EIGENFREQUENCIES OF GENERALLY RESTRAINED BEAMS

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Received 13 March 2002 and in revised form 15 May 2003

We deal with the exact determination of eigenfrequencies of a beam with intermediate elastic constraints and generally restrained ends. It is the purpose of this paper to use the calculus of variations to obtain the equations of motion and the natural boundary conditions, and particularly those at the intermediate constraints. Numerical values for the first five natural frequencies are presented in a tabular form for a wide range of values of the restraint parameters. Several particular cases are presented and some of these cases have been compared with those available in the literature.

1. Introduction

Several investigators have studied the influence of rotational and/or translational restraints at the ends of vibrating beams [2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20]. Rao and Mirza [22] have derived exact frequency and normal mode shape expressions for uniform beams with ends elastically restrained against rotation and translation. Nallim and Grossi [21] studied the dynamical behaviour of beams with complicating effects such as nonuniform cross sections, presence of an arbitrarily placed concentrated mass and an axial force, and ends elastically restrained against rotation and translation.

In contrast to the body of information described, there is only a limited amount of information for beams elastically restrained at intermediate points. Rutemberg [24] presented eigenfrequencies for a uniform cantilever beam with a rotational restraint at some position. Lau [13] extended Rutemberg's results including an additional spring to against

Copyright © 2003 Hindawi Publishing Corporation
Journal of Applied Mathematics 2003:10 (2003) 503–516
2000 Mathematics Subject Classification: 74H45, 35A15, 47A75
URL: http://dx.doi.org/10.1155/S1110757X03203065

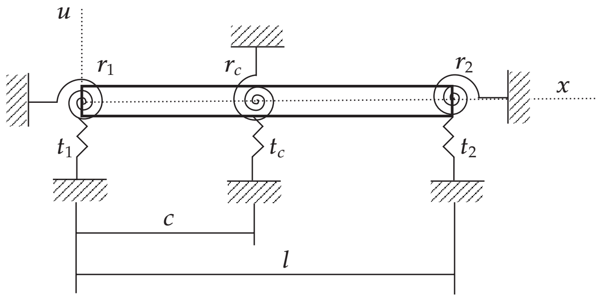

FIGURE 2.1. Vibrating system under study.

translation. Arenas and Grossi [1] presented exact and approximate frequencies of a uniform beam, with one end spring-hinged and a rotational restraint in a variable position.

The present paper is concerned with the general problem of free vibrations of a uniform beam with intermediate constraints and ends elastically restrained against rotation and translation.

Exact expressions for frequencies are presented. The generally restrained beam analysed includes the classical end conditions: clamped, simply supported, sliding, and free as simply particular cases. It also includes the cases with ends and/or intermediate points elastically restrained, previously analysed by other investigators and available in the literature. Some of these particular cases are discussed.

The eigenvalues have been calculated numerically by applying a bracketing method strategy to the corresponding frequency equation. Results for the first five eigenfrequencies for some typical cases are presented. A comparison with published results is included. A great number of problems were solved; and since this number of cases is prohibitively large, results are presented for only a few cases.

2. Variational derivation of the boundary and eigenvalue problem

We consider the uniform beam of length l, shown in Figure 2.1, which has elastically restrained ends and is constrained at an intermediate point with variable position. It has been assumed that the ends and the intermediate points are elastically restrained against rotation and translation. The rotational restraints are characterised by the spring constants r_1, r_2, and r_c and the translational restraints by the spring constants t_1, t_2, and t_c. Adopting the adequate values of the parameters r_i and t_i, $i = 1,2$, all the possible combinations of classical end conditions (i.e., clamped, pinned, sliding, and free) can be generated. On the other hand, adopting the adequate values of the parameters r_c and t_c, different constraints can be generated and also the case of a two-span beam.

It is the purpose of this paper to use the calculus of variations to obtain the equations of motion and the natural boundary conditions, particularly, those at the intermediate constraints.

