Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 213659, 7 pages
http://dx.doi.org/10.1155/2013/213659

Research Article

Block Preconditioned SSOR Methods for *H*-Matrices Linear Systems

Zhao-Nian Pu and Xue-Zhong Wang

School of Mathematics and Statistics, Hexi University, Zhangye, Gansu 734000, China

Correspondence should be addressed to Xue-Zhong Wang; mail2011wang@gmail.com

Received 9 January 2013; Accepted 12 March 2013

Academic Editor: Hak-Keung Lam

We present a block preconditioner and consider block preconditioned SSOR iterative methods for solving linear system $Ax = b$. When A is an H-matrix, the convergence and some comparison results of the spectral radius for our methods are given. Numerical examples are also given to illustrate that our methods are valid.

1. Introduction

For the linear system

$$Ax = b, \tag{1}$$

where A is an $n \times n$ square matrix and x and b are n-dimensional vectors. The basic iterative method for solving (1) is

$$Mx^{k+1} = Nx^k + b, \quad k = 0, 1, \ldots, \tag{2}$$

where $A = M - N$ and M is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \ldots, \tag{3}$$

where $T = M^{-1}N$, $c = M^{-1}b$.

Let us consider the following partition of A:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}, \tag{4}$$

where the blocks $A_{ii} \in C^{n_i \times n_i}$, $i = 1, \ldots, m$, are nonsingular and $n_1 + n_2 + \cdots + n_m = n$.

Usually we split A into

$$A = D - L - U, \tag{5}$$

where $D = \operatorname{diag}(A_{11}, \ldots, A_{mm})$, $-L$ and $-U$ are strictly block lower and strictly block upper triangular parts of A, respectively. Let $0 < \omega < 2$, and

$$M = \frac{1}{\omega(2-\omega)}(D - \omega L) D^{-1} (D - \omega U),$$
$$N = \frac{1}{\omega(2-\omega)}((1-\omega) D + \omega L) D^{-1} ((1-\omega) D + \omega U). \tag{6}$$

Then, the iteration matrix of the SSOR method for A is given by

$$\begin{aligned} \mathscr{L}_\omega &= M^{-1}N \\ &= (D - \omega U)^{-1} D (D - \omega L)^{-1} \\ &\quad \times ((1-\omega) D + \omega L) D^{-1} \\ &\quad \times ((1-\omega) D + \omega U). \end{aligned} \tag{7}$$

Transforming the original system (1) into the preconditioned form

$$PAx = Pb, \tag{8}$$

then we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb, \quad k = 0, 1, \ldots, \tag{9}$$

where $PA = M_p - N_p$ and M_p is nonsingular. Thus (9) can also be written as

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \ldots, \tag{10}$$

where $T = M_p^{-1} N_p$, $c = M_p^{-1} Pb$. Similar to the original system (1), we call the basic iterative methods corresponding to the preconditioned system the preconditioned iterative methods.

When A is an M-matrix, Alanelli and Hadjidimosin [1] considered the preconditioner $P = Q + S$, where $Q = \operatorname{diag}(L_{11}^{-1}, I_{22}, \ldots, I_{mm})$ and S is given by

$$S = \begin{pmatrix} O_{11} & O_{12} & \cdots & O_{1m} \\ -A_{21}L_{11}^{-1} & O_{22} & \cdots & O_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{m1}L_{11}^{-1} & O_{m2} & \cdots & O_{mm} \end{pmatrix}, \tag{11}$$

with L_{11} being the lower triangular matrix in the LU triangular decomposition of A_{11}.

We consider the preconditioner $P_1 = I + S_1$, where

$$I + S_1 = \begin{pmatrix} I_{11} & & -\alpha_1 A_{11}^{-1} A_{1s} & & \\ & I_{22} & & -\alpha_2 A_{22}^{-1} A_{2t} & \\ & & \ddots & & \\ & & & \ddots & \\ & -\alpha_m A_{mm}^{-1} A_{mu} & & & I_{mm} \end{pmatrix}. \tag{12}$$

Let

$$P_1 A = (I + S_1)(D - L - U) = \widetilde{D} - \widetilde{L} - \widetilde{U}, \tag{13}$$

where $\widetilde{D}$, $-\widetilde{L}$ and $-\widetilde{U}$ are block diagonally, strictly block lower, and strictly block upper triangular parts of $P_1 A$, respectively. If $\widetilde{D}$ is nonsingular, then $(\widetilde{D} - \omega\widetilde{L})^{-1}$ and $(\widetilde{D} - \omega\widetilde{U})^{-1}$ exist and it is possible to define the SSOR iteration matrix for $P_1 A$. Namely,

$$\begin{aligned} \widetilde{\mathscr{L}}_\omega &= \left(\widetilde{D} - \omega\widetilde{U}\right)^{-1} \widetilde{D} \left(\widetilde{D} - \omega\widetilde{L}\right)^{-1} \\ &\quad \times \left((1-\omega)\widetilde{D} + \omega\widetilde{L}\right) \widetilde{D}^{-1} \left((1-\omega)\widetilde{D} + \omega\widetilde{U}\right). \end{aligned} \tag{14}$$

Alanelli and Hadjidimos in [1] showed that the preconditioned Gauss-Seidel, the preconditioned SOR, and the preconditioned Jacobi methods with preconditioner P are better than original methods. Our work in the presentation is to prove convergence of the block preconditioned SSOR method with preconditioner P_1 and give some comparison results of the spectral radius for the case when A is an H-matrix.

