THE DUAL INTEGRAL EQUATION METHOD IN HYDROMECHANICAL SYSTEMS

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Some hydromechanical systems are investigated by applying the dual integral equation method. In developing this method we suggest from elementary appropriate solutions of Laplace's equation, in the domain under consideration, the introduction of a potential function which provides useful combinations in cylindrical and spherical coordinates systems. Since the mixed boundary conditions and the form of the potential function are quite general, we obtain integral equations with mth-order Hankel kernels. We then discuss a kind of approximate practicable solutions. We note also that the method has important applications in situations which arise in the determination of the temperature distribution in steady-state heat-conduction problems.

1. Introduction

In many circumstances the determination of the impulsive response of a fluid is of particular interest, that is, the determination of the jump of the velocity field, due to an impulsive pressure distribution acting on a part of the boundary surface of the fluid, or due to an impact excitation of some part of its rigid boundary. Since the acceleration of the boundaries and of the fluid particles takes on very large values over a short duration, it is natural to study these problems by means of the impulsive form of the equations of motion (see [1, page 471] or [8, page 91]), derived by integrating the usual equations over small time interval during which the impulsive forces are exerted. In many cases the effect of the compressibility and the viscous resistance on the impulsive response of the fluid can be neglected (see [3, page 272], [7, page 34], and [8, page 92]). Thus, the model of an ideal and incompressible liquid may be used for the study of the impulsive response of a fluid, regardless of the specific nature of the latter. This is not true as regards the evolution of the system after the initial impulsive excitation, where compressibility and viscosity may seriously affect the fluid motion.

In the present work, we will consider some impulsive problems for the hydromechanical system consisting of a fluid layer horizontally extending at infinity and a sphere totally

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submerged in the fluid. These problems are

(i) the impulsive response of the fluid-sphere system due to an underground explosion beneath the submerged sphere;
(ii) the impulsive response of the fluid due to impulsive expansion of the sphere;
(iii) the impulsive response of the fluid-sphere system due to an impulsive pressure acting on the free surface of the fluid layer.

It is also noted that steady-state heat-conduction and electrostatic interpretations of the solved boundary value problems are plausible.

2. Mathematical formulation of the problems

A Cartesian coordinate system $Oxyz$ is used with the Oxy plane on the bottom of the fluid layer and the Oz-axis directed vertically upwards. A sphere of radius $R > 0$ centered at the point $(0,0,h_1)$, $h_1 > R$, is totally submerged in the fluid layer, the quiescent free surface which is represented by the plane $z = h_1 + h_2$, $h_2 > R$.

Let S be the fluid domain, that is, the layer between the two planes $z = 0$ and $z = h_1 + h_2$ except the spherical cavity

$$S_C = \{(x,y,z) : x^2 + y^2 + (z-h)^2 \leq R^2\}. \tag{2.1}$$

The plane bottom $z = 0$ is divided into two parts by means of a circle of radius 1, centered at the origin 0. The total boundary ∂S of the fluid domain S consists of the following four parts:

$$\begin{aligned}
\partial S_1 &= \{(x,y,z) : x^2 + y^2 < 1,\ z = 0\},\\
\partial S_2 &= \{(x,y,z) : x^2 + y^2 > 1,\ z = 0\},\\
\partial S_3 &= \{(x,y,z) : x^2 + y^2 + (z-h_1)^2 = R^2\},\\
\partial S_4 &= \{(x,y,z) : -\infty < x,y < \infty,\ z = h_1 + h_2\},
\end{aligned} \tag{2.2}$$

and the infinite "boundary" ∂S_∞ is defined as

$$\partial S_\infty = \{(x,y,z) : (x^2 + y^2)^{1/2} \longrightarrow \infty,\ 0 < z < h_1 + h_2\}. \tag{2.3}$$

The plane bottom of the fluid layer is denoted by $\partial S_{1,2} = \partial S_1 \cup \partial S_2$.

We introduce cylindrical coordinates (ρ,ϕ,z) whose z-direction and origin coincide with the z-direction and the origin of the Cartesian coordinates, and spherical polar coordinates (r,θ,ϕ) with their origin at the center of the spherical cavity.

We will now state some "mixed" boundary value problems to distinguish this type of problems from problems of Dirichlet and Neumann type.

Problem (P_1). Find the impulsive response of the fluid-sphere hydromechanical system due to an underground explosion of a point charge located beneath the submerged sphere, in the soil, at some point $(0,0,-h)$, $h > 0$. The sphere is assumed to be rigid and freely moving under the action of the impulsive hydromechanic pressure. In this case the action of the underground explosion on the bottom $\partial S_{1,2}$ can be modeled as an axisymmetric impulsive pressure $\bar{p} = f_1(\rho)$ (an overbar denotes the impulse of the corresponding quantity defined as $\bar{p} = \int_0^\tau p\,dt$, where $[0,\tau]$ is the time interval during which the impulsive loads are applied) acting on the fluid through some disk $\partial S_1(b) = \{(x,y,z) : x^2 + y^2 < b^2,\ z = 0\}$; the remaining part of the bottom being at rest. The radius b and the impulse $f_1(\rho)$ can be related to the depth h and the energy emitted from the explosion by the aid of empirical or semi-empirical formulae [7, page 335].

