(s), (Hx)(s))\,ds.$$

**Theorem 7.** *Under the conditions $(H_1)$, $(H_2)$, $(H_3)$, and $(H_6)$, the control system (1) admits a mild solution on $[0, b]$, and here $M_B = \|B\|$ and $M_1 = \|\rho\|_{L^{p^2}(J, R^+)}$.*

*Proof.* The main aim in this section is to find conditions for solvability of systems (16) and (17) for $\alpha > 0$. In the Banach space $C(J, X_\gamma)$, consider a set

$$B_{r_0} = \left\{x(\cdot) \in C\left(J, X_\gamma\right) : \|x\|_\gamma \leq r\right\}, \quad (18)$$

where $r$ is the positive constant. Now, it will be shown that, using Schauder's fixed point theorem, for all $\alpha > 0$, the operator $F_\alpha : C(J, X_\gamma) \to C(J, X_\gamma)$ has a fixed point.

First, we prove that for an arbitrary $\alpha > 0$ and there is a positive constant $r_0 = r_0(\alpha)$ such that $F_\alpha : B_{r_0} \to B_{r_0}$.

Let $x \in B_{r_0}$, and then, for $t \in J$ and $1/p < q(\Rightarrow (pq-1)) > 0$, using Holders inequality, Lemma 6, and conditions $(H_1)$ and $(H_2)$ in (17), we obtain

$$
\begin{aligned}
\|v(t)\|_\gamma \leq\ & \left\| B^* \mathcal{S}^* (b-t) R\left(\alpha, \Gamma_0^b\right) \right\| \\
& \times \Bigg[ \|x_b\|_\gamma - \|\mathcal{T}(b)[x_0 + g(x)]\|_\gamma \\
& \quad + \int_0^b (b-s)^{q-1} s^n \\
& \qquad \times \|\mathcal{S}(b-s) f(s, x(s), (Hx)(s))\|_\gamma ds \Bigg] \\
\leq\ & \frac{M_B M q}{\alpha \Gamma(q+1)} \Bigg[ \|x_b\|_\gamma + M\left(\|x_0\|_\gamma + \beta_1 \|x\|_\infty + \beta_2\right) \\
& \quad + \frac{M_1 M q}{\Gamma(q+1)} \left(\frac{p-1}{pq-1}\right)^{(p-1)/p} \\
& \qquad \times \left(\frac{p-1}{np^2 + p - 1}\right)^{(p-1)/p^2} b^{n+q-1/p^2} \Bigg].
\end{aligned}
$$

(19)

It follows from (16) that

$$
\begin{aligned}
\|z(t)\|_\gamma \leq\ & \|\mathcal{T}(t)[x_0 + g(x)]\|_\gamma \\
& + \int_0^t (t-s)^{q-1} \|\mathcal{S}(t-s) Bv(s)\|_\gamma ds \\
& + \int_0^t (t-s)^{q-1} s^n \|\mathcal{S}(t-s) f(s, x(s), (Hx)(s))\|_\gamma ds \\
\leq\ & M\left(\|x_0\|_\gamma + \beta_1 \|x\|_\infty + \beta_2\right) \\
& + \frac{M_B M q}{\Gamma(q+1)} \left(\frac{p-1}{qp-1}\right)^{(p-1)/p} b^{((q-1)p+p-1)/p} b\|v(s)\|_\gamma \\
& + \frac{M_1 M q}{\Gamma(q+1)} \left(\frac{p-1}{pq-1}\right)^{(p-1)/p} \left(\frac{p-1}{np^2 + p - 1}\right)^{(p-1)/p^2} \\
& \times b^{n+q-1/p^2}.
\end{aligned}
$$

(20)

From the above two inequalities, we get that $\|(F_\alpha x)(t)\|_\gamma \leq r_0$. This follows that $F_\alpha$ maps $B_{r_0}$ into itself.