Let $|A|$ denote the matrix whose elements are the moduli of the elements of the given matrix. We call $\langle A \rangle = (\overline{a}_{ij})$ to comparison matrix if $\overline{a}_{ij} = |a_{ij}|$ for $i = j$, if $\overline{a}_{ij} = -|a_{ij}|$ for $i \neq j$. For (4), under the previous definition, we have

$$\langle A \rangle = \begin{pmatrix} \langle A_{11} \rangle & -|A_{12}| & \cdots & -|A_{1m}| \\ -|A_{21}| & \langle A_{22} \rangle & \cdots & -|A_{2m}| \\ \vdots & \vdots & \ddots & \vdots \\ -|A_{m1}| & -|A_{m2}| & \cdots & \langle A_{mm} \rangle \end{pmatrix}. \tag{15}$$

Let $\langle A \rangle = \langle D \rangle - |L| - |U|$, where $\langle D \rangle$, $-|L|$, and $-|U|$ are block diagonally, strictly block lower, and strictly block upper triangular parts of $\langle A \rangle$, respectively.

Notice that the preconditioner of the matrix $\langle A \rangle$ corresponding to P_1 is $P_2 = I + S_2$; namely,

$$I + S_2 = \begin{pmatrix} I_{11} & & |\alpha_1| \langle A_{11} \rangle^{-1} |A_{1s}| & & \\ & I_{22} & & |\alpha_2| \langle A_{22} \rangle^{-1} |A_{2t}| & \\ & & \ddots & & \\ & & & \ddots & \\ |\alpha_m| \langle A_{mm} \rangle^{-1} |A_{mu}| & & & & I_{mm} \end{pmatrix}. \tag{16}$$

Let $P_2\langle A\rangle = (I+S_2)\langle A\rangle = \overline{D}-\overline{L}-\overline{U}$, where $\overline{D}$, $-\overline{L}$, and $-\overline{U}$ are block diagonally, strictly block lower, and strictly block upper triangular parts of $P_2\langle A\rangle$, respectively.

If $\overline{D}$ is nonsingular, then $(\overline{D}-\omega\overline{L})^{-1}$ and $(\overline{D}-\omega\overline{U})^{-1}$ exist and the SSOR iteration matrix for $P_2\langle A\rangle$ is as follows:

$$\overline{\mathscr{L}}_\omega = \left(\overline{D}-\omega\overline{U}\right)^{-1}\overline{D}\left(\overline{D}-\omega\overline{L}\right)^{-1} \times \left((1-\omega)\overline{D}+\omega\overline{L}\right)\overline{D}^{-1}\left((1-\omega)\overline{D}+\omega\overline{U}\right). \tag{17}$$

2. Preliminaries

A matrix A is called nonnegative (positive) if each entry of A is nonnegative (positive). We denote it by $A \geq 0$ ($A > 0$). Similarly, for n-dimensional vector x, we can also define $x \geq 0$ ($x > 0$). Additionally, we denote the spectral radius of A by $\rho(A)$. A^T denotes the transpose of A. A matrix $A = (a_{ij})$ is called a *Z-matrix* if for any $i \neq j$, $a_{ij} \leq 0$. A *Z-matrix* is a nonsingular *M-matrix* if A is nonsingular and $A^{-1} \geq 0$, If $\langle A\rangle$ is a nonsingular *M-matrix* , then A is called an *H-matrix*. $A = M - N$ is said to be a *splitting* of A if M is nonsingular, $A = M - N$ is said to be *regular* if $M^{-1} \geq 0$ and $N \geq 0$, and *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, respectively.

Some basic properties on special matrices introduced previously are given to be used in this paper.

Lemma 1 (see [2]). *Let A be a Z-matrix. Then the following statements are equivalent.*

(a) *A is an M-matrix.*

(b) *There is a positive vector x such that $Ax > 0$.*

(c) $A^{-1} \geq 0$.

