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Research Article

Projection Algorithms for Variational Inclusions

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We present a projection algorithm for finding a solution of a variational inclusion problem in a real Hilbert space. Furthermore, we prove that the proposed iterative algorithm converges strongly to a solution of the variational inclusion problem which also solves some variational inequality.

1. Introduction

Let H be a real Hilbert space. Let $B : H \to H$ be a single-valued nonlinear mapping and $R : H \to 2^H$ be a set-valued mapping. Now we concern the following variational inclusion, which is to find a point $x \in H$ such that

$$\theta \in B(x) + R(x), \tag{1.1}$$

where θ is the zero vector in H. The set of solutions of problem (1.1) is denoted by $I(B, R)$. If $H = R^m$, then problem (1.1) becomes the generalized equation introduced by Robinson [1]. If $B = 0$, then problem (1.1) becomes the inclusion problem introduced by Rockafellar [2]. It is known that (1.1) provides a convenient framework for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, game theory, and so forth. Also various types of variational inclusions problems have been extended and generalized. Recently, Zhang et al. [3] introduced a new iterative scheme for finding a common element of the set of solutions to the problem (1.1) and the set of fixed points of nonexpansive mappings in Hilbert spaces. Peng et al. [4] introduced another iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping. For some related works, see [5–28] and the references therein.

Inspired and motivated by the works in the literature, in this paper, we present a projection algorithm for finding a solution of a variational inclusion problem in a real Hilbert space. Furthermore, we prove that the proposed iterative algorithm converges strongly to a solution of the variational inclusion problem which also solves some variational inequality.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Recall that a mapping $B : C \to C$ is said to be α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that $\langle Bx-By, x-y\rangle \geq \alpha\|Bx-By\|^2$, for all $x, y \in C$. A mapping A is strongly positive on H if there exists a constant $\mu > 0$ such that $\langle Ax, x\rangle \geq \mu\|x\|^2$ for all $x \in H$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \text{for all } y \in C. \tag{2.1}$$

Such a P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \Longleftrightarrow \langle x - x^*, x^* - y\rangle \geq 0 \quad \text{for all } y \in C. \tag{2.2}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let the set-valued mapping $R : H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}(x), \quad x \in H, \tag{2.3}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{R,\lambda}$ is single-valued, nonexpansive, and 1-inverse strongly monotone, and that a solution of problem (1.1) is a fixed point of the operator $J_{R,\lambda}(I - \lambda B)$ for all $\lambda > 0$, see for instance [29].

Lemma 2.1 (see [30]). *Let $R : H \to 2^H$ be a maximal monotone mapping and $B : H \to H$ be a Lipschitz-continuous mapping. Then the mapping $(R + B) : H \to 2^H$ is maximal monotone.*

Lemma 2.2 (see [8]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.3 (see [31]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that* $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n$, *where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Result

In this section, we will prove our main result. First, we give some assumptions on the operators and the parameters. Subsequently, we introduce our iterative algorithm for finding solutions of the variational inclusion (1.1). Finally, we will show that the proposed algorithm has strong convergence.

In the sequel, we will assume that

(A1) C is a nonempty closed convex subset of a real Hilbert space H;

(A2) A is a strongly positive bounded linear operator with coefficient $0 < \mu < 1$, $R : H \to 2^H$ is a maximal monotone mapping and $B : C \to C$ is an α-inverse strongly monotone mapping;

(A3) $\lambda > 0$ is a constant satisfying $\lambda < 2\alpha$.

Now we introduce the following iteration algorithm.

Algorithm 3.1. For given $x_0 \in C$ arbitrarily, compute the sequence $\{x_n\}$ as follows:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[(I-\alpha_n A)J_{R,\lambda}(I-\lambda B)x_n], \quad n \geq 0, \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$.

