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Research Article

Convergence Theorems for Equilibrium Problems and Fixed-Point Problems of an Infinite Family of k_i-Strictly Pseudocontractive Mapping in Hilbert Spaces

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We first extend the definition of W_n from an infinite family of nonexpansive mappings to an infinite family of strictly pseudocontractive mappings, and then propose an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of an infinite family of k_i-strictly pseudocontractive mappings in Hilbert spaces. The results obtained in this paper extend and improve the recent ones announced by many others. Furthermore, a numerical example is presented to illustrate the effectiveness of the proposed scheme.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let $F : C \times C \to R$ be a bifunction. We consider the following equilibrium problem (EP) which is to find $z \in C$ such that

$$\text{EP} : F(z, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

Denote the set of solutions of EP by $\text{EP}(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $z \in \text{EP}(F)$ if and only if $\langle Tx, y - x \rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem [1–13].

A mapping $B : C \to C$ is called θ-Lipschitzian if there exists a positive constant θ such that

$$\|Bx - By\| \leq \theta\|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

B is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Bx - By, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

A mapping $S : C \to C$ is said to be k-strictly pseudocontractive mapping if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \tag{1.4}$$

for all $x, y \in C$ and $F(S)$ denotes the set of fixed point of the mapping S, that is $F(S) = \{x \in C : Sx = x\}$.

If $k = 1$, then S is said to a pseudocontractive mapping, that is,

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|(I - S)x - (I - S)y\|^2, \tag{1.5}$$

is equivalent to

$$\langle (I - S)x - (I - S)y, x - y\rangle \geq 0, \tag{1.6}$$

for all $x, y \in C$.

The class of k-strict pseudo-contractive mappings extends the class of nonexpansive mappings (A mapping T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$). That is, S is nonexpansive if and only if S is a 0-strict pseudocontractive mapping. Clearly, the class of k-strictly pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mapping.

In 2006, Marino and Xu [14] introduced the general iterative method and proved that for a given $x_0 \in H$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Tx_n, \quad n \in N, \tag{1.7}$$

where T is a self-nonexpansive mapping on H, f is an α-contraction of H into itself (i.e., $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, for all $x, y \in H$ and $\alpha \in (0,1)$), $\{\alpha_n\} \subset (0,1)$ satisfies certain conditions, B is strongly positive bounded linear operator on H, and converges strongly to fixed point x^* of T which is the unique solution to the following variational inequality:

$$\langle (\gamma f - B)x^*, x^* - x\rangle \leq 0, \quad \forall x \in F(T). \tag{1.8}$$

Tian [15] considered the following iterative method, for a nonexpansive mapping $T:H \to H$ with $F(T) \neq \emptyset$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad n \in N, \tag{1.9}$$

where F is k-Lipschitzian and η-strongly monotone operator. The sequence $\{x_n\}$ converges strongly to fixed-point q in $F(T)$ which is the unique solution to the following variational inequality:

$$\langle (\gamma f - \mu F)q, p - q\rangle \leq 0, \quad p \in F(T). \tag{1.10}$$

For finding a common element of $\mathrm{EP}(F) \cap F(S)$, S. Takahashi and W. Takahashi [16] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $S : C \to H$ be a nonexpansive mapping. Starting with arbitrary initial point $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad \forall n \in N. \end{gathered} \tag{1.11}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap \mathrm{EP}(F)$, where $z = P_{F(S)\cap \mathrm{EP}(F)} f(z)$.

Liu [17] introduced the following scheme: $x_1 \in H$ and

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n u_n + (1 - \beta_n)Su_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n, \quad \forall n \in N, \end{gathered} \tag{1.12}$$

where S is a k-strict pseudo-contractive mapping and B is a strongly positive bounded linear operator. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$, the sequence $\{x_n\}$ converges strongly to $z \in F(S) \cap \mathrm{EP}(F)$, where $z = P_{F(S)\cap \mathrm{EP}(F)}(I - B + \gamma f)(z)$.

In [18], the concept of W mapping had been modified for a countable family $\{T_n\}_{n\in N}$ of nonexpansive mappings by defining the sequence $\{W_n\}_{n\in N}$ of W-mappings generated by $\{T_n\}_{n\in N}$ and $\{\lambda_n\} \subset (0,1)$, proceeding backward

$$\begin{gathered} U_{n,n+1} := I, \\ U_{n,n} := \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ \cdots \end{gathered}$$

$$U_{n,k} := \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,$$
$$\cdots$$
$$U_{n,2} := \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,$$
$$W_n = U_{n,1} := \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \tag{1.13}$$

Yao et al. [19] using this concept, introduced the following algorithm: $x_1 \in H$ and

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n u_n, \quad \forall n \in N. \end{gathered} \tag{1.14}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{EP}(F)$.

Colao and Marino [20] considered the following explicit viscosity scheme

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n r f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) W_n u_n, \quad \forall n \in N, \end{gathered} \tag{1.15}$$

where A is a strongly positive operator on H. Under certain appropriate conditions, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{EP}(F)$.

Motivated and inspired by these facts, in this paper, we first extend the definition of W_n from an infinite family of nonexpansive mappings to an infinite family of strictly pseudo-contractive mappings, and then propose the iteration scheme (3.2) for finding an element of $\mathrm{EP}(F) \bigcap_{i=1}^{\infty} F(S_i)$, where $\{S_i\}$ is an infinite family of k_i-strictly pseudo-contractive mappings of C into itself. Finally, the convergence theorem of the iteration scheme is obtained. Our results include Yao et al. [19], Colao and Marino [20] as some special cases.

2. Preliminaries

Throughout this paper, we always assume that C is a nonempty closed convex subset of a Hilbert space H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by N and R the sets of positive integers and real numbers, respectively. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C x \Longleftrightarrow \langle x - u, u - y\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

It is widely known that H satisfies Opial's condition [21], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.3}$$

holds for every $y \in H$ with $y \neq x$.

In order to solve the equilibrium problem for a bifunction $F : C \times C \to R$, we assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$.

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$.

(A3) $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$, for all $x, y, z \in C$.

(A4) For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let us recall the following lemmas which will be useful for our paper.

