Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 194509, 38 pages
doi:10.1155/2012/194509

Research Article

Well-Posedness by Perturbations of Generalized Mixed Variational Inequalities in Banach Spaces

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Received 24 September 2011; Accepted 3 November 2011

Academic Editor: Yeong-Cheng Liou

We consider an extension of the notion of well-posedness by perturbations, introduced by Zolezzi (1995, 1996) for a minimization problem, to a class of generalized mixed variational inequalities in Banach spaces, which includes as a special case the class of mixed variational inequalities. We establish some metric characterizations of the well-posedness by perturbations. On the other hand, it is also proven that, under suitable conditions, the well-posedness by perturbations of a generalized mixed variational inequality is equivalent to the well-posedness by perturbations of the corresponding inclusion problem and corresponding fixed point problem. Furthermore, we derive some conditions under which the well-posedness by perturbations of a generalized mixed variational inequality is equivalent to the existence and uniqueness of its solution.

1. Introduction

Let X be a real Banach space and $f : X \to \mathbf{R} \cup \{+\infty\}$ a real-valued functional on X. In 1966, Tikhonov [1] first introduced the classical notion of well-posedness for a minimization problem $\min_{x \in X} f(x)$, which has been known as the Tikhonov well-posedness. A minimization problem is said to be Tikhonov well-posed if it has a unique solution toward which every minimizing sequence of the problem converges. It is obvious that the notion of Tikhonov well-posedness is inspired by the numerical methods producing optimizing sequences for optimization problems and plays a crucial role in the optimization theory. The notion of generalized Tikhonov well-posedness is also introduced for a minimization problem having more than one solution, which requires the existence of solutions and the convergence of some subsequence of every minimizing sequence toward some solution. Another important notion of well-posedness for a minimization problem is the well-posedness by perturbations or extended well-posedness due to Zolezzi [2, 3]. The notion of well-posedness by perturbations establishes a form of continuous dependence of the solutions upon a parameter. There are many other notions of well-posedness in optimization problems. For more details, we refer the readers to [1–7] and the references therein.

On the other hand, the concept of well-posedness has been generalized to other variational problems, such as variational inequalities [4, 8–14], saddle point problems [15], Nash equilibrium problems [14, 16–18], equilibrium problems [19], inclusion problems [20, 21], and fixed point problems [20–22]. An initial notion of well-posedness for a variational inequality is due to Lucchetti and Patrone [4]. They introduced the notion of well-posedness for variational inequalities and proved some related results by means of Ekeland's variational principle. Since then, many papers have been devoted to the extensions of well-posedness of minimization problems to various variational inequalities. Lignola and Morgan [12] generalized the notion of well-posedness by perturbations to a variational inequality and established the equivalence between the well-posedness by perturbations of a variational inequality and the well-posedness by perturbations of the corresponding minimization problem. Lignola and Morgan [14] introduced the concepts of α-well-posedness for variational inequalities. Del Prete et al. [13] further proved that the α-well-posedness of variational inequalities is closely related to the well-posedness of minimization problems. Recently, Fang et al. [9] generalized the notions of well-posedness and α-well-posedness to a mixed variational inequality. In the setting of Hilbert spaces, Fang et al. [9] proved that under suitable conditions the well-posedness of a mixed variational inequality is equivalent to the existence and uniqueness of its solution. They also showed that the well-posedness of a mixed variational inequality has close links with the well-posedness of the corresponding inclusion problem and corresponding fixed point problem in the setting of Hilbert spaces. Subsequently, the notions of well-posedness and α-well-posedness for a mixed variational inequality in [9] are extended by Ceng and Yao [11] to a generalized mixed variational inequality in the setting of Hilbert spaces. Very recently, Fang et al. [10] generalized the notion of well-posedness by perturbations to a mixed variational inequality in Banach spaces. In the setting of Banach spaces, they established some metric characterizations and showed that the well-posedness by perturbations of a mixed variational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem and corresponding fixed point problem. They also derived some conditions under which the well-posedness by perturbations of the mixed variational inequality is equivalent to the existence and uniqueness of its solution.

In this paper, we further extend the notion of well-posedness by perturbations to a class of generalized mixed variational inequalities in Banach spaces, which includes as a special case the class of mixed variational inequalities in [10]. Under very mild conditions, we establish some metric characterizations for the well-posed generalized mixed variational inequality and show that the well-posedness by perturbations of a generalized mixed variational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem and corresponding fixed point problem. We also derive some conditions under which the well-posedness by perturbations of the generalized mixed variational inequality is equivalent to the existence and uniqueness of its solution.

2. Preliminaries

Throughout this paper, unless stated otherwise, we always suppose that X is a real reflexive Banach space with its dual X^* and the duality pairing $\langle \cdot, \cdot \rangle$ between X and X^*. For convenience, we denote strong (resp., weak) convergence by $\to$ (resp., $\rightharpoonup$). Let $F : X \to 2^X$ be a nonempty-valued multifunction, $A : X \to X^*$ a single-valued mapping, and $f : X \to \mathbf{R} \cup \{+\infty\}$ a proper, convex, and lower semicontinuous functional. Denote by $\operatorname{dom} f$ the domain of f, that is,

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}. \tag{2.1}$$

The generalized mixed variational inequality associated with (A, F, f) is formulated as follows:

$$\begin{gathered}\text{GMVI}(A, F, f)\text{: find } x \in X \text{ such that, for some } u \in F(x),\\ \langle Au, x - y\rangle + f(x) - f(y) \le 0, \quad \forall y \in X,\end{gathered} \tag{2.2}$$

which has been studied intensively (see, e.g., [11, 23–25]).

In the following, we give some special cases of GMVI(A, F, f).

(i) Whenever $F = I$, the identity mapping of X, GMVI(A, F, f) reduces to the following mixed variational inequality associated with (A, f):

$$\text{MVI}(A, f)\text{: find } x \in X \text{ such that } \langle Ax, x - y\rangle + f(x) - f(y) \le 0, \quad \forall y \in X, \tag{2.3}$$

which has been considered in [8–11, 26].

(ii) Whenever $f = \delta_K$, MVI(A, f) reduces to the following classical variational inequality:

$$\text{VI}(A, K)\text{: find } x \in K \text{ such that } \langle Ax, x - y\rangle \le 0, \quad \forall y \in K, \tag{2.4}$$

where δ_K denotes the indicator functional of a convex subset K of X.

(iii) Whenever $A = 0$, MVI(A, f) reduces to the global minimization problem:

$$\text{MP}(f, X)\text{: } \min_{x \in X} f(x). \tag{2.5}$$

Suppose that L is a parametric normed space, $P \subset L$ is a closed ball with positive radius, and $p^* \in P$ is a fixed point. The perturbed problem of GMVI(A, F, f) is always given by

$$\begin{gathered}\text{GMVI}_p(A, F, f)\text{: find } x \in X \text{ such that for some } u \in F(x),\\ \left\langle \widetilde{A}(p, u), x - y\right\rangle + \widetilde{f}(p, x) - \widetilde{f}(p, y) \le 0, \quad \forall y \in X,\end{gathered} \tag{2.6}$$

where $\widetilde{A} : P \times X \to X^*$ is such that $\widetilde{A}(p^*, \cdot) = A$ and $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ is such that $\widetilde{f}(p^*, \cdot) = f$.

Now we recall some concepts and results.

Definition 2.1 (see [26]). A mapping $M : X \to 2^{X^*}$ is said to be

(i) monotone if

$$\langle g - h, x - y \rangle \geq 0, \quad \forall x, y \in \operatorname{Dom} M,\ g \in M(x),\ h \in M(y); \tag{2.7}$$

(ii) maximal monotone if M is monotone and

$$\langle g - w, x - z \rangle \geq 0, \quad \forall x \in \operatorname{Dom} M,\ g \in M(x) \Longrightarrow w \in M(z), \tag{2.8}$$

where 2^{X^*} denotes the family of all subsets of X^* and $\operatorname{Dom} M = \{x \in X : M(x) \neq \emptyset\}$.

Definition 2.2 (see [11]). A nonempty-valued multifunction $F : X \to 2^X$ is said to be monotone with respect to a single-valued mapping $A : X \to X^*$ if, for all $x, y \in X$,

$$\langle Au - Av, x - y \rangle \geq 0, \quad \forall u \in F(x),\ v \in F(y). \tag{2.9}$$

Proposition 2.3 (Nadler's Theorem [27]). *Let $(X, \|\cdot\|)$ be a normed vector space and $\mathscr{H}(\cdot,\cdot)$ the Hausdorff metric on the collection $\mathrm{CB}(X)$ of all nonempty, closed, and bounded subsets of X, induced by a metric d in terms of $d(x,y) = \|x - y\|$, which is defined by $\mathscr{H}(U,V) = \max\{e(U,V), e(V,U)\}$ for U and V in $\mathrm{CB}(X)$, where $e(U,V) = \sup_{x\in U} d(x,V)$ with $d(x,V) = \inf_{y\in V}\|x - y\|$. If U and V lie in $\mathrm{CB}(X)$, then, for any $\varepsilon > 0$ and any $u \in U$, there exists $v \in V$ such that $\|u - v\| \leq (1 + \varepsilon)\mathscr{H}(U,V)$. In particular, whenever U and V are compact subsets in X, one has $\|u - v\| \leq \mathscr{H}(U,V)$.*

Definition 2.4. Let $\{U_n\}$ be a sequence of nonempty subsets of X. One says that U_n converges to U in the sense of Hausdorff metric if $\mathscr{H}(U_n, U) \to 0$. It is easy to see that $e(U_n, U) \to 0$ if and only if $d(x_n, U) \to 0$ for all selection $x_n \in U_n$. For more details on this topic, the reader is referred to [28].

Definition 2.5 (see [29]). A mapping $A : X \to X^*$ is said to be

(i) coercive if

$$\lim_{\|x\|\to\infty} \frac{\langle Ax, x\rangle}{\|x\|} = +\infty; \tag{2.10}$$

(ii) bounded if $A(B)$ is bounded for every bounded subset B of X;

(iii) hemicontinuous if, for any $x, y \in X$, the function $t \mapsto \langle A(x + t(y - x)), y - x\rangle$ from $[0,1]$ into $\mathbf{R}$ is continuous at 0^+;

(iv) uniformly continuous if, for any neighborhood V of 0 in X^*, there exists a neighborhood U of 0 in X such that $Ax - Ay \in V$ for all $x, y \in U$.

Clearly, the uniform continuity implies the continuity, and the continuity implies the hemicontinuity, but the converse is not true in general.

Definition 2.6. (i) A nonempty weakly compact-valued multifunction $F : X \to 2^X$ is said to be $\mathcal{H}$-hemicontinuous if, for any $x, y \in X$, the function $t \mapsto \mathcal{H}(F(x+t(y-x)), F(x))$ from $[0,1]$ into $\mathbf{R}^+ = [0,\infty)$ is continuous at 0^+, where $\mathcal{H}$ is the Hausdorff metric defined on $\mathrm{CB}(X)$.

(ii) A nonempty weakly compact-valued multifunction $F : X \to 2^X$ is said to be $\mathcal{H}$-continuous at a point $x \in X$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$ with $\|x-y\| < \delta$, one has $\mathcal{H}(F(x), F(y)) < \varepsilon$, where $\mathcal{H}$ is the Hausdorff metric defined on $\mathrm{CB}(X)$. If this multifunction $F : X \to 2^X$ is $\mathcal{H}$-continuous at each $x \in X$, then one says that $F : X \to 2^X$ is $\mathcal{H}$-continuous.

(iii) A nonempty weakly compact-valued multifunction $F : X \to 2^X$ is said to be $\mathcal{H}$-uniformly continuous if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$ with $\|x-y\| < \delta$, one has $\mathcal{H}(F(x), F(y)) < \varepsilon$, where $\mathcal{H}$ is the Hausdorff metric defined on $\mathrm{CB}(X)$.

Remark 2.7. If $X = H$ a real Hilbert space, then Definition 2.6(i)–(iii) reduce to Definition 2.3 (ii)–(iv) in [11], respectively.

Lemma 2.8. *Let $A : X \to X^*$ be weakly continuous (i.e., continuous from the weak topology of X to the weak topology of X^*), let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathcal{H}$-hemicontinuous and monotone with respect to A, and let $f : X \to \mathbf{R} \cup \{+\infty\}$ be proper and convex. Then, for a given $x \in X$, the following statements are equivalent:*

(i) *there exists $u \in F(x)$ such that $\langle Au, x-y\rangle + f(x) - f(y) \le 0$, for all $y \in X$;*

(ii) *$\langle Av, x-y\rangle + f(x) - f(y) \le 0$, for all $y \in X, v \in F(y)$.*

Proof. Suppose that, for some $u \in F(x)$,

$$\langle Au, x-y\rangle + f(x) - f(y) \le 0, \quad \forall y \in X. \tag{2.11}$$

Since F is monotone with respect to A, one has

$$\langle Av, x-y\rangle + f(x) - f(y) \le \langle Au, x-y\rangle + f(x) - f(y) \le 0, \quad \forall y \in X,\ v \in F(y). \tag{2.12}$$

Consequently,

$$\langle Av, x-y\rangle + f(x) - f(y) \le 0, \quad \forall y \in X,\ v \in F(y). \tag{2.13}$$

Conversely, suppose that the last inequality is valid. Given any $y \in X$, we define $y_t = x + t(y-x)$ for all $t \in (0,1)$. Replacing y by y_t in the left-hand side of the last inequality, one derives, for each $v_t \in F(y_t)$,

$$0 \ge \langle Av_t, x-y_t\rangle + f(x) - f(y_t) \ge t\left[\langle Av_t, x-y\rangle + f(x) - f(y)\right], \tag{2.14}$$

which hence implies that

$$\langle Av_t, x-y\rangle + f(x) - f(y) \le 0, \quad \forall v_t \in F(y_t),\ t \in (0,1). \tag{2.15}$$

Since $F : X \to 2^X$ is a nonempty weakly compact-valued multifunction, both $F(y_t)$ and $F(x)$ are nonempty weakly compact and hence are nonempty, weakly closed, and weakly bounded. Note that the weak closedness of sets in X implies the strong closedness and that the weak boundedness of sets in X is equivalent to the strong boundedness. Thus, it is known that both $F(y_t)$ and $F(x)$ lie in CB(X). From Proposition 2.3, it follows that, for each $t \in (0,1)$ and each fixed $v_t \in F(y_t)$, there exists a $u_t \in F(x)$ such that

$$\|v_t - u_t\| \leq (1+t)\mathscr{H}(F(y_t), F(x)). \tag{2.16}$$

Since $F(x)$ is weakly compact, it follows from the net $\{u_t : t \in (0,1)\} \subset F(x)$ that there exists some subnet which converges weakly to a point of $F(x)$. Without loss of generality, we may assume that $u_t \rightharpoonup u \in F(x)$ as $t \to 0^+$. Since F is $\mathscr{H}$-hemicontinuous, one deduces that as $t \to 0^+$

$$\|v_t - u_t\| \leq (1+t)\mathscr{H}(F(y_t), F(x)) = (1+t)\mathscr{H}(F(x+t(y-x)), F(x)) \longrightarrow 0. \tag{2.17}$$