Now we study the strong convergence of the algorithm (3.1)

Theorem 3.2. *Suppose that* $I(B,R) \neq \emptyset$. *Assume the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $\tilde{x} \in I(B,R)$ *which solves the following variational inequality:*

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in I(B,R). \tag{3.2}$$

Proof. Take $x^* \in I(B,R)$. It is clear that

$$J_{R,\lambda}(x^* - \lambda B x^*) = x^*. \tag{3.3}$$

We divide our proofs into the following five steps:

(1) the sequence $\{x_n\}$ is bounded.

(2) $\|x_{n+1} - x_n\| \to 0$.

(3) $\|Bx_n - Bx^*\| \to 0$.

(4) $\limsup_{n\to\infty}\langle A\tilde{x}, x_n - \tilde{x}\rangle \geq 0$ where $\tilde{x} = P_{I(B,R)}(I - A)(\tilde{x})$.

(5) $x_n \to \tilde{x}$.

□

Proof of (1.1). Since B is α-inverse strongly monotone, we have

$$\left\|(I - \lambda B)x - (I - \lambda B)y\right\|^2 \leq \left\|x - y\right\|^2 + \lambda(\lambda - 2\alpha)\left\|Bx - By\right\|^2. \tag{3.4}$$

It is clear that if $0 \leq \lambda \leq 2\alpha$, then $(I - \lambda B)$ is nonexpansive. Set $y_n = J_{R,\lambda}(x_n - \lambda Bx_n), n \geq 0$. It follows that

$$\begin{aligned} \left\|y_n - x^*\right\| &= \|J_{R,\lambda}(x_n - \lambda Bx_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\| \\ &\leq \|(x_n - \lambda Bx_n) - (x^* - \lambda Bx^*)\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.5}$$

Since A is linear bounded self-adjoint operator on H, then

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}. \tag{3.6}$$

Observe that

$$\begin{aligned} \langle (I - \alpha_n A)u, u\rangle &= 1 - \alpha_n\langle Au, u\rangle \\ &\geq 1 - \alpha_n\|A\| \\ &\geq 0, \end{aligned} \tag{3.7}$$

that is to say $I - \alpha_n A$ is positive. It follows that

$$\begin{aligned} \|(I - \alpha_n A)\| &= \sup\{\langle (I - \alpha_n A)u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \alpha_n\langle Au, u\rangle : u \in H, \|u\| = 1\} \\ &\leq 1 - \alpha_n\mu. \end{aligned} \tag{3.8}$$

From (3.1), we deduce that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|(1-\beta_n)x_n + \beta_n P_C\left[(I-\alpha_n A)y_n\right] - x^*\right\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\left\|\left[(I-\alpha_n A)(y_n - x^*)\right] - \alpha_n A x^*\right\| \\
&\le (1-\beta_n)\|x_n - x^*\| + (1-\alpha_n\mu)\beta_n\left\|y_n - x^*\right\| + \alpha_n\beta_n\|Ax^*\| \\
&\le (1-\alpha_n\beta_n\mu)\|x_n - x^*\| + \alpha_n\beta_n\|Ax^*\| \\
&\le \max\left\{\|x_0 - x^*\|, \frac{\|Ax^*\|}{\mu}\right\}.
\end{aligned} \tag{3.9}$$

Therefore, $\{x_n\}$ is bounded. □

Proof of (3.1). Set $z_n = P_C[(I-\alpha_n A)J_{R,\lambda}(I-\lambda B)x_n]$ for all $n \ge 0$. Then, we have

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \left\|P_C\left[(I-\alpha_n A)y_n\right] - P_C\left[(I-\alpha_{n-1}A)y_{n-1}\right]\right\| \\
&\le \left\|\left[(I-\alpha_n A)y_n\right] - \left[(I-\alpha_{n-1}A)y_{n-1}\right]\right\| \\
&= \left\|(I-\alpha_n A)(y_n - y_{n-1}) + (\alpha_{n-1} - \alpha_n)Ay_{n-1}\right\| \\
&\le \left\|(I-\alpha_n A)(y_n - y_{n-1})\right\| + \left\|(\alpha_{n-1} - \alpha_n)Ay_{n-1}\right\| \\
&\le (1-\alpha_n\mu)\left\|y_n - y_{n-1}\right\| + |\alpha_n - \alpha_{n-1}|\left\|Ay_{n-1}\right\|.
\end{aligned} \tag{3.10}$$