(d) *All principal submatrices of A are M-matrices.*

(e) *All principal minors are positive.*

Lemma 2 (see [3, 4]). *Let A be an M-matrix and let $A = M - N$ be a weak regular splitting. Then $\rho(M^{-1}N) < 1$.*

Lemma 3 (see [2]). *Let A and B be two $n \times n$ matrices with $0 \leq B \leq A$. Then $\rho(B) \leq \rho(A)$.*

Lemma 4 (see [5]). *If A is an H-matrix, then $|A^{-1}| \leq \langle A\rangle^{-1}$.*

Lemma 5 (see [6]). *Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splitting of monotone matrices A_1 and A_2, respectively, such that $M_2^{-1} \geq M_1^{-1}$. If there exists a positive vector x such that $0 \leq A_1 x \leq A_2 x$, then for the monotone norm associated with x,*

$$\left\|M_1^{-1}N_1\right\|_x \leq \left\|M_2^{-1}N_2\right\|_x. \tag{18}$$

In particular, if $M_1^{-1}N_1$ has a positive Perron vector, then

$$\rho\left(M_1^{-1}N_1\right) \leq \rho\left(M_2^{-1}N_2\right). \tag{19}$$

Moreover if x is a Perron vector of $M_1^{-1}N_1$ and strict inequality holds in (18)*, then strict inequality holds in* (19).

Lemma 6. *If A and B are two $n \times n$ matrices, then $\langle A - B\rangle \geq \langle A\rangle - |B|$.*

Proof. It is easy to see that $|a_{ij} - b_{ij}| \geq |a_{ij}| - |b_{ij}|$, for $i = j$, and $-|a_{ij}-b_{ij}| \geq -|a_{ij}|-|b_{ij}|$, for $i \neq j$. Therefore, $\langle A-B\rangle \geq \langle A\rangle-|B|$ is true. □

Lemma 7. *If A is an H-matrix with unit diagonal elements, then $\| \langle A\rangle^{-1}\|_\infty > 1$.*

Proof. Let $\langle A\rangle = I - B$, from $\langle A\rangle$ being an M-matrix; then $B \geq 0$ and $\rho(B) < 1$, and thus, we have

$$\langle A\rangle^{-1} = \sum_{K=0}^{\infty} B^k \geq I \tag{20}$$

and then $\| \langle A\rangle^{-1}\|_\infty > 1$. □

3. Convergence Results

Let $e_i = (1,\dots,1)^T \in R^{n_i}$, $i = 1,2,\dots,m$, $e = (e_1^T,\dots,e_m^T)^T$, $r = (r_1^T,\dots,r_m^T)^T = \langle A\rangle^{-1}e$, $O_i = (0,\dots,0)^T \in R^{n_i}$, where r and e are partitioned in accordance with the block partitioning of the matrix A, and let

$$s_i = \left\|\langle A_{ii}\rangle^{-1}\left|A_{ik}\right| e_k\right\|_\infty, \quad h_i = \frac{1}{s_i\left(2\left\|\langle A\rangle^{-1}\right\|_\infty - 1\right)}, \quad i = 1,2,\dots,m. \tag{21}$$

Theorem 8. *Let A be a nonsingular H-matrix; if $|\alpha_i| < h_i$, $i = 1,2,\dots,m$, then $P_1 A$ is also an H-matrix.*

Proof. From A being an H-matrix, we have $r > 0$, and $r_k \leq\| \langle A\rangle^{-1}\|_\infty e_k$. Let

$$\left((P_1A)_{ij}\right) = \begin{cases} A_{ij}-\alpha_i A_{ii}^{-1}A_{ik}A_{kj}, & i\neq j,\ i,j = 1,2,\dots,m,\ k\neq i, \\ A_{ii}-\alpha_i A_{ii}^{-1}A_{ik}A_{ki}, & i=j,\ i,j = 1,2,\dots,m,\ k\neq i. \end{cases} \tag{22}$$

Then

$$\begin{aligned} \left(\langle P_1A\rangle r\right)_i = {} & \left\langle A_{ii} - \alpha_i A_{ii}^{-1}A_{ik}A_{ki}\right\rangle r_i - \sum_{j\neq i,k}^{m}\left|A_{ij} - \alpha_i A_{ii}^{-1}A_{ik}A_{kj}\right| r_j - \left|A_{ik} - \alpha_i A_{ii}^{-1}A_{ik}A_{kk}\right| r_k \\ \geq {} & \langle A_{ii}\rangle r_i - |\alpha_i|\left|A_{ii}^{-1}\right|\left|A_{ik}\right|\left|A_{ki}\right| r_i - \sum_{j\neq i,k}^{m}\left|A_{ij}\right| - \sum_{j\neq i,k}^{m}|\alpha_i|\left|A_{ii}^{-1}\right|\left|A_{ik}\right|\left|A_{kj}\right| r_j \end{aligned}$$