Observe that, for each $\varphi \in X^*$,

$$\begin{aligned} |\langle \varphi, v_t - u\rangle| &= |\langle \varphi, v_t\rangle - \langle \varphi, u_t\rangle + \langle \varphi, u_t\rangle - \langle \varphi, u\rangle| \\ &\leq \|\varphi\|\|v_t - u_t\| + |\langle \varphi, u_t - u\rangle| \longrightarrow 0 \quad \text{as } t \longrightarrow 0^+, \end{aligned} \tag{2.18}$$

that is, $v_t \rightharpoonup u$ as $t \to 0^+$. Since A is weakly continuous, $Av_t \rightharpoonup Au$ and hence, for $y \in X$,

$$\langle Av_t, x-y\rangle \longrightarrow \langle Au, x-y\rangle \quad \text{as } t \longrightarrow 0^+. \tag{2.19}$$

Thus, letting $t \to 0^+$ in the left-hand side of (2.15), we obtain that

$$\langle Au, x-y\rangle + f(x) - f(y) \leq 0, \quad \forall y \in X. \tag{2.20}$$

Finally let us show that the vector u in the last inequality is not dependent on y, that is,

$$\langle Au, x-z\rangle + f(x) - f(z) \leq 0, \quad \forall z \in X. \tag{2.21}$$

Indeed, take a fixed $z \in X$ arbitrarily, and define $z_t = x + t(z-x)$ for all $t \in (0,1)$. Utilizing Proposition 2.3, for each $t \in (0,1)$ and $u_t \in F(x)$, there exists $w_t \in F(z_t)$ such that

$$\|u_t - w_t\| \leq (1+t)\mathscr{H}(F(x), F(z_t)). \tag{2.22}$$

Since F is $\mathscr{H}$-hemicontinuous, we deduce that as $t \to 0^+$

$$\|u_t - w_t\| \leq (1+t)\mathscr{H}(F(x), F(z_t)) = (1+t)\mathscr{H}(F(x), F(x+t(z-x))) \longrightarrow 0. \tag{2.23}$$

Thus, one has, for each $\varphi \in X^*$,

$$\begin{aligned} |\langle \varphi, w_t - u\rangle| &= |\langle \varphi, w_t\rangle - \langle \varphi, u_t\rangle + \langle \varphi, u_t\rangle - \langle \varphi, u\rangle| \\ &\leq \|\varphi\|\|u_t - w_t\| + |\langle \varphi, u_t - u\rangle| \longrightarrow 0 \quad \text{as } t \longrightarrow 0^+. \end{aligned} \tag{2.24}$$

This shows that $w_t \rightharpoonup u$ as $t \to 0^+$. Since A is weakly continuous, $Aw_t \rightharpoonup Au$ and hence, for $z \in X$,

$$\langle Aw_t, x - z\rangle \longrightarrow \langle Au, x - z\rangle \quad \text{as } t \longrightarrow 0^+. \tag{2.25}$$

Replacing y, y_t, and v_t in (2.15) by z, z_t, and w_t, respectively, one concludes that

$$\langle Aw_t, x - z\rangle + f(x) - f(z) \le 0, \quad \forall t \in (0,1). \tag{2.26}$$

This immediately implies that inequality (2.21) is valid. This completes the proof. □

Corollary 2.9 (see [11, Lemma 2.2]). *Let H be a real Hilbert space. Let $A : H \to H$ be weakly continuous (i.e., continuous from the weak topology of H to the weak topology of H), let $F : H \to 2^H$ be a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-hemicontinuous and monotone with respect to A, and let $f : H \to \mathbf{R} \cup \{+\infty\}$ be proper and convex. Then, for a given $x \in H$, the following statements are equivalent:*

(i) *there exists $u \in F(x)$ such that $\langle Au, x - y\rangle + f(x) - f(y) \le 0$, for all $y \in H$;*

(ii) *$\langle Av, x - y\rangle + f(x) - f(y) \le 0$, for all $y \in H, v \in F(y)$.*

Definition 2.10 (see [30]). Let K be a nonempty, closed, and convex subset of X. One says that K is well-positioned if there exist $x_0 \in X$ and $g \in X^*$ such that

$$\langle g, x - x_0\rangle \ge \|x - x_0\|, \quad \forall x \in K. \tag{2.27}$$

Remark 2.11 (see [10, Remark 2.1]). (i) If K is well-positioned, then $K + x^*$ is well-positioned for all $x^* \in X$.

(ii) As pointed out in [30, Remark 2.2], every nonempty compact convex set of a finite-dimensional space is well-positioned. Some useful properties and interesting applications have been discussed in [30, 31]. The following result is exacted from Proposition 2.1 of [30]. Also see [31, Proposition 2.1].

Lemma 2.12. *Let K be a nonempty, closed, and convex subset of a reflexive Banach space X. If K is well-positioned, then there is no sequence $\{x_n\} \subset K$ with $\|x_n\| \to +\infty$ such that origin is a weak limit of $\{x_n/\|x_n\|\}$.*

Definition 2.13 (see [28]). Let B be a nonempty subset of X. The measure of noncompactness μ of the set B is defined by

$$\mu(B) = \inf\left\{\varepsilon > 0 \colon A \subset \bigcup_{i=1}^{n} B_i,\ \operatorname{diam} B_i < \varepsilon,\ i = 1, 2, \ldots, n,\ \text{for some integer } n \ge 1\right\}, \tag{2.28}$$

where diam means the diameter of a set.

Lemma 2.14 (see [10, Lemma 2.3]). *Let A_n, A be nonempty, closed, and convex subsets of a real reflexive Banach space X, and let A be well-positioned. Suppose that $e(A_n, A) \to 0$ as $n \to \infty$ and*

$x \in X$. Then, there is no sequence $a_n \in A_n$ with $\|a_n\| \to +\infty$ such that origin is a weak limit of $\{(a_n - x)/\|a_n - x\|\}$.

3. Well-Posedness by Perturbations and Metric Characterizations

In this section, we generalize the concepts of well-posedness by perturbations to the generalized mixed variational inequality and establish their metric characterizations. In the sequel we always denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let $\alpha \geq 0$ be a fixed number.

Definition 3.1. Let $\{p_n\} \subset P$ be with $p_n \to p^*$. A sequence $\{x_n\} \subset X$ is called an α-approximating sequence corresponding to $\{p_n\}$ for GMVI(A, F, f) if there exists a sequence $\{u_n\} \subset X$ with $u_n \in F(x_n)$ (for all $n \geq 1$) and a sequence $\{\varepsilon_n\}$ of nonnegative numbers with $\varepsilon_n \to 0$ such that

$$\begin{gathered} x_n \in \operatorname{dom} \tilde{f}(p_n, \cdot), \\ \left\langle \tilde{A}(p_n, u_n), x_n - y \right\rangle + \tilde{f}(p_n, x_n) - \tilde{f}(p_n, y) \leq \frac{\alpha}{2} \|x_n - y\|^2 + \varepsilon_n, \quad \forall y \in X,\ n \geq 1. \end{gathered} \tag{3.1}$$

Whenever $\alpha = 0$, we say that $\{x_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for GMVI(A, F, f). Clearly, every α_2-approximating sequence corresponding to $\{p_n\}$ is α_1-approximating corresponding to $\{p_n\}$ provided $\alpha_1 > \alpha_2 \geq 0$.

Definition 3.2. One says that GMVI(A, F, f) is strongly (resp., weakly) α-well-posed by perturbations if GMVI(A, F, f) has a unique solution and, for any $\{p_n\} \subset P$ with $p_n \to p^*$, every α-approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution. In the sequel, strong (resp., weak) 0-well-posedness by perturbations is always called strong (resp., weak) well-posedness by perturbations. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) α_1-well-posedness by perturbations implies strong (resp., weak) α_2-well-posedness by perturbations.

Remark 3.3. (i) When X is a Hilbert space and $p_n = p^*$ (for all $n \geq 1$), Definitions 3.1 and 3.2 coincide with Definitions 3.1 and 3.2 of [11], respectively. (ii) When $f = \delta_K$ and $F = I$ the identity mapping of X, Definitions 3.1 and 3.2 reduce to the definitions of approximating sequences of the classical variational inequality (see [12, 13]).

Definition 3.4. One says that GMVI(A, F, f) is strongly (resp., weakly) generalized α-well-posed by perturbations if GMVI(A, F, f) has a nonempty solution set S and, for any $\{p_n\} \subset P$ with $p_n \to p^*$, every α-approximating sequence corresponding to $\{p_n\}$ has some subsequence which converges strongly (resp., weakly) to some point of S. Strong (resp., weak) generalized 0-well-posedness by perturbations is always called strong (resp., weak) generalized well-posedness by perturbations. Clearly, if $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) generalized α_1-well-posedness by perturbations implies strong (resp., weak) generalized α_2-well-posedness by perturbations.

Remark 3.5. (i) When X is a Hilbert space and $p_n = p^*$ (for all $n \geq 1$), Definition 3.4 coincides with Definition 3.3 of [11]. (ii) When $f = \delta_K$ and $F = I$ the identity mapping of X, Definition 3.4 reduces to the definition of strong (resp., weak) parametric α-well-posedness in the generalized sense for the classical variational inequality (see [12–14]). (iii) When $A = 0$ and $\alpha = 0$, Definition 3.4 coincides with the definition of well-posedness by perturbations introduced for a minimization problem [2, 3].

To derive the metric characterizations of α-well-posedness by perturbations, we consider the following approximating solution set of GMVI(A, F, f):

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(p^*, \varepsilon)} \Big\{x \in \operatorname{dom} \tilde{f}(p, \cdot)\text{: there exists } u \in F(x) \text{ such that } \Big\langle \tilde{A}(p, u), x - y \Big\rangle + \tilde{f}(p, x) - \tilde{f}(p, y) \leq \frac{\alpha}{2}\|x - y\|^2 + \varepsilon,\ \forall y \in X\Big\}, \quad \forall \varepsilon \geq 0, \tag{3.2}$$

where $B(p^*, \varepsilon)$ denotes the closed ball centered at p^* with radius ε. In this section, we always suppose that x^* is a fixed solution of GMVI(A, F, f). Define

$$\theta(\varepsilon) = \sup\{\|x - x^*\| : x \in \Omega_\alpha(\varepsilon)\}, \quad \forall \varepsilon \geq 0. \tag{3.3}$$

It is easy to see that $\theta(\varepsilon)$ is the radius of the smallest closed ball centered at x^* containing $\Omega_\alpha(\varepsilon)$. Now, we give a metric characterization of strong α-well-posedness by perturbations by considering the behavior of $\theta(\varepsilon)$ when $\varepsilon \to 0$.

Theorem 3.6. GMVI(A, F, f) *is strongly α-well-posed by perturbations if and only if* $\theta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. Repeating almost the same argument as in the proof of [10, Theorem 3.1], we can easily obtain the desired result. □

Remark 3.7. Theorem 3.6 improves Proposition 2.2 of [13], Theorem 3.1 of [9], and Theorem 3.1 of [10].

Now, we give an example to illustrate Theorem 3.6.

Example 3.8. Let $X = \mathbf{R}$, $P = [-1, 1]$, $p^* = 0$, $\alpha = 2$, $\tilde{A}(p, x) = x(p^2 + 1)$, $F(x) = \{x, 0\}$, and $\tilde{f}(p, x) = x^2$ for all $x \in X$, $p \in P$. Clearly, $x^* = 0$ is a solution of GMVI(A, F, f). For any $\varepsilon > 0$, it follows that

$$\begin{aligned} \Omega_\alpha^p(\varepsilon) &= \Big\{x \in \mathbf{R}\text{: } u\Big(p^2 + 1\Big)(x - y) + x^2 - y^2 \leq (x - y)^2 + \varepsilon,\ \forall y \in \mathbf{R},\ \text{for some } u \in F(x)\Big\} \\ &= \Delta \cup \Lambda, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \Delta &= \Big\{x \in \mathbf{R}\text{: } x\Big(p^2 + 1\Big)(x - y) + x^2 - y^2 \leq (x - y)^2 + \varepsilon,\ \forall y \in \mathbf{R}\Big\}, \\ \Lambda &= \Big\{x \in \mathbf{R}\text{: } 0\Big(p^2 + 1\Big)(x - y) + x^2 - y^2 \leq (x - y)^2 + \varepsilon,\ \forall y \in \mathbf{R}\Big\}. \end{aligned} \tag{3.5}$$

Observe that

$$\Delta = \left\{ x \in \mathbf{R}: -2\left(y + \frac{p^2-1}{4}x \right)^2 + \frac{(p^2+3)^2}{8}x^2 \le \varepsilon,\ \forall y \in \mathbf{R} \right\}$$
$$= \left[-\frac{2\sqrt{2\varepsilon}}{(p^2+3)}, \frac{2\sqrt{2\varepsilon}}{(p^2+3)} \right], \tag{3.6}$$
$$\Lambda = \left\{ x \in \mathbf{R}:\ \left(y - \frac{x}{2} \right)^2 - \frac{x^2}{4} + \frac{\varepsilon}{2} \ge 0,\ \forall y \in \mathbf{R} \right\} = \left[-\sqrt{2\varepsilon}, \sqrt{2\varepsilon} \right].$$

Thus, we obtain

$$\Omega_\alpha^p(\varepsilon) = \Delta \cup \Lambda = \left[-\frac{2\sqrt{2\varepsilon}}{(p^2+3)}, \frac{2\sqrt{2\varepsilon}}{(p^2+3)} \right] \cup \left[-\sqrt{2\varepsilon}, \sqrt{2\varepsilon} \right] = \left[-\sqrt{2\varepsilon}, \sqrt{2\varepsilon} \right]. \tag{3.7}$$

Therefore,

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega_\alpha^p(\varepsilon) = \left[-\sqrt{2\varepsilon}, \sqrt{2\varepsilon} \right] \tag{3.8}$$

for sufficiently small $\varepsilon > 0$. By trivial computation, we have

$$\theta(\varepsilon) = \sup\{ \|x - x^*\| : x \in \Omega_\alpha(\varepsilon) \} = \sqrt{2\varepsilon} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.9}$$

By Theorem 3.6, GMVI(A, F, f) is 2-well-posed by perturbations.

To derive a characterization of strong generalized α-well-posedness by perturbations, we need another function q which is defined by

$$q(\varepsilon) = e(\Omega_\alpha(\varepsilon), S), \quad \forall \varepsilon \ge 0, \tag{3.10}$$

where S is the solution set of GMVI(A, F, f) and e is defined as in Proposition 2.3.