Note that

$$\begin{aligned}
\left\|y_n - y_{n-1}\right\| &= \|J_{R,\lambda}(x_n - \lambda Bx_n) - J_{R,\lambda}(x_{n-1} - \lambda Bx_{n-1})\| \\
&\le \|(x_n - \lambda Bx_n) - (x_{n-1} - \lambda Bx_{n-1})\| \\
&\le \|x_n - x_{n-1}\|.
\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), we get

$$\|z_n - z_{n-1}\| \le (1-\alpha_n\mu)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left\|Ay_{n-1}\right\|. \tag{3.12}$$

Therefore,

$$\limsup_{n\to\infty}(\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|) \le 0. \tag{3.13}$$

This together with Lemma 2.2 imply that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.14}$$

Hence,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\beta_n\|z_n-x_n\| = 0. \tag{3.15}$$

□

Proof of (3.4). From (3.4), we get

$$\begin{aligned}\left\|y_n-x^*\right\|^2 &= \|J_{R,\lambda}(x_n-\lambda Bx_n)-J_{R,\lambda}(x^*-\lambda Bx^*)\|^2\\ &\le \|(x_n-\lambda Bx_n)-(x^*-\lambda Bx^*)\|^2\\ &\le \|x_n-x^*\|^2+\lambda(\lambda-2\alpha)\|Bx_n-Bx^*\|^2.\end{aligned} \tag{3.16}$$

By (3.1), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\left\|P_C\left[(I-\alpha_nA)y_n\right]-x^*\right\|^2\\ &\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\left\|(I-\alpha_nA)y_n-x^*\right\|^2\\ &= (1-\beta_n)\|x_n-x^*\|^2+\beta_n\left(\left\|y_n-x^*-\alpha_nAy_n\right\|^2\right)\\ &= (1-\beta_n)\|x_n-x^*\|^2+\beta_n\left(\left\|y_n-x^*\right\|^2-2\alpha_n\langle y_n-x^*,Ay_n\rangle+\alpha_n^2\left\|Ay_n\right\|^2\right)\\ &\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\left(\left\|y_n-x^*\right\|^2+\alpha_n\left(2\left\|y_n-x^*\right\|\left\|Ay_n\right\|+\left\|Ay_n\right\|^2\right)\right)\\ &\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\left(\left\|y_n-x^*\right\|^2+\alpha_nM\right),\end{aligned} \tag{3.17}$$

where $M>0$ is some constant satisfying $\sup_n\{2\|y_n-x^*\|\|Ay_n\|+\|Ay_n\|^2\}\le M$. From (3.16) and (3.17), we have

$$\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2+\lambda(\lambda-2\alpha)\beta_n\|Bx_n-Bx^*\|^2+\alpha_nM. \tag{3.18}$$

Thus,

$$\begin{aligned}\lambda(2\alpha-\lambda)\beta_n\|Bx_n-Bx^*\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM\\ &\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|+\alpha_nM,\end{aligned} \tag{3.19}$$

which imply that

$$\lim_{n\to\infty}\|Bx_n-Bx^*\| = 0. \tag{3.20}$$

□

Proof of (3.10). Since $J_{R,\lambda}$ is 1-inverse strongly monotone, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|J_{R,\lambda}(x_n - \lambda Bx_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2 \\
&\le \langle x_n - \lambda Bx_n - (x^* - \lambda Bx^*), y_n - x^* \rangle \\
&= \frac{1}{2}\Big(\|x_n - \lambda Bx_n - (x^* - \lambda Bx^*)\|^2 + \|y_n - x^*\|^2 \\
&\qquad - \|x_n - \lambda Bx_n - (x^* - \lambda Bx^*) - (y_n - x^*)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|y_n - x^*\|^2 - \|x_n - y_n - \lambda(Bx_n - Bx^*)\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|y_n - x^*\|^2 - \|x_n - y_n\|^2 \\
&\qquad + 2\lambda\langle Bx_n - Bx^*, x_n - y_n\rangle - \lambda^2\|Bx_n - Bx^*\|^2\Big),
\end{aligned} \tag{3.21}$$

which implies that

$$\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + 2\lambda\|Bx_n - Bx^*\|\|x_n - y_n\|. \tag{3.22}$$