Lemma 2.1 (see [22]). *Let F be a bifunction from $C \times C$ into R satisfying (A1), (A2), (A3), and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \forall y \in C. \tag{2.4}$$

Furthermore, if $T_r x = \{z \in C : F(z, y) + (1/r)(y - z, z - x) \geq 0, \forall y \in C\}$, then the following hold:

(1) *T_r is single-valued.*

(2) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H. \tag{2.5}$$

(3) *$F(T_r) = EP(F)$.*

(4) *$EP(F)$ is closed and convex.*

Lemma 2.2 (see [23]). *Let $S : C \to H$ be a k-strictly pseudo-contractive mapping. Define $T : C \to H$ by $Tx = \lambda x + (1-\lambda)Sx$ for each $x \in C$. Then, as $\lambda \in [k, 1)$, T is nonexpansive mapping such that $F(T) = F(S)$.*

Lemma 2.3 (see [24]). *In a Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.6}$$

Lemma 2.4 (see [25]). *Let H be a Hilbert space and C be a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.5 (see [26]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ be a sequence in $[0,1]$ satisfying the following condition*

$$0 < \lim_{n\to\infty} \inf \gamma_n \leq \lim_{n\to\infty} \sup \gamma_n < 1. \tag{2.7}$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n) z_n$, $n \geq 0$ and $\lim_{n\to\infty} \sup(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 2.6 (see [27]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-b_n)a_n + b_n\delta_n, \quad n \geq 0, \tag{2.8}$$

where $\{b_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R, such that

(i) $\sum_{i=1}^{\infty} b_i = \infty$.

(ii) $\lim_{n\to\infty} \sup \delta_n \leq 0$ *or* $\sum_{i=1}^{\infty} |b_n\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Let $\{S_i\}$ be an infinite family of k_i-strictly pseudo-contractive mappings of C into itself, we define a mapping W_n of C into itself as follows,

$$\begin{aligned} U_{n,n+1} &:= I, \\ U_{n,n} &:= \tau_n S'_n U_{n,n+1} + (1-\tau_n)I, \\ &\cdots \\ U_{n,k} &:= \tau_k S'_k U_{n,k+1} + (1-\tau_k)I, \\ &\cdots \\ U_{n,2} &:= \tau_2 S'_2 U_{n,3} + (1-\tau_2)I, \\ W_n = U_{n,1} &:= \tau_1 S'_1 U_{n,2} + (1-\tau_1)I, \end{aligned} \tag{2.9}$$

where $0 \leq \tau_i \leq 1$, $S'_i = \sigma_i I + (1-\sigma_i)S_i$ and $\sigma_i \in [k_i, 1)$ for $i \in N$. We can obtain S'_i is a nonexpansive mapping and $F(S_i) = F(S'_i)$ by Lemma 2.2. Furthermore, we obtain that W_n is a nonexpansive mapping.

Remark 2.7. If $k_i = 0$, and $\sigma_i = 0$ for $i \in N$, then the definition of W_n in (2.9) reduces to the definition of W_n in (1.13).

To establish our results, we need the following technical lemmas.

Lemma 2.8 (see [18]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{S'_i\}$ be an infinite family of nonexpansive mappings of C into itself and let $\{\tau_i\}$ be a real sequence such that $0 < \tau_i \leq b < 1$ for every $i \in N$. Then, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

In view of the previous lemma, we will define

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C. \tag{2.10}$$

Lemma 2.9 (see [18]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{S_i'\}$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(S_i') \neq \emptyset$ and let $\{\tau_i\}$ be a real sequence such that $0 < \tau_i \leq b < 1$ for every $i \in N$. Then, $F(W) = \bigcap_{i=1}^{\infty} F(S_i') \neq \emptyset$.*

The following lemmas follow from Lemmas 2.2, 2.8, and 2.9.

Lemma 2.10. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{S_i\}$ be an infinite family of k_i-strictly pseudo-contractive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Define $S_i' = \sigma_i I + (1-\sigma_i)S_i$ and $\sigma_i \in [k_i, 1)$ and let $\{\tau_i\}$ be a real sequence such that $0 < \tau_i \leq b < 1$ for every $i \in N$. Then, $F(W) = \bigcap_{i=1}^{\infty} F(S_i) = \bigcap_{i=1}^{\infty} F(S_i') \neq \emptyset$.*

Lemma 2.11 (see [28]). *Let C be a nonempty closed convex subset of a Hilbert space. Let $\{S_i'\}$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(S_i') \neq \emptyset$ and let $\{\tau_i\}$ be a real sequence such that $0 < \tau_i \leq b < 1$ for every $i \in N$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0. \tag{2.11}$$

3. Main Results

Let H be a real Hilbert space and F be a k-Lipschitzian and η-strongly monotone operator with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$ and $0 < t < 1$. Then, for $t \in \min\{0, \{1, 1/\tau\}\}$, $S = (I - t\mu F) : H \to H$ is a contraction with contractive coefficient $1 - t\tau$ and $\tau = (1/2)\mu(2\eta - \mu k^2)$.

In fact, from (1.2) and (1.3), we obtain

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|x - y - t\mu(Fx - Fy)\|^2 \\
&= \|x - y\|^2 + t^2\mu^2\|Fx - Fy\|^2 - 2t\mu\langle Fx - Fy, x - y\rangle \\
&\leq \|x - y\|^2 + k^2t^2\mu^2\|x - y\|^2 - 2t\eta\mu\|x - y\|^2 \\
&\leq \left(1 - t\mu\left(2\eta - \mu k^2\right)\right)\|x - y\|^2 \\
&\leq (1 - t\tau)^2\|x - y\|^2.
\end{aligned} \tag{3.1}$$

Thus, $S = (1 - t\mu F)$ is a contraction with contractive coefficient $1 - t\tau \in (0, 1)$.