In order to analyse the transverse planar displacements of the system under study, we suppose that the vertical position of the beam at any time t is described by the function $u = u(x,t)$, $x \in [0,l]$.

It is well known that at time t, the kinetic energy of the beam, the potential energy due to elastic deformation of the beam, and the springs are given by (see [7, 25])

$$T = \frac{1}{2}\int_0^l \rho A\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx,$$
$$U = \frac{1}{2}\left\{\int_0^l \mathrm{EI}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 dx + r_1\left(\frac{\partial u(0,t)}{\partial x}\right)^2 + t_1 u(0,t)^2 + r_c\left(\frac{\partial u(c,t)}{\partial x}\right)^2 + t_c u(c,t)^2 + r_2\left(\frac{\partial u(l,t)}{\partial x}\right)^2 + t_2 u(l,t)^2\right\}, \tag{2.1}$$

where ρ is the mass per unit length, A the cross-sectional area, and EI the flexural rigidity of the beam.

Hamilton's principle requires that between times t_a and t_b, at which the positions are known, the motion will make the action integral $F(u) = \int_{t_a}^{t_b} L\,dt$ on the space of admissible functions stationary, where the Lagrangian is given by $L = T - U$ (see [26]).

In consequence, the energy functional to be considered is given by

$$\begin{aligned} F(u) = {} & \frac{1}{2}\int_{t_a}^{t_b}\int_0^l \left(\rho A\left(\frac{\partial u(x,t)}{\partial t}\right)^2 - \mathrm{EI}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2\right) dx\,dt \\ & - \frac{1}{2}\int_{t_a}^{t_b} r_1\left(\frac{\partial u(0,t)}{\partial x}\right)^2 dt - \frac{1}{2}\int_{t_a}^{t_b} t_1 u(0,t)^2 dt - \frac{1}{2}\int_{t_a}^{t_b} r_c\left(\frac{\partial u(c,t)}{\partial x}\right)^2 dt \\ & - \frac{1}{2}\int_{t_a}^{t_b} t_c u(c,t)^2 dt - \frac{1}{2}\int_{t_a}^{t_b} r_2\left(\frac{\partial u(l,t)}{\partial x}\right)^2 dt - \frac{1}{2}\int_{t_a}^{t_b} t_2 u(l,t)^2 dt. \end{aligned} \tag{2.2}$$

The stationary condition for the functional (2.2) requires that $\delta F(u,v) = 0$, for all $v \in \mathbf{D}_0$, where $\mathbf{D}_0$ is the space of admissible directions at u for the domain $\mathbf{D}$ of the functional.

In order to make the mathematical developments required by the application of the techniques of the calculus of variations, we assume that the domain $\mathbf{D}$ is the set of functions

$$u(x,\cdot)\in C^2[t_a,t_b],\qquad u(\cdot,t)\in C^1[0,l]\cap\hat{C}^4[0,l], \tag{2.3}$$

where $\hat{C}^4[0,l]$ denotes the space of functions with piecewise continuous derivatives up to order four with only one corner point c. At this point c, at least the one-sided derivatives, with respect to x of order greater than one, exist. For instance, $\partial^2u(x,t)/\partial x^2$ is continuous on $[0,l]$ except at the point c, where it has the one-sided derivatives $\partial^2u(c^-,t)/\partial x^2$ and $\partial^2u(c^+,t)/\partial x^2$. The same situation occurs for $\partial^3u(x,t)/\partial x^3$ and $\partial^4u(x,t)/\partial x^4$. Consequently, if $u(\cdot,t)\in C^1[0,l]\cap\hat{C}^4[0,l]$, then $u(\cdot,t)\in C^1[0,l]$ and also $u\in C^4[0,c]$ and $u\in C^4[c,l]$.