Substitute (3.22) into (3.17) to get

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - \beta_n\|x_n - y_n\|^2 + 2\lambda\|Bx_n - Bx^*\|\|x_n - y_n\| + \alpha_n M. \tag{3.23}$$

Then we derive

$$\begin{aligned}
\beta_n\|x_n - y_n\|^2 &\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&\quad + 2\lambda\|Bx_n - Bx^*\|\|x_n - y_n\| + \alpha_n M.
\end{aligned} \tag{3.24}$$

So, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.25}$$

We note that $P_{I(B,R)}(I - A)$ is a contraction. As a matter of fact,

$$\begin{aligned}
\|P_{I(B,R)}(I - A)x - P_{I(B,R)}(I - A)y\| &\le \|(I - A)x - (I - A)y\| \\
&\le \|I - A\|\|x - y\| \\
&\le (1 - \mu)\|x - y\|,
\end{aligned} \tag{3.26}$$

for all $x, y \in H$. Hence $P_{I(B,R)}(I - A)$ has a unique fixed point, say $\tilde{x} \in I(B,R)$. That is $\tilde{x} = P_{I(B,R)}(I - A)(\tilde{x})$. This implies that $\langle A\tilde{x}, y - \tilde{x} \rangle \geq 0$ for all $y \in I(B,R)$. Next, we prove that

$$\limsup_{n \to \infty} \langle A\tilde{x}, x_n - \tilde{x} \rangle \geq 0. \tag{3.27}$$

First, we note that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \to \infty} \langle A\tilde{x}, x_n - \tilde{x} \rangle = \lim_{j \to \infty} \left\langle A\tilde{x}, x_{n_j} - \tilde{x} \right\rangle. \tag{3.28}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{ji}}\}$ of $\{x_{n_j}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup w$.

Next, we show that $w \in I(B,R)$. In fact, since B is α-inverse strongly monotone, B is Lipschitz-continuous monotone mapping. It follows from Lemma 2.1 that $R + B$ is maximal monotone. Let $(v, g) \in G(R + B)$, that is, $g - Bv \in R(v)$. Again since $y_{n_i} = J_{R,\lambda}(x_{n_i} - \lambda B x_{n-i})$, we have $x_{n_i} - \lambda B x_{n_i} \in (I + \lambda R)(y_{n_i})$, that is, $(1/\lambda)(x_{n_i} - y_{n_i} - \lambda B x_{n_i}) \in R(y_{n_i})$. By virtue of the maximal monotonicity of $R + B$, we have

$$\left\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(x_{n_i} - y_{n_i} - \lambda B x_{n_i}) \right\rangle \geq 0, \tag{3.29}$$

and so

$$\begin{aligned} \langle v - y_{n_i}, g \rangle &\geq \left\langle v - y_{n_i}, Bv + \frac{1}{\lambda}(x_{n_i} - y_{n_i} - \lambda B x_{n_i}) \right\rangle \\ &= \left\langle v - y_{n_i}, Bv - By_{n_i} + By_{n_i} - Bx_{n_i} + \frac{1}{\lambda}(x_{n_i} - y_{n_i}) \right\rangle \\ &\geq \langle v - y_{n_i}, By_{n_i} - Bx_{n_i} \rangle + \left\langle v - y_{n_i}, \frac{1}{\lambda}(x_{n_i} - y_{n_i}) \right\rangle. \end{aligned} \tag{3.30}$$

