Illinois Journal of Mathematics
Volume 59, Number 3, Fall 2015, Pages 545–575
S 0019-2082

EXISTENCE RESULT FOR A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS WITH (*p*-*q*)-LAPLACIAN AND VANISHING POTENTIALS

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Abstract. The main purpose of this paper is to establish the existence of positive solutions to a class of quasilinear elliptic equations involving the $(p\text{-}q)$-Laplacian operator. We consider a nonlinearity that can be subcritical at infinity and supercritical at the origin; we also consider potential functions that can vanish at infinity. The approach is based on variational arguments dealing with the mountain-pass lemma and an adaptation of the penalization method. In order to overcome the lack of compactness, we modify the original problem and the associated energy functional. Finally, to show that the solution of the modified problem is also a solution of the original problem we use an estimate obtained by the Moser iteration scheme.

1. Introduction and main result

In this paper, we consider a class of quasilinear elliptic equations involving the $(p\text{-}q)$-Laplacian operator of the form

$$\begin{cases} -\Delta_p u - \Delta_q u + a(x)|u|^{p-2}u + b(x)|u|^{q-2}u = f(u), & x \in \mathbb{R}^N; \\ u(x) > 0, \quad u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N), & x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

The m-Laplacian operator $\Delta_m u(x)$ is defined by

$$\Delta_m u(x) \equiv \operatorname{div}\big(\big|\nabla u(x)\big|^{m-2}\nabla u(x)\big),$$

Received August 28, 2015; received in final form March 17, 2016.
O. H. Miyagaki was partially supported by CNPq/Brasil and INCTMAT/Brasil.
2010 *Mathematics Subject Classification.* Primary 35J20, 35J92. Secondary 35J10, 35B09, 35B38, 35B45.

for $m \in \{p,q\}$, where $2 \leq q \leq p < N$. The Sobolev space $D^{1,m}(\mathbb{R}^N)$ is defined by

$$D^{1,m}\big(\mathbb{R}^N\big) \equiv \big\{u \in L^{m^*}\big(\mathbb{R}^N\big) \colon (\partial u/\partial x_i)(x) \in L^m\big(\mathbb{R}^N\big), 1 \leq i \leq N\big\},$$

and the critical Sobolev exponent is given by $m^* \equiv Nm/(N-m)$, also for $m \in \{p,q\}$.

The nonlinearity $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous and nonnegative function that is not a pure power and can be subcritical at infinity and supercritical at the origin. More precisely, the following set of hypotheses on the nonlinearity f is used.

- (f_1) $\limsup_{s\to 0^+} sf(s)/s^{p^*} < +\infty$.
- (f_2) There exists $\tau \in (p,p^*)$ such that $\limsup_{s\to+\infty} sf(s)/s^{\tau} = 0$.
- (f_3) There exists $\theta > p$ such that $0 \leq \theta F(s) \leq sf(s)$ for every $s \in \mathbb{R}^+$, where we use the notation $F(s) \equiv \int_0^s f(t)\,\mathrm{d}t$.
- (f_4) $f(t) = 0$ for every $t \leq 0$.

The following properties are easily seen: under hypothesis (f_1) there exists $c_1 \in \mathbb{R}^+$ such that $|sf(s)| \leq c_1|s|^{p^*}$ for s close to zero; and under hypothesis (f_2) there exists $c_2 \in \mathbb{R}^+$ such that $|sf(s)| \leq c_2|s|^{\tau}$ for s large enough. Combining these results and defining $c_0 \equiv \max\{c_1, c_2\}$, we have the pair of inequalities

$$\big|sf(s)\big| \leq c_0|s|^{p^*} \quad \text{and} \quad \big|sf(s)\big| \leq c_0|s|^{\tau} \quad (s \in \mathbb{R}). \tag{1.2}$$

It is worth noticing that hypothesis (f_3) extends a well known condition which was first formulated by Ambrosetti and Rabinowitz [5]. It states a sufficient condition to ensure that the energy functional, associated in a natural way to this type of problem, verifies the Palais–Smale condition. Recall that a functional $J \colon D^{1,m}(\mathbb{R}^N) \to \mathbb{R}$ is said to verify the Palais–Smale condition at the level c if any sequence $(u_n)_{n\in\mathbb{N}} \subset D^{1,m}(\mathbb{R}^N)$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$, as $n \to +\infty$, possess a convergent subsequence. Hypothesis (f_3) also allows us to study the asymptotic behavior of the solution to the problem.

As an example of a nonlinearity f verifying the above set of hypotheses, for $\sigma > p^*$ and for $\tau \in (p,p^*)$ given in hypothesis (f_2), we define

$$f(t) = \begin{cases} t^{\sigma-1}, & \text{if } 0 \leq t \leq 1; \\ t^{\tau-1}, & \text{if } 1 \leq t. \end{cases}$$

We also assume that the functions $a, b \colon \mathbb{R}^N \to \mathbb{R}$ are continuous and nonnegative. Moreover, the following set of hypotheses on the potential functions a and b is used.

- (P_1) $a \in L^{N/p}(\mathbb{R}^N)$ and $b \in L^{N/q}(\mathbb{R}^N)$.
- (P_2) $a(x) \leq a_\infty$ and $b(x) \leq b_\infty$ for every $x \in B_1(0)$, where $a_\infty, b_\infty \in \mathbb{R}^+$ are positive constants and $B_1(0)$ denotes the unitary ball centered at the origin.