In view of all these observations and since Hamilton's principle requires that between times t_a and t_b, the positions are known, the domain of the functional (2.2) is given by

$$\begin{aligned}\mathbf{D}=\{u:u(x,\cdot)\in C^2[t_a,t_b],\ u(\cdot,t)\in C^1[0,l]\cap\hat{C}^4[0,l],\\ u(x,t_a),\ u(x,t_b)\ \text{prescribed}\}.\end{aligned} \tag{2.4}$$

Since $u(\cdot,t)\in C^1[0,l]$, there exists continuity of deflection and slope at the point $x=c$ and this generates the following conditions:

$$\begin{aligned}u(c^-,t)=u(c^+,t)=u(c,t),\\ \frac{\partial u(c^-,t)}{\partial x}=\frac{\partial u(c^+,t)}{\partial x}=\frac{\partial u(c,t)}{\partial x}.\end{aligned} \tag{2.5}$$

The only admissible directions v at $u\in\mathbf{D}$ are those for which $u+\varepsilon v\in\mathbf{D}$ for sufficiently small ε; in consequence, in view of (2.4), v is an admissible direction at u for $\mathbf{D}$ if and only if $v\in\mathbf{D}_0$, where

$$\begin{aligned}\mathbf{D}_0=\{v:v(x,\cdot)\in C^2[t_a,t_b],\ v(\cdot,t)\in C^1[0,l]\cap\hat{C}^4[0,l],\\ v(x,t_a)=v(x,t_b)=0,\ \forall x\in(0,l)\}.\end{aligned} \tag{2.6}$$

Using the definition of variation of F at u in the direction v

$$\delta F(u;v)=\left.\frac{dF(u+\varepsilon v)}{d\varepsilon}\right|_{\varepsilon=0}, \tag{2.7}$$

we obtain

$$\begin{aligned}\delta F(u;v) = &\int_{t_a}^{t_b}\int_0^l\left(\rho A\frac{\partial u}{\partial t}\frac{\partial v}{\partial t} - \mathrm{EI}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\right)dx\,dt\\ &-\int_{t_a}^{t_b} r_1\frac{\partial u(0,t)}{\partial x}\frac{\partial v(0,t)}{\partial x}dt - \int_{t_a}^{t_b} t_1 u(0,t)v(0,t)dt\\ &-\int_{t_a}^{t_b} r_c\frac{\partial u(c,t)}{\partial x}\frac{\partial v(c,t)}{\partial x}dt - \int_{t_a}^{t_b} t_c u(c,t)v(c,t)dt\\ &-\int_{t_a}^{t_b} r_2\frac{\partial u(l,t)}{\partial x}\frac{\partial v(l,t)}{\partial x}dt - \int_{t_a}^{t_b} t_2 u(l,t)v(l,t)dt.\end{aligned} \tag{2.8}$$

Now we consider the integral $\int_{t_a}^{t_b}\int_0^l(\rho A(\partial u/\partial t)(\partial v/\partial t))dx\,dt$. Since $u(x,\cdot)$, $v(x,\cdot)\in C^2[t_a,t_b]$, we can integrate by parts with respect to t; and if we apply the conditions $v(x,t_a)=v(x,t_b)=0$ for all $x\in(0,l)$, imposed in (2.6), we obtain

$$\int_{t_a}^{t_b}\int_0^l \rho A\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}dx\,dt = -\int_{t_a}^{t_b}\int_0^l \rho A\frac{\partial^2 u}{\partial t^2}v\,dx\,dt. \tag{2.9}$$

On the other hand, in $\int_{t_a}^{t_b}\int_0^l \mathrm{EI}(\partial^2 u/\partial x^2)(\partial^2 v/\partial x^2)dx\,dt$, the integrand may not be continuous at the corner point c, but since

$$u(\cdot,t),v(\cdot,t)\in C^4[0,c],\qquad u(\cdot,t),v(\cdot,t)\in C^4[c,l], \tag{2.10}$$

the integral may be represented as the sum of two integrals on $[0,c]$ and $[c,l]$, respectively. Thus if we integrate twice by parts, with respect to x, we obtain