Theorem 3.9. GMVI (A, F, f) *is strongly generalized α-well-posed by perturbations if and only if S is nonempty compact and $q(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Proof. Repeating almost the same argument as in the proof of [10, Theorem 3.2], we can readily derive the desired result. □

Example 3.10. Let $X = \mathbf{R}$, $P = [-1, 1]$, $p^* = 0$, $\alpha = 2$, $\widetilde{A}(p, x) = x(p^2 + 1)$, $F(x) = \{x, 0\}$, and $\widetilde{f}(p, x) = x^2$ for all $x \in X$, $p \in P$. Clearly, $x^* = 0$ is a solution of GMVI(A, F, f). Repeating the same argument as in Example 3.8, we obtain that, for any $\varepsilon > 0$,

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega_\alpha^p(\varepsilon) = \left[-\sqrt{2\varepsilon}, \sqrt{2\varepsilon} \right] \tag{3.11}$$

for sufficiently small $\varepsilon > 0$. By trivial computation, we have

$$q(\varepsilon) = e(\Omega_\alpha(\varepsilon), S) = \sup_{x(\varepsilon)\in\Omega_\alpha(\varepsilon)} d(x(\varepsilon), S) \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.12}$$

By Theorem 3.9, GMVI(A, F, f) is generalized α-well-posed by perturbations.

The strong generalized α-well-posedness by perturbations can be also characterized by the behavior of the noncompactness measure $\mu(\Omega_\alpha(\varepsilon))$.

Theorem 3.11. *Let L be finite dimensional, $\widetilde{A} : P \times X \to X^*$ weakly continuous (i.e., continuous from the product of the norm topology of P and weak topology of X to the weak topology of X^*), $F : X \to 2^X$ a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-continuous, and $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ a continuous functional such that $\widetilde{f}(p, \cdot)$ is proper and convex. Then,* GMVI(A, F, f) *is strongly generalized α-well-posed by perturbations if and only if $\Omega_\alpha(\varepsilon) \neq \emptyset$, for all $\varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$.*

Proof. First, we will prove that $\Omega_\alpha(\varepsilon)$ is closed for all $\varepsilon \geq 0$. Let $\{x_n\} \subset \Omega_\alpha(\varepsilon)$ with $x_n \to \overline{x}$. Then, there exist $\{p_n\} \subset B(p^*, \varepsilon)$ and $\{u_n\} \subset X$ with $u_n \in F(x_n)$ (for all $n \geq 1$) such that

$$\left\langle \widetilde{A}(p_n, u_n), x_n - y \right\rangle + \widetilde{f}(p_n, x_n) - \widetilde{f}(p_n, y) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \varepsilon, \quad \forall y \in X,\ n \geq 1. \tag{3.13}$$

Without loss of generality, we may assume $p_n \to \overline{p} \in B(p^*, \varepsilon)$ since L is finite dimensional. Since $F : X \to 2^X$ is a nonempty weakly compact-valued multifunction, $F(x_n)$ and $F(\overline{x})$ are nonempty weakly compact and hence are nonempty, weakly closed, and weakly bounded. Note that the weak closedness of sets in X implies the strong closedness and that the weak boundedness of sets in X is equivalent to the strong boundedness. Thus, it is known that $F(x_n)$ and $F(\overline{x})$ lie in CB(X). According to Proposition 2.3, for each $n \geq 1$ and $u_n \in F(x_n)$, there exists $v_n \in F(\overline{x})$ such that

$$\|u_n - v_n\| \leq \left(1 + \frac{1}{n}\right)\mathscr{H}(F(x_n), F(\overline{x})). \tag{3.14}$$

Since F is $\mathscr{H}$-continuous, one deduces that

$$\|u_n - v_n\| \leq \left(1 + \frac{1}{n}\right)\mathscr{H}(F(x_n), F(\overline{x})) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

Also, since $F(\overline{x})$ is weakly compact, it follows from $\{v_n\} \subset F(\overline{x})$ that there exists some subsequence of $\{v_n\}$ which converges weakly to a point of $F(\overline{x})$. Without loss of generality, we may assume that

$$v_n \rightharpoonup \overline{u} \in F(\overline{x}) \text{ as } n \longrightarrow \infty. \tag{3.16}$$

Consequently, one has, for each $\varphi \in X^*$,

$$\begin{aligned} \left|\langle \varphi, u_n - \overline{u}\rangle\right| &\le \left|\langle \varphi, u_n - v_n\rangle\right| + \left|\langle \varphi, v_n - \overline{u}\rangle\right| \\ &\le \|\varphi\| \|u_n - v_n\| + \left|\langle \varphi, v_n - \overline{u}\rangle\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.17}$$

This implies that $u_n \rightharpoonup \overline{u}$ as $n \to \infty$. Taking into account the weak continuity of $\widetilde{A}$, we immediately obtain that

$$\widetilde{A}(p_n, u_n) \rightharpoonup \widetilde{A}(\overline{p}, \overline{u}) \quad \text{as } n \longrightarrow \infty, \tag{3.18}$$

and hence, for each $y \in X$,

$$\begin{aligned} &\left|\left\langle \widetilde{A}(p_n, u_n), x_n - y\right\rangle - \left\langle \widetilde{A}(\overline{p}, \overline{u}), \overline{x} - y\right\rangle\right| \\ &\quad\le \left|\left\langle \widetilde{A}(p_n, u_n), x_n - y\right\rangle - \left\langle \widetilde{A}(p_n, u_n), \overline{x} - y\right\rangle\right| \\ &\qquad + \left|\left\langle \widetilde{A}(p_n, u_n), \overline{x} - y\right\rangle - \left\langle \widetilde{A}(\overline{p}, \overline{u}), \overline{x} - y\right\rangle\right| \\ &\quad\le \left\|\widetilde{A}(p_n, u_n)\right\| \|x_n - \overline{x}\| + \left|\left\langle \widetilde{A}(p_n, u_n) - \widetilde{A}(\overline{p}, \overline{u}), \overline{x} - y\right\rangle\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{3.19}$$

that is,

$$\left\langle \widetilde{A}(p_n, u_n), x_n - y\right\rangle \longrightarrow \left\langle \widetilde{A}(\overline{p}, \overline{u}), \overline{x} - y\right\rangle \quad \text{as } n \longrightarrow \infty. \tag{3.20}$$

Therefore, it follows from (3.13) and the continuity of $\widetilde{f}$ that

$$\left\langle \widetilde{A}(\overline{p}, \overline{u}), \overline{x} - y\right\rangle + \widetilde{f}(\overline{p}, \overline{x}) - \widetilde{f}(\overline{p}, y) \le \frac{\alpha}{2}\|\overline{x} - y\|^2 + \varepsilon, \quad \forall y \in X. \tag{3.21}$$

This shows that $\overline{x} \in \Omega_\alpha(\varepsilon)$ and so $\Omega_\alpha(\varepsilon)$ is closed.

Second, we show that

$$S = \bigcap_{\varepsilon > 0} \Omega_\alpha(\varepsilon). \tag{3.22}$$

It is obvious that $S \subset \bigcap_{\varepsilon>0} \Omega_\alpha(\varepsilon)$. Let $x^* \in \bigcap_{\varepsilon>0} \Omega_\alpha(\varepsilon)$. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \to 0$. Then, $x^* \in \Omega_\alpha(\varepsilon_n)$ and so there exist $p_n \in B(p^*, \varepsilon_n)$ and $u_n^* \in F(x^*)$ such that

$$\left\langle \widetilde{A}(p_n, u_n^*), x^* - y\right\rangle + \widetilde{f}(p_n, x^*) - \widetilde{f}(p_n, y) \le \frac{\alpha}{2}\|x^* - y\|^2 + \varepsilon_n, \quad \forall y \in X,\ n \ge 1. \tag{3.23}$$

It is clear that $p_n \to p^*$ as $n \to \infty$. Since $F(x^*)$ is weakly compact, it follows from $\{u_n^*\} \subset F(x^*)$ that there exists some subsequence of $\{u_n^*\}$ which converges weakly to a point of $F(x^*)$. Without loss of generality, we may assume that

$$u_n^* \rightharpoonup u^* \in F(x^*) \quad \text{as } n \longrightarrow \infty. \tag{3.24}$$

Note that $\widetilde{A}$ is weakly continuous. Thus,

$$\widetilde{A}(p_n, u_n^*) \rightharpoonup \widetilde{A}(p^*, u^*), \tag{3.25}$$

and hence, letting $n \to \infty$ in the last inequality, we get

$$\begin{aligned} \langle A(u^*), x^* - y\rangle + f(x^*) - f(y) &= \left\langle \widetilde{A}(p^*, u^*), x^* - y \right\rangle + \widetilde{f}(p^*, x^*) - \widetilde{f}(p^*, y) \\ &\le \frac{\alpha}{2}\|x^* - y\|^2, \quad \forall y \in X. \end{aligned} \tag{3.26}$$

For any $z \in X$ and $t \in (0,1)$, putting $y = x^* + t(z - x^*)$ in (3.26), we have

$$\begin{aligned} t\{\langle A(u^*), x^* - z\rangle + f(x^*) - f(z)\} &\le t\langle A(u^*), x^* - z\rangle + f(x^*) - f(x^* + t(z - x^*)) \\ &\le \frac{\alpha t^2}{2}\|x^* - z\|^2. \end{aligned} \tag{3.27}$$

This implies that

$$\langle A(u^*), x^* - z\rangle + f(x^*) - f(z) \le \frac{\alpha t}{2}\|x^* - z\|^2, \quad \forall z \in X. \tag{3.28}$$

Letting $t \to 0$ in the last inequality, we get

$$\langle A(u^*), x^* - z\rangle + f(x^*) - f(z) \le 0, \quad \forall z \in X. \tag{3.29}$$

Consequently, $x^* \in S$ and so (3.22) is proved.

Now, we suppose that GMVI(A, F, f) is strongly generalized α-well-posed by perturbations. By Theorem 3.9, S is nonempty compact and $q(\varepsilon) \to 0$. Then, $\Omega_\alpha(\varepsilon) \neq \emptyset$ since $S \subset \Omega_\alpha(\varepsilon)$ for all $\varepsilon > 0$. Observe that, for all $\varepsilon > 0$,

$$\mathscr{H}(\Omega_\alpha(\varepsilon), S) = \max\{e(\Omega_\alpha(\varepsilon), S), e(S, \Omega_\alpha(\varepsilon))\} = e(\Omega_\alpha(\varepsilon), S). \tag{3.30}$$

Taking into account the compactness of S, we get

$$\mu(\Omega_\alpha(\varepsilon)) \le 2\mathscr{H}(\Omega_\alpha(\varepsilon), S) = 2e(\Omega_\alpha(\varepsilon), S) = 2q(\varepsilon) \longrightarrow 0. \tag{3.31}$$

Conversely, we suppose that $\Omega_\alpha(\varepsilon) \neq \emptyset$, for all $\varepsilon > 0$, and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Since $\Omega_\alpha(\varepsilon)$ is increasing with respect to $\varepsilon > 0$, by the Kuratowski theorem [28, page 318], we have from (3.22)

$$q(\varepsilon) = \mathscr{H}(\Omega_\alpha(\varepsilon), S) \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0 \tag{3.32}$$

and S is nonempty compact. By Theorem 3.9, GMVI(A, F, f) is strongly generalized α-well-posed by perturbations. □

Remark 3.12. Theorem 3.3 of [10] generalizes Theorem 3.2 of [9]. Theorem 3.11 generalizes Theorem 3.2 of [11] from the case of strong generalized α-well-posedness in the setting of Hilbert spaces to the case of strong generalized α-well-posedness by perturbations in the setting of Banach spaces. Furthermore, Theorem 3.11 improves, extends, and develops [10, Theorem 3.3] in the following aspects.

(i) The mixed variational inequality problem (MVI) in [10, Theorem 3.3] is extended to develop the more general problem, that is, the generalized mixed variational inequality problem (GMVI) with a nonempty weakly compact-valued multifunction in the setting of Banach spaces. Moreover, the concept of strong generalized α-well-posedness by perturbations for MVI in [10, Theorem 3.3] is extended to develop the concept of strong generalized α-well-posedness by perturbations for GMVI.

(ii) Since the generalized mixed variational inequality problem (GMVI) is more general and more complicated than the mixed variational inequality problem (MVI), the assumptions in Theorem 3.11 are very different from the ones in [10, Theorem 3.3]; for instance, in Theorem 3.11, let L be finite dimensional, $\widetilde{A} : P \times X \to X^*$ weakly continuous, and $F : X \to 2^X$ a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-continuous, but, in [10, Theorem 3.3], let L be finite dimensional, $\widetilde{A} : P \times X \to X^*$ a continuous mapping.

(iii) The technique of proving strong generalized α-well-posedness by perturbations for GMVI in Theorem 3.11 is very different from the one for MVI in [10, Theorem 3.3] because our technique depends on the well-known Nadler's Theorem [27], the $\mathscr{H}$-continuity of nonempty weakly compact-valued multifunction F and the property of the Hausdorff metric $\mathscr{H}$.

Remark 3.13. Clearly, any solution of GMVI(A, F, f) is a solution of the α problem: find $x \in X$ such that, for some $u \in F(x)$,

$$\langle A(u), x - y \rangle + f(x) - f(y) \leq \frac{\alpha}{2} \|x - y\|^2, \quad \forall y \in X, \tag{3.33}$$

but the converse is not true in general. To show this, let $X = \mathbf{R}$, $A(x) = x$, $F(x) = \{x, 0\}$, and $f(x) = -x^2$ for all $x \in X$. It is easy to verify that the solution set of GMVI(A, F, f) is empty and 0 is the unique solution of the corresponding α problem with $\alpha = 2$. If f is proper and convex, then GMVI(A, F, f) and α problem have the same solution (this fact has been shown in the proof of Theorem 3.11).

4. Links with the Well-Posedness by Perturbations of Inclusion Problems

Lemaire et al. [20] introduced the concept of well-posedness by perturbations for an inclusion problem. In this section, we will show that the well-posedness by perturbations of a generalized mixed variational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem. Let us recall some concepts. Let $M : X \to 2^{X^*}$. The inclusion problem associated with M is defined by

$$\text{IP}\,(M)\text{: find } x \in X \text{ such that } 0 \in M(x). \tag{4.1}$$

The perturbed problem of IP(M) is given by

$$\text{IP}_p(M)\text{: find } x \in X \text{ such that } 0 \in \widetilde{M}(p, x), \tag{4.2}$$

where $\widetilde{M} : P \times X \to 2^{X^*}$ is such that $\widetilde{M}(p^*, \cdot) = M$.