(P_3) There exist constants $\Lambda \in \mathbb{R}^+$ and $R_0 > 1$ such that

$$\frac{1}{R_0^{p^2/(p-1)}} \inf_{|x| \geq R_0} |x|^{p^2/(p-1)} a(x) \geq \Lambda.$$

As an example of a potential function a verifying this set of hypotheses, for $\Lambda \in \mathbb{R}^+$ and $R_0 > 1$ given in hypothesis (P_3) we define

$$a(x) = \begin{cases} 0, & \text{if } |x| \leq R_0 - 1; \\ \Lambda R_0^{-p^2/(p-1)}(|x| - R_0 + 1), & \text{if } R_0 - 1 < |x| < R_0; \\ \Lambda |x|^{-p^2/(p-1)}, & \text{if } R_0 \leq |x|. \end{cases}$$

An example of a potential function b can be obtained in a similar way with minor modifications.

The (p-q)-Laplacian operator generalizes several types of problems. For example, in the case $2 = q = p$ with $a(x) = b(x) = V(x)$ and $f(u) = 2g(u)$, problem (1.1) can be written in the form $-\Delta u + V(x)u = g(u)$, which appears in the study of stationary solutions of Schrödinger equation and has been extensively studied by several authors; and in the case $2 \leq q = p$ with $a(x) = b(x) = -V(x)$ and $f(u) = 0$, problem (1.1) assumes the form of the eigenvalue problem $-\Delta_p u = V(x)|u|^{p-2}u$.

The interest in the study of this type of problem is twofold. On the one hand, we have the physical motivations, since the quasilinear operator (p-q)-Laplacian has been used to model steady-state solutions of reaction-diffusion problems arising in biophysics, in plasma physics and in the study of chemical reactions. More precisely, the prototype for these models can be written in the form

$$u_t = -\operatorname{div}\big[D(u)\nabla u\big] + f(x,u),$$

where $D(u) = a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}$ and $a_p, b_q \in \mathbb{R}^+$ are positive constants. In this framework, the function u generally stands for a concentration, the term $\operatorname{div}[D(u)\nabla u]$ corresponds to the diffusion with coefficient $D(u)$, and $f(x,u)$ is the reaction term related to source and loss processes. See Cherfils and Il'yasov [20], Figueiredo [26], [27], Benouhiba and Belyacine [15], Mercuri and Squassina [31], Wu and Yang [41], Yin and Yang [44], Chaves, Ercole and Miyagaki [18], [19], and references therein for more details. In addition, a model of elementary particle physics was studied by Benci, D'Avenia, Fortunato and Pisani [12] which yields an equation of the same class as that in problem (1.1).

On the other hand, we have the purely mathematical interest in these type of problems, mainly regarding the existence of nonnegative nontrivial solutions as well as multiplicity results. In what follows, we present a very brief historical sketch to show some hypotheses on the nonlinearity that have been used by several authors in recent years as sufficient conditions to guarantee the existence of solutions.

We begin by considering the case $2 \leq q = p < p^*$, which includes both the Laplacian operator with $p = 2$ or the p-Laplacian operator with $p > 2$; we also mention some papers dealing with bounded domains and others dealing with the entire space $\mathbb{R}^N$.

Berestycki and Lions [16] considered a positive, constant potential function to show an existence result. Coti Zelati and Rabinowitz [22], Pankov [33], Pankov and Pflüger [34], and Kryszewski and Szulkin [29] considered periodic potential functions with a positive infimum. Zhu and Yang [42], [45] assumed that the potential is asymptotic to a positive constant. Alves, Carrião and Miyagaki [3] studied a problem involving an asymptotically periodic potential. The case of a coercive potential was treated, among others, by Costa [21] and Miyagaki [32]. For a weakened coercivity condition, we refer the reader to Bartsch and Wang [10]. The case of radially symmetric potentials were considered by Alves, de Morais Filho and Souto [2] and Su, Wang and Willem [39], where these authors established some embedding results of weighted Sobolev spaces to obtain ground state solutions. Rabinowitz [36] introduced a hypothesis where the limit inferior of the potential outside a bounded domain is strictly greater than its infimum on the whole space. Afterwards, del Pino and Felmer [23] weakened this condition by considering a situation where the minimum of the potential on the boundary of an open bounded set is strictly greater than its minimum on the closure of this set. The case of sign-changing potentials related to singular perturbation problems were considered by Ding and Szulkin [25] and by Alves, Assunção, Carrião and Miyagaki [4].

As we have seen, most of the papers cited assume that the potential is positive at infinity. However, the case where the potential can vanish at infinity was also studied, among others, by Berestycki and Lions [16], Yang and Zhu [43], Benci, Grisanti and Micheletti [13], Ambrosetti and Wang [6], Ambrosetti, Felli and Malchiodi [7], Alves and Souto [1] and Bastos, Miyagaki and Vieira [11]. In particular, we cite the work by Barile and Figueiredo [9], where it is proved that a problem involving a differential operator more general than that of problem (1.1), with a different perturbation, and with vanishing potential functions $a(x) \equiv b(x)$, has at least three weak solutions, one of which with precisely two nodal domains.

In problem (1.1), we consider the exponents $2 \leq q \leq p < N$ and we allow the particular conditions $\liminf_{|x| \to +\infty} a(x) = 0$ and $\liminf_{|x| \to +\infty} b(x) = 0$, called the zero mass cases. These constitute the main features of our work.

Our result reads as follows.