For each $\alpha > 0$, we prove the operator $F_\alpha$ maps $B_{r_0}$ into a relatively compact subset of $B_{r_0}$. First, we show that $V(t) = \{(F_\alpha x)(t) : x \in B_{r_0}\}$ is relatively compact in $X_\gamma$ for every $t \in J$. The case $t = 0$ is obvious. Let $t \in (0, b]$ be fixed, and for each

$\varepsilon \in (0, t)$, arbitrary $\delta > 0$, and $x \in B_{r_0}$, we define the operator $F_\alpha^{\varepsilon, \delta}$ by

$$
\begin{aligned}
& \left(F_\alpha^{\varepsilon,\delta} x\right)(t) \\
& = \mathcal{T}[x_0 + g(x)] \\
& \quad + q \int_0^{t-\varepsilon} \int_\delta^\infty \theta(t-s)^{q-1} \xi_q(\theta) T\left((t-s)^q \theta\right) \\
& \qquad\qquad \times Bv(s)\, d\theta\, ds \\
& \quad + q \int_0^{t-\varepsilon} \int_\delta^\infty \theta(t-s)^{q-1} s^n \xi_q(\theta) T\left((t-s)^q \theta\right) \\
& \qquad\qquad \times f(s, x(s), (Hx)(s))\, d\theta\, ds \\
& = T\left(\varepsilon^q \delta\right) \\
& \quad \times \int_\delta^\infty \xi_q(\theta)\left(T\left(t^q \theta\right) - T\left(\varepsilon^q \delta\right)\right) d\theta\, [x_0 + g(x)] \\
& \quad + \left(T\left(\varepsilon^q \delta\right)\right) q \\
& \quad \times \int_0^{t-\varepsilon} \int_\delta^\infty \theta(t-s)^{q-1} \xi_q(\theta) \\
& \qquad \times \left(T\left((t-s)^q \theta\right) - T\left(\varepsilon^q \delta\right)\right) Bv(s)\, d\theta\, ds \\
& \quad + T\left(\varepsilon^q \delta\right) q \int_0^{t-\varepsilon} \int_\delta^\infty \theta(t-s)^{q-1} s^n \xi_q(\theta) \\
& \qquad \times \left(T\left((t-s)^q \theta\right) - T\left(\varepsilon^q \delta\right)\right) \\
& \qquad \times f(s, x(s), (Hx)(s))\, d\theta\, ds \\
& = T\left(\varepsilon^q \delta\right) y(t, \varepsilon).
\end{aligned}
$$

(21)

Since $T(\varepsilon^q \delta)$ is compact in $X_\gamma$ and $y(t, \varepsilon)$ is bounded in $B_{r_0}$, the set $V_\varepsilon(t) = \{(F_\alpha^{\varepsilon,\delta} x)(t) : x \in B_{r_0}\}$ is relatively compact in $X_\gamma$ [4]. On the other hand, using $(H_1)$ and Holder's inequality, we have

$$
\begin{aligned}
& \left\|(F_\alpha x)(t) - (F_\alpha^{\varepsilon,\delta} x)(t)\right\|_\gamma \\
& \leq q \int_0^t \int_0^\delta \theta(t-s)^{q-1} \xi_q(\theta) \\
& \qquad \times \left\| T((t-s)^q \theta)[Bv(s) + s^n f(s, x(s), (Hx)(s))] \right\|_\gamma d\theta\, ds \\
& \quad + q \int_{t-\varepsilon}^t \int_\delta^\infty \theta(t-s)^{q-1} \xi_q(\theta) \\
& \qquad \times \left\| T((t-s)^q \theta)[Bv(s) + s^n f(s, x(s), (Hx)(s))] \right\|_\gamma d\theta\, ds \\
& \leq M_B M q \left(\frac{p-1}{qp-1}\right)^{(p-1)/p} t^{1+q-1/p} \|v(s)\|_\gamma \int_0^\delta \theta \xi_q(\theta)\, d\theta
\end{aligned}
$$

$$+ M_1 M q \left(\frac{p-1}{pq-1}\right)^{(p-1)/p} \left(\frac{p-1}{np^2+p-1}\right)^{(p-1)/p^2} b^{n+q-1/p^2}$$

$$\times \int_0^\delta \theta \xi_q(\theta) d\theta$$

$$+ \frac{M_B M q}{\Gamma(q+1)} \left(\frac{p-1}{qp-1}\right)^{(p-1)/p} \varepsilon^{1+q-1/p} \|v(s)\|_\gamma$$

$$+ \frac{M_1 M q}{\Gamma(q+1)} \left(\frac{p-1}{pq-1}\right)^{(p-1)/p} \left(\frac{p-1}{np^2+p-1}\right)^{(p-1)/p^2} \varepsilon^{n+q-1/p^2}. \tag{22}$$

This implies that there are relatively compact sets $V_\varepsilon(t)$ arbitrarily close to the set $V(t)$ for each $t \in (0, b]$. Thus, $V(t)$ is relatively compact in $X_\gamma$ for all $t \in (0, b]$ since it is compact at $t = 0$. Therefore, we have the relatively compactness in $X_\gamma$ for all $t \in [0, b]$.