$$\begin{aligned}
&- |A_{ik}| r_k - |\alpha_i| \left|A_{ii}^{-1}\right| |A_{ik}| |A_{kk}| r_k \\
&\geq e_i - |\alpha_i| \langle A_{ii}\rangle^{-1} |A_{ik}| |A_{ki}| r_i \\
&\quad - \sum_{j\neq i,k}^{m} |\alpha_i| \langle A_{ii}\rangle^{-1} |A_{ik}| \left|A_{kj}\right| r_j \\
&\quad - |\alpha_i| \langle A_{ii}\rangle^{-1} |A_{ik}| |A_{kk}| r_k \\
&= e_i + |\alpha_i| \langle A_{ii}\rangle^{-1} |A_{ik}| \\
&\quad \times \left(- \sum_{j\neq k}^{m} \left|A_{kj}\right| r_j - |A_{kk}| r_k \right. \\
&\qquad \left. + \langle A_{kk}\rangle r_k - \langle A_{kk}\rangle r_k \right) \\
&= e_i + |\alpha_i| \langle A_{ii}\rangle^{-1} |A_{ik}| (e_k - 2r_k) \\
&\geq e_i - |\alpha_i| \left(2\left\|\langle A\rangle^{-1}\right\|_\infty - 1\right) \langle A_{ii}\rangle^{-1} |A_{ik}| e_k \\
&\geq e_i - |\alpha_i| s_i \left(2\left\|\langle A\rangle^{-1}\right\|_\infty - 1\right) e_i \\
&> O_i.
\end{aligned} \tag{23}$$

Therefore, $\langle P_1 A\rangle$ is an M-matrix, and then $P_1 A$ is an H-matrix. □

Theorem 9. *If A is a nonsingular H-matrix with unit diagonal elements, $0 < \omega \leq 1$ and $|\alpha_i| < h_i$, $i = 1, 2, \ldots, m$. Then $\rho(\widetilde{\mathscr{L}}_\omega) < 1$.*

Proof. From Theorem 8, we know $\langle P_1 A\rangle = \langle\widetilde{D}\rangle - |\widetilde{L}| - |\widetilde{U}|$ is an M-matrix; if we let

$$\begin{aligned}
\langle P_1 A\rangle = {} & \frac{1}{\omega(2-\omega)} \left(\langle\widetilde{D}\rangle - \omega\left|\widetilde{L}\right|\right) \langle\widetilde{D}\rangle^{-1} \left(\langle\widetilde{D}\rangle - \omega\left|\widetilde{U}\right|\right) \\
& - \frac{1}{\omega(2-\omega)} \left((1-\omega)\langle\widetilde{D}\rangle + \omega\left|\widetilde{L}\right|\right) \langle\widetilde{D}\rangle^{-1} \\
& \times \left((1-\omega)\langle\widetilde{D}\rangle + \omega\left|\widetilde{U}\right|\right),
\end{aligned} \tag{24}$$

then the SSOR iteration matrix for $\langle P_1 A\rangle$ is as follows:

$$\begin{aligned}
\ddot{\mathscr{L}}_\omega = {} & \left(\langle\widetilde{D}\rangle - \omega\left|\widetilde{U}\right|\right)^{-1} \langle\widetilde{D}\rangle \left(\langle\widetilde{D}\rangle - \omega\left|\widetilde{L}\right|\right)^{-1} \\
& \times \left((1-\omega)\langle\widetilde{D}\rangle + \omega\left|\widetilde{L}\right|\right) \langle\widetilde{D}\rangle^{-1} \\
& \times \left((1-\omega)\langle\widetilde{D}\rangle + \omega\left|\widetilde{U}\right|\right).
\end{aligned} \tag{25}$$

Since $\langle P_1 A\rangle$ is an M-matrix; we have $\langle\widetilde{D}\rangle$, $\langle\widetilde{D}\rangle - \omega|\widetilde{L}|$ and $\langle\widetilde{D}\rangle - \omega|\widetilde{U}|$ are M-matrices; by simple calculation, we obtain that (24) is a weak regular splitting; from Lemma 2, we know that $\rho(\ddot{\mathscr{L}}_\omega) < 1$. Since

$$\begin{aligned}
\left|\widetilde{\mathscr{L}}_\omega\right| &= \left|\left(\widetilde{D} - \omega\widetilde{U}\right)^{-1} D \left(\widetilde{D} - \omega\widetilde{L}\right)^{-1} \right. \\
&\qquad \left. \times \left((1-\omega)\widetilde{D} + \omega\widetilde{L}\right) D^{-1} \left((1-\omega)\widetilde{D} + \omega\widetilde{U}\right)\right| \\
&= \left|\left(I - \omega\widetilde{D}^{-1}\widetilde{U}\right)^{-1} \left(I - \omega\widetilde{D}^{-1}\widetilde{L}\right)^{-1} \right. \\
&\qquad \left. \times \left((1-\omega) I + \omega D^{-1}\widetilde{L}\right) \left((1-\omega) I + \omega\widetilde{D}^{-1}\widetilde{U}\right)\right| \\
&\leq \left|\left(I - \omega\widetilde{D}^{-1}\widetilde{U}\right)^{-1}\right| \left|\left(I - \omega\widetilde{D}^{-1}\widetilde{L}\right)^{-1}\right| \\
&\qquad \times \left|\left((1-\omega) I + \omega D^{-1}\widetilde{L}\right)\right| \left|\left((1-\omega) I + \omega\widetilde{D}^{-1}\widetilde{U}\right)\right| \\
&\leq \left(I - \omega\langle\widetilde{D}\rangle^{-1} |\widetilde{U}|\right)^{-1} \left(I - \omega\langle\widetilde{D}\rangle^{-1} \left|\widetilde{L}\right|\right)^{-1} \\
&\qquad \times \left((1-\omega) I + \omega\langle\widetilde{D}\rangle^{-1} \left|\widetilde{L}\right|\right) \left((1-\omega) I + \omega\langle\widetilde{D}\rangle^{-1} \left|\widetilde{U}\right|\right) \\
&= \rho\left(\ddot{\mathscr{L}}_\omega\right)
\end{aligned} \tag{26}$$