THEOREM 1.1. *Consider* $2 \leq q \leq p < N$ *and suppose that the potential functions* a *and* b *verify the hypotheses* (P_1), (P_2) *and* (P_3) *and that the nonlinearity* f *verifies the hypotheses* (f_1), (f_2), (f_3), *and* (f_4). *Then there exists a* *constant* $\Lambda^* = \Lambda^*(a_\infty, b_\infty, \theta, \tau, c_0)$ *such that problem* (1.1) *has a positive solution for every* $\Lambda \geq \Lambda^*$.

Usually, a solution to problem (1.1) is obtained as a critical point of the corresponding energy functional defined in some appropriate Sobolev space. To do this, one uses critical point theory, mainly of minimax type; see Mawhin and Willem [30], Struwe [38], and Willem [40]. A well-known result concerning the existence of a nontrivial weak solution is that if the energy functional verifies the geometry of the mountain-pass lemma near the origin and also verifies the Palais–Smale condition, then problem (1.1) has at least one solution. The main difficulty in proving the existence of solution to problem (1.1) resides in the fact that the embedding of the Sobolev space $D^{1,m}(\mathbb{R}^N)$ in the Lebesgue space $L^{Nm/(N-m)}(\mathbb{R}^N)$ is not compact due to the action of a group of homoteties and translations. Besides, the Palais–Smale condition for the corresponding energy functional cannot be obtained directly. Adding to these difficulties, we have to consider the presence of both operators $\Delta_p u$ and $\Delta_q u$. When $q < p$ the study of problem (1.1) does not allow the use of the Lagrange's multipliers method due to the lack of homogeneity; moreover, the first eigenvalue of the $-\Delta_p u$ operator brings no valuable information on the eigenvalue of the $-\Delta_q u$ operator; finally, the method of sub- and super-solutions cannot be applied. Therefore, to study problem (1.1) we are required to make a careful analysis of the energy level of the Palais–Smale sequences in order to obtain their boundedness and also to overcome the lack of compactness. Furthermore, we have to adapt the Moser iteration scheme to our setting, since this is a crucial step to obtain an estimate for the solution.

Inspired mainly by Wu and Yang [41] regarding the (p-q)-Laplacian type operator, and by Alves and Souto [1], with respect to the set of hypotheses, we adapt the penalization method developed by del Pino and Felmer [23] to show our existence result. The basic idea can be described in the following way. In Section 2, we modify the original problem and study its corresponding energy functional, showing that it verifies the geometry of the mountain-pass lemma and that every Palais–Smale sequence is bounded in an appropriate Sobolev space. Using the standard theory this implies that the modified problem has a solution. In Section 3 we show, using the Moser iteration scheme, that the solution of the auxiliary problem verifies an estimate involving the $L^\infty(\mathbb{R}^N)$ norm. Finally, in Section 4 we use this estimate to show that the solution of the modified problem is also a solution of the original problem (1.1).

2. An auxiliary problem

In order to prove the existence of a positive solution to problem (1.1), we establish a variational setting and apply the mountain-pass lemma. Using hypothesis (P_1), we define the space

$$E \equiv \Big\{ u \in D_a^{1,p}(\mathbb{R}^N) \cap D_b^{1,q}(\mathbb{R}^N) : \\ \int_{\mathbb{R}^N} a(x)|u|^p \, \mathrm{d}x < +\infty \text{ and } \int_{\mathbb{R}^N} b(x)|u|^q \, \mathrm{d}x < +\infty \Big\},$$

which can be endowed with the norm $\|u\| = \|u\|_{1,p} + \|u\|_{1,q}$, where we denote

$$\|u\|_{1,p} \equiv \left(\int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x + \int_{\mathbb{R}^N} a(x)|u|^p \, \mathrm{d}x \right)^{1/p}$$

and

$$\|u\|_{1,q} \equiv \left(\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x + \int_{\mathbb{R}^N} b(x)|u|^q \, \mathrm{d}x \right)^{1/q}.$$

Now we define the Euler–Lagrange energy functional $I \colon E \to \mathbb{R}$ associated to problem (1.1) by

$$I(u) \equiv \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x + \frac{1}{p} \int_{\mathbb{R}^N} a(x)|u|^p \, \mathrm{d}x \\ + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} b(x)|u|^q \, \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \, \mathrm{d}x.$$

Using the hypotheses on the nonlinearity f, we can deduce that $I \in C^1(E; \mathbb{R})$; moreover, for every $u, v \in E$ its Gâteaux derivative can be computed by

$$I'(u)v = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x + \int_{\mathbb{R}^N} a(x)|u|^{p-2} uv \, \mathrm{d}x \\ + \int_{\mathbb{R}^N} |\nabla u|^{q-2} \nabla u \cdot \nabla v \, \mathrm{d}x + \int_{\mathbb{R}^N} b(x)|u|^{q-2} uv \, \mathrm{d}x - \int_{\mathbb{R}^N} f(u) v \, \mathrm{d}x.$$

It is a well-known fact that if u is a critical point of the energy functional I, then u is a weak solution to problem (1.1). This means that

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, \mathrm{d}x + \int_{\mathbb{R}^N} a(x)|u|^{p-2} u\phi \, \mathrm{d}x \\ + \int_{\mathbb{R}^N} |\nabla u|^{q-2} \nabla u \cdot \nabla \phi \, \mathrm{d}x + \int_{\mathbb{R}^N} b(x)|u|^{q-2} u\phi \, \mathrm{d}x - \int_{\mathbb{R}^N} f(u)\phi \, \mathrm{d}x = 0$$

for every $v \in E$.