Definition 4.1 (see [20]). Let $\{p_n\} \subset P$ be with $p_n \to p^*$. A sequence $\{x_n\} \subset X$ is called an approximating sequence corresponding to $\{p_n\}$ for IP (M) if $x_n \in \text{Dom } \widetilde{M}(p_n, \cdot)$ for all $n \geq 1$ and $d(0, \widetilde{M}(p_n, x_n)) \to 0$, or, equivalently, there exists $y_n \in \widetilde{M}(p_n, x_n)$ such that $\|y_n\| \to 0$ as $n \to \infty$.

Definition 4.2 (see [20]). One says that IP(M) is strongly (resp., weakly) well-posed by perturbations if it has a unique solution and, for any $\{p_n\} \subset P$ with $p_n \to p^*$, every approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution of IP(M). IP(M) is said to be strongly (resp., weakly) generalized well posed by perturbations if the solution set S of IP(M) is nonempty and, for any $\{p_n\} \subset P$ with $p_n \to p^*$, every approximating sequence corresponding to $\{p_n\}$ has a subsequence which converges strongly (resp., weakly) to a point of S.

Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous functional. Denote by ∂f and $\partial_\varepsilon f$ the subdifferential and ε-subdifferential of f, respectively, that is,

$$\begin{aligned} \partial f(x) &= \{x^* \in X^*\colon f(y) - f(x) \geq \langle x^*, y - x\rangle,\ \forall y \in X\}, \quad \forall x \in \text{dom } f, \\ \partial_\varepsilon f(x) &= \{x^* \in X^*\colon f(y) - f(x) \geq \langle x^*, y - x\rangle - \varepsilon,\ \forall y \in X\}, \quad \forall x \in \text{dom } f. \end{aligned} \tag{4.3}$$

It is known that ∂f is maximal monotone and $\partial_\varepsilon f(x) \supset \partial f(x) \neq \emptyset$ for all $x \in \text{dom } f$ and for all $\varepsilon > 0$. In terms of ∂f, GMVI(A, F, f) is equivalent to the following inclusion problem:

$$\text{IP}(AF + \partial f)\text{: find } x \in X \text{ such that } 0 \in AF(x) + \partial f(x). \tag{4.4}$$

In other words, we have the following lemma.

Lemma 4.3. *Let $x \in X$ be a fixed point, and let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous functional. Then, the following statements are equivalent:*

(i) $\langle Au, x - y\rangle + f(x) - f(y) \leq 0$ *for all* $y \in X$ *and some* $u \in F(x)$;

(ii) $0 \in AF(x) + \partial f(x)$.

Proof. Observe that

$$\begin{aligned}\langle Au, x - y\rangle + f(x) - f(y) \leq 0 &\Longleftrightarrow f(y) - f(x) \geq \langle -Au, y - x\rangle \\ &\Longleftrightarrow -Au \in \partial f(x) \\ &\Longleftrightarrow 0 \in Au + \partial f(x) \\ &\Longleftrightarrow 0 \in AF(x) + \partial f(x),\end{aligned} \tag{4.5}$$

for all $y \in X$ and some $u \in F(x)$. The desired result follows immediately from the above relations. □

Naturally, we consider the perturbed problem of $\mathrm{IP}(AF + \partial f)$ as follows:

$$\mathrm{IP}_p(AF + \partial f)\text{: find } x \in X \text{ such that } 0 \in \widetilde{A}(p, F(x)) + \partial \widetilde{f}(p, \cdot)(x), \tag{4.6}$$

where $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ is such that $\widetilde{f}(p, \cdot)$ is proper, convex, and lower semicontinuous for all $p \in P$, and $\widetilde{f}(p^*, \cdot) = f$.

The following theorems establish the relations between the strong (resp., weak) well-posedness by perturbations of generalized mixed variational inequalities and the strong (resp., weak) well-posedness by perturbations of inclusion problems.

Theorem 4.4. *Let $\widetilde{A}(\cdot, y) : P \to X^*$ be continuous for each $y \in X$, let $A(= A(p^*, \cdot)) : X \to X^*$ be weakly continuous, let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-hemicontinuous and monotone with respect to $\widetilde{A}(p, \cdot)$ for each $p \in P$, and let $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ be a continuous functional with respect to the product of the norm topology of P and weak topology of X such that the following conditions hold:*

(i) $\widetilde{f}(p, \cdot)$ *is proper and convex for all* $p \in P$;

(ii) $\operatorname{dom} f$ *is well-positioned and* $\operatorname{dom} f \subset \bigcap_{p \in P} \operatorname{dom} \widetilde{f}(p, \cdot)$;

(iii) $e(\operatorname{dom} \widetilde{f}(p_n, \cdot), \operatorname{dom} f) \to 0$ *whenever* $p_n \to p^*$, *where e is defined as in Proposition 2.3.*

Then, $\mathrm{IP}(AF + \partial f)$ *is weakly well-posed by perturbations whenever* $\mathrm{GMVI}(A, F, f)$ *has a unique solution.*

Proof. Suppose that $\mathrm{GMVI}(A, F, f)$ has a unique solution x^*. Let $\{p_n\} \subset P$ be with $p_n \to p^*$, and let $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$ for $\mathrm{IP}(AF + \partial f)$. Then, there exists $w_n \in \widetilde{A}(p_n, F(x_n)) + \partial \widetilde{f}(p_n, \cdot)(x_n)$ such that $\|w_n\| \to 0$. Further, there exists $u_n \in F(x_n)$ such that $w_n \in \widetilde{A}(p_n, u_n) + \partial \widetilde{f}(p_n, \cdot)(x_n)$ with $\|w_n\| \to 0$. It follows that

$$\widetilde{f}(p_n, y) - \widetilde{f}(p_n, x_n) \geq \left\langle w_n - \widetilde{A}(p_n, u_n), y - x_n \right\rangle, \quad \forall y \in X,\ n \geq 1. \tag{4.7}$$

We claim that $\{x_n\}$ is bounded. Indeed, if $\{x_n\}$ is unbounded, without loss of generality, we may assume that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \qquad z_n = x^* + t_n(x_n - x^*). \tag{4.8}$$

By conditions (i)-(ii), we get $z_n \in \operatorname{dom} \widetilde{f}(p_n, \cdot)$. Note that

$$\|z_n\| \le \|x^*\| + t_n\|x_n - x^*\| = \|x^*\| + 1, \quad \forall n \ge 1. \tag{4.9}$$

So, $\{z_n\}$ is bounded. Since X is reflexive, it follows from the boundedness of $\{z_n\}$ that there exists some subsequence of $\{z_n\}$ which converges weakly to a point of X. Hence, without loss of generality, we may assume that $t_n \in (0,1)$ and $z_n \rightharpoonup z$. It follows from Lemma 2.14 and conditions (ii)-(iii) that $z \neq x^*$. For any $y \in X$, observe that, for all $v \in F(y)$,

$$\begin{aligned}
\left\langle \widetilde{A}(p_n, v), z - y \right\rangle &= \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + \left\langle \widetilde{A}(p_n, v), z_n - x^* \right\rangle + \left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle \\
&= \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + t_n\left\langle \widetilde{A}(p_n, v), x_n - x^* \right\rangle + \left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle \\
&= \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + t_n\left\langle \widetilde{A}(p_n, v), x_n - y \right\rangle + (1 - t_n)\left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle.
\end{aligned} \tag{4.10}$$

Since x^* is the unique solution of GMVI(A, F, f), there exists some $u^* \in F(x^*)$ such that

$$\langle Au^*, x^* - y\rangle + f(x^*) - f(y) \le 0, \quad \forall y \in X. \tag{4.11}$$

Also, since F is monotone with respect to $\widetilde{A}(p_n, \cdot)$, we deduce that, for $u^* \in F(x^*)$, $v \in F(y)$, and $u_n \in F(x_n)$,

$$\left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle \le \left\langle \widetilde{A}(p_n, u^*), x^* - y \right\rangle, \qquad \left\langle \widetilde{A}(p_n, v), x_n - y \right\rangle \le \left\langle \widetilde{A}(p_n, u_n), x_n - y \right\rangle. \tag{4.12}$$

In addition, we have

$$\widetilde{f}(p_n, z_n) \le t_n\widetilde{f}(p_n, x_n) + (1 - t_n)\widetilde{f}(p_n, x^*) \tag{4.13}$$

by virtue of the convexity of $\tilde{f}(p_n,\cdot)$. It follows from (4.7)–(4.13) that

$$\begin{aligned}\left\langle \tilde{A}(p_n,v), z-y\right\rangle &\le \left\langle \tilde{A}(p_n,v), z-z_n\right\rangle + t_n\left\langle \tilde{A}(p_n,u_n), x_n-y\right\rangle \\ &\quad + (1-t_n)\left\langle \tilde{A}(p_n,u^*), x^*-y\right\rangle \\ &\le \left\langle \tilde{A}(p_n,v), z-z_n\right\rangle + (1-t_n)\left\langle \tilde{A}(p_n,u^*), x^*-y\right\rangle \\ &\quad + t_n\left(\tilde{f}(p_n,y)-\tilde{f}(p_n,x_n)\right) + t_n\langle w_n, x_n-y\rangle \\ &\le \left\langle \tilde{A}(p_n,v), z-z_n\right\rangle + (1-t_n)\left\langle \tilde{A}(p_n,u^*), x^*-y\right\rangle \\ &\quad + t_n\langle w_n, x_n-y\rangle + t_n\tilde{f}(p_n,y) + (1-t_n)\tilde{f}(p_n,x^*) - \tilde{f}(p_n,z_n).\end{aligned} \tag{4.14}$$

Moreover, it is easy to see from $w_n \to 0$ that $t_n\langle w_n, x_n-y\rangle \to 0$. Further, since $\tilde{A}(\cdot,y) : P \to X^*$ is continuous for each $y \in X$ and $z_n \rightharpoonup z$, it is known that $\tilde{A}(p_n,v) \to \tilde{A}(p^*,v)$ and $\{z_n\}$ is bounded. Consequently,

$$\begin{aligned}\left|\left\langle \tilde{A}(p_n,v), z-z_n\right\rangle\right| &= \left|\left\langle \tilde{A}(p_n,v), z-z_n\right\rangle - \left\langle \tilde{A}(p^*,v), z-z_n\right\rangle + \left\langle \tilde{A}(p^*,v), z-z_n\right\rangle\right| \\ &\le \left|\left\langle \tilde{A}(p_n,v)-\tilde{A}(p^*,v), z-z_n\right\rangle\right| + \left|\left\langle \tilde{A}(p^*,v), z-z_n\right\rangle\right| \\ &\le \left\|\tilde{A}(p_n,v)-\tilde{A}(p^*,v)\right\|\|z_n-z\| \\ &\quad + \left|\left\langle \tilde{A}(p^*,v), z_n-z\right\rangle\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{4.15}$$

In the meantime, since $\tilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$is a continuous functional with respect to the product of the norm topology of P and weak topology of X, we conclude from $p_n \to p^*$ and $z_n \rightharpoonup z$ that $\tilde{f}(p_n,z_n) \to \tilde{f}(p^*,z)$ and $\tilde{f}(p_n,x^*) \to \tilde{f}(p^*,x^*)$ as $n \to \infty$. Now, letting $n \to \infty$ in (4.14) we get

$$\begin{aligned}\langle Av, z-y\rangle &= \left\langle \tilde{A}(p^*,v), z-y\right\rangle \\ &\le \left\langle \tilde{A}(p^*,u^*), x^*-y\right\rangle + \tilde{f}(p^*,x^*) - \tilde{f}(p^*,z) \\ &= \langle Au^*, x^*-y\rangle + f(x^*) - f(z), \quad \forall y \in X.\end{aligned} \tag{4.16}$$

Since x^* is the unique solution of GMVI(A,F,f), from (4.16) we get

$$\begin{aligned}\langle Av, z-y\rangle &\le \langle Au^*, x^*-y\rangle + f(x^*) - f(z) \\ &= \langle Au^*, x^*-y\rangle + f(x^*) - f(y) + f(y) - f(z) \\ &\le f(y) - f(z), \quad \forall y \in X,\end{aligned} \tag{4.17}$$

which implies that

$$\langle Av, z - y \rangle + f(z) - f(y) \leq 0, \quad \forall y \in X, \ v \in F(y). \tag{4.18}$$

Note that A is weakly continuous, that f is proper and convex, and that F is a nonempty weakly compact-valued multifunction which is $\mathcal{H}$-hemicontinuous and monotone with respect to A. Hence, all conditions of Lemma 2.8 are satisfied. Thus, it follows from Lemma 2.8 that there exists $w \in F(z)$ such that

$$\langle Aw, z - y \rangle + f(z) - f(y) \leq 0, \quad \forall y \in X. \tag{4.19}$$

Therefore, z is a solution of GMVI(A, F, f), a contradiction. This shows that $\{x_n\}$ is bounded.

Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \overline{x}$ as $k \to \infty$. It follows from (4.7) that

$$\left\langle \widetilde{A}(p_{n_k}, u_{n_k}), x_{n_k} - y \right\rangle + \widetilde{f}(p_{n_k}, x_{n_k}) - \widetilde{f}(p_{n_k}, y) \leq \langle w_{n_k}, x_{n_k} - y \rangle, \quad \forall y \in X, \ k \geq 1. \tag{4.20}$$

Since $\widetilde{A}(\cdot, y) : P \to X^*$ is continuous for each $y \in X$ and $x_{n_k} \rightharpoonup \overline{x}$, it is known that $\widetilde{A}(p_{n_k}, v) \to \widetilde{A}(p^*, v)$ and $\{x_{n_k}\}$ is bounded. Consequently,

$$\begin{aligned}
&\left| \left\langle \widetilde{A}(p_{n_k}, v), x_{n_k} - y \right\rangle - \left\langle \widetilde{A}(p^*, v), \overline{x} - y \right\rangle \right| \\
&\quad = \left| \left\langle \widetilde{A}(p_{n_k}, v), x_{n_k} - y \right\rangle - \left\langle \widetilde{A}(p^*, v), x_{n_k} - y \right\rangle + \left\langle \widetilde{A}(p^*, v), x_{n_k} - y \right\rangle - \left\langle \widetilde{A}(p^*, v), \overline{x} - y \right\rangle \right| \\
&\quad \leq \left| \left\langle \widetilde{A}(p_{n_k}, v) - \widetilde{A}(p^*, v), x_{n_k} - y \right\rangle \right| + \left| \left\langle \widetilde{A}(p^*, v), x_{n_k} - \overline{x} \right\rangle \right| \\
&\quad \leq \left\| \widetilde{A}(p_{n_k}, v) - \widetilde{A}(p^*, v) \right\| \| x_{n_k} - y \| + \left| \left\langle \widetilde{A}(p^*, v), x_{n_k} - \overline{x} \right\rangle \right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.21}$$

Moreover, since $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$is a continuous functional with respect to the product of the norm topology of P and weak topology of X, we conclude from $p_{n_k} \to p^*$ and $x_{n_k} \rightharpoonup \overline{x}$ that $\widetilde{f}(p_{n_k}, x_{n_k}) \to \widetilde{f}(p^*, \overline{x})$ and $\widetilde{f}(p_{n_k}, y) \to \widetilde{f}(p^*, y)$ as $k \to \infty$. Note that F is monotone with respect to $\widetilde{A}(p_{n_k}, \cdot)$. Hence, it follows that for, $u_{n_k} \in F(x_{n_k})$ and $v \in F(y)$,

$$\begin{aligned}
\langle Av, \overline{x} - y \rangle + f(\overline{x}) - f(y) &= \left\langle \widetilde{A}(p^*, v), \overline{x} - y \right\rangle + \widetilde{f}(p^*, \overline{x}) - \widetilde{f}(p^*, y) \\
&= \lim_{k \to \infty} \left\{ \left\langle \widetilde{A}(p_{n_k}, v), x_{n_k} - y \right\rangle + \widetilde{f}(p_{n_k}, x_{n_k}) - \widetilde{f}(p_{n_k}, y) \right\} \\
&\leq \lim_{k \to \infty} \left\{ \left\langle \widetilde{A}(p_{n_k}, u_{n_k}), x_{n_k} - y \right\rangle + \widetilde{f}(p_{n_k}, x_{n_k}) - \widetilde{f}(p_{n_k}, y) \right\} \\
&\leq \lim_{k \to \infty} \langle w_{n_k}, x_{n_k} - y \rangle = 0, \quad \forall y \in X.
\end{aligned} \tag{4.22}$$

This together with Lemma 2.8 yields that there exists $\overline{u} \in F(\overline{x})$ such that

$$\langle A\overline{u}, \overline{x} - y \rangle + f(\overline{x}) - f(y) \leq 0, \quad \forall y \in X. \tag{4.23}$$

Consequently, $\overline{x}$ solves GMVI(A, F, f). We must have $\overline{x} = x^*$ since GMVI(A, F, f) has a unique solution x^*. Therefore, $\{x_n\}$ converges weakly to x^* and so IP$(AF + \partial f)$ is weakly well-posed by perturbations. □

Remark 4.5. Theorem 4.4 improves, extends, and develops [10, Theorem 4.1] in the following aspects.