Using the model of an ideal and incompressible liquid, the impulsive response of the system is described by means of a velocity potential $u(x,y,z)$ which is harmonic in S and satisfies the boundary conditions

$$u = -\frac{f_1(\rho)}{d} \quad \text{on } \partial S_1, \tag{2.4}$$

$$\frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial S_2, \tag{2.5}$$

$$\frac{\partial u}{\partial \eta} = U\cos\theta \quad \text{on } \partial S_3, \tag{2.6}$$

$$u = 0 \quad \text{on } \partial S_4, \tag{2.7}$$

$$u = 0 \quad \text{on } \partial S_\infty, \tag{2.8}$$

$$mU = -d\int_{\partial S_3} u\cos\theta\,ds, \tag{2.9}$$

where d is the density of the fluid, U is the vertical velocity that will be gotten by the sphere just after the impulsive excitation, and m is the mass of the rigid sphere. If other (i.e., of nonhydrodynamic origin) vertical impulsive forces act simultaneously on the sphere, their impulse must be added to the right-hand side of (2.9). Since u as well as U are unknown, it is convenient to divide u into two parts:

$$u = u_1 + Uu_2, \tag{2.10}$$

where u_1 satisfies (2.4), (2.5), (2.7), and (2.8) together with $\partial u_1/\partial\eta = 0$ on ∂S_3, while u_2 satisfies (2.5), (2.7), (2.8), and $u_2 = 0$ on ∂S_1, $\partial u_2/\partial\eta = \cos\theta$ on ∂S_3. Thus u_1 and u_2 are now independent of U and (2.9) is written in the form

$$(m + m_I)U = -d\int_{\partial S_3} u_1\cos\theta\,ds, \tag{2.11}$$

where

$$m_I = d\int_{\partial S_3} u_2\cos\theta\,ds, \tag{2.12}$$

from which U is obtained immediately by the determination of the potentials u_1 and u_2. The quantity m_I is an impulsive added mass of the rigid sphere.

Problem (P_2). Find the impulsive response of the fluid S due to an impulsive expansion of the sphere. The impulsive response of the fluid is described by means of a velocity potential $u(x,y,z)$ which is harmonic in S and satisfies the boundary conditions

$$\begin{gathered}
\frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial S_{1,2} = \partial S_2 \ (\partial S_1 = \varnothing), \\
\frac{\partial u}{\partial \eta} = U_\eta \quad \text{on } \partial S_3, \\
u = 0 \quad \text{on } \partial S_4, \\
u = 0 \quad \text{on } \partial S_\infty,
\end{gathered} \tag{2.13}$$

where U_η is the radial velocity of the expanded sphere. This classical problem is of particular interest in the theory of underwater explosions and has been treated in the past by the method of images.

The mathematical model corresponding to problems (P_1) and (P_2) can be readily adapted to the following mixed boundary value problem.

Problem (P). Suppose that the potential function $u(x,y,z)$ must satisfy Laplace's equation in the region S. Find u under the boundary conditions

$$u = f^{(1)}(\rho,\phi), \quad \rho < 1 \text{ on } \partial S_1, \tag{2.14}$$

$$\frac{\partial u}{\partial \eta} = f^{(2)}(\rho,\phi), \quad \rho > 1 \text{ on } \partial S_2, \tag{2.15}$$

$$\frac{\partial u}{\partial \eta} = f^{(3)}(\theta,\phi) \quad \text{on } \partial S_3, \tag{2.16}$$

$$u = f^{(4)}(\rho,\phi) \quad \text{on } \partial S_4, \tag{2.17}$$

$$u \longrightarrow 0, \quad \text{as } \left(\rho^2 + z^2\right)^{1/2} \longrightarrow \infty. \tag{2.18}$$

The functions $f^{(m)}$, $m = 1,2,4$, are considered in cylindrical coordinates while the boundary function $f^{(3)}$ is considered in spherical coordinates. We make the assumption that $f^{(m)}$ $(m = 1,\dots,4)$ are continuous functions of both variables in the appropriate regions ∂S_m $(m = 1,\dots,4)$ and that

$$f^{(m)}(\rho,\phi) = O\left(\rho^{-2-\varepsilon}\right) \quad \text{as } \rho \longrightarrow \infty \text{ uniformly with respect to } \phi,\ m = 2,4,\ \varepsilon > 0. \tag{2.19}$$