Next we show that $V(t) = \{(F_\alpha x)(t) : x \in B_{r_0}\}$ is an equicontinuous family of functions on $[0, b]$. By the compactness of the set, we can prove that the functions $F_\alpha x, x \in B_{r_0}$ are equicontinuous at $t = 0$. For any $x \in B_{r_0}$ and $0 < t_1 < t_2 \le b$, we have

$$\|z(t_2) - z(t_1)\|_\gamma$$

$$\le \|(\mathcal{T}(t_2) - \mathcal{T}(t_1))[x_0 + g(x)]\|_\gamma$$

$$+ \left\|\int_0^{t_1} (t_2 s)^{q-1} [\mathcal{S}(t_2 - s) - \mathcal{S}(t_1 - s)] Bv(s) ds\right\|_\gamma$$

$$+ \left\|\int_0^{t_1} \left[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\right] \mathcal{S}(t_1 - s) Bv(s) ds\right\|_\gamma$$

$$+ \left\|\int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{S}(t_2 - s) Bv(s) ds\right\|_\gamma$$

$$+ \left\|\int_0^{t_1} (t_2 - s)^{q-1} s^n \left[\mathcal{S}(t_2 - s) - \mathcal{S}(t_1 - s)\right]\right.$$

$$\left.\times f(s, x(s), (Hx)(s)) ds\right\|_\gamma \tag{23}$$

$$+ \left\|\int_0^{t_1} \left[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\right] s^n \mathcal{S}(t_2 - s)\right.$$

$$\left.\times f(s, x(s), (Hx)(s)) ds\right\|_\gamma$$

$$+ \left\|\int_{t_1}^{t_2} (t_2 - s)^{q-1} s^n \mathcal{S}(t_2 - s) f(s, x(s), (Hx)(s)) ds\right\|_\gamma$$

$$\le I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.$$

Now, we have to prove that $I_1, I_2, I_3, I_4, I_5, I_6$, and $I_7$ tend to 0 independently of $x \in B_{r_0}$ as $t_2 - t_1 \to 0$. By Lemma 6, one can easily show that $\lim_{t_1 \to t_2} I_1 = 0$. By using Lemmas 5 and

6 and the Lagrange mean value theorem and following the similar procedure as in the proof of Theorem 3.2 of [4], one can deduce that $\lim_{t_1 \to t_2} I_5, I_6, I_7 = 0$. In a similar way, we can obtain

$$I_2 \le \frac{\Gamma(2)}{\Gamma(q+1)} \frac{M_{1+\gamma}}{\gamma} \left(\frac{(p-1)^2}{(q-1)p^2+(p-1)^2}\right)^{(p-1)^2/p^2}$$

$$\times b^{((q-1)p^2+(p-1)^2)/p^2}$$

$$\times \left(\int_0^b \left[(t_2-s)^{-q\gamma} - (t_1-s)^{-q\gamma}\right]^{p^2/(p-1)} ds\right)^{(p-1)/p^2}$$

$$\times b M_B \|v(s)\|,$$

$$I_3 \le \frac{Mq}{\Gamma(q+1)} \left(\int_0^b \left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right]^{p/(p-1)} ds\right)^{(p-1)/p}$$

$$\times b M_B \|v(s)\|,$$

$$I_4 \le \frac{Mq}{\Gamma(q+1)} \left(\frac{p-1}{pq-1}\right)^{(p-1)/p} (t_2-t_1)^{((q-1)p+p-1)/p}$$

$$\times M_B \|v(s)\| (t_2 - t_1). \tag{24}$$

By the Lagrange mean value theorem and Lemmas 5 and 6, it can be easily seen that $I_2, I_3$, and $I_4$ tend to 0 as $t_2 - t_1 \to 0$. Thus, the right-hand side of (23) tends to 0 as $t_2 - t_1 \to 0$ which means that $\{(F_\alpha x) : x \in B_{r_0}\}$ is a family of equicontinuous functions. It can be easily seen that for all $\alpha > 0$, $F_\alpha$ is continuous on $C(J, X_\gamma)$. Hence, by Arzela-Ascoli's theorem, $F_\alpha$ is compact. By Schauder's fixed point theorem, $F_\alpha$ has a fixed point $x \in B_{r_0}$. Thus, the control system (1) has at least one mild solution on $[0, b]$. □