(i) The mixed variational inequality problem (MVI) in [10, Theorem 4.1] is extended to develop the more general problem, that is, the generalized mixed variational inequality problem (GMVI) with a nonempty weakly compact-valued multifunction in the setting of Banach spaces. Moreover, the inclusion problem corresponding to MVI in [10, Theorem 4.1] is extended to develop the more general problem, that is, the inclusion problem corresponding to GMVI.

(ii) Since the generalized mixed variational inequality problem (GMVI) is more general and more complicated than the mixed variational inequality problem (MVI), the assumptions in Theorem 4.4 are very different from the ones in [10, Theorem 4.1], for instance, in Theorem 4.4, let $\widetilde{A}(\cdot, y) : P \to X^*$ be continuous for each $y \in X$, let $A(= A(p^*, \cdot)) : X \to X^*$ be weakly continuous, and let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-hemicontinuous and monotone with respect to $\widetilde{A}(p, \cdot)$ for each $p \in P$, but, in [10, Theorem 4.1], let $\widetilde{A} : P \times X \to X^*$ be a continuous mapping such that $\widetilde{A}(p, \cdot) : X \to X^*$ is monotone for all $p \in P$.

(iii) The technique of proving weak well-posedness by perturbations for inclusion problem IP$(AF + \partial f)$ in Theorem 4.4 is very different from the one for inclusion problem IP$(A + \partial f)$ in [10, Theorem 4.1] because our technique depends on Lemma 2.8. Note that A is weakly continuous, that f is proper and convex, and that F is a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-hemicontinuous and monotone with respect to A. Hence, all the conditions of Lemma 2.8 are satisfied. Recall that the proof of Lemma 2.8 depends on the well-known Nadler's Theorem [27]. Thus, our technique depends essentially on the well-known Nadler's Theorem [27], the $\mathscr{H}$-hemicontinuity of nonempty weakly compact-valued multifunction F and the monotonicity of F with respect to $\widetilde{A}(p, \cdot)$ for each $p \in P$.

Theorem 4.6. *Let $\widetilde{A}(\cdot, y) : P \to X^*$ be continuous for each $y \in X$, let $A : X \to X^*$ be weakly continuous, let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-hemicontinuous and monotone with respect to $\widetilde{A}(p, \cdot)$ for each $p \in P$, and let $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ be a continuous functional with respect to the product of the norm topology of P and weak topology of X such that the following conditions hold:*

(i) $\widetilde{f}(p, \cdot)$ *is proper and convex for all* $p \in P$;

(ii) $\operatorname{dom} f$ *is well-positioned and* $\operatorname{dom} f \subset \bigcap_{p \in P} \operatorname{dom} \widetilde{f}(p, \cdot)$;

(iii) $e(\operatorname{dom} \widetilde{f}(p_n, \cdot), \operatorname{dom} f) \to 0$ *whenever* $p_n \to p^*$.

Then, GMVI(A, F, f) *is weakly well-posed by perturbations whenever* IP$(AF + \partial f)$ *has a unique solution.*

Proof. Let IP$(AF+\partial f)$ have a unique solution x^*. By Lemma 4.3, x^* is also the unique solution of GMVI(A, F, f). Let $\{p_n\} \subset P$ be with $p_n \to p^*$, and let $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$ for GMVI(A, F, f). Then, there exist $\{u_n\} \subset X$ with $u_n \in F(x_n)$ (for all $n \geq 1$) and $0 < \varepsilon_n \to 0$ such that

$$\left\langle \widetilde{A}(p_n, u_n), x_n - y \right\rangle + \widetilde{f}(p_n, x_n) - \widetilde{f}(p_n, y) \leq \varepsilon_n, \quad \forall y \in X, \ n \geq 1. \tag{4.24}$$

We claim that $\{x_n\}$ is bounded. Indeed, if $\{x_n\}$ is unbounded, without loss of generality, we may assume that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \qquad z_n = x^* + t_n(x_n - x^*). \tag{4.25}$$

By conditions (i)-(ii), we get $z_n \in \operatorname{dom} \widetilde{f}(p_n, \cdot)$. Without loss of generality we may assume that $t_n \in (0, 1)$ and $z_n \rightharpoonup z$. From Lemma 2.14 and conditions (ii)-(iii) we obtain that $z \neq x^*$. For any $y \in X$, observe that, for all $v \in F(y)$,

$$\begin{aligned}\left\langle \widetilde{A}(p_n, v), z - y \right\rangle &= \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + \left\langle \widetilde{A}(p_n, v), z_n - x^* \right\rangle + \left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle \\ &= \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + t_n \left\langle \widetilde{A}(p_n, v), x_n - y \right\rangle + (1 - t_n) \left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle.\end{aligned} \tag{4.26}$$

Since x^* is the unique solution of GMVI(A, F, f), there exists some $u^* \in F(x^*)$ such that

$$\left\langle \widetilde{A}u^*, x^* - y \right\rangle + f(x^*) - f(y) \leq 0, \quad \forall y \in X. \tag{4.27}$$

Also, since F is monotone with respect to $\widetilde{A}(p_n, \cdot)$, we deduce that, for $u^* \in F(x^*)$, $v \in F(y)$, and $u_n \in F(x_n)$,

$$\left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle \leq \left\langle \widetilde{A}(p_n, u^*), x^* - y \right\rangle, \qquad \left\langle \widetilde{A}(p_n, v), x_n - y \right\rangle \leq \left\langle \widetilde{A}(p_n, u_n), x_n - y \right\rangle. \tag{4.28}$$

In addition, since $\widetilde{f}(p_n, \cdot)$ is convex, we get

$$\widetilde{f}(p_n, z_n) \leq t_n \widetilde{f}(p_n, x_n) + (1 - t_n) \widetilde{f}(p_n, x^*). \tag{4.29}$$

It follows from (4.24)–(4.29) that

$$\begin{aligned}\left\langle \widetilde{A}(p_n,v),z-y\right\rangle &\leq \left\langle \widetilde{A}(p_n,v),z-z_n\right\rangle + t_n\left\langle \widetilde{A}(p_n,u_n),x_n-y\right\rangle \\ &\quad + (1-t_n)\left\langle \widetilde{A}(p_n,u^*),x^*-y\right\rangle \\ &\leq \left\langle \widetilde{A}(p_n,v),z-z_n\right\rangle + (1-t_n)\left\langle \widetilde{A}(p_n,u^*),x^*-y\right\rangle \\ &\quad + t_n\left(\widetilde{f}(p_n,y)-\widetilde{f}(p_n,x_n)\right)+t_n\varepsilon_n \\ &\leq \left\langle \widetilde{A}(p_n,v),z-z_n\right\rangle + (1-t_n)\left\langle \widetilde{A}(p_n,u^*),x^*-y\right\rangle \\ &\quad + t_n\varepsilon_n + t_n\widetilde{f}(p_n,y)+(1-t_n)\widetilde{f}(p_n,x^*)-\widetilde{f}(p_n,z_n).\end{aligned} \tag{4.30}$$

Letting $n \to \infty$ in the last inequality we get

$$\begin{aligned}\langle Av,z-y\rangle &= \left\langle \widetilde{A}(p^*,v),z-y\right\rangle \\ &\leq \left\langle \widetilde{A}(p^*,u^*),x^*-y\right\rangle + \widetilde{f}(p^*,x^*)-\widetilde{f}(p^*,z) \\ &= \langle Au^*,x^*-y\rangle + f(x^*)-f(z), \quad \forall y\in X.\end{aligned} \tag{4.31}$$

By using (4.31) and the same argument as in the proof of Theorem 4.4, we can prove that z is a solution of $\mathrm{GMVI}(A,F,f)$, a contradiction. Thus, $\{x_n\}$ is bounded.

The rest follows from the similar argument to that in the proof of Theorem 4.4 and so is omitted. □

By Lemma 4.3 and Theorems 4.4 and 4.6, we have the following result.

Theorem 4.7. *Let $\widetilde{A}(\cdot,y) : P \to X^*$ be continuous for each $y \in X$, let $A : X \to X^*$ be weakly continuous, let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathcal{H}$-hemicontinuous and monotone with respect to $\widetilde{A}(p,\cdot)$ for each $p \in P$, and let $\widetilde{f} : P\times X \to \mathbf{R}\cup\{+\infty\}$ be a continuous functional with respect to the product of the norm topology of P and weak topology of X such that the following conditions hold:*

(i) $\widetilde{f}(p,\cdot)$ *is proper and convex for all* $p \in P$;

(ii) $\operatorname{dom} f$ *is well-positioned and* $\operatorname{dom} f \subset \bigcap_{p\in P}\operatorname{dom}\widetilde{f}(p,\cdot)$;

(iii) $e(\operatorname{dom}\widetilde{f}(p_n,\cdot),\operatorname{dom} f) \to 0$ *whenever* $p_n \to p^*$.

Then, the following statements are equivalent:

(i) $\mathrm{GMVI}(A,F,f)$ *is weakly well posed by perturbations;*

(ii) $\mathrm{IP}(AF+\partial f)$ *is weakly well posed by perturbations;*

(iii) $\mathrm{GMVI}(A,F,f)$ *has a unique solution;*

(iv) $\mathrm{IP}(AF+\partial f)$ *has a unique solution.*

Remark 4.8. Theorem 4.7 improves Theorems 4.1, 4.2, and 6.1 of [9], Theorems 4.1–4.3 of [10], and Theorems 4.1, 4.2, and 6.1 of [11].

Now we give the following example as an application of Theorem 4.7.

Example 4.9. Let $X = \mathbf{R}$, $P = [-1,1]$, and $p^* = 0$. Let $\widetilde{A}(p,x) = x + p$, $F(x) = x^3$ for all $x \in X$, $p \in P$, and

$$\widetilde{f}(p,x) = \begin{cases} (p^2+1)x^2, & x \in [-1,1], \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.32}$$

for all $p \in P$. Clearly, $\operatorname{Dom}\widetilde{A}(p,\cdot) = \mathbf{R}$ and $\operatorname{Dom} f = \operatorname{Dom}\widetilde{f}(p,\cdot) = [-1,1]$ for all $p \in P$. It is easy to see that $\widetilde{A}$ and $\widetilde{f}$ are continuous, $\widetilde{f}(p,\cdot)$ is proper and convex, and F is $\mathscr{H}$-hemicontinuous and monotone with respect to $\widetilde{A}(p,\cdot)$ for each $p \in P$. By (ii) of Remark 2.7, $\operatorname{Dom} f$ is well-positioned. Hence, all the assumptions of Theorem 4.7 are satisfied. Let S be the solution set of $\mathrm{GMVI}(A,F,f)$. It follows that

$$\begin{aligned} S &= \left\{x \in [-1,1] : x^3(x-y) + x^2 - y^2 \le 0,\ \forall y \in [-1,1]\right\} \\ &= \left\{x \in [-1,1] : -\left(y + \frac{x^3}{2}\right)^2 + \frac{x^6}{4} + x^4 + x^2 \le 0,\ \forall y \in [-1,1]\right\} \\ &= \{0\}. \end{aligned} \tag{4.33}$$

So $x^* = 0$ is the unique solution of $\mathrm{GMVI}(A,F,f)$. By Theorem 4.7, $\mathrm{GMVI}(A,F,f)$ is well-posed by perturbations.

Next, we discuss the relationships between the generalized well-posedness by perturbations of $\mathrm{GMVI}(A,F,f)$ and the generalized well-posedness by perturbations of $\mathrm{IP}(AF + \partial f)$.