then, by Lemma 3, $\rho(\widetilde{\mathscr{L}}_\omega) \leq \rho(|\widetilde{\mathscr{L}}_\omega|) \leq \rho(\ddot{\mathscr{L}}_\omega) < 1$. □

4. Comparison Results of Spectral Radius

Theorem 10. *Let A be a nonsingular H-matrix with unit diagonal elements, $0 < \omega \leq 1$ and $|\alpha_i| < h_i$, $i = 1, 2, \ldots, m$. Then $P_2\langle A\rangle$ is an M-matrix and $\rho(\widetilde{\mathscr{L}}_\omega) < 1$.*

Proof. Similar to the proof of Theorems 8 and 9, it is easy to get the proof of this theorem. □

In what follows we will give some comparison results on the spectral radius of preconditioned SSOR iteration matrices with different preconditioner.

Let

$$\begin{aligned}
\langle A\rangle &= \widehat{M} - \widehat{N} \\
&= \frac{1}{\omega(2-\omega)} (\langle D\rangle - \omega|L|) \langle D\rangle^{-1} (\langle D\rangle - \omega|U|) \\
&\quad - \frac{1}{\omega(2-\omega)} ((1-\omega)\langle D\rangle + \omega|L|) \langle D\rangle^{-1} \\
&\quad \times ((1-\omega)\langle D\rangle + \omega|U|),
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\widehat{M} &= \frac{1}{\omega(2-\omega)} (\langle D\rangle - \omega|L|) \langle D\rangle^{-1} (\langle D\rangle - \omega|U|), \\
\widehat{N} &= \frac{1}{\omega(2-\omega)} ((1-\omega)\langle D\rangle + \omega|L|) \langle D\rangle^{-1} \\
&\quad \times ((1-\omega)\langle D\rangle + \omega|U|).
\end{aligned} \tag{28}$$

Then the SSOR iteration matrix for $\langle A\rangle$ is as follows:

$$\begin{aligned}\widehat{\mathscr{L}}_{\omega} &= \widehat{M}^{-1}\widehat{N}\\ &= (\langle D\rangle - \omega\,|U|)^{-1}\,\langle D\rangle\,(\langle D\rangle - \omega\,|L|)^{-1}\\ &\quad\times((1-\omega)\,\langle D\rangle + \omega\,|L|)\,\langle D\rangle^{-1}\,((1-\omega)\,\langle D\rangle + \omega\,|U|),\end{aligned} \tag{29}$$

and let

$$\begin{aligned}P_2\,\langle A\rangle &= \overline{M} - \overline{N}\\ &= \frac{1}{\omega(2-\omega)}\left(\overline{D} - \omega\overline{L}\right)\overline{D}^{-1}\left(\overline{D} - \omega\overline{U}\right)\\ &\quad - \frac{1}{\omega(2-\omega)}\left((1-\omega)\,\overline{D} + \omega\overline{L}\right)\overline{D}^{-1}\\ &\quad\times\left((1-\omega)\,\overline{D} + \omega\overline{U}\right),\end{aligned} \tag{30}$$

where

$$\begin{aligned}\overline{M} &= \frac{1}{\omega(2-\omega)}\left(\overline{D} - \omega\overline{L}\right)\overline{D}^{-1}\left(\overline{D} - \omega\overline{U}\right),\\ \overline{N} &= \frac{1}{\omega(2-\omega)}\left((1-\omega)\,\overline{D} + \omega\overline{L}\right)\overline{D}^{-1}\left((1-\omega)\,\overline{D} + \omega\overline{U}\right).\end{aligned} \tag{31}$$

Then the AOR iteration matrix for $P_2\langle A\rangle$ is (17).

Theorem 11. *If A is a nonsingular H-matrix with unit diagonal elements, $0 < \omega \le 1$ and $|\alpha_i| < h_i$, $i = 1,2,\ldots,m$. Then $\rho(\overline{\mathscr{L}}_{\omega}) \le \rho(\widehat{\mathscr{L}}_{\omega})$.*

Proof. Since $\langle A\rangle$ is a nonsingular M-matrix, by Theorem 10, $P_2\langle A\rangle$ is a nonsingular M-matrix, and thus $\langle A\rangle$ and $P_2\langle A\rangle$ are two monotone matrices.