$$\begin{aligned}&\int_{t_a}^{t_b}\int_0^c \mathrm{EI}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}dx\,dt\\ &\quad= \int_{t_a}^{t_b}\int_0^c \mathrm{EI}\frac{\partial^4 u}{\partial x^4}v\,dx\,dt\\ &\quad\quad+\mathrm{EI}\int_{t_a}^{t_b}\left(\frac{\partial^2 u(c^-,t)}{\partial x^2}\frac{\partial v(c,t)}{\partial x} - \frac{\partial^2 u(0,t)}{\partial x^2}\frac{\partial v(0,t)}{\partial x}\right.\\ &\quad\quad\quad\left.-\frac{\partial^3 u(c^-,t)}{\partial x^3}v(c,t) + \frac{\partial^3 u(0,t)}{\partial x^3}v(0,t)\right)dt.\end{aligned} \tag{2.11}$$

Similarly, we obtain

$$\begin{aligned}
\int_{t_a}^{t_b}\int_c^l \mathrm{EI}\,\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\,dx\,dt
&= \int_{t_a}^{t_b}\int_c^l \mathrm{EI}\,\frac{\partial^4 u}{\partial x^4}v\,dx\,dt \\
&\quad + \mathrm{EI}\int_{t_a}^{t_b}\Bigg(-\frac{\partial^2 u(c^+,t)}{\partial x^2}\frac{\partial v(c,t)}{\partial x}+\frac{\partial^2 u(l,t)}{\partial x^2}\frac{\partial v(l,t)}{\partial x} \\
&\qquad +\frac{\partial^3 u(c^+,t)}{\partial x^3}v(c,t)-\frac{\partial^3 u(l,t)}{\partial x^3}v(l,t)\Bigg)dt.
\end{aligned}\tag{2.12}$$

Replacing (2.9), (2.11), and (2.12) in (2.8), we obtain

$$\begin{aligned}
\delta F(u;v) = &-\int_{t_a}^{t_b}\int_0^c\left(\rho A\frac{\partial^2 u}{\partial t^2}+\mathrm{EI}\frac{\partial^4 u}{\partial x^4}\right)v\,dx\,dt \\
&-\int_{t_a}^{t_b}\int_c^l\left(\rho A\frac{\partial^2 u}{\partial t^2}+\mathrm{EI}\frac{\partial^4 u}{\partial x^4}\right)v\,dx\,dt \\
&+\int_{t_a}^{t_b}\left(-r_1\frac{\partial u(0,t)}{\partial x}+\mathrm{EI}\frac{\partial^2 u(0,t)}{\partial x^2}\right)\frac{\partial v(0,t)}{\partial x}dt \\
&+\int_{t_a}^{t_b}\left(-t_1 u(0,t)-\mathrm{EI}\frac{\partial^3 u(0,t)}{\partial x^3}\right)v(0,t)dt \\
&+\int_{t_a}^{t_b}\left(-r_c\frac{\partial u(c,t)}{\partial x}-\mathrm{EI}\frac{\partial^2 u(c^-,t)}{\partial x^2}+\mathrm{EI}\frac{\partial^2 u(c^+,t)}{\partial x^2}\right)\frac{\partial v(c,t)}{\partial x}dt \\
&+\int_{t_a}^{t_b}\left(-t_c u(c,t)+\mathrm{EI}\frac{\partial^3 u(c^-,t)}{\partial x^3}-\mathrm{EI}\frac{\partial^3 u(c^+,t)}{\partial x^3}\right)v(c,t)dt \\
&+\int_{t_a}^{t_b}\left(-r_2\frac{\partial u(l,t)}{\partial x}-\mathrm{EI}\frac{\partial^2 u(l,t)}{\partial x^2}\right)\frac{\partial v(l,t)}{\partial x}dt \\
&+\int_{t_a}^{t_b}\left(-t_2 u(l,t)+\mathrm{EI}\frac{\partial^3 u(l,t)}{\partial x^3}\right)v(l,t)dt.
\end{aligned}\tag{2.13}$$