**Theorem 8.** *Assume that $(H_1)$–$(H_6)$ hold. Then, system (1) is approximately controllable on $[0, b]$.*

*Proof.* Linear system (8) is approximately controllable, by Lemma 4, and it can be seen that approximate controllability of (8) is equivalent to convergence of the operator $\alpha R(\alpha, \Gamma_0^b)$ to zero operator in the strong operator topology as $\alpha \to 0^+$. Let $\hat{x}_\alpha(\cdot)$ be a fixed point of $F_\alpha$ in $B_r$. Any fixed point of $F_\alpha$ is a mild solution of (1) under the control $\hat{u}_\alpha(t) = B^* S^*(b - t) R(\alpha, \Gamma_0^b) p(\hat{x}_\alpha)$ and satisfies $\hat{x}_\alpha(b) = x_b - \alpha R(\alpha, \Gamma_0^b) p(\hat{x}_\alpha)$. By the condition $(H_5)$, we have $\int_0^b \|f(s, \hat{x}_\alpha(s), (H\hat{x}_\alpha)(s))\|^2 ds \le bN^2$, and consequently, the sequence $\{f(s, \hat{x}_\alpha(s), (H\hat{x}_\alpha)(s))\}$ is bounded in $L^2(J, X_\gamma)$. Then there is a subsequence denoted by $\{f(s, \hat{x}_\alpha(s), (H\hat{x}_\alpha)(s))\}$ that converges weakly to say $f(s)$ in $L^2(J, X_\gamma)$. Now, the compactness of $\mathcal{S}(t)$ implies that

$\mathcal{S}(b-s)f(s, \hat{x}_\alpha(s), (H\hat{x}_\alpha)(s)) \rightarrow \mathcal{S}(b-s)f(s)$ in $L^2(J, X_\gamma)$, and we obtain

$$\|p(\hat{x}_\alpha) - w\|$$

$$= \left\| \int_0^b (b-s)^{q-1}s^n\mathcal{S}(b-s) \right.$$

$$\left. \times \left[ f(s, \hat{x}_\alpha(s), (H\hat{x}_\alpha)(s)) - f(s) \right] ds \right\|$$

$$\le \sup_{0 \le t \le b} \left\| \int_0^t (t-s)^{q-1}s^n\mathcal{S}(t-s) \right.$$

$$\left. \times \left[ f(s, \hat{x}_\alpha(s), (H\hat{x}_\alpha)(s)) - f(s) \right] ds \right\|$$

$$\longrightarrow 0 \quad \text{as } \alpha \longrightarrow 0^+, \tag{25}$$

where

$$w = x_b - \mathcal{T}(b)\left[ x_0 + g(\hat{x}_\alpha) \right]$$

$$- \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)s^n f(s) ds. \tag{26}$$

Then, we obtain

$$\|\hat{x}_\alpha(b) - x_b\| \le \|\alpha R(\alpha, \Gamma_0^b)(w)\|$$

$$+ \|\alpha R(\alpha, \Gamma_0^b)\| \|p(\hat{x}_\alpha) - w\| \tag{27}$$

$$\le \|\alpha R(\alpha, \Gamma_0^b)(w)\| + \|p(\hat{x}_\alpha) - w\|.$$

It follows from Lemma 4 and estimation (25) that $\|\hat{x}_\alpha(b) - x_b\| \rightarrow 0$ as $\alpha \rightarrow 0^+$. This proves the approximate controllability of (1). The proof is completed. □