Now we define the energy functional $I_\infty \colon D_0^{1,p}(B_1(0)) \cap D_0^{1,q}(B_1(0)) \to \mathbb{R}$ by

$$I_\infty(u) \equiv \frac{1}{p} \int_{B_1(0)} |\nabla u|^p \, \mathrm{d}x + \frac{1}{p} \int_{B_1(0)} a_\infty |u|^p \, \mathrm{d}x \\ + \frac{1}{q} \int_{B_1(0)} |\nabla u|^q \, \mathrm{d}x + \frac{1}{q} \int_{B_1(0)} b_\infty |u|^q \, \mathrm{d}x - \int_{B_1(0)} F(u) \, \mathrm{d}x.$$

Using the hypotheses (P_1) and (P_2) it can be shown that it is well defined. Our first lemma concerns the geometry of this functional.

LEMMA 2.1. *The functional I_∞ verifies the geometry of the mountain-pass lemma. More precisely, the following claims are valid.*

(1) *There exist $r_0, \mu_0 \in \mathbb{R}^+$ such that $I_\infty(u) \geq \mu_0$ for $\|u\| = r_0$.*
(2) *There exists $e_0 \in [D^{1,p}(B_1(0)) \cap D^{1,q}(B_1(0))] \setminus \{0\}$ such that $\|e_0\| \geq r_0$ and $I_\infty(e_0) < 0$.*

Proof. By using the hypotheses (f_1), (f_2), and (f_3) it is standard to verify item (1).

By hypothesis (f_3), it follows that there exist $\theta > p$ and $C_0 \in \mathbb{R}^+$ such that $F(s) \geq C_0|s|^\theta$. Now, if $u \in [D_0^{1,p}(B_1(0)) \cap D_0^{1,q}(B_1(0))] \setminus \{0\}$, then

$$\begin{aligned} I_\infty(tu) \leq{}& \frac{1}{p}|t|^p \int_{B_1(0)} |\nabla u|^p \,\mathrm{d}x + \frac{a_\infty}{p}|t|^p \int_{B_1(0)} |u|^p \,\mathrm{d}x \\ &+ \frac{1}{q}|t|^q \int_{B_1(0)} |\nabla u|^q \,\mathrm{d}x + \frac{b_\infty}{q}|t|^q \int_{B_1(0)} |u|^q \,\mathrm{d}x \\ &- C_0|t|^\theta \int_{B_1(0)} |u|^\theta \,\mathrm{d}x. \end{aligned}$$

Using this inequality, we deduce that there exist $t_u \in \mathbb{R}^+$ large enough such that, taking $e_0 = t_u u$, we have $\|e_0\| \geq r_0$ and $I_\infty(e_0) < 0$. This concludes the proof of item (2). $\square$

We denote by d the mountain-pass level associated to the functional I_∞, that is,

$$d \equiv \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\infty\big(\gamma(t)\big),$$

where

$$\Gamma \equiv \big\{\gamma \in C\big([0,1]; D^{1,p}\big(B_1(0)\big) \cap D^{1,q}\big(B_1(0)\big)\big) : \gamma(0) = 0 \text{ and } \gamma(1) = e_0\big\}$$

and the function $e_0 \in [D^{1,p}(B_1(0)) \cap D^{1,q}(B_1(0))] \setminus \{0\}$ is given in Lemma 2.1. It is standard to verify that the mountain-pass level d depends only on a_∞, on b_∞, on θ, and on the function f.

For $R > 1$ and for $\theta > p$ given in hypothesis (f_3), we set $k \equiv \theta p/(\theta - p) > p$ and we define a new nonlinearity $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$g(x,t) \equiv \begin{cases} f(t), & \text{if } |x| \leq R \text{ or if } |x| > R \text{ and } f(t) \leq \frac{a(x)}{k}|t|^{p-2}t; \\ \frac{a(x)}{k}|t|^{p-2}t, & \text{if } |x| > R \text{ and } f(t) > \frac{a(x)}{k}|t|^{p-2}t. \end{cases}$$

Using the notation $G(x,t) \equiv \int_0^t g(x,s)\,\mathrm{d}s$, by direct computations we get the set of inequalities

$$g(x,t) \leq \frac{a(x)}{k}|t|^{p-2}t, \quad \text{for all } |x| \geq R; \tag{2.1}$$

$$G(x,t) = F(t), \quad \text{if } |x| \leq R; \tag{2.2}$$

$$G(x,t) \leq \frac{a(x)}{kp}|t|^{p-1}t, \quad \text{if } |x| > R > 1. \tag{2.3}$$

Now we define the auxiliary problem

$$\begin{cases} -\Delta_p u - \Delta_q u + a(x)|u|^{p-2}u + b(x)|u|^{q-2}u = g(x,u), & x \in \mathbb{R}^N; \\ u(x) > 0, \quad u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N), & x \in \mathbb{R}^N. \end{cases} \tag{2.4}$$

The Euler–Lagrange energy functional $J\colon E \to \mathbb{R}$ associated to the auxiliary problem (2.4) is given by

$$\begin{aligned} J(u) \equiv \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} a(x)|u|^p \,\mathrm{d}x \\ + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} b(x)|u|^q \,\mathrm{d}x - \int_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x. \end{aligned}$$

Using the hypotheses on the nonlinearity f and on the potential functions a and b we can show that $J \in C^1(E;\mathbb{R})$; moreover, for every $u, v \in E$ its Gâteaux derivative can be computed by

$$\begin{aligned} J'(u)v = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} a(x)|u|^{p-2}uv \,\mathrm{d}x \\ + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} b(x)|u|^{q-2}uv \,\mathrm{d}x \\ - \int_{\mathbb{R}^N} g(x,u)v \,\mathrm{d}x. \end{aligned}$$

As before, critical points of the energy functional J are weak solutions to problem (2.4).