In the following we will consider the truncated problem, that is, we suppose that

$$f^{(m)}(x_m,\phi) = \sum_{k=0}^{N} f_k^{(m)}(x_m)e^{ik\phi}, \quad x_m = \rho,\ m = 1,2,4;\ x_3 = \theta. \tag{2.20}$$

In fact, it can be shown that the functions $f^{(m)}$ can be approximated uniformly, with respect to ρ (or θ for $m = 3$) and ϕ, as functions of ϕ by trigonometric polynomials in the appropriate regions (Weierstrass approximation theorem). By Harnack's convergence theorem we can also write

$$u^N(\rho,z,\phi) = \sum_{m=0}^{N} u_m(\rho,z)e^{im\phi}, \tag{2.21}$$

where $\lim_{N\to\infty} u^N = u$ (uniformly), and introduce a practicable solution.

3. Reduction of problem (P) to a system of dual integral equations

We consider a potential solution of problem (P) of the form

$$u(\rho,z,\phi) = \sum_{m=0}^{N} u_m(\rho,z)e^{im\phi}, \tag{3.1}$$

in the cavity $0 \le z \le h_1 + h_2$, where

$$\begin{aligned} u_m(\rho,z) = {} & \int_0^\infty [\alpha(\xi)\sinh(\xi z) + \beta(\xi)\cosh(\xi z)]J_m(\xi\rho)d\xi \\ & + \sum_{k=0}^{\infty} d_k^{(1)} \left(\frac{R}{\sqrt{\rho^2+(z-h_1)^2}}\right)^{k+m+1} P_{k+m}^m\left(\frac{z-h_1}{\sqrt{\rho^2+(z-h_1)^2}}\right). \end{aligned} \tag{3.2}$$

In fact, the functions u_m are elementary solutions of Laplace's equation in the domain S; also, P_n^m is an associated Legendre polynomial and $J_m(x)$ is a Bessel function of the first kind. In order to find the solution $u(\rho,z,\phi)$ we have to compute the unknowns $\alpha(\xi)$, $\beta(\xi)$, and $d_k^{(1)}$, which must be chosen in such a way that the functions u_m satisfy certain boundary conditions on S. At this stage, we make use of the definition of Hankel's transform [9], to transform formula (3.2) in the form

$$\begin{aligned} u_m(\rho,z) = {} & H_m(\xi^{-1}[\alpha(\xi)\sinh(\xi z) + \beta(\xi)\cosh(\xi z)];\rho) \\ & + \sum_{k=0}^{\infty} d_k^{(1)} \left(\frac{R}{\sqrt{\rho^2+(z-h_1)^2}}\right)^{k+m+1} P_{k+m}^m\left(\frac{z-h_1}{\sqrt{\rho^2+(z-h_1)^2}}\right), \end{aligned} \tag{3.3}$$

where H_m is Hankel's transform of order m. We show (see Appendix A) that the series and integral in (3.3) are absolutely and uniformly convergent, and that our subsequent operations with them are justified.

Using (3.1) and (3.2) in condition (2.16) and transforming to spherical coordinates with origin at the center of the sphere, we obtain a relation between $\alpha(\xi)$, $\beta(\xi)$, and $d_k^{(1)}$:

$$\begin{aligned} & -\frac{\partial}{\partial r}\left(\int_0^\infty [\alpha(\xi)\sinh\xi(h_1+r\cos\theta) + \beta(\xi)\cosh\xi(h_1+r\cos\theta)J_m(\xi r\sin\theta)d\xi]\right)\Bigg|_{r=R} \\ & \quad + \sum_{k=0}^{\infty}\frac{(k+m+1)}{R}d_k^{(1)}P_{k+m}^m(\cos\theta) = f_m^{(3)}(\theta), \quad 0\le\theta\le\pi,\ m=0,1,\dots,N. \end{aligned} \tag{3.4}$$

It is known (see [12]) that Bessel functions can be expressed through Gegenbauer polynomials:

$$e^{z\cos\theta}J_{\nu-1/2}(z\sin\theta)=\frac{\Gamma(\nu)}{\Gamma(1/2)}(2\sin\theta)^{\nu-1/2}\sum_{n=0}^{\infty}\frac{z^{\nu+n-1/2}}{\Gamma(2\nu+n)}C_n^{\nu}(\cos\theta),\quad 0\le\theta\le\pi,\tag{3.5}$$

where $C_n^{\nu}(x)$ represents the Gegenbauer polynomials. Now employing the relation between Gegenbauer polynomials and Legendre polynomials, we obtain the desired expansion of cylindrical solutions of problem (P) in terms of spherical solutions:

$$e^{\pm\xi z}J_m(\xi\rho)=(-1)^m\sum_{k=0}^{\infty}\frac{(\pm1)^k(\xi r)^{m+k}}{(2m+k)!}P_{m+k}^m(\cos\theta),\quad 0\le\theta\le\pi,\tag{3.6}$$

where $z=r\cos\theta$, $0<\theta<\pi$; see [4].