Theorem 4.10. *Let $\widetilde{A} : P \times X \to X^*$ be a uniformly continuous mapping, let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-uniformly continuous, and let $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ be a functional such that $\widetilde{f}(p,\cdot)$ is proper, convex, and lower semicontinuous for each $p \in P$. Then,* $\mathrm{GMVI}(A,F,f)$ *is strongly (resp., weakly) generalized well-posed by perturbations whenever* $\mathrm{IP}(AF + \partial f)$ *is strongly (resp., weakly) generalized well-posed by perturbations.*

Proof. Let $\{p_n\} \subset P$ be with $p_n \to p^*$, and let $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$ for $\mathrm{GMVI}(A,F,f)$. Then, there exist $\{u_n\} \subset X$ with $u_n \in F(x_n)$ (for all $n \ge 1$) and $0 < \varepsilon_n \to 0$ such that

$$\left\langle \widetilde{A}(p_n,u_n), x_n - y \right\rangle + \widetilde{f}(p_n,x_n) - \widetilde{f}(p_n,y) \le \varepsilon_n, \quad \forall y \in X,\ n \ge 1. \tag{4.34}$$

Define $\varphi_n : X \to \mathbf{R} \cup \{+\infty\}$ as follows:

$$\varphi_n(y) = \tilde{f}(p_n, y) + \left\langle \tilde{A}(p_n, u_n), y - x_n \right\rangle, \quad \forall y \in X, \ n \geq 1. \tag{4.35}$$

Clearly, φ_n is proper, convex, and lower semicontinuous and $0 \in \partial_{\varepsilon_n}\varphi_n(x_n)$ for all $n \geq 1$. By the Brondsted-Rockafellar theorem [32], there exist $\overline{x}_n \in X$ and

$$w_n \in \partial\varphi_n(\overline{x}_n) = \tilde{A}(p_n, u_n) + \partial\tilde{f}(p_n, \cdot)(\overline{x}_n) \tag{4.36}$$

such that

$$\|x_n - \overline{x}_n\| \leq \sqrt{\varepsilon_n}, \qquad \|w_n\| \leq \sqrt{\varepsilon_n}. \tag{4.37}$$

Since $F : X \to 2^X$ is a nonempty weakly compact-valued multifunction, both $F(x_n)$ and $F(\overline{x}_n)$ are nonempty weakly compact and hence are nonempty, weakly closed, and weakly bounded. Note that the weak closedness of sets in X implies the strong closedness and that the weak boundedness of sets in X is equivalent to the strong boundedness. Thus, it is known that both $F(x_n)$ and $F(\overline{x}_n)$ lie in $CB(X)$ for each $n \geq 1$. By Proposition 2.3, for each $n \geq 1$ and $u_n \in F(x_n)$, there exists $\overline{u}_n \in F(\overline{x}_n)$ such that

$$\|u_n - \overline{u}_n\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(F(x_n), F(\overline{x}_n)). \tag{4.38}$$

Since F is $\mathcal{H}$-uniformly continuous, we have from (4.37)

$$\|u_n - \overline{u}_n\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(F(x_n), F(\overline{x}_n)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.39}$$

It follows from (4.36) that

$$w_n + \tilde{A}(p_n, \overline{u}_n) - \tilde{A}(p_n, u_n) \in \tilde{A}(p_n, \overline{u}_n) + \partial\tilde{f}(p_n, \cdot)(\overline{x}_n). \tag{4.40}$$

Since $\tilde{A}$ is a uniformly continuous, from (4.37) we get

$$\left\| w_n + \tilde{A}(p_n, \overline{u}_n) - \tilde{A}(p_n, u_n) \right\| \leq \|w_n\| + \left\| \tilde{A}(p_n, \overline{u}_n) - \tilde{A}(p_n, u_n) \right\| \longrightarrow 0. \tag{4.41}$$

So $\{\overline{x}_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for $\text{IP}(AF + \partial f)$.

By the strong (resp., weak) generalized well-posedness by perturbations of $\text{IP}(AF + \partial f)$, there exists some subsequence $\{\overline{x}_{n_k}\}$ of $\{\overline{x}_n\}$ such that $\overline{x}_{n_k} \to x^*$ (resp., $\overline{x}_{n_k} \rightharpoonup x^*$), where x^* is some solution of $\text{IP}(AF + \partial f)$. By Lemma 4.3, x^* is also a solution of $\text{GMVI}(A, F, f)$.

Case i. $\mathrm{IP}(AF+\partial f)$ is strongly generalized well-posed by perturbations. It follows from (4.37) that

$$\|x_{n_k} - x^*\| \le \|x_{n_k} - \overline{x}_{n_k}\| + \|\overline{x}_{n_k} - x^*\| \longrightarrow 0 \tag{4.42}$$

and so $\mathrm{GMVI}(A, F, f)$ is strongly generalized well-posed by perturbations.

Case ii. $\mathrm{IP}(AF+\partial f)$ is weakly generalized well-posed by perturbations. For any $\varphi \in X^*$, from (4.37) we have

$$\begin{aligned} \left|\langle \varphi, x_{n_k} - x^*\rangle\right| &\le \left|\langle \varphi, x_{n_k} - \overline{x}_{n_k}\rangle\right| + \left|\langle \varphi, \overline{x}_{n_k} - x^*\rangle\right| \\ &\le \|\varphi\|\sqrt{\varepsilon_{n_k}} + \left|\langle \varphi, \overline{x}_{n_k} - x^*\rangle\right| \longrightarrow 0. \end{aligned} \tag{4.43}$$

Thus, $\mathrm{GMVI}(A, F, f)$ is weakly generalized well-posed by perturbations. □

Theorem 4.11. *Let $\widetilde{A} : P \times X \to X^*$ and $F : X \to 2^X$, and let $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ be a functional such that $\widetilde{f}(p, \cdot)$ is proper, convex, and lower semicontinuous for each $p \in P$. Then,* $\mathrm{IP}(AF+\partial f)$ *is strongly (resp., weakly) generalized well-posed by perturbations whenever* $\mathrm{GMVI}(A, F, f)$ *is strongly (resp., weakly) generalized 1-well-posed by perturbations.*

Proof. Let $\{p_n\} \subset P$ be with $p_n \to p^*$, and let $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$ for $\mathrm{IP}(AF + \partial f)$. Then, there exists $w_n \in \widetilde{A}(p_n, F(x_n)) + \partial \widetilde{f}(p_n, \cdot)(x_n)$ such that $\|w_n\| \to 0$. It follows that there exists a sequence $\{u_n\} \subset X$ with $u_n \in F(x_n)$ (for all $n \ge 1$) such that

$$w_n \in \widetilde{A}(p_n, u_n) + \partial \widetilde{f}(p_n, \cdot)(x_n), \quad \forall n \ge 1, \tag{4.44}$$

and hence

$$\begin{aligned} \left\langle \widetilde{A}(p_n, u_n), x_n - y \right\rangle + \widetilde{f}(p_n, x_n) - \widetilde{f}(p_n, y) &\le \langle w_n, x_n - y\rangle \\ &\le \frac{1}{2}\left\|x_n - y\right\|^2 + \frac{1}{2}\|w_n\|^2, \quad \forall y \in X,\ n \ge 1. \end{aligned} \tag{4.45}$$

This together with $\|w_n\| \to 0$ implies that $\{x_n\}$ is a 1-approximating sequence corresponding to $\{p_n\}$ for $\mathrm{GMVI}(A, F, f)$. Since $\mathrm{GMVI}(A, F, f)$ is strongly (resp., weakly) generalized 1-well-posed by perturbations, $\{x_n\}$ converges strongly (resp., weakly) to some solution x^* of $\mathrm{GMVI}(A, F, f)$. By Lemma 4.3, x^* is also a solution of $\mathrm{IP}(AF + \partial f)$. So $\mathrm{IP}(AF + \partial f)$ is strongly (resp., weakly) generalized well-posed by perturbations. □

Remark 4.12. When $p_n = p^*$ (for all $n \ge 1$) and $F = I$ the identity mapping of X, Theorems 4.10 and 4.11 coincide with Theorems 4.3 and 4.4 of [9], respectively. Also, when $F = I$, Theorems 4.10 and 4.11 coincide with Theorems 4.4 and 4.5 of [10], respectively. Furthermore, it can be found that Theorems 4.10 and 4.11 also improve and extend Theorems 4.3 and 4.4 of [11], respectively. In the meantime, Theorems 4.10 and 4.11 partially generalize Theorem 2.1 of Lemaire et al. [20].

5. Links with the Well-Posedness by Perturbations of Fixed Point Problems

Lemaire et al. [20] also considered the concepts of well-posedness by perturbations for a (single-valued) fixed point problem. In this section, we consider the concepts of well-posedness by perturbations for a (set-valued) fixed point problem. Let $T : X \to 2^X$ be a set-valued mapping. The fixed point problem associated with T is defined by

$$\text{FP}(T)\text{: find } x \in X \text{ such that } x \in T(x). \tag{5.1}$$

The perturbed problem of FP(T) is given by

$$\text{FP}_p(T)\text{: find } x \in X \text{ such that } x \in \widetilde{T}(p, x), \tag{5.2}$$

where $\widetilde{T} : P \times X \to 2^X$ is such that $\widetilde{T}(p^*, \cdot) = T$.

Definition 5.1. Let $\{p_n\} \subset P$ be with $p_n \to p^*$. A sequence $\{x_n\} \subset X$ is called an approximating sequence corresponding to $\{p_n\}$ for FP(T) if there exists a sequence $\{y_n\} \subset X$ with $y_n \in T(x_n)$ (for all $n \geq 1$) such that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Definition 5.2. One says that FP(T) is strongly (resp., weakly) well-posed by perturbations if FP(T) has a unique solution and, for any $\{p_n\} \subset P$ with $p_n \to p^*$, every approximating sequence corresponding to $\{p_n\}$ for FP(T) converges strongly (resp., weakly) to the unique solution. FP(T) is said to be strongly (resp., weakly) generalized well-posed by perturbations if FP(T) has a nonempty solution set S and, for any $\{p_n\} \subset P$ with $p_n \to p^*$, every approximating sequence corresponding to $\{p_n\}$ for FP(T) has a subsequence which converges strongly (resp., weakly) to some point of S.

In particular, whenever T is a single-valued mapping, we can readily see that Definitions 5.1 and 5.2 reduce to the corresponding definitions in [20]. It is known that in the setting of Hilbert spaces a generalized mixed variational inequality can be transformed into a fixed point problem (see [11, Proposition 2.1]). Utilizing this result, Ceng and Yao [11] proved that in the setting of Hilbert spaces the well-posedness of a generalized mixed variational inequality is equivalent to the well-posedness of the corresponding fixed point problem. In this section, we will further show that the well-posedness by perturbations of a generalized mixed variational inequality is closely related to the well-posedness by perturbations of the corresponding fixed point problem in the setting of Banach spaces. Let us first recall some concepts.

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere. A Banach space X is said to be

(a) strictly convex if, for any $x, y \in U$,

$$x \neq y \Longrightarrow \left\| \frac{x+y}{2} \right\| < 1, \tag{5.3}$$

(b) smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{5.4}$$

exists for all $x, y \in U$.

The modulus of convexity of X is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in U, \|x + y\| \ge \varepsilon\right\}, \tag{5.5}$$

and the modulus of smoothness of X is defined by

$$\rho_X(\tau) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in U, \|y\| \le \tau\right\}. \tag{5.6}$$

In the sequel we always suppose that $q > 1$ and $s > 1$ are fixed numbers. A Banach space X is said to be

(c) uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2)$,

(d) q-uniformly convex if there exists a constant $c > 0$ such that $\delta_X(\varepsilon) \ge c\varepsilon^q$ for all $\varepsilon \in [0, 2]$,

(e) uniformly smooth if

$$\frac{\rho_X(\tau)}{\tau} \longrightarrow 0 \quad \text{as } \tau \longrightarrow 0, \tag{5.7}$$

(f) q-uniformly smooth if there exists a constant $k > 0$ such that

$$\rho_X(\tau) \le k\tau^q. \tag{5.8}$$

It is well known that the Lebesgue $L^q(q \ge 2)$ spaces are q-uniformly convex and 2-uniformly smooth and $L^q(1 < q < 2)$ is 2-uniformly convex and q-uniformly smooth.

The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{j_q(x) \in X^* : \langle j_q(x), x\rangle = \|x\|^q, \left\|j_q(x)\right\| = \|x\|^{q-1}\right\}. \tag{5.9}$$

In particular, $J = J_2$ is called the normalized duality mapping. J_q has the following properties:

(i) J_q is bounded;

(ii) if X is smooth, then J_q is single-valued;

(iii) if X is strictly convex, then J_q is one-to-one and strictly monotone.

For more details, we refer the readers to [29, 33] and the references therein.

Lemma 5.3 (see [34]). *Let X be a q-uniformly smooth Banach space. Then, there exists a constant $L_q > 0$ such that*

$$\left\| J_q(x) - J_q(y) \right\| \leq L_q \|x - y\|^{q-1}, \quad \forall x, y \in X. \tag{5.10}$$

Lemma 5.4 (see [34]). *Let X be a q-uniformly convex Banach space. Then, there exists a constant $k_q > 0$ such that*

$$\langle J_q(x) - J_q(y), x - y \rangle \leq k_q \|x - y\|^{q}, \quad \forall x, y \in X. \tag{5.11}$$

Lemma 5.5 (see [10, Lemma 5.3]). *Let X be a q-uniformly convex Banach space and $M : X \to 2^{X^*}$ a maximal monotone operator. Then, for every $\lambda > 0$ and $s > 1$, $(M + \lambda J_s)^{-1}$ is well-defined and single-valued.*

The following result indicates that, under suitable conditions, the mapping $\Pi_\lambda^f : X^* \to X$ is Lipschitz continuous, where $\Pi_\lambda^f = (J_q + \lambda \partial f)^{-1}$.

Lemma 5.6 (see [10, Lemma 5.4]). *Let X be a q-uniformly convex Banach space and $M : X \to 2^{X^*}$ a maximal monotone operator. Then, for every $\lambda > 0$,*

$$\left\| \Pi_\lambda^f(w_1) - \Pi_\lambda^f(w_2) \right\| \leq \left(\frac{1}{\lambda k_q} \right)^{1/(q-1)} \|w_1 - w_2\|^{1/(q-1)}, \quad \forall w_1, w_2 \in X^*. \tag{5.12}$$

By means of Lemma 5.5, we can transform $\mathrm{GMVI}(A, F, f)$ into a (set-valued) fixed point problem.

Lemma 5.7. *Let X be a q-uniformly convex Banach space, and let $A : X \to X^*$ and $F : X \to 2^X$. Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous functional. Then, $x \in X$ is a solution of $\mathrm{GMVI}(A, F, f)$ if and only if it is a solution of the following fixed point problem:*

$$\mathrm{FP}\left(\Pi_\lambda^f (J_q - \lambda AF)\right) \text{: find } x \in X \text{ such that } x \in \Pi_\lambda^f \left(J_q(x) - \lambda AF(x)\right). \tag{5.13}$$

Proof. The conclusion follows directly from the definitions of ∂f and Π_λ^f and Lemma 5.5. □

Naturally, the perturbed problem of $\mathrm{FP}(\Pi_\lambda^f (J_q - \lambda AF))$ is given by

$$\mathrm{FP}_p\left(\Pi_\lambda^f (J_q - \lambda AF)\right) \text{: find } x \in X \text{ such that } x \in \Pi_\lambda^{\tilde{f}(p,\cdot)} \left(J_q(x) - \lambda \tilde{A}(p, F(x))\right). \tag{5.14}$$

Theorem 5.8. *Let X be an s-uniformly convex and q-uniformly smooth Banach space. Let $\tilde{A} : P \times X \to X^*$ be uniformly continuous, and let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-uniformly continuous and monotone with respect to $\tilde{A}(p, \cdot)$ for each $p \in P$.*

Let $\tilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ be a continuous functional with respect to the product of the norm topology of P and weak topology of X such that the following conditions hold:

(i) $\tilde{f}(p,\cdot)$ *is proper and convex for all* $p \in P$*;*

(ii) $\operatorname{dom} f$ *is well-positioned and* $\operatorname{dom} f \subset \bigcap_{p\in P} \operatorname{dom} \tilde{f}(p,\cdot)$*;*

(iii) $e(\operatorname{dom} \tilde{f}(p_n,\cdot), \operatorname{dom} f) \to 0$ *whenever* $p_n \to p^*$.