Our next goal is to apply the mountain-pass lemma to show that problem (2.4) has a positive solution.

LEMMA 2.2. *The functional J verifies the geometry of the mountain-pass lemma. More precisely, the following claims are valid.*

(1) *There exist $r_1, \mu_1 \in \mathbb{R}^+$ such that $J(u) \geq \mu_1$ for $\|u\| = r_1$.*
(2) *There exists $e_1 \in [D^{1,p}(B_1(0)) \cap D^{1,q}(B_1(0))] \backslash \{0\}$ such that $\|e_1\| \geq r_1$ and $J(e_1) < 0$.*

Proof. Using the equality (2.2) and the inequality (2.3) together with the hypotheses (f_1) and (f_3) and the first inequality in (1.2), we obtain

$$\begin{aligned} J(u) \geq \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q - \int_{|x|\leq R} F(u)\,\mathrm{d}x \\ - \int_{|x|>R} \frac{a(x)|u|^p}{kp}\,\mathrm{d}x \end{aligned}$$

$$\geq \frac{1}{p}\|u\|_{1,p}^{p} + \frac{1}{q}\|u\|_{1,q}^{q} - \frac{c_0}{\theta}\int_{\mathbb{R}^N} |u|^{p^*}\,\mathrm{d}x - \frac{1}{kp}\|u\|_{1,p}^{p}$$
$$= \left(\frac{1}{p} - \frac{1}{kp}\right)\|u\|_{1,p}^{p} + \frac{1}{q}\|u\|_{1,q}^{q} - \frac{c_0}{\theta}|u|_{L^{p^*}}^{p^*}.$$

Now we apply the Sobolev inequality

$$\|u\|_{L^{m^*}(\mathbb{R}^N)}^{m} \leq S_m \int_{\mathbb{R}^N} |\nabla u|^m\,\mathrm{d}x \quad \text{for all } u \in D^{1,m}\big(\mathbb{R}^N\big)\ \big(m \in \{p,q\}\big) \tag{2.5}$$

in the computations above and set $S \equiv \max\{S_p, S_q\}$ to get

$$\begin{aligned} J(u) &\geq \left(\frac{1}{p} - \frac{1}{kp}\right)\|u\|_{1,p}^{p} + \frac{1}{q}\|u\|_{1,q}^{q} \\ &\quad - \frac{c_0}{\theta} S^{p^*/p}\left(\int_{\mathbb{R}^N} |\nabla u|^p\,\mathrm{d}x\right)^{p^*/p} \\ &\geq \min\left\{\frac{1}{p} - \frac{1}{kp}, \frac{1}{q}\right\}\big(\|u\|_{1,p}^{p} + \|u\|_{1,q}^{q}\big) \\ &\quad - \frac{c_0}{\theta} S^{p^*/p}\big(\|u\|_{1,p}^{p} + \|u\|_{1,q}^{q}\big)^{p^*/p}. \end{aligned}$$

If we take $\|u\|_{1,p}$ and $\|u\|_{1,q}$ small enough, it follows that $\|u\|_{1,p}^{p}$ and $\|u\|_{1,q}^{q}$ are also small enough. For that reason, we obtain the existence of r_1, $\mu_1 \in \mathbb{R}^+$ such that $J(u) \geq \mu_1$ for $\|u\| = r_1$. This concludes the proof of item (1).

By definition, for all $u \in [D^{1,p}(B_1(0)) \cap D^{1,q}(B_1(0))]\backslash\{0\}$ we have that $G(x,u) = F(u)$. Arguing as in the proof of Lemma 2.1 we conclude that there exist $r_1, t_u \in \mathbb{R}^+$ such that $e_1 \equiv t_u u$ verify the inequalities $\|e_1\| \leq r_1$ and $J(e_1) < 0$. This concludes the proof of item (2). The lemma is proved. □

Since the functional J has the geometry of the mountain-pass lemma, using Willem [40, Theorem 1.15] we obtain a Palais–Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to +\infty$. Here $c \in \mathbb{R}^+$ is the mountain-pass level associated to the energy functional J, that is,

$$c \equiv \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J\big(\gamma(t)\big),$$

where

$$\Gamma \equiv \big\{\gamma \in C([0,1]; D^{1,p}\big(B_1(0)\big) \cap D^{1,q}\big(B_1(0)\big) : \gamma(0) = 0 \text{ and } \gamma(1) = e_1\big\}$$

and $e_1 \in [D^{1,p}(B_1(0)) \cap D^{1,q}(B_1(0))]\backslash\{0\}$ is the same function verifying inequality $J(e_1) < 0$ in Lemma 2.2. Using the hypothesis (f_4), without loss of generality we can suppose that the sequence $(u_n)_{n\in\mathbb{N}} \subset E$ consists of nonnegative functions.

We note that for all $u \in [D^{1,p}(B_1(0)) \cap D^{1,q}(B_1(0))]\backslash\{0\}$ the inequality $J(u) \leq I_\infty(u)$ is valid, and this implies that

$$c \leq d. \tag{2.6}$$

Now we prove the boundedness of the Palais–Smale sequences for the functional J.