It follows from (3.6), the orthogonality of associated Legendre polynomials, and (3.4) that coefficients $d_k^{(1)}$ satisfy

$$\begin{aligned}d_k^{(1)}&-\frac{(-1)^m(m+k)R^{m+k}}{(m+k+1)(2m+k)!}\int_0^{\infty}\left[\alpha(\xi)\sinh_k(\xi h_1)+\beta(\xi)\cosh_k(\xi h_1)\right]\xi^{m+k}d\xi\\&=\frac{R}{(m+k+1)}\sqrt{\frac{(k+m+1/2)k!}{(k+2m)!}}f_{mk}^{(3)},\quad k=0,1,\ldots,\ m=0,1,\ldots,N,\end{aligned}\tag{3.7}$$

where $\cosh_k(x)=(e^x+(-1)^ke^{-x})/2$, $\sinh_k(x)=(e^x-(-1)^ke^{-x})/2$, and $f_{mk}^{(3)}$ are the coefficients in the expansion of $f_m^{(3)}$ in a series of normalized associated Legendre polynomials. Using now the well-known relation (see [2]) between Legendre polynomials and Bessel functions

$$\frac{P_n^m\left(z/\sqrt{\rho^2+z^2}\right)}{\left(\sqrt{\rho^2+z^2}\right)^{n+1}}=\frac{(-1)^m}{(n-m)!}\int_0^{\infty}e^{-\xi z}\xi^nJ_m(\xi\rho)d\xi,\quad z>0,\tag{3.8}$$

and taking also into account the boundary conditions (2.14), (2.15), and (2.17), we obtain the following relations between the unknowns $\alpha(\xi)$, $\beta(\xi)$, and $d_k^{(1)}$:

$$\begin{aligned}&\alpha(\xi)\sinh\xi(h_1+h_2)+\beta(\xi)\cosh\xi(h_1+h_2)+e^{-\xi h_2}Q^+(\xi)=\xi H_m\left(f_m^{(4)}(\rho)\right),\\&\alpha(\xi)\sinh\xi(h_1+h_2)+\beta(\xi)\cosh\xi(h_1+h_2)+e^{-\xi h_2}Q^-(\xi)=\xi H_m\left(f_m^{(4)}(\rho)\right),\end{aligned}\tag{3.9}$$

where

$$\begin{aligned}Q^+(\xi)&=(-1)^m\xi^mR^{m+1}\sum_{k=0}^{\infty}\frac{d_k^{(1)}}{k!}(\xi R)^k,\\Q^-(\xi)&=(-1)^m\xi^mR^{m+1}\sum_{k=0}^{\infty}\frac{(-1)^kd_k^{(1)}}{k!}(\xi R)^k,\end{aligned}\tag{3.10}$$

and

$$H_m[\alpha(\xi)+e^{-\xi h_1}Q^-(\xi)] = f_m^{(2)}(\rho), \quad \rho > 1,$$
$$H_m[\alpha(\xi)+e^{-\xi h_1}Q^+(\xi)] = f_m^{(1)}(\rho), \quad 0<\rho<1. \tag{3.11}$$

We now eliminate $\beta(\xi)$ from (3.9)–(3.11) and obtain the following system of dual integral equations [9, 12]:

$$H_m[p(\xi)] = f_m^{(2)}(\rho), \quad \rho > 1,$$
$$H_m[\xi^{-1}p(\xi)\tanh\xi(h_1+h_2)] = f(\rho), \quad 0<\rho<1, \tag{3.12}$$

where

$$p(\xi) = -\alpha(\xi) - e^{-\xi h_1}Q^-(\xi) \tag{3.13}$$

and

$$f(\rho) = f_m^{(1)}(\rho) - \int_0^\infty \frac{e^{\xi h_2}Q^-(\xi) - e^{-\xi h_2}Q^+(\xi) + g_m^{(4)}(\xi)}{\cosh\xi(h_1+h_2)} J_m(\xi\rho)d\xi,$$
$$g_m^{(4)}(\xi) = \xi\int_0^\infty f_m^{(4)}(\rho)\rho J_m(\xi\rho)d\rho. \tag{3.14}$$