From $\langle A\rangle$ and $P_2\langle A\rangle$ being M-matrices, we can get $\langle D\rangle$, $\overline{D}$, $\widehat{M}$, and $\overline{M}$ are M-matrices, together with

$$\begin{aligned}&\left((1-\omega)\,I + \omega\langle D\rangle^{-1}\,|L|\right)\langle D\rangle^{-1}\left((1-\omega)\,I + \omega\langle D\rangle^{-1}\,|U|\right) > 0,\\ &\left((1-\omega)\,I + \omega\overline{D}^{-1}\overline{L}\right)\overline{D}^{-1}\left((1-\omega)\,I + \omega\overline{D}^{-1}\overline{U}\right) > 0.\end{aligned} \tag{32}$$

We obtain that $\langle A\rangle = \widehat{M} - \widehat{N}$ and $P_2\langle A\rangle = \overline{M} - \overline{N}$ are two weak regular splittings. By simple calculation, we have

$$\begin{aligned}\overline{M} &= \frac{1}{\omega(2-\omega)}\left(\overline{D} - \omega\overline{L}\right)\overline{D}^{-1}\left(\overline{D} - \omega\overline{U}\right)\\ &\le \frac{1}{\omega(2-\omega)}\,(\langle D\rangle - \omega\,|L|)\,\langle D\rangle^{-1}\,(\langle D\rangle - \omega\,|U|) = \widehat{M}\end{aligned} \tag{33}$$

and thus $\overline{M}^{-1} \ge \widehat{M}^{-1} \ge 0$; letting $x = \langle A\rangle^{-1}e > 0$, then $(P_2\langle A\rangle - \langle A\rangle)x = (I + S_2)e > 0$; since $\overline{M}^{-1} \ge \widehat{M}^{-1} \ge 0$, we have

$$\overline{M}^{-1}\,(P_2\,\langle A\rangle)\,x = \left(I - \overline{M}^{-1}\overline{N}\right)x \ge \widehat{M}^{-1}\,\langle A\rangle\,x = \left(I - \widehat{M}^{-1}\widehat{N}\right)x. \tag{34}$$

It follows that

$$\left\|\overline{M}^{-1}\overline{N}\right\|_x \le \left\|\widehat{M}^{-1}\widehat{N}\right\|_x. \tag{35}$$

As $\langle A\rangle = \widehat{M} - \widehat{N}$ is a weak regular splitting, there exists a positive perron vector y; by Lemma 5, the following inequality holds:

$$\rho\left(\overline{M}^{-1}\overline{N}\right) \le \rho\left(\widehat{M}^{-1}\widehat{N}\right), \tag{36}$$

that is,

$$\rho\left(\overline{\mathscr{L}}_{\omega}\right) \le \rho\left(\widehat{\mathscr{L}}_{\omega}\right). \tag{37}$$

□

When A is a nonsingular M-matrix, we have $A = \langle A\rangle$. If $\alpha_i > 0$, $i = 1,2,\ldots,m$, then $P_2 = P_1$. Furthermore, we have $\overline{\mathscr{L}}_{\omega} = \widetilde{\mathscr{L}}_{\omega}$ and $\mathscr{L}_{\omega} = \widehat{\mathscr{L}}_{\omega}$; therefore, we get the following result.

Corollary 12. *Let A be a nonsingular M-matrix with unit diagonal elements, $0 < \alpha_i < |h_i|$, $i = 1,2,\ldots,m$, and $0 < \omega \le 1$. Then*

$$\rho\left(\overline{\mathscr{L}}_{\omega}\right) = \rho\left(\widetilde{\mathscr{L}}_{\omega}\right) \le \rho\left(\widehat{\mathscr{L}}_{\omega}\right) = \rho\left(\mathscr{L}_{\omega}\right). \tag{38}$$

Theorem 13. *Let A be a nonsingular H-matrix with unit diagonal elements, $0 < \omega \le 1$ and $|\alpha_i| < h_i$, $i = 1,2,\ldots,m$. Then $\rho(\ddot{\mathscr{L}}_{\omega}) \le \rho(\overline{\mathscr{L}}_{\omega})$.*

Proof. Let

$$\begin{aligned}\langle P_1 A\rangle &= \frac{1}{\omega(2-\omega)}\left(\left\langle\widetilde{D}\right\rangle - \omega\left|\widetilde{L}\right|\right)\left\langle\widetilde{D}\right\rangle^{-1}\left(\left\langle\widetilde{D}\right\rangle - \omega\left|\widetilde{U}\right|\right)\\ &\quad - \frac{1}{\omega(2-\omega)}\left((1-\omega)\left\langle\widetilde{D}\right\rangle + \omega\left|\widetilde{L}\right|\right)\left\langle\widetilde{D}\right\rangle^{-1}\\ &\quad\times\left((1-\omega)\left\langle\widetilde{D}\right\rangle + \omega\left|\widetilde{U}\right|\right).\end{aligned} \tag{39}$$