Now, we show the strong convergence results for an infinite family k_i-strictly pseudo-contractive mappings in Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and F be a bifunction from $C \times C \to R$ satisfying (A1)–(A4). Let $S_i : C \to C$ be a k_i-strictly pseudo-contractive mapping with $\bigcap_{i=1}^{\infty} F(S_i) \cap EP \neq \emptyset$ and $\{\tau_i\}$ be a real sequence such that $0 < \tau_i \leq b < 1$, $i \in N$. Let f be a contraction of H into itself with $\beta \in (0, 1)$ and B be k-Lipschitzian and η-strongly monotone* *operator on H with coefficients* $k, \eta > 0$, $0 < \mu < 2\eta/k^2$, $0 < r < (1/2)\mu(2\eta - \mu k^2)/\beta = (\tau/\beta)$ *and* $\tau < 1$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{gathered} F(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \delta_n u_n + (1-\delta_n)W_n u_n, \\ x_{n+1} = \alpha_n r f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \mu\alpha_n B\big)y_n, \quad \forall n \in N, \end{gathered} \tag{3.2}$$

where $u_n = T_{\lambda_n} x_n$ *and* $\{W_n : C \to C\}$ *is the sequence defined by* (2.9). *If* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, *and* $\{\lambda_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{i=1}^{\infty}\alpha_n = \infty$,

(ii) $0 < \lim_{n\to\infty}\inf\beta_n \leq \lim_{n\to\infty}\sup\beta_n < 1$,

(iii) $0 < \lim_{n\to\infty}\inf\delta_n \leq \lim_{n\to\infty}\sup\delta_n < 1$, $\lim_{n\to\infty}|\delta_{n+1} - \delta_n| = 0$,

(iv) $\{\lambda_n\} \subset (0,\infty)$, $\lim_{n\to\infty}\lambda_n > 0$, $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$.

Then $\{x_n\}$ *converges strongly to* $z \in \bigcap_{i=1}^{\infty} F(S_i) \cap EP \neq \emptyset$, *where z is the unique solution of variational inequality*

$$\lim_{n\to\infty}\sup\big\langle (rf - \mu B)z, p - z\big\rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{EP} \neq \emptyset, \tag{3.3}$$

that is, $z = P_{F(W)\cap EP(F)}(I - \mu B + rf)z$, *which is the optimality condition for the minimization problem*

$$\min_{z \in \bigcap_{i=1}^{\infty} F(S_i)\cap EP} \frac{1}{2}\langle \mu B z, z\rangle - h(z), \tag{3.4}$$

where h is a potential function for rf *(i.e.,* $h'(z) = rf(z)$ *for* $z \in H$*).*

Proof. We divide the proof into five steps.
Step 1. We prove that $\{x_n\}$ is bounded.

Noting the conditions (i) and (ii), we may assume, without loss of generality, that $\alpha_n/(1-\beta_n) \leq \min\{1, 1/\tau\}$. For $x, y \in C$, we obtain

$$\begin{aligned} &\big\|\big((1-\beta_n)I - \alpha_n\mu B\big)x - \big((1-\beta_n)I - \alpha_n\mu B\big)y\big\| \\ &\quad \leq (1-\beta_n)\left\|\left(I - \frac{\alpha_n}{1-\beta_n}\mu B\right)x - \left(I - \frac{\alpha_n}{1-\beta_n}\mu B\right)y\right\| \\ &\quad \leq (1-\beta_n)\left(1 - \frac{\alpha_n}{1-\beta_n}\tau\right)\|x - y\| \\ &\quad = (1 - \beta_n - \alpha_n\tau)\|x - y\|. \end{aligned} \tag{3.5}$$

Take $p \in \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{EP} \neq \emptyset$. Since $u_n = T_{\lambda_n} x_n$ and $p = T_{\lambda_n} p$, then from Lemma 2.1, we know that, for any $n \in N$,

$$\|u_n - p\| = \|T_{\lambda_n} x_n - T_{\lambda_n} p\| \leq \|x_n - p\|. \tag{3.6}$$

Furthermore, since $W_n p = p$ and (3.6), we have

$$\begin{aligned}
\|y_n - p\| &= \|\delta_n u_n + (1-\delta_n) W_n u_n - p\| \\
&= \|\delta_n (u_n - p) + (1-\delta_n)(W_n u_n - p)\| \\
&\leq \delta_n \|u_n - p\| + (1-\delta_n)\|W_n u_n - p\| \\
&\leq \|u_n - p\| \leq \|x_n - p\|.
\end{aligned} \tag{3.7}$$

Thus, it follows from (3.7) that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n r f(x_n) + \beta_n x_n + ((1-\beta_n) I - \mu \alpha_n B) y_n - p\| \\
&= \|\alpha_n r (f(x_n) - f(p)) + \alpha_n (r f(p) - \mu B p) \\
&\qquad + \beta_n (x_n - p) + ((1-\beta_n) I - \mu \alpha_n B)(y_n - p)\| \\
&\leq \alpha_n r \beta \|x_n - p\| + \alpha_n \|r f(p) - \mu B p\| + \beta_n \|x_n - p\| \\
&\quad + (1 - \beta_n - \tau \alpha_n)\|y_n - p\| \\
&\leq (1 - \alpha_n(\tau - r\beta))\|x_n - p\| + \alpha_n \|r f(p) - \mu B p\| \\
&\leq \max\left\{ \|x_n - p\|, \frac{\|r f(p) - \mu B p\|}{\tau - r\beta} \right\}.
\end{aligned} \tag{3.8}$$

By induction, we have

$$\|x_n - p\| \leq \max\left\{ \|x_1 - p\|, \frac{\|r f(p) - \mu B p\|}{\tau - r\beta} \right\}, \quad n \geq 1. \tag{3.9}$$

Hence, $\{x_n\}$ is bounded and we also obtain that $\{u_n\}$, $\{W_n u_n\}$, $\{y_n\}$, $\{B y_n\}$, and $\{f(x_n)\}$ are all bounded. Without loss of generality, we can assume that there exists a bounded set $K \subset C$ such that $\{u_n\}, \{W_n u_n\}, \{y_n\}, \{B y_n\}, \{f(x_n)\} \in K$, for all $n \in N$.