4. Reduction to a Fredholm equation

We can reduce the problem of solving the pair of dual integral equations (3.12) to that of solving a Fredholm equation of the second kind. Therefore we seek a solution of system (3.12) in the form

$$p(\xi) = \sqrt{\frac{2}{\pi}}\int_0^1 \phi(t)\sqrt{\xi t}J_{m-1/2}(\xi t)dt + \int_1^\infty \rho f_m^{(2)}(\rho)J_m(\xi\rho)d\rho, \tag{4.1}$$

where $\phi(t)$ is an unknown function to be computed below. In fact, using the exact solution of the equations

$$\int_0^\infty K(y)J_\nu(xy)dy = G(x), \quad 0<x<1,$$
$$\int_0^\infty yK(y)J_\nu(xy)dy = F(x), \quad x>1, \tag{4.2}$$

and following some ideas from [11], we transform system (3.12) to the advantageous form of the following Fredholm equation of the second kind:

$$\begin{aligned}
&\phi(t) - \sqrt{t}\int_0^1 \phi(\rho)\int_0^\infty \xi\frac{\exp[-\xi(h_1+h_2)]}{\cosh\xi(h_1+h_2)}J_{m-1/2}(\xi t)\,J_{m-1/2}(\xi\rho)\sqrt{\rho}\,d\xi\,d\rho \\
&= \int_0^1 \frac{d}{dx}\big[x^m f_m^{(1)}(tx)\big]\frac{dx}{\sqrt{1-x^2}} - t^m\int_1^\infty \frac{f_m^{(2)}(\rho)}{\rho^{m-1}}\frac{d\rho}{\sqrt{\rho^2-t^2}} \\
&\quad+\sqrt{\frac{\pi}{2}}\int_0^\infty\left(\frac{\exp[-\xi(h_1+h_2)]}{\cosh\xi(h_1+h_2)}\sqrt{\xi t}\,J_{m-1/2}(\xi t)\int_1^\infty \rho f_m^{(2)}(\rho)J_m(\xi\rho)d\rho\right)d\xi \\
&\quad-\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{e^{\xi h_2}Q^-(\xi)-e^{-\xi h_2}Q^+(\xi)+g_m^{(4)}(\xi)}{\cosh\xi(h_1+h_2)}\sqrt{\xi t}\,J_{m-1/2}(\xi t)d\xi, \quad m=1,2,\dots,N,\ 0<t<1.
\end{aligned} \tag{4.3}$$

5. Reduction to a linear algebraic system

Now we expand the function $\phi(t)$ in a Fourier series in $[-1,1]$, assuming that it is continued as an even function to the negative part of the interval:

$$\begin{gathered}
\phi(t) = \sum_{n=0}^{\infty}\varepsilon_n d_n^{(2)}\cos(n\pi t), \quad d_n^{(2)} = 2\int_0^1 \phi(t)\cos(n\pi t)dt,\ n=1,2,\dots, \\
\varepsilon_0 = \frac{1}{2}, \qquad \varepsilon_n = 1, \quad n = 1,2.
\end{gathered} \tag{5.1}$$

Expressing $\alpha(\xi)$, $\beta(\xi)$ in terms of $p(\xi)$ and using (3.9) and (3.13), we obtain that

$$\begin{aligned}
\alpha(\xi) &= -p(\xi) - e^{-\xi h_1}Q^-(\xi), \\
\beta(\xi) &= p(\xi)\tanh\xi(h_1+h_2) + \frac{e^{-\xi h_1}\sinh\xi(h_1+h_2)Q^-(\xi) - e^{-\xi h_2}Q^+(\xi) + g_m^{(4)}(\xi)}{\cosh\xi(h_1+h_2)}.
\end{aligned} \tag{5.2}$$

Employing now the previous relations in (3.7) and taking also into account (4.1) and the Fourier expansion of $\phi(t)$, we obtain the following infinite system of linear algebraic equations:

$$d_n^{(i)} + \sum_{k=0}^{\infty} d_k^{(2)}t_{nk}^{(i)} + \sum_{k=0}^{\infty} d_k^{(1)}t_{nk}^{(i+2)} = q_n^{(i)}, \quad n = 1,2,\dots,\ i=1,2; \tag{5.3}$$

here

$$t_{nk}^{(1)} = -\frac{\varepsilon_k}{\sqrt{2\pi}} \frac{(-1)^m (m+n) R^{m+n}}{(n+m+1)(2m+n)!} \int_0^\infty \frac{(-1)^n \sinh(\xi h_2)}{\cosh \xi (h_1+h_2)} \xi^{n+m} r_k^{(m)}(\xi) d\xi, \tag{5.4}$$

$$t_{nk}^{(2)} = -\frac{\varepsilon_k}{2} \int_0^\infty \frac{e^{-\xi(h_1+h_2)}}{\cosh \xi (h_1+h_2)} r_k^{(m)}(\xi) r_k^{(m)}(\xi) d\xi, \tag{5.5}$$