LEMMA 2.3. *Suppose that the potential functions a, b verify the hypothesis (P_1), and that the nonlinearity f verifies the hypotheses (f_1), (f_2), (f_3), and (f_4). If $(u_n)_{n\in\mathbb{N}} \subset E$ is a Palais–Smale sequence for the energy functional J, then the sequence $(u_n)_{n\in\mathbb{N}} \subset E$ is bounded in E.*

Proof. To prove that the sequence $(u_n)_{n\in\mathbb{N}} \subset E$ is bounded in E it is sufficient to prove that both sequences $(\|u_n\|_{1,q}^q)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}} \subset \mathbb{R}$ are bounded, which we do in the two claims below.

Before that, however, we remark that there exist constants $c_1 > 0$ and $n_0 \in \mathbb{N}$ such that $J(u_n) \leq c_1$ and $|J'(u_n u_n)| \leq \min\{\|u_n\|_{1,q}, \|u_n\|_{1,p}\}$ for all $n \in \mathbb{N}$ such that $n \geq n_0$; and since $\theta > p > 1$, for all $n \geq n_0$ we have

$$J(u_n) - \frac{1}{\theta}J'(u_n)u_n \leq c_1 + \frac{1}{\theta}\|u_n\| \leq c_1 + \min\big\{\|u_n\|_{1,q}, \|u_n\|_{1,p}\big\}. \tag{2.7}$$

CLAIM 1. *The sequence $(\|u_n\|_{1,q}^q)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded.*

Proof. We divide our analysis into cases that mirror the definition of the nonlinearity g. If $|x| > R$ and $f(t) > a(x)|t|^{p-2}t/k$, then

$$\int_{\mathbb{R}^N} G(x, u_n)\,\mathrm{d}x = \frac{1}{p}\int_{\mathbb{R}^N} g(x, u_n)u_n\,\mathrm{d}x,$$

and this implies that

$$J(u_n) - \frac{1}{p}J'(u_n)u_n = \left(\frac{1}{q} - \frac{1}{p}\right)\|u_n\|_{1,q}^q. \tag{2.8}$$

Combining inequalities (2.7) and (2.8), we conclude that

$$\left(\frac{1}{q} - \frac{1}{p}\right)\|u_n\|_{1,q}^q \leq c_1 + \|u_n\|_{1,q}.$$

So, in this case the sequence $(\|u_n\|_{1,q}^q)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded, say $\|u_n\|_{1,q}^q \leq c_q$ for every $n \in \mathbb{N}$.

If $|x| \leq R$ or if $|x| > R$ and $f(t) \leq a(x)|t|^{p-2}t/k$, the boundedness of the sequence can be proved using the same ideas as that of the previous case with some minor changes. This concludes the proof of the claim. $\square$

CLAIM 2. *The sequence $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded.*

Proof. We also divide our analysis into the same cases. If $|x| > R$ and $f(t) > a(x)|t|^{p-2}t/k$, then we have

$$\begin{aligned} J(u_n) &- \frac{1}{\theta}J'(u_n)u_n \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_{1,p}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u_n\|_{1,q}^q - \frac{1}{kp}\left\{\int_{\mathbb{R}^N} a(x)|u_n|^p\,\mathrm{d}x\right\} \end{aligned} \tag{2.9}$$

$$\geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_n\|_{1,p}^p + \left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_n\|_{1,p}^q - \frac{1}{kp}\big\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\big\}$$
$$= \frac{(p-1)}{kp}\big\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\big\}.$$

Combining inequalities (2.7) and (2.9) and using Claim 1, we obtain

$$\frac{(p-1)}{kp}\|u_n\|_{1,p}^p \leq c_1 + \|u_n\|_{1,p}.$$

This means that in this case the sequence $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded.

If $|x| \leq R$ or if $|x| > R$ and $f(t) \leq a(x)|t|^{p-2}t/k$, then

$$\int_{\mathbb{R}^N} G(x,u_n)\,\mathrm{d}x + \frac{1}{\theta}\int_{\mathbb{R}^N} g(x,u_n)u_n\,\mathrm{d}x \geq 0.$$

Hence,

$$\begin{aligned}
J(u_n) &- \frac{1}{\theta}J'(u_n)u_n \\
&\geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_n\|_{1,p}^p + \left(\frac{1}{q}-\frac{1}{\theta}\right)\|u_n\|_{1,q}^q - \int_{\mathbb{R}^N} G(x,u_n)\,\mathrm{d}x \\
&\quad + \frac{1}{\theta}\int_{\mathbb{R}^N} g(x,u_n)u_n\,\mathrm{d}x \\
&\geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\big\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\big\} - \int_{\mathbb{R}^N} G(x,u_n)\,\mathrm{d}x \\
&\quad + \frac{1}{\theta}\int_{\mathbb{R}^N} g(x,u_n)u_n\,\mathrm{d}x \\
&\geq \frac{1}{k}\big\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\big\} \\
&\geq \frac{(p-1)}{kp}\big\{\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q\big\}.
\end{aligned} \tag{2.10}$$

Combining inequalities (2.7) and (2.10), we get

$$\frac{1}{k}\|u_n\|_{1,p}^p \leq c_1 + \|u_n\|_{1,p}.$$

This means that also in this case the sequence $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded. This concludes the proof of the claim. □

Using Claims 1 and 2, we deduce the proof of the lemma. □

The following result shows that the functional J verifies the Palais–Smale condition.

LEMMA 2.4. *Suppose that the potential functions a, b verify the hypotheses (P_1), (P_2), and (P_3) and that the nonlinearity f verifies the hypotheses (f_1),*

(f_2), (f_3), and (f_4). Then the Palais–Smale condition is valid for the energy functional J.