Step 2. We show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Let $x_{n+1} = (1-\beta_n) z_n + \beta_n x_n$. We note that

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\alpha_n r f(x_n) + ((1-\beta_n) I - \mu \alpha_n B) y_n}{1-\beta_n}, \tag{3.10}$$

and then

$$z_{n+1} - z_n = \frac{\alpha_{n+1} r f(x_{n+1}) + ((1-\beta_{n+1})I - \mu\alpha_{n+1}B)y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n r f(x_n) + ((1-\beta_n)I - \mu\alpha_n B)y_n}{1-\beta_n} = \frac{\alpha_{n+1}}{1-\beta_{n+1}}(rf(x_{n+1}) - \mu B y_{n+1}) - \frac{\alpha_n}{1-\beta_n}(rf(x_n) - \mu B y_n) + y_{n+1} - y_n. \tag{3.11}$$

Therefore,

$$\|z_{n+1} - z_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|rf(x_{n+1})\| + \|\mu B y_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|rf(x_n)\| + \|\mu B y_n\|) + \|y_{n+1} - y_n\|. \tag{3.12}$$

It follows from (3.2) that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|\delta_{n+1}u_{n+1} + (1-\delta_{n+1})W_{n+1}u_{n+1} - (\delta_n u_n + (1-\delta_n)W_n u_n)\| \\ &\le |\delta_{n+1} - \delta_n|\|u_n\| + \delta_{n+1}\|u_{n+1} - u_n\| + (1-\delta_{n+1})\|W_{n+1}u_{n+1} - W_n u_n\| \\ &\quad + |\delta_{n+1} - \delta_n|\|W_n u_n\|.\end{aligned} \tag{3.13}$$

We will estimate $\|u_{n+1} - u_n\|$. From $u_{n+1} = T_{\lambda_{n+1}} x_{n+1}$ and $u_n = T_{\lambda_n} x_n$, we obtain

$$F(u_{n+1}, y) + \frac{1}{\lambda_{n+1}}\langle y - u_{n+1}, u_{n+1} - y_{n+1}\rangle \ge 0, \quad \forall y \in C, \tag{3.14}$$

$$F(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \quad \forall y \in C. \tag{3.15}$$

Taking $y = u_n$ in (3.14) and $y = u_{n+1}$ in (3.15), we have

$$\begin{aligned}F(u_{n+1}, u_n) + \frac{1}{\lambda_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle &\ge 0, \\ F(u_n, u_{n+1}) + \frac{1}{\lambda_n}\langle u_{n+1} - u_n, u_n - x_n\rangle &\ge 0.\end{aligned} \tag{3.16}$$

So, from (A2), one has

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_{\mathrm{n}}}{\lambda_n} - \frac{u_{n+1} - x_{n+1}}{\lambda_{n+1}}\right\rangle \ge 0, \tag{3.17}$$

furthermore,

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} - x_n - \frac{\lambda_n}{\lambda_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.18}$$

Since $\lim_{n\to\infty}\lambda_n > 0$, we assume that there exists a real number such that $\lambda_n > a > 0$ for all $n \in N$. Thus, we obtain

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}, \end{aligned} \tag{3.19}$$

which means

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{a}|\lambda_{n+1} - \lambda_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + L_1|\lambda_{n+1} - \lambda_n|, \end{aligned} \tag{3.20}$$

where $L_1 = \sup\{\|u_{n+1} - x_{n+1}\| : n \in N\}$.

Next, we estimate $\|W_{n+1}u_{n+1} - W_n u_n\|$. Notice that

$$\begin{aligned} \|W_{n+1}u_{n+1} - W_n u_n\| &= \|W_{n+1}u_{n+1} - W_{n+1}u_n + W_{n+1}u_n - W_n u_n\| \\ &\leq \|u_{n+1} - u_n\| + \|W_{n+1}u_n - W_n u_n\|. \end{aligned} \tag{3.21}$$

From (2.9), we obtain

$$\begin{aligned} \|W_{n+1}u_n - W_n u_n\| &= \left\|\tau_1 S_1' U_{n+1,2}u_n - \tau_1 S_1' U_{n,2}u_n\right\| \\ &\leq \tau_1 \|U_{n+1,2}u_n - U_{n,2}u_n\| \\ &= \tau_1 \left\|\tau_2 S_2' U_{n+1,3}u_n - \tau_2 S_2' U_{n,3}u_n\right\| \\ &\leq \tau_1\tau_2 \|U_{n+1,3}u_n - U_{n,3}u_n\| \\ &\leq \cdots \\ &\leq \tau_1\tau_2\cdots\tau_n \|U_{n+1,n+1}u_n - U_{n,n+1}u_n\| \\ &\leq L_2\prod_{i=1}^{n}\tau_i, \end{aligned} \tag{3.22}$$

where $L_2 \geq 0$ is a constant such that $\|U_{n+1,n+1}u_n - U_{n,n+1}u_n\| \leq L_2$, for all $n \in N$.

Substituting (3.20) and (3.22) into (3.21), we obtain

$$\|W_{n+1}u_{n+1} - W_n u_n\| \le \|x_{n+1} - x_n\| + L_1|\lambda_{n+1} - \lambda_n| + L_2\prod_{i=1}^{n}\tau_i. \tag{3.23}$$

Hence, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &\le |\delta_{n+1} - \delta_n|(\|u_n\| + \|W_n u_n\|) + \|x_{n+1} - x_n\| \\ &\quad + (1-\delta_{n+1})L_2\prod_{i=1}^{n}\tau_i + L_1|\lambda_{n+1} - \lambda_n| \\ &\le L_3|\delta_{n+1} - \delta_n| + \|x_{n+1} - x_n\| + (1-\delta_{n+1})L_2\prod_{i=1}^{n}\tau_i + L_1|\lambda_{n+1} - \lambda_n|,\end{aligned} \tag{3.24}$$

where $L_3 = \sup\{\|u_n\| + \|W_n u_n\| : n \in N\}$.

Furthermore,

$$\begin{aligned}\|z_{n+1} - z_n\| \le{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|rf(x_{n+1})\| + \|\mu B y_{n+1}\|) \\ &+ \frac{\alpha_n}{1-\beta_n}(\|rf(x_n)\| + \|\mu B y_n\|) \\ &+ \|x_{n+1} - x_n\| + L_1|\lambda_{n+1} - \lambda_n| + L_2(1-\delta_{n+1})\prod_{i=1}^{n}\tau_i \\ &+ L_3|\delta_{n+1} - \delta_n|.\end{aligned} \tag{3.25}$$

It follows from (3.25) that

$$\begin{aligned}&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|rf(x_{n+1})\| + \|\mu B y_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|rf(x_n)\| + \|\mu B y_n\|) \\ &\quad+ L_1|\lambda_{n+1} - \lambda_n| + L_2(1-\delta_{n+1})\prod_{i=1}^{n}\tau_i + L_3|\delta_{n+1} - \delta_n|.\end{aligned} \tag{3.26}$$

By the conditions (i), (iii), and (iv), we obtain

$$\lim_{n\to\infty}\sup(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.27}$$

Hence, by Lemma 2.5, one has

$$\lim_{n\to\infty}\|z_n - x_n\| = 0, \tag{3.28}$$

which implies

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\| = 0. \tag{3.29}$$

Step 3. We claim that $\lim_{n\to\infty}\|Wu_n-u_n\| = 0$.