$$t_{nk}^{(3)} = -\frac{(m+n) R^{n+k+2m+1}}{(n+m+1)(2m+n)!k!} \int_0^\infty \frac{(-1)^{n+k} \sinh_n(\xi h_2) e^{-\xi h_1} - \cosh_n(\xi h_1) e^{-\xi h_2}}{\cosh \xi (h_1+h_2)} \xi^{2m+n+k+1} d\xi, \tag{5.6}$$

$$t_{nk}^{(4)} = -\frac{\pi R}{\varepsilon_n} \frac{(k+m+1)(2m+k)!}{(m+k)k!} t_{kn}^{(1)}, \tag{5.7}$$

$$\begin{aligned} q_n^{(1)} = \sqrt{\frac{(n+m+1/2)n!}{(n+2m)!}} & \frac{R f_{mn}^{(3)}}{(n+m+1)} \\ & + \frac{(m+n)R^{m+n}}{(m+n+1)(2m+n)!} \int_0^\infty \left(\frac{g_m^{(4)}(\xi) \cosh_n(\xi h_1) + (-1)^n \sinh_n(\xi h_2) g_m^{(2)}(\xi)}{\cosh \xi (h_1+h_2)} \right) \xi^{n+m} d\xi, \end{aligned} \tag{5.8}$$

$$\begin{aligned} q_n^{(2)} = & \, 2\int_0^1 \cos(n\pi t) \int_0^1 \frac{d}{dx} \left[\xi^m f_m^{(1)}(tx) \right] \frac{dx}{\sqrt{1-x^2}} - 2\int_0^1 t^m \cos(n\pi t) \int_1^\infty \frac{f_m^{(2)}(\rho)}{\rho^{m-1}} \frac{d\rho}{\sqrt{\rho^2 - t^2}} dt \\ & - \sqrt{\frac{\pi}{2}} \int_0^\infty \frac{g_m^{(4)}(\xi)}{\cosh \xi (h_1+h_2)} r_n^m(\xi) d\xi + \sqrt{\frac{\pi}{2}} \int_0^\infty \frac{e^{-\xi(h_1+h_2)} r_k^{(m)}(\xi) g_m^{(2)}(\xi)}{\cosh \xi (h_1+h_2)} d\xi, \end{aligned} \tag{5.9}$$

$$r_n^{(m)}(\xi) = 2\int_0^1 \sqrt{\xi t} \, J_{m-1/2}(\xi t) \cos(n\pi t) dt, \tag{5.10}$$

$$g_m^{(2)}(\xi) = \int_1^\infty f_m^{(2)}(\rho) \rho J_m(\xi \rho) d\rho. \tag{5.11}$$

Appendices

A. Investigation of the linear algebraic system

Equations (5.3)–(5.11) can be written in the vector form

$$\vec{x} + L\vec{x} = \vec{c}, \tag{A.1}$$

where $\vec{x}$ and $\vec{c}$ are column vectors formed of the components of the unknowns and the right-hand side of (A.1), respectively, while L is the coefficient matrix of the system. We will prove that the double series formed of the squares of the components of L is convergent, and so the infinite matrix L defines a completely continuous operator mapping the Hilbert space ℓ_2 into itself.

Lemma A.1. *For $m \geq 1$, the following inequality holds on the positive semiaxis:*

$$|r_n^{(m)}(\xi)| \leq \sqrt{\frac{2}{\pi}} \frac{\gamma^m \xi^m}{\pi^2 n^2 2^{m-1}(m-1)!} U_2^m(\gamma^2 \xi^2), \tag{A.2}$$

where $U_k^{(m)}$ is a kth-degree polynomial in x, with nonnegative coefficients depending on m.

Proof. We note that the function $\psi_m(z) = \sqrt{z} J_{m-1/2}(z)$, for $m \geq 0$, is continuously differentiable an arbitrary number of times on the positive semiaxis. Introducing now the notation

$$W_m(z) = \sqrt{\frac{2}{\pi}} \frac{1}{2^{m-1}(m-1)!} \int_0^{\pi/2} \cos(z \sin\theta)(\cos\theta)^{2m-1} d\theta, \tag{A.3}$$

it follows from the first Sonine integral [12] that $\psi_m(z) = z^m W_m(z)$. Taking the z-derivative of (A.3) and integrating by parts, we obtain the recurrence formula $W_m'(z) = -zW_{m+1}(z)$; hence, on the positive semiaxis,

$$\begin{aligned} |\psi_m'(z)| &\leq \sqrt{\frac{2}{\pi}} \frac{z^{m-1}}{2^{m-1}(m-1)!} U_1^{(m)}(z^2), \\ |\psi_m''(z)| &\leq \sqrt{\frac{2}{\pi}} \left[(1-\delta_{m1}) \frac{z^{m-2}}{2^{m-1}(m-1)!} U_2^{(m)}(z^2) + \delta_{m1} \right], \end{aligned} \tag{A.4}$$

where δ_{mk} is the Kronecker delta. The desired result is obtained now by two integrations by parts of the $r_n^{(m)}$. □