Proof. Let $(u_n)_{n\in N} \subset E$ be a Palais–Smale sequence at the level c; this means that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0$$

as $n \to \infty$. By Lemma 2.3, this sequence is bounded. Then there exist a subsequence of $(u_n)_{n\in N} \subset E$, which we still denote in the same way, and there exists a function $u \in E$ such that $u_n \rightharpoonup u$ weakly in E as $n \to +\infty$.

For each $\epsilon > 0$, there exist $r > R > 1$ such that

$$\begin{aligned} 2\big(2^N - 1\big)^{1/N} \omega_N^{\frac{1}{N}} \left(1 - \frac{1}{k}\right)^{-1} \Bigg\{ &\left(\int_{r\le|x|\le 2r} |u|^{p^*} \,\mathrm{d}x \right)^{1/p^*} \|u\|^{p-1} \\ &+ \left(\int_{r\le|x|\le 2r} |u|^{q^*} \,\mathrm{d}x \right)^{1/q^*} \|u\|^{q-1} \Bigg\} < \epsilon. \end{aligned} \tag{2.11}$$

Let $\eta = \eta_r \in C^\infty(B_r^c(0))$ be a cut off function such that $0 \le \eta \le 1$, with $\eta = 1$ in $B_{2r}^c(0)$ and also $|\nabla\eta| \le 2/r$ for all $x \in \mathbb{R}^N$. Since the sequence $(u_n)_{n\in N} \subset E$ is bounded, it follows that the sequence $(\eta u_n)_{n\in N} \subset E$ is bounded also. Therefore, $J'(u_n)(\eta u_n) = o_n(1)$, that is,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla(\eta u_n) \,\mathrm{d}x + \int_{\mathbb{R}^N} a(x)|u_n|^{p-2} u_n(\eta u_n) \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla(\eta u_n) \,\mathrm{d}x + \int_{\mathbb{R}^N} b(x)|u|_n^{q-2} u_n(\eta u_n) \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} g(x, u_n)(\eta u_n) \,\mathrm{d}x + o(1). \end{aligned} \tag{2.12}$$

The previous expression and the properties of the cut off function η imply that

$$\begin{aligned} &\int_{|x|\ge r} \eta |\nabla u_n|^p \,\mathrm{d}x + \int_{|x|\ge r} |\nabla u_n|^{p-2} u_n \nabla u_n \cdot \nabla\eta \,\mathrm{d}x \\ &\quad + \int_{|x|\ge r} \eta a(x)|u_n|^p \,\mathrm{d}x + \int_{|x|\ge r} \eta |\nabla u_n|^q \,\mathrm{d}x \\ &\quad + \int_{|x|\ge r} |\nabla u_n|^{q-2} u_n \nabla u_n \cdot \nabla\eta \,\mathrm{d}x + \int_{|x|\ge r} \eta b(x)|u_n|^q \,\mathrm{d}x \\ &= \int_{|x|\ge r} \eta g(x, u_n) u_n \,\mathrm{d}x + o(1). \end{aligned}$$

By the inequality (2.1), it follows that

$$\int_{|x|\ge r} \eta g(x, u_n) u_n \,\mathrm{d}x \le \int_{|x|\ge r} \eta \frac{a(x)}{k} |u_n|^p \,\mathrm{d}x;$$

thus, we obtain

$$\begin{aligned}
&\int_{|x|\geq r} \eta|\nabla u_n|^p \,\mathrm{d}x + \int_{|x|\geq r} \eta a(x)|u_n|^p \,\mathrm{d}x \\
&\quad + \int_{|x|\geq r} \eta|\nabla u_n|^q \,\mathrm{d}x + \int_{|x|\geq r} \eta b(x)|u_n|^q \,\mathrm{d}x - \int_{|x|\geq r} \eta \frac{a(x)}{k}\,|u_n|^p \\
&\quad \leq \int_{|x|\geq r} |\nabla u_n|^{p-1}|u_n||\nabla \eta| \,\mathrm{d}x + \int_{|x|\geq r} |\nabla u_n|^{q-1}|u_n||\nabla \eta| \,\mathrm{d}x + o(1) \\
&\quad \leq \frac{2}{r}\left\{\int_{r\leq|x|\leq 2r} |\nabla u_n|^{p-1}|u_n| \,\mathrm{d}x + \int_{r\leq|x|\leq 2r} |\nabla u_n|^{q-1}|u_n| \,\mathrm{d}x\right\} + o(1).
\end{aligned}$$

Subtracting the terms

$$\frac{1}{k}\int_{|x|\geq r} \eta|\nabla u_n|^p \,\mathrm{d}x + \frac{1}{k}\int_{|x|\geq r} \eta|\nabla u_n|^q \,\mathrm{d}x + \frac{1}{k}\int_{|x|\geq r} \eta b(x)|u_n|^q \,\mathrm{d}x$$

from the left-hand side of the previous inequality and grouping the several integrals, we deduce that

$$\begin{aligned}
&\left(1-\frac{1}{k}\right)\left\{\int_{|x|\geq r} \eta|\nabla u_n|^p \,\mathrm{d}x + \int_{|x|\geq r} \eta a(x)|u_n|^p \,\mathrm{d}x \right.\\
&\quad \left. + \int_{|x|\geq r} \eta|\nabla u_n|^q \,\mathrm{d}x + \int_{|x|\geq r} \eta b(x)|u_n|^q \,\mathrm{d}x\right\} \\
&\quad \leq \frac{2}{r}\left\{\int_{r\leq|x|\leq 2r} |u_n||\nabla u_n|^{p-1} \,\mathrm{d}x + \int_{r\leq|x|\leq 2r} |u_n||\nabla u_n|^{q-1} \,\mathrm{d}x\right\} + o(1).
\end{aligned}$$