Notice that

$$\begin{aligned}\|Wu_n-u_n\| &= \|Wu_n-W_nu_n+W_nu_n-u_n\| \\ &\le \|Wu_n-W_nu_n\|+\|W_nu_n-u_n\| \\ &\le \sup_{u\in K}\|Wu-W_nu\|+\|W_nu_n-u_n\|.\end{aligned} \tag{3.30}$$

It follows from (3.2) that

$$\begin{aligned}\|W_nu_n-u_n\| &= \|W_nu_n-y_n+y_n-u_n\| \\ &\le \|y_n-u_n\|+\|W_nu_n-y_n\| \\ &= \|y_n-u_n\|+\delta_n\|W_nu_n-u_n\| \\ &\le \|x_n-u_n\|+\|y_n-x_n\|+\delta_n\|W_nu_n-u_n\|.\end{aligned} \tag{3.31}$$

By the condition (iii), we obtain

$$\|W_nu_n-u_n\| \le \frac{1}{1-\delta_n}\left(\|x_n-u_n\|+\|y_n-x_n\|\right). \tag{3.32}$$

First, we show $\lim_{n\to\infty}\|x_n-u_n\| = 0$. From (3.2), for all $p\in\bigcap_{i=1}^{\infty}F(S_i)\cap\mathrm{EP}(F)$, applying Lemma 2.3 and noting that $\|\cdot\|$ is convex, we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n rf(x_n)+\beta_nx_n+((1-\beta_n)I-\mu\alpha_nB)y_n-p\|^2 \\ &= \|\alpha_n(rf(x_n)+\mu By_n)+\beta_n(x_n-p)+(1-\beta_n)(y_n-p)\|^2 \\ &\le \|\beta_n(x_n-p)+(1-\beta_n)(y_n-p)\|^2+2\alpha_n\langle rf(x_n)+\mu By_n, x_{n+1}-p\rangle \\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2+2\alpha_n\|rf(x_n)+\mu By_n\|\|x_{n+1}-p\| \\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|u_n-p\|^2+2\alpha_n\|rf(x_n)+\mu By_n\|\|x_{n+1}-p\|.\end{aligned} \tag{3.33}$$

Since $u_n = T_{\lambda_n}x_n$, $p = T_{\lambda_n}p$, we have

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{\lambda_n}x_n-T_{\lambda_n}\mathrm{p}\|^2 \le \langle x_n-p, u_n-p\rangle \\ &= \frac{1}{2}\left(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2\right),\end{aligned} \tag{3.34}$$

which implies

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.35}$$

Substituting (3.35) into (3.33), we have

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2 + 2\alpha_n \|rf(x_n) + \mu B y_n\| \|x_{n+1} - p\|, \tag{3.36}$$

which means

$$\begin{aligned}(1-\beta_n)\|x_n - u_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n \|rf(x_n) + \mu B y_n\| \|x_{n+1} - p\| \\ &\leq \|x_{n+1} - x_n\|\left(\|x_n - p\| + \|x_{n+1} - p\|\right) + 2\alpha_n \|rf(x_n) + \mu B y_n\| \|x_{n+1} - p\|.\end{aligned} \tag{3.37}$$

Noticing $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\inf(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.38}$$

Second, we show $\lim_{n\to\infty}\|y_n - x_n\| = 0$. It follows from (3.2) that

$$\begin{aligned}\|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &= \|\alpha_n r f(x_n) + \beta_n x_n + ((1-\beta_n)I - \mu\alpha_n B)y_n - y_n\| + \|x_{n+1} - x_n\| \\ &\leq \alpha_n \|rf(x_n) + \mu B y_n\| + \beta_n \|x_n - y_n\| + \|x_{n+1} - x_n\|.\end{aligned} \tag{3.39}$$

This implies that

$$(1-\beta_n)\|y_n - x_n\| \leq \alpha_n \|rf(x_n) + \mu B y_n\| + \|x_{n+1} - x_n\|. \tag{3.40}$$

Noticing $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\inf(1-\beta_n) > 0$ and (3.30), we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.41}$$

Thus, substituting (3.41) and (3.38) into (3.32), we obtain

$$\lim_{n\to\infty}\|W_n u_n - u_n\| = 0. \tag{3.42}$$

Furthermore, (3.42), (3.30), and Lemma 2.11 lead to

$$\lim_{n\to\infty}\|W u_n - u_n\| = 0. \tag{3.43}$$

Step 4. Letting $z = P_{F(W)\cap \mathrm{EP}(F)}(I - \mu B + rf)z$, we show

$$\lim_{n\to\infty} \sup\langle (rf - \mu B)z, x_n - z\rangle \leq 0. \tag{3.44}$$

We know that $P_{F(W)\cap \mathrm{EP}(F)}(I - \mu B + rf)$ is a contraction. Indeed, for any $x, y \in H$, we have

$$\begin{aligned} &\left\| P_{F(W)\cap \mathrm{EP}(F)}(I - \mu B + rf)x - P_{F(W)\cap \mathrm{EP}(F)}(I - \mu B + rf)y\right\| \\ &\quad\leq \left\|(I - \mu B + rf)x - (I - \mu B + rf)y\right\| \\ &\quad\leq (1 - \tau + r\beta)\|x - y\|, \end{aligned} \tag{3.45}$$

and hence $P_{F(W)\cap \mathrm{EP}(F)}(I - \mu B + rf)$ is a contraction due to $(1 - \tau + r\beta) \in (0,1)$. Thus, Banach's Contraction Mapping Principle guarantees that $P_{F(W)\cap \mathrm{EP}(F)}(I - \mu B + rf)$ has a unique fixed point, which implies $z = P_{F(W)\cap \mathrm{EP}(F)}(I - \mu B + rf)z$.