According to Hamilton's principle, the expression (2.13) must vanish for the function u corresponding to the actual motion of the beam. If we first suppose that both ends of the beam and the restraint at $x = c$ are rigidly clamped, the directions v must satisfy

$$v(0,t)=\frac{\partial v(0,t)}{\partial x}=v(l,t)=\frac{\partial v(l,t)}{\partial x}=v(c,t)=\frac{\partial v(c,t)}{\partial x}=0\quad \forall t\in(t_a,t_b).\tag{2.14}$$

Using (2.14) in (2.13) leads to

$$\delta F(u;v) = -\int_{t_a}^{t_b}\int_0^c \left(\rho A\frac{\partial^2 u}{\partial t^2} + \mathrm{EI}\frac{\partial^4 u}{\partial x^4}\right) v\,dx\,dt - \int_{t_a}^{t_b}\int_c^l \left(\rho A\frac{\partial^2 u}{\partial t^2} + \mathrm{EI}\frac{\partial^4 u}{\partial x^4}\right) v\,dx\,dt. \tag{2.15}$$

Setting (2.15) to zero since v is an arbitrary smooth function satisfying conditions (2.14), the fundamental lemma of the calculus of variations can be applied, and we obtain that u must satisfy the following differential equations:

$$\mathrm{EI}\frac{\partial^4 u(x,t)}{\partial x^4} + \rho A\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \quad \forall t,\ \forall x \in (0,c), \tag{2.16}$$

$$\mathrm{EI}\frac{\partial^4 u(x,t)}{\partial x^4} + \rho A\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \quad \forall t,\ \forall x \in (c,l). \tag{2.17}$$

Next we remove the restrictions (2.14); and since u must satisfy (2.16) and (2.17), (2.13) reduces to

$$\begin{aligned}
\delta F(u;v) = &\int_{t_a}^{t_b}\left(-r_1\frac{\partial u(0,t)}{\partial x} + \mathrm{EI}\frac{\partial^2 u(0,t)}{\partial x^2}\right)\frac{\partial v(0,t)}{\partial x}dt \\
&+ \int_{t_a}^{t_b}\left(-t_1 u(0,t) - \mathrm{EI}\frac{\partial^3 u(0,t)}{\partial x^3}\right)v(0,t)dt \\
&+ \int_{t_a}^{t_b}\left(-r_c\frac{\partial u(c,t)}{\partial x} - \mathrm{EI}\frac{\partial^2 u(c^-,t)}{\partial x^2} + \mathrm{EI}\frac{\partial^2 u(c^+,t)}{\partial x^2}\right)\frac{\partial v(c,t)}{\partial x}dt \\
&+ \int_{t_a}^{t_b}\left(-t_c u(c,t) + \mathrm{EI}\frac{\partial^3 u(c^-,t)}{\partial x^3} - \mathrm{EI}\frac{\partial^3 u(c^+,t)}{\partial x^3}\right)v(c,t)dt \\
&+ \int_{t_a}^{t_b}\left(-r_2\frac{\partial u(l,t)}{\partial x} - \mathrm{EI}\frac{\partial^2 u(l,t)}{\partial x^2}\right)\frac{\partial v(l,t)}{\partial x}dt \\
&+ \int_{t_a}^{t_b}\left(-t_2 u(l,t) + \mathrm{EI}\frac{\partial^3 u(l,t)}{\partial x^3}\right)v(l,t)dt.
\end{aligned} \tag{2.18}$$