*Example 9.* As an application of the obtained theory, we consider a control system which is represented by fractional partial differential equation. Let $X = U = L^2([0,1])$, and let $A : D(A) \subset X \rightarrow X$ be the operator defined by $Az = -z''$ with domain

$$D(A) = \left\{ z \in L^2([0,1]) : z, z' \text{ are absolutely continuous,} \right.$$

$$z'' \in L^2([0,1]), \ z(0) = z(1) = 0 \right\}. \tag{28}$$

Let $D(A^{1/2}) = \{z(\cdot) \in X : \sum_{n=1}^\infty (z, e_n)e_n \in X\}$, and let $X_{1/2} = D(A^{1/2}, \|\cdot\|_{1/2})$, where $\|\cdot\|_{1/2} = \|A^{1/2}\|_2$. For each $z \in D(A^{1/2})$ and $\|A^{-1/2}\| = 1$, the operator $A^{1/2}$ is given by $A^{1/2}z = \sum_{n=1}^\infty n^2(z, e_n)e_n$, where $e_n(y) = \sqrt{2}\sin(ny)$ and $0 \le y \le 1$, $n = 1, 2, \ldots$ is the orthogonal set of eigenvectors of $A$. It is well known that $-A$ generates a compact analytic semigroup of uniformly bounded linear operator.

Consider the fractional partial differential equation with control in the following form:

$$\frac{\partial^q}{\partial t^q}x(t, y)$$

$$= \frac{\partial^2}{\partial y^2}x(t, y) + \mu(t, y)$$

$$+ \frac{e^{-t}}{e^t + e^{-t}}\cos\left[ x(t, y) + \int_0^b \cos(ts)x(s, y)ds \right] + e^{-t},$$

$$t \in [0, 1], \quad y \in [0, 1],$$

$$x(t, 0) = x(t, 1) = 0, \quad t > 0,$$

$$x(0, y) = \sum_{i=0}^\sigma \int_0^1 k_1(y, \tau)x(t_i, \tau)d\tau$$

$$+ \sum_{i=0}^\sigma \int_0^1 k_2(y, \tau)\frac{\partial}{\partial \tau}x(t_i, \tau)d\tau, \tag{29}$$

where $0 \le q \le 1$; $\mu : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous; $k_1, k_2 \in L^2([0, 1] \times [0, 1])$; $\sigma \in N$.

Let $x(t)(y) = x(t, y)$, and define the bounded linear operator $B : U \rightarrow X$ by $(Bu)(t)(y) = \mu(t, y)$, $0 \le y \le 1$. Let $C([0, 1], X_{1/2})$ be a Banach space equipped with sup norm. Further, we define the operators $f : [0, 1] \times X_{1/2} \rightarrow X_{1/2}$ and $g : C([0, 1], X_{1/2}) \rightarrow X_{1/2}$ by

$$f(t, x(t), (Hx)(t))(y)$$

$$= \frac{e^{-t}}{e^t + e^{-t}}\cos\left[ x(t) + \int_0^b \cos(ts)x(s)ds \right](y) + e^{-t},$$

$$g(x)(y) = \left( \sum_{i=0}^\sigma (\mathcal{K}x)(t_i) \right)(y), \quad \text{for } x \in C(J, X_{1/2}), \tag{30}$$

where $(Hx)(t)(y) = \int_0^b \cos(ts)x(s, y)ds$, $(\mathcal{K}\phi)(\tau) = \int_0^1 k_1(y, \tau)\phi(\tau)d\tau + \int_0^1 k_2(y, \tau)(\partial/\partial\tau)\phi(\tau)d\tau$ for all $\phi \in X_{1/2}$. Moreover, the linear fractional control system corresponding to (29) is approximately controllable. Thus, with the above choices of $A$, $B$, $f$, and $g$, the system (29) can be written to the abstract form of (1). Therefore, all the conditions of Theorem 8 are satisfied. Hence, by Theorem 8, the fractional nonlinear integrodifferential system (29) is approximately controllable on $[0, 1]$.

*Note.* The considered system (1) is of the more general form, and in particular, if functions $f$ and $h$ have various physical meanings, it is important to note that (1) has a great diversity. The result in this paper assumes that the linear system has a compact semigroup and consequently is not completely controllable. Moreover, the functions with Lipschitz condition are considerably strong when one discusses various applications in the real-world problems. Such an assumption is removed from this paper.

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