Lemma A.2. *The series formed of the components of the matrix T and the right-hand sides of the system (5.3)–(5.11) converge absolutely, that is,*

$$\sum_{n,k=0}^{\infty} |t_{nk}^{(i)}| < \infty, \quad i = 1, \ldots, 4, \tag{A.5}$$

$$\sum_{n=0}^{\infty} |q_n^{(i)}| < \infty, \quad i = 1, 2. \tag{A.6}$$

Proof. Using Lemma A.1 we find that

$$\sum_{n,k=0}^{\infty} |t_{nk}^{(1)}| \leq c_m^{(1)}(\gamma) \frac{1}{(m-1)!} \left(\frac{\gamma}{2h}\right)^m \sum_{n=1}^{\infty} \frac{(n+2m+4)!}{(n+2m)!} \left(\frac{R}{h_1}\right)^{n+m} \sum_{h=1}^{\infty} \frac{1}{k^2}, \tag{A.7}$$

where $c_m^{(1)}(\gamma)$ is a positive constant depending on m and γ. Since $R < h_1$, the series on the right-hand side of the previous relation converge, hence the series of components $t_{nk}^{(1)}$ converge absolutely. Using similar arguments we can prove that $\sum_{n,k=0}^{\infty} t_{nk}^{(4)}$ is absolute convergent as well. Concerning the $t_{nk}^{(2)}$, the required bound follows from Lemma A.1 and the convergence of the integral

$$\int_0^{\infty} \frac{e^{-\xi(h_1+h_2)}}{\cosh\xi(h_1+h_2)} \xi^{2m} [U_2^{(m)}(\gamma^2\xi^2)]^2 d\xi. \tag{A.8}$$

Finally, for the integrand in the expression of $t_{nk}^{(3)}$, we obtain

$$\begin{aligned} |t_{nk}^{(3)}| &\le \frac{R^{k+n+2m+1}}{(2m+n)!k!}\int_0^\infty \left[e^{-2\xi h_1}+e^{-2\xi h_2}\right]\xi^{2m+n+k}d\xi \\ &\le 2\frac{(2m+n+k)!}{(2m+n)!k!}\left(\frac{R}{2h^*}\right)^{k+n+2m+1}, \qquad h^*=\min\{h_1,h_2\}; \end{aligned} \tag{A.9}$$

this proves (A.5), because the double series formed of the quantities on the right-hand side of (A.9) is convergent for $R<h_i$, $i=1,2$. In order to take an estimate for $q_n^{(i)}$, $i=1,2$, we have to impose extra conditions on the boundary conditions of the original problem. Therefore, we assume that the functions

$$s_1(t)=\int_0^1 \frac{d}{dx}\left[x^m f_m^{(1)}(tx)\right]\frac{dx}{\sqrt{1-x^2}}, \qquad s_2(t)=\int_\gamma^\infty \frac{f_m^{(2)}(\rho)}{\rho^{m-1}}\frac{1}{\sqrt{\rho^2-t^2}}d\rho \tag{A.10}$$

are twice continuously differentiable with respect to t on $[0,\gamma]$; to ensure that this assumption holds, it is sufficient to impose that the $f_m^{(i)}(\rho)$, $i=1,2,3$, are C^3-functions and $f_m^{(2)}$ together with its derivatives satisfies relation (2.19). Also, let

$$\sum_{n=0}^\infty \sqrt{n}\,|f_{mn}^{(3)}|<\infty. \tag{A.11}$$

It follows from the conditions imposed on $s_1(t)$ and $s_2(t)$ that their Fourier coefficients decrease like $1/n^2$; hence the series formed of the first two terms on the right-hand side of (5.9) and the first term on the right-hand side of (5.8) converge absolutely. Bounds for the other terms in (5.8) and (5.9) can be obtained similarly, and bounds for the components of matrix T have already been established. □

Lemma A.2 implies the convergence of the series formed of the squares of the elements of matrix T. Thus we have established that the infinite system (A.1) has a completely continuous form, and that the nonhomogeneous term $\vec{c}$ is in ℓ_1 and so in ℓ_2. Hence, by virtue of the existence and uniqueness of a solution of the original problem and the Hilbert alternative, system (A.1) has a unique solution in ℓ_2. This solution can be calculated by the method described in [5, 6]. This result and Lemma A.2 imply that

$$|d_n^{(i)}|\le \gamma^{(2)}\sum_{k=0}^\infty |t_{nk}^{(i)}|+\gamma^{(1)}\sum_{k=0}^\infty |t_{nk}^{(i+2)}|+|q_n^{(i)}|, \quad n=0,1,\dots,\ i=1,2, \tag{A.12}$$

where the $\gamma^{(i)}$ are positive numbers depending on m. It follows from our assumptions concerning $f_m^{(3)}(\theta)$ that

$$n\frac{(n+2m)!}{(n+m)!}|d_n^{(1)}|<\mu_n, \qquad \sum_{n=0}^\infty \mu_n<\infty. \tag{A.13}$$