The expression (2.18) must vanish for the function u corresponding to the actual motion of the mechanical system under study, and as the functions $v(0,t)$, $\partial v(0,t)/\partial x$, $v(l,t)$, $\partial v(l,t)/\partial x$, $v(c,t)$, $\partial v(c,t)/\partial x$, and the interval $[t_a,t_b]$ are arbitrary, equating (2.18) to zero leads to the natural boundary conditions of the problem:

$$\begin{aligned}
r_1 \frac{\partial u(0,t)}{\partial x} &= \mathrm{EI}\,\frac{\partial^2 u(0,t)}{\partial x^2},\\
t_1 u(0,t) &= -\mathrm{EI}\,\frac{\partial^3 u(0,t)}{\partial x^3},\\
r_c \frac{\partial u(c,t)}{\partial x} &= \mathrm{EI}\left(-\frac{\partial^2 u(c^-,t)}{\partial x^2} + \frac{\partial^2 u(c^+,t)}{\partial x^2}\right),\\
t_c u(c,t) &= \mathrm{EI}\left(\frac{\partial^3 u(c^-,t)}{\partial x^3} - \frac{\partial^3 u(c^+,t)}{\partial x^3}\right),\\
r_2 \frac{\partial u(l,t)}{\partial x} &= -\mathrm{EI}\,\frac{\partial^2 u(l,t)}{\partial x^2}, \qquad t_2 u(l,t) = \mathrm{EI}\,\frac{\partial^3 u(l,t)}{\partial x^3}.
\end{aligned} \tag{2.19}$$

3. Determination of the exact solution

Using the well-known method of separation of variables, one assumes as a solution of (2.16) the expression of the form

$$u^-(x,t) = \sum_{n=1}^{\infty} u_n^-(x) T(t). \tag{3.1}$$

Similarly, for (2.17), we write

$$u^+(x,t) = \sum_{n=1}^{\infty} u_n^+(x) T(t). \tag{3.2}$$

The functions $u_n^-(x)$ and $u_n^+(x)$ denote the corresponding nth mode of natural vibration and are, respectively, given by

$$\begin{aligned}
u_n^-(x) &= A_1 \cosh kx + A_2 \sinh kx + A_3 \cos kx + A_4 \sin kx,\\
u_n^+(x) &= A_5 \cosh kx + A_6 \sinh kx + A_7 \cos kx + A_8 \sin kx,
\end{aligned} \tag{3.3}$$

where the parameter k is given by $k = (\sqrt{\rho A/\mathrm{EI}}\omega_n)^{1/2}$.

Substituting (3.3) in (3.1) and (3.2) and then in the boundary conditions (2.19) and in the continuity conditions (2.5), one obtains a set of eight homogeneous equations in the constants A_i. Since the system is homogeneous for the existence of a nontrivial solution, the determinant of coefficients must be equal to zero. This procedure yields the frequency equation

$$G(R_i, T_i, R_c, T_c, \lambda, c) = 0, \tag{3.4}$$

where

$$R_i = \frac{r_i l}{(\mathrm{EI})}, \quad T_i = \frac{t_i l^3}{(\mathrm{EI})}, \quad i = 1,2,$$
$$R_c = \frac{r_c l}{(\mathrm{EI})}, \qquad T_c = \frac{t_c l^3}{(\mathrm{EI})}, \tag{3.5}$$
$$\lambda = kl, \qquad c = \left(\sqrt{\frac{\rho A}{\mathrm{EI}}}\omega l\right)^{1/2}.$$

4. Analysis of particular cases

Since the analytical expression of (3.4) is extremely complex, it is not included, but since it can be used to obtain special cases by substituting limiting values of the restraint parameters R_i, T_i, $i = 1,2$, R_c, and T_c, some particular analytical expressions will be included.

(i) Boundary conditions: RR-F (one end rotationally restrained and the other free, $T_1 \to \infty$, $R_2 \to 0$, $T_2 \to 0$, $R_c \to 0$, $T_c \to 0$).

Frequency equation:

$$\lambda(\sinh\lambda\cos\lambda - \sin\lambda\cosh\lambda) + R_1(1 + \cos\lambda\cosh\lambda) = 0. \tag{4.1}$$

(ii) Boundary conditions: TR-TR (both ends translationally restrained $R_1 \to 0$, $R_2 \to 0$, $R_c \to 0$, $T_c \to 0$).