Then the SSOR iteration matrix for $\langle P_1 A\rangle$ is $\ddot{\mathscr{L}}_{\omega}$ which is defined in the proof of Theorem 9, and let

$$\begin{aligned}P_2\,\langle A\rangle &= \frac{1}{\omega(2-\omega)}\left(\overline{D} - \omega\overline{L}\right)\overline{D}^{-1}\left(\overline{D} - \omega\overline{U}\right)\\ &\quad - \frac{1}{\omega(2-\omega)}\left((1-\omega)\,\overline{D} + \omega\overline{L}\right)\overline{D}^{-1}\\ &\quad\times\left((1-\omega)\,\overline{D} + \omega\overline{U}\right).\end{aligned} \tag{40}$$

Then the SSOR iteration matrix for $P_2\langle A\rangle$ is (17). It is easy to know that the previous two splittings are weak regular splittings. Furthermore, by Lemma 6, we have the following result, for any i, $i = 1,2,\ldots,m$,

$$\begin{aligned}\left\langle\widetilde{D}_{ii}\right\rangle &= \left\langle A_{ii} - \alpha_i A_{ii}^{-1} A_{ik} A_{ki}\right\rangle\\ &\ge \langle A_{ii}\rangle - |\alpha_i|\,\langle A_{ii}\rangle^{-1}\,|A_{ik}|\,|A_{ki}| = \left\langle\overline{D}_{ii}\right\rangle.\end{aligned} \tag{41}$$

From $\langle P_1 A\rangle$ and $P_2\langle A\rangle$ being two M-matrices, we have

$$0 \le \left\langle\widetilde{D}\right\rangle^{-1} \le \overline{D}^{-1} \tag{42}$$

Table 1: Comparison of spectral radius with preconditioner P_1.

ω, r	N	$\rho(\widetilde{\mathscr{L}}_\omega)$	$\rho(\overline{\mathscr{L}}_\omega)$	$\rho(\ddot{\mathscr{L}}_\omega)$	$\rho(\widehat{\mathscr{L}}_\omega)$	$\rho(\mathscr{L}_\omega)$
	100	0.5636	0.8635	0.7404	0.8999	0.6288
$\omega = 0.8$	200	0.6030	0.9195	0.7698	0.9473	0.7059
	500	0.6120	0.9751	0.7844	0.9847	0.7103
	100	0.3650	0.7906	0.5923	0.8530	0.4770
$\omega = 0.6$	200	0.4510	0.8832	0.6507	0.9240	0.5769
	500	0.4345	0.9730	0.6766	0.9835	0.5609
	100	0.2387	0.7512	0.6647	0.8087	0.3602
	200	0.3494	0.8491	0.5800	0.9009	0.4899
$\omega = 0.9$	500	0.3569	0.9561	0.6284	0.9731	0.5019
	1000	0.3674	0.9738	0.6316	0.9840	0.5173

and then

$$\begin{aligned}
\ddot{\mathscr{L}}_\omega &= \left(\langle\widetilde{D}\rangle - \omega\left|\widetilde{U}\right|\right)^{-1}\langle\widetilde{D}\rangle\left(\langle\widetilde{D}\rangle - \omega\left|\widetilde{L}\right|\right)^{-1} \\
&\quad\times\left((1-\omega)\langle\widetilde{D}\rangle+\omega\left|\widetilde{L}\right|\right)\langle\widetilde{D}\rangle^{-1}\left((1-\omega)\langle\widetilde{D}\rangle+\omega\left|\widetilde{U}\right|\right) \\
&= \left(I-\omega\langle\widetilde{D}\rangle^{-1}\left|\widetilde{U}\right|\right)^{-1}\left(I-\omega\langle\widetilde{D}\rangle^{-1}\left|\widetilde{L}\right|\right)^{-1} \\
&\quad\times\left((1-\omega)I+\omega\langle\widetilde{D}\rangle^{-1}\left|\widetilde{L}\right|\right)\left((1-\omega)I+\omega\langle\widetilde{D}\rangle^{-1}\left|\widetilde{U}\right|\right) \\
&\le \left(I-\omega\overline{D}^{-1}\left|\widetilde{U}\right|\right)^{-1}\left(I-\omega\overline{D}^{-1}\left|\widetilde{L}\right|\right)^{-1} \\
&\quad\times\left((1-\omega)I+\omega\overline{D}^{-1}\left|\widetilde{L}\right|\right)\left((1-\omega)I+\omega\overline{D}^{-1}\left|\widetilde{U}\right|\right) \\
&= \overline{\mathscr{L}}_\omega.
\end{aligned} \tag{43}$$

Therefore, by Lemma 3, $\rho(\ddot{\mathscr{L}}_\omega) \le \rho(\overline{\mathscr{L}}_\omega)$. □

Combining the previous Theorems, we can obtain the following conclusion.