Since $\{u_{n_i}\} \subset \{u_n\}$ is bounded in C, without loss of generality, we can assume that $\{u_{n_i}\} \rightharpoonup \omega$, it follows from (3.43) that $Wu_{n_i} \rightharpoonup \omega$. Since C is closed and convex, C is weakly closed. Thus we have $\omega \in C$.

Let us show $\omega \in F(W)$. For the sake of contradiction, suppose that $\omega \notin F(W)$, that is, $W\omega \neq \omega$. Since $\{u_{n_i}\} \rightharpoonup \omega$, by the Opial condition, we have

$$\begin{aligned} \lim_{n\to\infty} \inf\|u_{n_i} - \omega\| &< \lim_{n\to\infty} \inf\|u_{n_i} - W\omega\| \\ &\leq \lim_{n\to\infty} \inf\{\|u_{n_i} - Wu_{n_i}\| + \|Wu_{n_i} - W\omega\|\} \\ &\leq \lim_{n\to\infty} \inf\{\|u_{n_i} - Wu_{n_i}\| + \|u_{n_i} - \omega\|\}. \end{aligned} \tag{3.46}$$

It follows (3.43) that

$$\lim_{n\to\infty} \inf\|u_{n_i} - \omega\| < \lim_{n\to\infty} \inf\|u_{n_i} - \omega\|. \tag{3.47}$$

This is a contradiction, which shows that $\omega \in F(W)$.

Next, we prove that $\omega \in \mathrm{EP}(F)$. By (3.2), we obtain

$$F(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0. \tag{3.48}$$

It follows from (A2) that

$$\frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n). \tag{3.49}$$

Replacing n by n_i, we have

$$\left\langle y - u_{n_i}, \frac{1}{\lambda_{n_i}}(u_{n_i} - x_{n_i})\right\rangle \geq F(y, u_{n_i}). \tag{3.50}$$

Since $(1/\lambda_{n_i})(u_{n_i}-x_{n_i}) \to 0$ and $\{u_{n_i}\} \rightharpoonup \omega$, it follows from (A4) that $F(y,\omega) \geq 0$ for all $y \in C$. Put $z_t = ty + (1-t)\omega$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$ and then $F(z_t,\omega) \geq 0$. Hence, from (A1) and (A4), we have

$$0 = F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, y) \leq tF(z_t, y), \tag{3.51}$$

which means $F(z_t, y) \geq 0$. From (A3), we obtain $F(\omega, y) \geq 0$ for $y \in C$ and then $\omega \in \mathrm{EP}(F)$. Therefore, $\omega \in F(W) \cap \mathrm{EP}(F)$.

Since $z = P_{F(W)\cap \mathrm{EP}(F)}(I - \mu B + rf)z$, it follows from (3.38), (3.42), and Lemma 2.11 that

$$\begin{aligned}
\limsup_{i\to\infty}\langle (rf-\mu B)z, x_n - z\rangle &\leq \lim_{n\to\infty}\langle (rf-\mu B)z, x_{n_i} - z\rangle \\
&= \lim_{i\to\infty}\langle (rf-\mu B)z, x_{n_i} - u_{n_i}\rangle \\
&\quad + \lim_{i\to\infty}\langle (rf-\mu B)z, u_{n_i} - W_{n_i}u_{n_i}\rangle \\
&\quad + \lim_{i\to\infty}\langle (rf-\mu B)z, W_{n_i}u_{n_i} - Wu_{n_i}\rangle \\
&\quad + \lim_{i\to\infty}\langle (rf-\mu B)z, Wu_{n_i} - z\rangle \\
&= \langle (rf-\mu B)z, \omega - z\rangle \leq 0.
\end{aligned} \tag{3.52}$$

Step 5. Finally we prove that $x_n \to \omega$ as $n \to \infty$. In fact, from (3.2) and (3.7), we obtain

$$\begin{aligned}
\|x_{n+1}-\omega\|^2 &= \left\|\alpha_n rf(x_n) + \beta_n x_n + ((1-\beta_n)I - \mu\alpha_n B)y_n - \omega\right\|^2 \\
&= \big\|\alpha_n r(f(x_n) - f(\omega)) + \alpha_n(rf(\omega) - \mu B\omega) \\
&\qquad + \beta_n(x_n-\omega) + ((1-\beta_n)I - \mu\alpha_n B)(y_n-\omega)\big\|^2 \\
&= \alpha_n r\langle f(x_n) - f(\omega), x_{n+1}-\omega\rangle + \alpha_n\langle rf(\omega) - \mu B\omega, x_{n+1}-\omega\rangle \\
&\quad + \beta_n\langle x_n - \omega, x_{n+1}-\omega\rangle + \langle ((1-\beta_n)I - \mu\alpha_n B)(y_n-\omega), x_{n+1}-\omega\rangle \\
&\leq \alpha_n r\beta\frac{\|x_n-\omega\|^2 + \|x_{n+1}-\omega\|^2}{2} + \alpha_n\langle rf(\omega) - \mu B\omega, x_{n+1}-\omega\rangle \\
&\quad + \beta_n\frac{\|x_n-\omega\|^2 + \|x_{n+1}-\omega\|^2}{2} + (1-\beta_n-\alpha_n\tau)\frac{\|y_n-\omega\|^2 + \|x_{n+1}-\omega\|^2}{2} \\
&\leq \frac{1-\alpha_n(\tau - r\beta)}{2}\left(\|x_n-\omega\|^2 + \|x_{n+1}-\omega\|^2\right) + \alpha_n\langle rf(\omega) - \mu B\omega, x_{n+1}-\omega\rangle,
\end{aligned} \tag{3.53}$$

which implies

$$\begin{aligned}\|x_{n+1}-\omega\|^2 &\leq \frac{1-\alpha_n(\tau-r\beta)}{1+\alpha_n(\tau-r\beta)}\|x_n-\omega\|^2 \\ &\quad+\frac{2\alpha_n(\tau-r\beta)}{(1+\alpha_n(\tau-r\beta))(\tau-r\beta)}\langle rf(\omega)-\mu B\omega, x_{n+1}-\omega\rangle \\ &\leq (1-\alpha_n(\tau-r\beta))\|x_n-\omega\|^2 \\ &\quad+\frac{2\alpha_n(\tau-r\beta)}{(1+\alpha_n(\tau-r\beta))(\tau-r\beta)}\langle rf(\omega)-\mu B\omega, x_{n+1}-\omega\rangle.\end{aligned} \tag{3.54}$$