Frequency equation:

$$\begin{aligned}\lambda^6(1 - \cos\lambda\cosh\lambda) &+ T_2\lambda^3(-\sin\lambda\cosh\lambda + \sinh\lambda\cos\lambda) \\ &+ T_1\lambda^3(\sinh\lambda\cos\lambda - \sin\lambda\cosh\lambda) + 2T_1T_2\sinh\lambda\sin\lambda = 0.\end{aligned} \tag{4.2}$$

(iii) Boundary conditions: SLIDING-ER (one end sliding and the other elastically restrained against rotation and translation, $R_1 \to \infty$, $T_1 \to 0$, $R_c \to 0$, $T_c \to 0$).

Frequency equation:

$$\begin{aligned}2&\left(\lambda T_2\cos\lambda\cosh\lambda - R_2\lambda^3\sinh\lambda\sin\lambda\right) \\ &+ (\sinh\lambda\cos\lambda + \cosh\lambda\sin\lambda)\left(R_2T_2 - \lambda^4\right) = 0.\end{aligned} \tag{4.3}$$

5. Numerical results

The first five natural frequencies of free vibration of beams with several complicating effects were obtained by using the following strategy. When the values of the parameters R_i, T_i, R_c, T_c, and c are given, (3.4) reduces to $\bar{G}(\lambda) = 0$. A first approximation of the roots of this equation was

TABLE 5.1. Values of the coefficient λ_1 of a cantilever beam with an intermediate point elastically restrained against rotation and translation, $T_1 = \infty$, $R_1 = \infty$, $T_2 = 0$, $R_2 = 0$, and $c = 0.6$.

		T_c					
		0	1	10	100	1000	10000
R_c	0	1.87510407	1.90645722	2.13028597	2.93657102	3.57232073	3.67174004
	1	2.06654909	2.09118908	2.27624824	3.02368609	3.65356416	3.75239457
	10	2.60875718	2.62383401	2.74617885	3.37789697	4.03322217	4.13616565
	100	2.94991835	2.96250788	3.06733852	3.67937709	4.44669639	4.56946848
	1000	3.00457788	3.01689906	3.11984386	3.73240031	4.53274975	4.66229891
	10000	3.01037153	3.02266576	3.12542146	3.73807988	4.54227057	4.67263743

TABLE 5.2. Values of coefficients λ_1 for a beam with both ends and the intermediate point elastically restrained against rotation and translation.

		T_c				
		0	1	10	100	1000
R_c	0	1.72043695	1.76837312	2.08199639	3.08486980	3.20873280
	1	1.73326078	1.78041921	2.09082061	3.12149362	3.33630498
	10	1.76617200	1.81126557	2.11253599	3.15326549	3.89970682
	100	1.78185396	1.82592282	2.12240538	3.15839242	4.34988176
	1000	1.78402861	1.82795307	2.12374977	3.15892488	4.35002422

determined by means of a graphical procedure. The corresponding numerical values with an accuracy of 15 digits were obtained by applying the classical bisection method and then rounded to eight decimal digits.

Some of these were compared with those available in the literature. The results are presented in a tabular form in Tables 5.1, 5.2, 5.3, 5.4, 5.5, and 5.6.

Translationally and rotationally constrained cantilever beam

Table 5.1 depicts the values of the coefficient λ_1 of a cantilever beam with an intermediate point elastically restrained against rotation and translation. The values obtained with the present approach, when rounded to five decimal digits, show a complete agreement with the values reported by Lau [13].

TABLE 5.3. Values of coefficients λ_2 for a beam with both ends and the intermediate point elastically restrained against rotation and translation.

		T_c				
		0	1	10	100	1000
	0	3.22334788	3.22346332	3.22460001	3.27769943	4.35024819
	1	3.34730310	3.34737250	3.34804509	3.36944527	4.35025349
R_c	10	3.90227826	3.90228315	3.90232854	3.90296435	4.35031637
	100	4.51301613	4.51301773	4.51303213	4.51318339	4.51602489
	1000	4.64793800	4.64794190	4.64797701	4.64833088	4.65211572

TABLE 5.4. Values of coefficients λ_3 for a beam with both ends and the intermediate point elastically restrained against rotation and translation.