Therefore it follows from (A.12) that the Fourier and Fourier-Legendre series in the foregoing formula are uniformly convergent; it also follows that our formal term-by-term differentiations and integrations of these series are justified. The boundedness of the $d_n^{(1)}$ ensures the uniform convergence of the series

$$\sum_{n=0}^{\infty} \frac{|d_n^{(1)}|(R\xi)^{n+m}}{n!} \leq c_1 (R\xi)^m e^{\xi R}, \quad m = 0,1,\ldots,N, \tag{A.14}$$

on each compact subset of the positive semiaxis. Then (5.2) imply that

$$\begin{aligned} |\alpha(\xi) \pm \beta(\xi)| \leq c_2 &\frac{e^{[\mp\xi(h_1+h_2)]}}{\cosh\xi(h_1+h_2)} + c_3 \frac{e^{[\mp\xi(h_1+h_2)]}}{\cosh\xi(h_1+h_2)} e^{-\xi(h_1-R)} \\ &+ c_4 \frac{e^{[-\xi(h_2-R)]}}{\cosh\xi(h_1+h_2)} + c_5 \frac{1}{\cosh\xi(h_1+h_2)}, \end{aligned} \tag{A.15}$$

and this guarantees the absolute and uniform convergence in the domain under consideration of the integrals we have used.

B. The case $m = 0$

The integral equation (4.3), in the case where $m = 0$, can take the form

$$\phi(t) - \int_0^1 K_0(t,\xi)\phi(\xi)d\xi = \Psi_0(t), \quad 0 < t < 1, \tag{B.1}$$

where

$$\begin{aligned} K_0(t,\xi) &= \frac{4}{\pi} t^{-1} \xi \int_0^{\infty} U(s)\cos(\xi s)\cos(st)ds, \\ U(t) &= \frac{e^{-2ht}}{1+e^{-2ht}}, \quad 0 < t < \infty,\ h = h_1 + h_2, \\ \Psi_0(t) &= 2\pi^{-1/2} t^{-1} \frac{d}{dt} \int_0^t \frac{\xi F_0(\xi)d\xi}{(t^2-\xi^2)^{1/2}}, \end{aligned} \tag{B.2}$$

and F_0 is the Hankel zero-order transform of a specific function. The case $m = 0$ is of interest since it arises in the discussion of certain contact problems [10].

Setting now $t\phi(t) = \nu(t)$, we derive the integral equation

$$\nu(t) - \int_0^1 M(t,\xi)\nu(\xi)d\xi = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{\xi F_0(\xi)d\xi}{(t^2-\xi^2)^{1/2}}, \tag{B.3}$$

where

$$M(t,\xi) = \frac{4}{\pi} \int_0^{\infty} U(s)\cos(\xi s)\cos(ts)ds. \tag{B.4}$$

We can find a sufficient condition which has a physical meaning for the integral equation (B.3) to have a solution. In fact, if we consider the Hilbert space $L^2(0,1)$ and the bounded operator M which corresponds to the kernel $M(x,\xi)$, we get the estimate

$$\begin{aligned}\|M\| &< \left\{\int_0^1\int_0^1 |M(x,\xi)|^2 dx\,d\xi\right\}^{1/2}\\ &< \frac{4}{\pi}\left\{\int_0^\infty \frac{e^{-2ht}}{1+e^{-2ht}}\cos^2(xt)dt\right\}^{1/2}\left\{\int_0^\infty \frac{e^{-2ht}}{1+e^{-2ht}}\cos^2(\xi t)dt\right\}^{1/2}\\ &< \frac{2}{\pi h}+\frac{1}{\pi(h^2+1)},\end{aligned} \tag{B.5}$$

and a sufficient condition for M to be a contraction operator is that

$$\frac{2}{\pi h}+\frac{1}{\pi(h^2+1)} < 1. \tag{B.6}$$

Summary

Impulsive problems for a system consisting of a fluid layer and a sphere totally submerged in the fluid have been examined by the method of dual integral equations. It has been shown that a suitable representation of the field can be derived from simple solutions of Laplace's equation in the domain under consideration. By this representation, which is a combination of Legendre polynomials and Bessel functions, mixed boundary conditions have been transformed to the solution of infinite systems and Fredholm integral equations, in which the kernel is in general expressed as an integral combination of exponentials and Bessel functions of order m. This leads to the investigation of approximating solutions under various assumptions, and some $L^2(0,1)$-estimates have been investigated.

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