Then, $\mathrm{FP}(\Pi_\lambda^f(J_q - \lambda AF))$ *is weakly well-posed by perturbations whenever* $\mathrm{GMVI}(A,F,f)$ *has a unique solution.*

Proof. By Lemma 5.5, $\Pi_\lambda^{\tilde{f}(p_n,\cdot)} = (J_q + \lambda\partial\tilde{f}(p_n,\cdot))^{-1}$ is well-defined and single-valued. Suppose that $\mathrm{GMVI}(A,F,f)$ has a unique solution x^*. Then, by Lemma 5.7, x^* is also the unique solution of $\mathrm{FP}(\Pi_\lambda^f(J_q-\lambda AF))$. Let $\{p_n\} \subset P$ be with $p_n \to p^*$, and let $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$ for $\mathrm{FP}(\Pi_\lambda^f(J_q-\lambda AF))$. Then, there exists a sequence $\{y_n\} \subset X$ with $y_n \in \Pi_\lambda^{\tilde{f}(p_n,\cdot)}(J_q(x_n) - \lambda\tilde{A}(p_n, F(x_n)))$ (for all $n \geq 1$) such that $\|x_n - y_n\| \to 0$. Further, it is known that there exists a sequence $\{u_n\} \subset X$ with $u_n \in F(x_n)$ (for all $n \geq 1$) such that

$$y_n = \Pi_\lambda^{\tilde{f}(p_n,\cdot)}\left(J_q(x_n) - \lambda\tilde{A}(p_n,u_n)\right), \quad \forall n \geq 1. \tag{5.15}$$

By the definition of $\Pi_\lambda^{\tilde{f}(p_n,\cdot)}$,

$$\frac{J_q(x_n) - J_q(y_n)}{\lambda} - \tilde{A}(p_n,u_n) \in \partial\tilde{f}(p_n,\cdot)(y_n). \tag{5.16}$$

It follows that

$$\tilde{f}(p_n,y) - \tilde{f}(p_n,y_n) \geq \left\langle \frac{J_q(x_n) - J_q(y_n)}{\lambda} - \tilde{A}(p_n,u_n), y - y_n \right\rangle, \quad \forall y \in X,\ n \geq 1. \tag{5.17}$$

From (5.17) we get $y_n \in \operatorname{dom} \tilde{f}(p_n,\cdot)$. We claim that $\{y_n\}$ is bounded. Indeed, if $\{y_n\}$ is unbounded, without loss of generality, we may assume that $\|y_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|y_n - x^*\|}, \qquad z_n = x^* + t_n(y_n - x^*). \tag{5.18}$$

From conditions (i)-(ii), we have $z_n \in \operatorname{dom} \tilde{f}(p_n,\cdot)$. Note that

$$\|z_n\| \leq \|x^*\| + t_n\|y_n - x^*\| = \|x^*\| + 1, \quad \forall n \geq 1. \tag{5.19}$$

So, $\{z_n\}$ is bounded. Since X is reflexive, it follows from the boundedness of $\{z_n\}$ that there exists some subsequence of $\{z_n\}$ which converges weakly to a point of X. Hence, without loss of generality, we may assume that $t_n \in (0,1)$ and $z_n \rightharpoonup z$. By Lemma 2.14 and conditions (ii)-(iii), we get $z \neq x^*$. For any $y \in X$, observe that, for all $v \in F(y)$,

$$\begin{aligned}\left\langle \widetilde{A}(p_n, v), z - y \right\rangle &= \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + \left\langle \widetilde{A}(p_n, v), z_n - x^* \right\rangle + \left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle \\ &= \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + t_n \left\langle \widetilde{A}(p_n, v), y_n - y \right\rangle + (1 - t_n) \left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle.\end{aligned} \tag{5.20}$$

Since x^* is the unique solution of GMVI(A, F, f), there exists some $u^* \in F(x^*)$ such that

$$\langle Au^*, x^* - y \rangle + f(x^*) - f(y) \leq 0, \quad \forall y \in X. \tag{5.21}$$

Also, since $F : X \to 2^X$ is a nonempty weakly compact-valued multifunction, both $F(x_n)$ and $F(y_n)$ are nonempty weakly compact and hence are nonempty, weakly closed, and weakly bounded. Note that the weak closedness of sets in X implies the strong closedness and that the weak boundedness of sets in X is equivalent to the strong boundedness. So, it is known that both $F(x_n)$ and $F(y_n)$ lie in CB(X). According to Proposition 2.3, for each $n \geq 1$ and $u_n \in F(x_n)$, there exists $v_n \in F(y_n)$ such that

$$\|u_n - v_n\| \leq \left(1 + \frac{1}{n}\right) \mathscr{H}(F(x_n), F(y_n)). \tag{5.22}$$

Note that F is $\mathscr{H}$-uniformly continuous. Thus, it follows that

$$\|u_n - v_n\| \leq \left(1 + \frac{1}{n}\right) \mathscr{H}(F(x_n), F(y_n)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.23}$$

Furthermore, since F is monotone with respect to $\widetilde{A}(p_n, \cdot)$, one concludes that, for $u^* \in F(x^*)$, $v \in F(y)$, and $v_n \in F(y_n)$,

$$\left\langle \widetilde{A}(p_n, v), x^* - y \right\rangle \leq \left\langle \widetilde{A}(p_n, u^*), x^* - y \right\rangle, \qquad \left\langle \widetilde{A}(p_n, v), y_n - y \right\rangle \leq \left\langle \widetilde{A}(p_n, v_n), y_n - y \right\rangle. \tag{5.24}$$

It follows from (5.20) and (5.24) that

$$\begin{aligned}\left\langle \widetilde{A}(p_n, v), z - y \right\rangle &\leq \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + t_n \left\langle \widetilde{A}(p_n, v_n), y_n - y \right\rangle + (1 - t_n) \left\langle \widetilde{A}(p_n, u^*), x^* - y \right\rangle \\ &= \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + t_n \left\langle \widetilde{A}(p_n, v_n) - \widetilde{A}(p_n, u_n), y_n - y \right\rangle \\ &\quad + t_n \left\langle \widetilde{A}(p_n, u_n), y_n - y \right\rangle + (1 - t_n) \left\langle \widetilde{A}(p_n, u^*), x^* - y \right\rangle.\end{aligned} \tag{5.25}$$

In addition, we have

$$\widetilde{f}(p_n, z_n) \leq t_n \widetilde{f}(p_n, y_n) + (1 - t_n)\widetilde{f}(p_n, x^*) \tag{5.26}$$

since $\widetilde{f}(p_n, \cdot)$ is convex. It follows from (5.17), (5.25), and (5.26) that

$$\begin{aligned}
\left\langle \widetilde{A}(p_n, v), z - y \right\rangle &\leq \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + t_n \left\langle \widetilde{A}(p_n, v_n) - \widetilde{A}(p_n, u_n), y_n - y \right\rangle \\
&\quad + \frac{t_n}{\lambda} \langle J_q(x_n) - J_q(y_n), y_n - y \rangle + t_n \widetilde{f}(p_n, y) - t_n \widetilde{f}(p_n, y_n) \\
&\quad + (1 - t_n) \left\langle \widetilde{A}(p_n, u^*), x^* - y \right\rangle \\
&= \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + t_n \left\langle \widetilde{A}(p_n, v_n) - \widetilde{A}(p_n, u_n), y_n - y \right\rangle \\
&\quad + \frac{t_n}{\lambda} \langle J_q(x_n) - J_q(y_n), y_n - y \rangle + (1 - t_n) \left\langle \widetilde{A}(p_n, u^*), x^* - y \right\rangle \\
&\quad + t_n \widetilde{f}(p_n, y) + (1 - t_n) \widetilde{f}(p_n, x^*) - \left\{ t_n \widetilde{f}(p_n, y_n) + (1 - t_n) \widetilde{f}(p_n, x^*) \right\} \\
&\leq \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle + t_n \left\langle \widetilde{A}(p_n, v_n) - \widetilde{A}(p_n, u_n), y_n - y \right\rangle \\
&\quad + \frac{t_n}{\lambda} \langle J_q(x_n) - J_q(y_n), y_n - y \rangle + (1 - t_n) \left\langle \widetilde{A}(p_n, u^*), x^* - y \right\rangle \\
&\quad + t_n \widetilde{f}(p_n, y) + (1 - t_n) \widetilde{f}(p_n, x^*) - \widetilde{f}(p_n, z_n).
\end{aligned} \tag{5.27}$$

Note that $\|y_n\| \to +\infty$ and

$$\left\| t_n(y_n - y) \right\| \leq \frac{\left\| y_n - x^* \right\| + \left\| x^* - y \right\|}{\left\| y_n - x^* \right\|} = 1 + \frac{\left\| x^* - y \right\|}{\left\| y_n - x^* \right\|}. \tag{5.28}$$

It is easy to see that $\{t_n(y_n - y)\}$ is bounded. Since $\|x_n - y_n\| \to 0$, by Lemma 5.3,

$$\left\| J_q(x_n) - J_q(y_n) \right\| \leq L_q \left\| x_n - y_n \right\|^{q-1} \longrightarrow 0. \tag{5.29}$$

In the meantime, on account of $\|u_n - v_n\| \to 0$, we have

$$\left\| \widetilde{A}(p_n, v_n) - \widetilde{A}(p_n, u_n) \right\| \longrightarrow 0 \tag{5.30}$$

by means of the uniformly continuity of $\widetilde{A}$. Consequently, letting $n \to \infty$ we obtain that

$$\left| t_n \left\langle \widetilde{A}(p_n, v_n) - \widetilde{A}(p_n, u_n), y_n - y \right\rangle \right| \le \left\| A(p_n, v_n) - \widetilde{A}(p_n, u_n) \right\| \| t_n (y_n - y) \| \longrightarrow 0,$$
$$\left| \frac{t_n}{\lambda} \langle J_q(x_n) - J_q(y_n), y_n - y \rangle \right| \le \frac{\| J_q(x_n) - J_q(y_n) \|}{\lambda} \| t_n (y_n - y) \| \longrightarrow 0,$$
$$(1 - t_n) \left\langle \widetilde{A}(p_n, u^*), x^* - y \right\rangle \longrightarrow \left\langle \widetilde{A}(p^*, u^*), x^* - y \right\rangle. \quad (5.31)$$

Moreover, also observe that

$$\begin{aligned} \left| \left\langle \widetilde{A}(p_n, v), z - z_n \right\rangle \right| &= \left| \left\langle \widetilde{A}(p_n, v) - \widetilde{A}(p^*, v), z - z_n \right\rangle + \left\langle \widetilde{A}(p^*, v), z - z_n \right\rangle \right| \\ &\le \left\| \widetilde{A}(p_n, v) - \widetilde{A}(p^*, v) \right\| \| z_n - z \| \\ &\quad + \left| \left\langle \widetilde{A}(p^*, v), z_n - z \right\rangle \right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (5.32)$$

Further, since $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ is a continuous functional with respect to the product of the norm topology of P and weak topology of X, we get that, as $n \to \infty$,

$$t_n \widetilde{f}(p_n, y) \longrightarrow 0, \qquad (1 - t_n) \widetilde{f}(p_n, x^*) \longrightarrow \widetilde{f}(p^*, x^*), \qquad \widetilde{f}(p_n, z_n) \longrightarrow \widetilde{f}(p^*, z). \quad (5.33)$$

Therefore, letting $n \to \infty$ in (5.27) we conclude that

$$\begin{aligned} \langle Av, z - y \rangle &= \left\langle \widetilde{A}(p^*, v), z - y \right\rangle \\ &\le \left\langle \widetilde{A}(p^*, u^*), x^* - y \right\rangle + \widetilde{f}(p^*, x^*) - \widetilde{f}(p^*, z) \\ &= \langle Au^*, x^* - y \rangle + f(x^*) - f(z), \quad \forall y \in X, \ v \in F(y). \end{aligned} \quad (5.34)$$

By using (5.34) and the same argument as in the proof of Theorem 4.4, we can prove that z is a solution of GMVI(A, F, f), a contradiction. Thus, $\{y_n\}$ is bounded and so is $\{x_n\}$.

By using (5.17) and the similar argument to that in the proof of Theorem 4.4, we can prove that $\{x_n\}$ converges weakly to x^*. Since x^* is the unique solution of FP$(\Pi_\lambda^f(J_q - \lambda AF))$, FP$(\Pi_\lambda^f(J_q - \lambda AF))$ is weakly well-posed by perturbations. □

Remark 5.9. Theorem 5.8 generalizes Theorem 5.1 of [9] and Theorem 5.1 of [11] since every Hilbert space is 2-uniformly convex and 2-uniformly smooth. Theorem 5.8 improves, extends, and develops [10, Theorem 5.1] in the following aspects.

(i) The mixed variational inequality problem (MVI) in [10, Theorem 5.1] is extended to develop the more general problem, that is, the generalized mixed variational inequality problem (GMVI) with a nonempty weakly compact-valued multifunction in the setting of Banach spaces. In the meantime, the (single-valued) fixed point problem corresponding to MVI in [10, Theorem 5.1] is extended to develop the more general problem, that is, the (set-valued) fixed point problem corresponding to GMVI. Furthermore, the concept of weak well-posedness by perturbations for the (single-valued) fixed point problem corresponding to MVI is extended to develop the concept of weak well-posedness by perturbations for the (set-valued) fixed point problem corresponding to GMVI.

(ii) Since the generalized mixed variational inequality problem (GMVI) is more general and more complicated than the mixed variational inequality problem (MVI), the assumptions in Theorem 5.8 are very different from the ones in [10, Theorem 5.1]; for instance, in Theorem 5.8, let $\widetilde{A} : P \times X \to X^*$ be uniformly continuous, and let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathscr{H}$-uniformly continuous and monotone with respect to $\widetilde{A}(p, \cdot)$ for each $p \in P$, but, in [10, Theorem 5.1], let $\widetilde{A} : P \times X \to X^*$ be a uniformly continuous mapping such that $\widetilde{A}(p, \cdot) : X \to X^*$ is monotone for all $p \in P$.

(iii) The technique of proving weak well-posedness by perturbations for (set-valued) fixed point problem $\mathrm{FP}(\Pi_\lambda^f(J_q - \lambda AF))$ in Theorem 5.8 is very different from the one for (single-valued) fixed point problem $\mathrm{FP}(\Pi_\lambda^f(J_q - \lambda A))$ in [10, Theorem 5.1] because our technique depends on the well-known Nadler's Theorem [27], the $\mathscr{H}$-uniformly continuity of nonempty weakly compact-valued multifunction F and the monotonicity of F with respect to $\widetilde{A}(p, \cdot)$ for each $p \in P$.