It follows from $\|x_n - y_n\| \to 0$, $\|Bx_n - By_n\| \to 0$ and $y_{n_i} \rightharpoonup w$ that

$$\lim_{n_i \to \infty} \langle v - y_{n_i}, g \rangle = \langle v - w, g \rangle \geq 0. \tag{3.31}$$

It follows from the maximal monotonicity of $B + R$ that $\theta \in (R + B)(w)$, that is, $w \in I(B,R)$. Therefore, $w \in I(B,R)$. It follows that

$$\begin{aligned} \limsup_{n \to \infty} \langle A\tilde{x}, x_n - \tilde{x} \rangle &= \lim_{j \to \infty} \left\langle A\tilde{x}, x_{n_j} - \tilde{x} \right\rangle \\ &= \langle A\tilde{x}, w - \tilde{x} \rangle \\ &\geq 0. \end{aligned} \tag{3.32}$$

□

Proof of (3.11). First, we note that $z_n = P_C[(I-\alpha_n A)y_n]$, then for all $x \in C$, we have $\langle z_n - (I - \alpha_n A)y_n, z_n - x\rangle \le 0$. Thus,

$$
\begin{aligned}
\|z_n - \tilde{x}\|^2 &= \langle z_n - \tilde{x}, z_n - \tilde{x}\rangle \\
&= \langle z_n - (I-\alpha_n A)y_n + (I-\alpha_n A)y_n - \tilde{x}, z_n - \tilde{x}\rangle \\
&= \langle z_n - (I-\alpha_n A)y_n, z_n - \tilde{x}\rangle + \langle (I-\alpha_n A)y_n - \tilde{x}, z_n - \tilde{x}\rangle \\
&\le \langle (I-\alpha_n A)(y_n - \tilde{x}) - \alpha_n A\tilde{x}, z_n - \tilde{x}\rangle \\
&= \langle (I-\alpha_n A)(y_n - \tilde{x}), z_n - \tilde{x}\rangle + \alpha_n\langle -A\tilde{x}, z_n - \tilde{x}\rangle \\
&\le \left\|(I-\alpha_n A)(y_n - \tilde{x})\right\| \|z_n - \tilde{x}\| + \alpha_n\langle -A\tilde{x}, z_n - \tilde{x}\rangle \\
&\le \frac{(1-\alpha_n\mu)}{2}\left(\|x_n - \tilde{x}\|^2 + \|z_n - \tilde{x}\|^2\right) + \alpha_n\langle -A\tilde{x}, z_n - \tilde{x}\rangle,
\end{aligned}
\tag{3.33}
$$

that is,

$$
\|z_n - \tilde{x}\|^2 \le (1-\alpha_n\mu)\|x_n - \tilde{x}\|^2 + \frac{\alpha_n}{1+\alpha_n\mu}\langle -A\tilde{x}, z_n - \tilde{x}\rangle. \tag{3.34}
$$

So,

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 \\
&\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n(1-\alpha_n\mu)\|x_n - \tilde{x}\|^2 + \frac{\alpha_n\beta_n}{1+\alpha_n\mu}\langle -A\tilde{x}, z_n - \tilde{x}\rangle \\
&= (1-\alpha_n\beta_n\mu)\|x_n - x^*\|^2 + \frac{\alpha_n\beta_n}{1+\alpha_n\mu}\langle -A\tilde{x}, z_n - \tilde{x}\rangle \\
&= (1-\delta_n)\|x_n - x^*\|^2 + \delta_n\sigma_n,
\end{aligned}
\tag{3.35}
$$

where $\delta_n = \alpha_n\beta_n\mu$ and $\sigma_n = (1/(1+\alpha_n\mu)\mu)\langle -A\tilde{x}, z_n - \tilde{x}\rangle$. It is easy to see that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. Hence, by Lemma 2.3, we conclude that the sequence $\{x_n\}$ converges strongly to $\tilde{x}$. This completes the proof. □

4. Conclusion

The results proved in this paper may be extended for multivalued variational inclusions and related optimization problems.

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