Theorem 14. *Let A be a nonsingular H-matrix with unit diagonal elements, $0 < \omega \le 1$ and $|\alpha_i| < h_i$, $i = 1, 2, \ldots, m$. Then*

$$\rho\left(\widetilde{\mathscr{L}}_\omega\right) \le \rho\left(\ddot{\mathscr{L}}_\omega\right) \le \rho\left(\overline{\mathscr{L}}_\omega\right) \le \rho\left(\widehat{\mathscr{L}}_\omega\right) < 1. \tag{44}$$

5. Numerical Example

For randomly generated nonsingular H-matrices for $n = 100, 200, 500, 1000$ with $n_1 = n_2 = \cdots = n_m = 5$, we have determined the spectral radius of the iteration matrices of SSOR method mentioned previously with preconditioner P_1. We report the spectral radius of the corresponding iteration matrix by ρ. The m parameters α_i, $i = 1, 2, \ldots, m$, are taken from the m equal-partitioned points of the interval $[0, 1]$. We take

$$P_1 = \begin{pmatrix}
A_{11}^{-1} & A_{11}^{-1}A_{12} & O_{13} & \cdots & O_{1m} \\
O_{21} & A_{22}^{-1} & A_{11}^{-1}A_{23} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
O_{m-1,1} & \cdots & O_{m-1,m-2} & A_{m-1,m-1}^{-1} & A_{m-1,m-1}^{-1}A_{m-1,m} \\
A_{mm}^{-1}A_{m1} & O_{m2} & \cdots & O_{m,m-1} & A_{mm}^{-1}
\end{pmatrix}. \tag{45}$$

For P_1, we make two groups of experiments. In Figure 1, we test the relation between ω and ρ, when $N = 100$, $\omega = 0.6$, where "×", "+", "∗", "·" and "∘" denote the spectral radius of $\langle A\rangle$, $P_2\langle A\rangle$, $\langle P_1A\rangle$, A, and P_1A, respectively. In Table 1, the meaning of notations $\rho(\widetilde{\mathscr{L}}_\omega)$, $\rho(\overline{\mathscr{L}}_\omega)$, $\rho(\ddot{\mathscr{L}}_\omega)$, $\rho(\widehat{\mathscr{L}}_\omega)$, and $\rho(\mathscr{L}_\omega)$ denotes the spectral radius of P_1A, $P_2\langle A\rangle$, $\langle P_1A\rangle$, $\langle A\rangle$, and A, respectively.

From Figure 1 and Table 1, we can conclude that the spectral radius of the preconditioned SSOR method with preconditioner P_1 is the best among others, which further illustrates that, Theorem 14 is true.

Acknowledgments

The authors express their thanks to the editor Professor Hak-Keung Lam and the anonymous referees who made much useful and detailed suggestions that helped them to correct some minor errors and improve the quality of the paper.

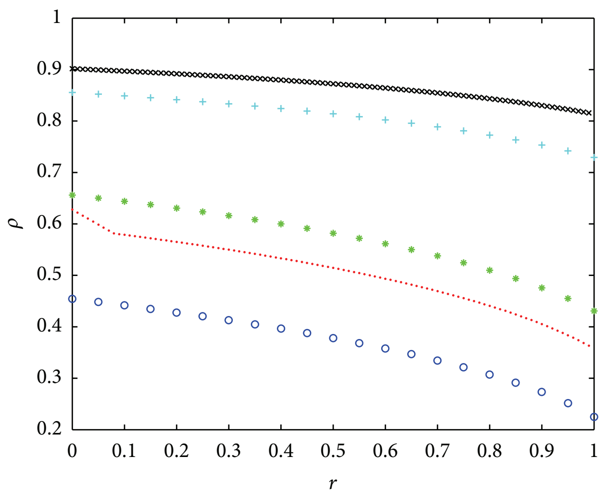

Figure 1: The relation between ω and ρ, when $N = 100,\ \omega = 0.6$.

References

[1] M. Alanelli and A. Hadjidimos, "Block Gauss elimination followed by a classical iterative method for the solution of linear systems," *Journal of Computational and Applied Mathematics*, vol. 163, no. 2, pp. 381–400, 2004.

[2] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, vol. 9 of *Classics in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 1994.

[3] W. Li and Z.-y. You, "The multi-parameters overrelaxation method," *Journal of Computational Mathematics*, vol. 16, no. 4, pp. 367–374, 1998.

[4] R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1962.

[5] L. Yu. Kolotilina, "Two-sided bounds for the inverse of an H-matrix," *Linear Algebra and Its Applications*, vol. 225, pp. 117–123, 1995.

[6] M. Neumann and R. J. Plemmons, "Convergence of parallel multisplitting iterative methods for M-matrices," *Linear Algebra and Its Applications*, vol. 88-89, pp. 559–573, 1987.