From condition (i) and (3.7), we know that $\sum_{i=1}^{n}\alpha_n(\tau-r\beta)=\infty$ and $\lim_{i\to\infty}\sup(2/(1+\alpha_n(\tau-r\beta))(\tau-r\beta))\langle rf(\omega)-\mu B\omega, x_{n+1}-\omega\rangle \leq 0$. we can conclude from Lemma 2.6 that $x_n \to \omega$ as $n\to\infty$. This completes the proof of Theorem 3.1. □

Remark 3.2. If $r=1$, $\mu=1$, $B=I$ and $\delta_i=0$, $k_i=0$, $\sigma_i=0$ for $i\in N$, then Theorem 3.1 reduces to Theorem 3.5 of Yao et al. [19]. Furthermore, we extend the corresponding results of Yao et al. [19] from one infinite family of nonexpansive mapping to an infinite family of strictly pseudo-contractive mappings.

Remark 3.3. If $\mu=1$ and $\delta_i=0$, $k_i=0$, $\sigma_i=0$ for $i\in N$, then Theorem 3.1 reduces to Theorem 10 of Colao and Marino [20]. Furthermore, we extend the corresponding results of Colao and Marino [20] from one infinite family of nonexpansive mapping to an infinite family of strictly pseudo-contractive mappings, and from a strongly positive bounded linear operator A to a k-Lipschitzian and η-strongly monotone operator B.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and F be a bifunction from $C\times C \to R$ satisfying (A1)–(A4). Let $S: C\to C$ be a nonexpansive mapping with $F(S)\cap EP\neq\emptyset$. Let f be a contraction of H into itself with $\beta\in(0,1)$ and B be k-Lipschitzian and η-strongly monotone operator on H with coefficients $k,\eta>0$, $0<\mu<2\eta/k^2$, $0<r<(1/2)\mu(2\eta-\mu k^2)/\beta=\tau/\beta$ and $\tau<1$. Let $\{x_n\}$ be sequence generated by*

$$\begin{gathered}F(u_n,y)+\frac{1}{\lambda_n}\langle y-u_n, u_n-x_n\rangle\geq 0,\quad \forall y\in C,\\ y_n=\delta_n u_n+(1-\delta_n)S_n u_n,\\ x_{n+1}=\alpha_n r f(x_n)+\beta_n x_n+((1-\beta_n)I-\mu\alpha_n B)y_n,\quad \forall n\in N,\end{gathered} \tag{3.55}$$

where $u_n=T_{\lambda_n}x_n$. If $\{\alpha_n\},\{\beta_n\},\{\delta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\}\subset(0,1)$, $\lim_{n\to\infty}\alpha_n=0$, $\sum_{i=1}^{\infty}\alpha_n=\infty$,

(ii) $0<\lim_{n\to\infty}\inf\beta_n\leq\lim_{n\to\infty}\sup\beta_n<1$,

(iii) $0<\lim_{n\to\infty}\inf\delta_n\leq\lim_{n\to\infty}\sup\delta_n<1$, $\lim_{n\to\infty}|\delta_{n+1}-\delta_n|=0$,

(iv) $\{\lambda_n\}\subset(0,\infty)$, $\lim_{n\to\infty}\lambda_n>0$, $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n|=0$.

Then $\{x_n\}$ *converges strongly to* $z \in F(S) \cap EP \neq \emptyset$, *where* z *is the unique solution of variational inequality*

$$\lim_{n\to\infty} \sup\langle (rf - \mu B)z, p - z\rangle \leq 0, \quad \forall p \in F(S) \cap EP \neq \emptyset, \tag{3.56}$$

that is, $z = P_{F(S)\cap EP(F)}(I - \mu B + rf)z$.

Proof. By Theorem 3.1, letting $k_i = 0$, $\sigma_i = 0$, $\tau_i = 1$ and $S_i = S$ for $i \in N$, we can obtain Theorem 3.4. □

4. Numerical Example

Now, we present a numerical example to illustrate our theoretical analysis results obtained in Section 3.

Example 4.1. Let $H = R$, $C = [-1,1]$, $S_n = I$, $\tau_n = \tau \in (0,1)$, $\lambda_n = 1$, $n \in N$, $F(x,y) = 0$, for all $x, y \in C$, $B = I$, $r = \mu = 1$, $f(x) = (1/10)x$, for all x, with contraction coefficient $\beta = 1/5$, $\delta_n = 1/2$, $\alpha_n = 1/n$, $\beta_n = 1/4 + 1/2n$ for every $n \in N$. Then $\{x_n\}$ is the sequence generated by

$$x_{n+1} = \left(1 - \frac{9}{10n}\right)x_n, \tag{4.1}$$

and $\{x_n\} \to 0$, as $n \to \infty$, where 0 is the unique solution of the minimization problem

$$\min_{x\in C} \frac{9}{20}x^2 + c. \tag{4.2}$$

Proof. We divide the proof into four steps.
Step 1. We show

$$T_{\lambda_n}x = P_C x, \quad \forall x \in H, \tag{4.3}$$

where

$$P_C x = \begin{cases} \dfrac{x}{|x|}, & x \notin C, \\ x, & x \in C. \end{cases} \tag{4.4}$$

Since $F(x,y) = 0$, for all $x, y \in C$, due to the definition of $T_{\lambda_n}(x)$, for all $x \in H$, by Lemma 2.1, we obtain

$$T_{\lambda_n}x = \{z \in C : \langle y - z, z - x\rangle \geq 0, \forall y \in C\}. \tag{4.5}$$