		T_c				
		0	1	10	100	1000
	0	6.06090936	6.06297669	6.08160554	6.26861751	7.70206094
	1	6.06131847	6.06338589	6.08201554	6.26903564	7.70295117
R_c	10	6.06395142	6.06601882	6.08464827	6.27166813	7.70772262
	100	6.06943070	6.07149472	6.09009401	6.27684185	7.71423506
	1000	6.07129751	6.07335937	6.09193939	6.27851772	7.71581204

TABLE 5.5. Values of coefficients λ_4 for a beam with both ends and the intermediate point elastically restrained against rotation and translation.

		T_c				
		0	1	10	100	1000
	0	9.08972148	9.08973195	9.08982653	9.09080408	9.10521528
	1	9.14024985	9.14026019	9.14035354	9.14131701	9.15524739
R_c	10	9.48041713	9.48042702	9.48051627	9.48142974	9.49324713
	100	10.25960952	10.25962165	10.25973098	10.26083269	10.27273276
	1000	10.53799639	10.53801108	10.53814340	10.53946999	10.55305237

Translationally and rotationally constrained beam at both ends and at an intermediate point

Tables 5.2, 5.3, 5.4, 5.5, and 5.6 depict the values of coefficients λ_i, $i = 1,\ldots,5$, for a general restrained beam. Both ends and the intermediate

Table 5.6. Values of coefficients λ_5 for a beam with both ends and the intermediate point elastically restrained against rotation and translation.

		T_c				
		0	1	10	100	1000
	0	12.15273465	12.15300155	12.15540541	12.17961523	12.43664656
	1	12.15362210	12.15388894	12.15629225	12.18049668	12.43748474
R_c	10	12.16024052	12.16050678	12.16290491	12.18705823	12.44361796
	100	12.18208455	12.18234743	12.18471518	12.20856803	12.46251085
	1000	12.19397508	12.19423521	12.19657823	12.22018715	12.47203118

point are elastically restrained against rotation and translation ($T_1 = 1$, $R_1 = 100$, $T_2 = 10$, $R_2 = 10$, and $c = 1/2$).

6. Conclusions

Exact frequency expressions for generally restrained beams with intermediate elastic constraints were derived. Numerical results for the first five natural frequencies have been presented in tabular form.

Several particular cases were solved and the results obtained were compared with previously published results to demonstrate the accuracy and flexibility of the present approach. Excellent agreement was obtained between the present results and the comparison exact values.

It can also be seen, from the results presented, that both the rotational and the translational restraints at the intermediate point have a significant effect on the frequencies and that the translational restraint generally has greater influence on these frequencies than the rotational restraint.

The procedure presented has a great flexibility and excellent accuracy and constitutes an efficient tool for the rapid and inexpensive determination of natural frequencies in an important number of beam vibrating problems being, in consequence, of interest in design works.

Boundary conditions containing the function u and derivatives of u of orders not greater than $m-1$ are called *stable* or *geometric* for a differential equation of order $2m$, and those containing derivatives of orders higher than $m-1$ are called *unstable* or *natural* [23]. Thus, if $0 \leq r_i, r_c < \infty$, and if $0 \leq t_i, t_c < \infty$, $i = 1,2$, conditions (2.19) are unstable. It is well known that when using the Ritz method, we choose a sequence of functions v_i which constitute a base in the space of homogeneous stable boundary conditions (see [23]), so in this case there is no need to subject the functions v_i to the natural boundary conditions (2.19). This is a very important characteristic of the mentioned variational method in the determination of approximate solutions of the problem under study.

Acknowledgments

The present study has been sponsored by Consejo de Investigación, Project no. 916 (UNSA). The authors are grateful to the referee for valuable comments and suggestions. Ricardo Oscar Grossi is a research member of CONICET, Argentina.

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