Now we give the following example as an application of Theorem 5.8.

Example 5.10. Let $X = \mathbf{R}$, $P = [-1, 1]$, and $p^* = 0$. Let $\widetilde{A}(p, x) = x + p$, $F(x) = x - \sin x$ for all $x \in X$, $p \in P$, and let

$$\widetilde{f}(p, x) = \begin{cases} 0, & x \in K(p), \\ +\infty, & \text{otherwise}, \end{cases} \tag{5.35}$$

for all $p \in P$, where $K(p) = [-1, 1] + \{p\}$. Then, $\mathrm{Dom}\, \widetilde{A}(p, \cdot) = \mathbf{R}$, $\mathrm{Dom}\, f = [-1, 1]$, $\mathrm{Dom}\, \widetilde{f}(p, \cdot) = K(p)$ for all $p \in P$. Clearly, if $p_n \to p^*$, then $e(\mathrm{Dom}\, \widetilde{f}(p_n, \cdot), \mathrm{Dom}\, f) \to 0$. By the above definitions, it is easy to see that $\widetilde{A}$ is uniformly continuous, F is $\mathscr{H}$-uniformly continuous and monotone with respect to $\widetilde{A}(p, \cdot)$ for each $p \in P$, and $\widetilde{f}(p, \cdot)$ is proper and convex. By (ii) of Remark 2.7, $\mathrm{Dom}\, f$ is well-positioned. It is known that $\mathbf{R}$ is 2-uniformly convex and 2-uniformly smooth. Hence, all the assumptions of Theorem 5.8 are satisfied. Let S be the solution set of $\mathrm{GMVI}(A, F, f)$. It follows that

$$S = \{x \in [-1, 1] : (x - \sin x)(x - y) \le 0,\ \forall y \in [-1, 1]\} = \{0\}. \tag{5.36}$$

So $x^* = 0$ is the unique solution of $\mathrm{GMVI}(A, F, f)$. By Theorem 5.8, $\mathrm{FP}(\Pi_\lambda^f(J_q - \lambda AF))$ is well-posed by perturbations.

Based on Theorems 4.7 and 5.8 and Lemma 5.7, we have the following result.

Theorem 5.11. *Let X be an s-uniformly convex and q-uniformly smooth Banach space. Let $\widetilde{A} : P \times X \to X^*$ be uniformly continuous, let $A : X \to X^*$ be weakly continuous, and let $F : X \to 2^X$*

be a nonempty weakly compact-valued multifunction which is $\mathcal{H}$-uniformly continuous and monotone with respect to $\tilde{A}(p,\cdot)$ for each $p \in P$. Let $\tilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ be a continuous functional with respect to the product of the norm topology of P and weak topology of X such that the following conditions hold:

(i) $\tilde{f}(p,\cdot)$ *is proper and convex for all* $p \in P$;

(ii) $\operatorname{dom} f$ *is well-positioned and* $\operatorname{dom} f \subset \bigcap_{p\in P} \operatorname{dom} \tilde{f}(p,\cdot)$;

(iii) $e(\operatorname{dom} \tilde{f}(p_n,\cdot), \operatorname{dom} f) \to 0$ *whenever* $p_n \to p^*$.

Then, the following statements are equivalent:

(i) $\mathrm{GMVI}(A,F,f)$ *is weakly well-posed by perturbations;*

(ii) $\mathrm{IP}(AF + \partial f)$ *is weakly well-posed by perturbations;*

(iii) $\mathrm{FP}(\Pi_\lambda^f(J_q - \lambda AF))$ *is weakly well-posed by perturbations;*

(iv) $\mathrm{GMVI}(A,F,f)$ *has a unique solution;*

(v) $\mathrm{IP}(AF + \partial f)$ *has a unique solution;*

(vi) $\mathrm{FP}(\Pi_\lambda^f(J_q - \lambda AF))$ *has a unique solution.*

Next we consider the case of generalized well-posedness by perturbations.

Theorem 5.12. *Let X be an s-uniformly convex and q-uniformly smooth Banach space. Let $\tilde{A} : P \times X \to X^*$ be uniformly continuous, and let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathcal{H}$-uniformly continuous and monotone with respect to $\tilde{A}(p,\cdot)$ for each $p \in P$. Let $\tilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ be a continuous functional with respect to the product of the norm topology of P and weak topology of X such that $\tilde{f}(p,\cdot)$ is proper and convex. Then,* $\mathrm{FP}(\Pi_\lambda^f(J_q - \lambda AF))$ *is strongly (resp., weakly) generalized well-posed by perturbations whenever* $\mathrm{GMVI}(A,F,f)$ *is strongly (resp., weakly) generalized $(1 + 1/\lambda)$-well-posed by perturbations.*

Proof. Suppose that $\mathrm{GMVI}(A,F,f)$ is strongly (resp., weakly) generalized $(1 + 1/\lambda)$-well-posed by perturbations. Let $\{p_n\} \subset P$ be with $p_n \to p^*$, and let $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$ for $\mathrm{FP}(\Pi_\lambda^f(J_q - \lambda AF))$. Then, there exists a sequence $\{y_n\} \subset X$ with $y_n \in \Pi_\lambda^{\tilde{f}(p_n,\cdot)}(J_q(x_n) - \lambda\tilde{A}(p_n, F(x_n)))$ (for all $n \geq 1$) such that $\|x_n - y_n\| \to 0$. Further, it is known that there exists a sequence $\{u_n\} \subset X$ with $u_n \in F(x_n)$ (for all $n \geq 1$) such that

$$y_n = \Pi_\lambda^{\tilde{f}(p_n,\cdot)}\left(J_q(x_n) - \lambda\tilde{A}(p_n,u_n)\right), \quad \forall n \geq 1. \tag{5.37}$$

By the definition of $\Pi_\lambda^{\tilde{f}(p_n,\cdot)}$,

$$\frac{J_q(x_n) - J_q(y_n)}{\lambda} - \tilde{A}(p_n,u_n) \in \partial\tilde{f}(p_n,\cdot)(y_n). \tag{5.38}$$

It follows that

$$\tilde{f}(p_n,y) - \tilde{f}(p_n,y_n) \geq \left\langle \frac{J_q(x_n) - J_q(y_n)}{\lambda} - \tilde{A}(p_n,u_n), y - y_n \right\rangle, \quad \forall y \in X,\ n \geq 1. \tag{5.39}$$

Furthermore, since $F : X \to 2^X$ is a nonempty weakly compact-valued multifunction, both $F(x_n)$ and $F(y_n)$ are nonempty weakly compact and hence are nonempty, weakly closed, and weakly bounded. Note that the weak closedness of sets in X implies the strong closedness and that the weak boundedness of sets in X is equivalent to the strong boundedness. So, it is known that both $F(x_n)$ and $F(y_n)$ lie in CB(X). According to Proposition 2.3, for each $n \geq 1$ and $u_n \in F(x_n)$ there exists $v_n \in F(y_n)$ such that

$$\|u_n - v_n\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(F(x_n), F(y_n)). \tag{5.40}$$

Note that F is $\mathcal{H}$-uniformly continuous. Thus, one deduces that

$$\|u_n - v_n\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(F(x_n), F(y_n)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.41}$$

Now utilizing (5.39) we have

$$\begin{aligned} &\left\langle \widetilde{A}(p_n, v_n), y_n - y \right\rangle + \widetilde{f}(p_n, y_n) - \widetilde{f}(p_n, y) \\ &\quad \leq \left\langle \frac{J_q(x_n) - J_q(y_n)}{\lambda}, y_n - y \right\rangle + \left\langle \widetilde{A}(p_n, v_n) - \widetilde{A}(p_n, u_n), y_n - y \right\rangle \\ &\quad \leq \frac{1}{2}\left(1 + \frac{1}{\lambda}\right)\|y_n - y\|^2 + \frac{1}{2}\varepsilon_n, \quad \forall y \in X, \ n \geq 1, \end{aligned} \tag{5.42}$$

where

$$\begin{aligned} \varepsilon_n &= \frac{1}{\lambda}\|J_q(x_n) - J_q(y_n)\|^2 + \left\|\widetilde{A}(p_n, v_n) - \widetilde{A}(p_n, u_n)\right\|^2 \\ &\leq \frac{L_q}{\lambda}\|x_n - y_n\|^{2(q-1)} + \left\|\widetilde{A}(p_n, v_n) - \widetilde{A}(p_n, u_n)\right\|^2 \longrightarrow 0. \end{aligned} \tag{5.43}$$

Therefore, $\{y_n\}$ is a $(1 + 1/\lambda)$-approximating sequence corresponding to $\{p_n\}$ for GMVI(A, F, f). By the strong (resp., weak) generalized $(1 + 1/\lambda)$-well-posedness by perturbations, $\{y_n\}$ has some subsequence which converges strongly (resp., weakly) to a solution x^* of GMVI(A, F, f). By Lemma 5.7, x^* is also a solution of FP$(\Pi_\lambda^f(J_q - \lambda AF))$. Consequently, FP$(\Pi_\lambda^f(J_q - \lambda AF))$ is strongly (resp., weakly) generalized well-posed by perturbations. □

Remark 5.13. Theorem 5.12 generalizes Theorem 5.3 of [9] and Theorem 5.3 of [11]. In addition, whenever $F = I$ the identity mapping of X, Theorem 5.12 reduces to Theorem 5.3 of [10].

Theorem 5.14. *Let X be a q-uniformly convex Banach space. Let $\widetilde{A} : P \times X \to X^*$ be uniformly continuous, and let $F : X \to 2^X$ be a nonempty weakly compact-valued multifunction which is $\mathcal{H}$-uniformly continuous. Let $\widetilde{f} : P \times X \to \mathbf{R} \cup \{+\infty\}$ be a functional such that $\widetilde{f}(p, \cdot)$ is proper, convex,* *and lower semicontinuous for each* $p \in P$*. Then,* $\mathrm{GMVI}(A, F, f)$ *is strongly (resp., weakly) generalized well-posed by perturbations whenever* $\mathrm{FP}(\Pi_\lambda^f(J_q - \lambda AF))$ *is strongly (resp., weakly) generalized well-posed by perturbations.*

Proof. Let $\{p_n\} \subset P$ be with $p_n \longrightarrow p^*$, and let $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$ for $\mathrm{GMVI}(A, F, f)$. Then, there exist $\{u_n\} \subset X$ with $u_n \in F(x_n)$ (for all $n \geq 1$) and $0 < \varepsilon_n \longrightarrow 0$ such that

$$\left\langle \widetilde{A}(p_n, u_n), x_n - y \right\rangle + \widetilde{f}(p_n, x_n) - \widetilde{f}(p_n, y) \leq \varepsilon_n, \quad \forall y \in X,\ n \geq 1. \tag{5.44}$$

Define $\varphi_n : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ as follows:

$$\varphi_n(y) = \widetilde{f}(p_n, y) + \left\langle \widetilde{A}(p_n, u_n), y - x_n \right\rangle, \quad \forall y \in X,\ n \geq 1. \tag{5.45}$$

Clearly, φ_n is proper, convex, and lower semicontinuous and $0 \in \partial_{\varepsilon_n}\varphi_n(x_n)$ for all $n \geq 1$. By the Brondsted-Rockafellar theorem [32], there exist $\overline{x}_n \in X$ and

$$w_n \in \partial\varphi_n(\overline{x}_n) = \widetilde{A}(p_n, u_n) + \partial\widetilde{f}(p_n, \cdot)(\overline{x}_n) \tag{5.46}$$

such that

$$\|x_n - \overline{x}_n\| \leq \sqrt{\varepsilon_n}, \qquad \|w_n\| \leq \sqrt{\varepsilon_n}. \tag{5.47}$$

Since F is a nonempty weakly compact-valued multifunction, both $F(x_n)$ and $F(\overline{x}_n)$ lie in $\mathrm{CB}(X)$ for each $n \geq 1$. By Proposition 2.3, for each $n \geq 1$ and $u_n \in F(x_n)$ there exists $\overline{u}_n \in F(\overline{x}_n)$ such that

$$\|u_n - \overline{u}_n\| \leq \left(1 + \frac{1}{n}\right)\mathscr{H}(F(x_n), F(\overline{x}_n)). \tag{5.48}$$

Since F is $\mathscr{H}$-uniformly continuous, we obtain from (5.47) that

$$\|u_n - \overline{u}_n\| \leq \left(1 + \frac{1}{n}\right)\mathscr{H}(F(x_n), F(\overline{x}_n)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.49}$$

Utilizing (5.46) we have

$$\overline{x}_n = \Pi_\lambda^{\widetilde{f}(p_n,\cdot)}\left[\lambda w_n + J_q(\overline{x}_n) - \lambda\widetilde{A}(p_n, u_n)\right]. \tag{5.50}$$

Since $\widetilde{A}$ is uniformly continuous, it follows from (5.47) and (5.50) that

$$\begin{aligned} &\left\| \left[\lambda w_n + J_q(\overline{x}_n) - \lambda\widetilde{A}(p_n, u_n)\right] - \left[J_q(\overline{x}_n) - \lambda\widetilde{A}(p_n, \overline{u}_n)\right] \right\| \\ &\quad \leq \lambda\|w_n\| + \lambda\left\|\widetilde{A}(p_n, \overline{u}_n) - \widetilde{A}(p_n, u_n)\right\| \longrightarrow 0. \end{aligned} \tag{5.51}$$

By Lemma 5.6 and (5.50) and (5.51),

$$\left\| \overline{x}_n - \Pi_{\lambda}^{\widetilde{f}(p_n,\cdot)} \left[J_q(\overline{x}_n) - \lambda \widetilde{A}(p_n, \overline{u}_n) \right] \right\| \longrightarrow 0. \tag{5.52}$$

Thus, $\{\overline{x}_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for $\mathrm{FP}(\Pi_{\lambda}^{f}(J_q - \lambda AF))$. Repeating the same argument as in the proof of Theorem 4.10, we can deduce that $\{\overline{x}_n\}$ has some subsequence $\{\overline{x}_{n_k}\}$ which converges strongly (resp., weakly) to some solution x^* of $\mathrm{FP}(\Pi_{\lambda}^{f}(J_q - \lambda AF))$. By Lemma 5.7, x^* is also a solution of $\mathrm{GMVI}(A, F, f)$. Thus, $\mathrm{GMVI}(A, F, f)$ is strongly (resp., weakly) generalized well-posed by perturbations. □

Remark 5.15. Theorem 5.14 generalizes Theorem 5.4 of [9] and Theorem 5.4 of [11]. In addition, whenever $F = I$ the identity mapping of X, Theorem 5.14 reduces to Theorem 5.4 of [10].

Acknowledgments

This research was partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133), and Leading Academic Discipline Project of Shanghai Normal University (DZL707) and by a grant from the National Science Council of ROC (NSC 100-2115-M-037-001).

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