By the property of P_C, for $x \in C$, we have $T_{\lambda_n}x = P_C x = Ix$. Furthermore, it follows from (3) in Lemma 2.1 that

$$\mathrm{EP}(F) = C. \tag{4.6}$$

Step 2. We show that

$$W_n = I. \tag{4.7}$$

It follows from (2.9) that

$$\begin{aligned} W_1 &= U_{1,1} = \tau_1 S_1' U_{1,2} + (1-\tau_1)I = \tau_1 S_1' + (1-\tau_1)I, \\ W_2 &= U_{2,1} = \tau_1 S_1' U_{2,2} + (1-\tau_1)I \\ &= \tau_1 S_1' \left(\tau_2 S_2' U_{2,3} + (1-\tau_2)I\right) + (1-\tau_1)I \\ &= \tau_1\tau_2 S_1' S_2' + \tau_1(1-\tau_2)S_1' + (1-\tau_1)I, \\ W_3 &= U_{3,1} = \tau_1 S_1' U_{3,2} + (1-\tau_1)I \\ &= \tau_1 S_1' \left(\tau_2 S_2' U_{3,3} + (1-\tau_2)I\right) + (1-\tau_1)I \\ &= \tau_1\tau_2 S_1' S_2' U_{3,3} + \tau_1(1-\tau_2)S_1' + (1-\tau_1)I \\ &= \tau_1\tau_2 S_1' S_2' \left(\tau_3 S_3' U_{3,4} + (1-\tau_3)I\right) + \tau_1(1-\tau_2)S_1' + (1-\tau_1)I \\ &= \tau_1\tau_2\tau_3 S_1' S_2' S_3' + \tau_1\tau_2(1-\tau_3)S_1' S_2' + \tau_1(1-\tau_2)S_1' + (1-\tau_1)I. \end{aligned} \tag{4.8}$$

Furthermore, we obtain

$$\begin{aligned} W_n = U_{n,1} &= \tau_1\tau_2\tau_3 \cdots \tau_n S_1' S_2' S_3' \cdots S_n' + \tau_1\tau_2 \cdots \tau_{n-1}(1-\tau_n)S_1' S_2' \cdots S_{n-1}' \\ &\quad + \tau_1\tau_2 \cdots \tau_{n-2}(1-\tau_{n-1})S_1' S_2' \cdots S_{n-2}' + \cdots + \tau_1(1-\tau_2)S_1' + (1-\tau_1)I. \end{aligned} \tag{4.9}$$

Since $S_i' = I$, $\tau_i = \tau$ for $i \in N$, one has

$$W_n = \left[\tau^n + \tau^{n-1}(1-\tau) + \cdots + \tau(1-\tau) + (1-\tau)\right]I = I. \tag{4.10}$$

Step 3. We show that

$$x_{n+1} = \left(1 - \frac{9}{10n}\right)x_n, \tag{4.11}$$

$\{x_n\} \to 0$, as $n \to \infty$, where 0 is the unique solution of the minimization problem

$$\min_{x\in C} \frac{9}{20}x^2 + c. \tag{4.12}$$

Table 1: This table shows the value of sequence $\{x_n\}$ on each iteration step (initial value $x_1 = 0.2$).

n	x_n	n	x_n
1	0.2000	17	0.0017
2	0.0200	18	0.0016
3	0.0110	19	0.0016
4	0.0077	20	0.0015
5	0.0060	21	0.0014
⋮	⋮	⋮	⋮
9	0.0032	26	0.0012
10	0.0029	27	0.0011
⋮	⋮	⋮	⋮
14	0.0021	30	0.0010
15	0.0019	31	0.0009
16	0.0018	32	0.0009

In fact, we can see that $B = I$ is k-Lipschitzian and η-strongly monotone operator on H with coefficient $k = 1$, $\eta = 3/4$ such that $0 < \mu < 2\eta/k^2$, $0 < r < (1/2)\mu(2\eta - \mu k^2)/\beta = \tau/\beta$, so we take $r = \mu = 1$. Since $S'_n = I$, $n \in N$, we have

$$\bigcap_{i=1}^{\infty} F(S_i) = H. \tag{4.13}$$

Furthermore, we obtain

$$\bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{EP}(F) = C = [-1, 1]. \tag{4.14}$$

Next, we need prove $\{x_n\} \to 0$, as $n \to \infty$. Since $y_n = u_n$ for all $n \in N$, we have

$$\begin{aligned} x_{n+1} &= \alpha_n r f(x_n) + \beta_n x_n + ((1-\beta_n)I - \mu\alpha_n B)y_n \\ &= \left(1 - \frac{9}{10n}\right)x_n, \end{aligned} \tag{4.15}$$

for all $n \in N$.

Thus, we obtain a special sequence $\{x_n\}$ of (3.2) in Theorem 3.1 as follows

$$x_{n+1} = \left(1 - \frac{9}{10n}\right)x_n. \tag{4.16}$$

By Lemma 2.6, it is obviously that $x_n \to 0$, 0 is the unique solution of the minimization problem

$$\min_{x \in C} \frac{9}{20}x^2 + c, \tag{4.17}$$

where c is a constant number.

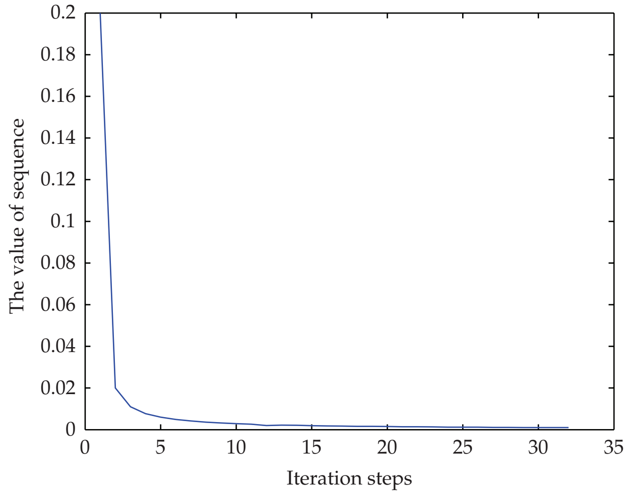

Figure 1: The corresponding graph at $x = 0.2$.

Step 4. Finally, we use software Matlab 7.0 to give the numerical experiment results and then obtain Table 1 which show that the iteration process of the sequence $\{x_n\}$ is a monotonedecreasing sequence and converges to 0. From Table 1 and the corresponding graph Figure 1, we show that the more the iteration steps are, the more slowly the sequence $\{x_n\}$ converges to 0. □

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