Now we use Hölder's inequality to get

$$\begin{aligned}
&\int_{r\leq|x|\leq 2r} |u_n||\nabla u_n|^{p-1} \,\mathrm{d}x \\
&\quad \leq \left(\int_{r\leq|x|\leq 2r} |u_n|^p \,\mathrm{d}x\right)^{1/p} \left\{\left(\int_{r\leq|x|\leq 2r} |\nabla u_n|^p \,\mathrm{d}x\right)^{1/p}\right\}^{p-1} \\
&\quad \leq \left(\int_{r\leq|x|\leq 2r} |u_n|^p \,\mathrm{d}x\right)^{1/p} \|u_n\|^{p-1}.
\end{aligned}$$

And in a similar way, we obtain

$$\int_{r\leq|x|\leq 2r} |u_n||\nabla u_n|^{q-1} \,\mathrm{d}x \leq \left(\int_{r\leq|x|\leq 2r} |u_n|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \|u_n\|^{q-1}.$$

By the compactness of the embedding $W^{1,p}(\overline{B}_{2r}\backslash B_r) \hookrightarrow L^p(\overline{B}_{2r}\backslash B_r)$, we infer that $u_n \to u$ strongly in $L^p(\overline{B}_{2r}\backslash B_r)$ as $n\to\infty$. Since $(\eta u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)\cap$

$W^{1,q}(\mathbb{R}^N)$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty}\Big(1-\frac{1}{k}\Big)\Big\{&\int_{|x|\geq r}\eta|\nabla u_n|^p\,\mathrm{d}x+\int_{|x|\geq r}\eta a(x)|u_n|^p\,\mathrm{d}x\\
&+\int_{|x|\geq r}\eta|\nabla u_n|^q\,\mathrm{d}x+\int_{|x|\geq r}\eta b(x)|u_n|^q\,\mathrm{d}x\Big\}\\
&\leq\frac{2}{r}\limsup_{n\to\infty}\Big\{\Big(\int_{r\leq|x|\leq 2r}|u_n|^p\,\mathrm{d}x\Big)^{1/p}\|u_n\|^{p-1}\\
&+\Big(\int_{r\leq|x|\leq 2r}|u_n|^q\,\mathrm{d}x\Big)^{1/q}\|u_n\|^{q-1}\Big\}\\
&=\frac{2}{r}\Big\{\Big(\int_{r\leq|x|\leq 2r}|u|^p\,\mathrm{d}x\Big)^{1/p}\|u\|^{p-1}\\
&+\Big(\int_{r\leq|x|\leq 2r}|u|^q\,\mathrm{d}x\Big)^{1/q}\|u\|^{q-1}\Big\}.
\end{aligned}\tag{2.13}$$

Applying Hölder's inequality once more and denoting the volume of the unitary ball by $|B_1(0)|=\omega_N$, we obtain

$$\begin{aligned}
&\Big(\int_{r\leq|x|\leq 2r}|u|^p\,\mathrm{d}x\Big)^{1/p}\\
&\leq\big((2^N-1)\omega_N r^N\big)^{1/N}\Big(\int_{r\leq|x|\leq 2r}|u|^{p^*}\,\mathrm{d}x\Big)^{1/p^*}.
\end{aligned}\tag{2.14}$$

And in a similar way, we obtain

$$\begin{aligned}
&\Big(\int_{r\leq|x|\leq 2r}|u|^q\,\mathrm{d}x\Big)^{1/q}\\
&\leq\big((2^N-1)\omega_N r^N\big)^{1/N}\Big(\int_{r\leq|x|\leq 2r}|u|^{q*}\,\mathrm{d}x\Big)^{1/q^*}.
\end{aligned}\tag{2.15}$$

Replacing inequalities (2.14) and (2.15) in (2.13), we get

$$\begin{aligned}
\limsup_{n\to\infty}\Big(1-\frac{1}{k}\Big)\Big\{&\int_{|x|\geq r}\eta|\nabla u_n|^p\,\mathrm{d}x+\int_{|x|\geq r}\eta a(x)|u_n|^p\,\mathrm{d}x\\
&+\int_{|x|\geq r}\eta|\nabla u_n|^q\,\mathrm{d}x+\int_{|x|\geq r}\eta b(x)|u_n|^q\,\mathrm{d}x\Big\}\\
&\leq 2\big((2^N-1)\omega_N\big)^{1/N}\Big\{\Big(\int_{r\leq|x|\leq 2r}|u|^{p*}\,\mathrm{d}x\Big)^{1/p^*}\|u\|^{p-1}\\
&+\Big(\int_{r\leq|x|\leq 2r}|u|^{q*}\,\mathrm{d}x\Big)^{1/q^*}\|u\|^{q-1}\Big\}.
\end{aligned}\tag{2.16}$$

In particular, since $\eta = 1$ outside the ball of radius $2r$, by inequalities (2.13) and (2.16) we obtain

$$\begin{aligned}(2.17)\qquad &\limsup_{n\to\infty}\left(1-\frac{1}{k}\right)\left\{\int_{|x|\geq 2r}|\nabla u_n|^p\,\mathrm{d}x+\int_{|x|\geq 2r}a(x)|u_n|^p\,\mathrm{d}x\right.\\ &\qquad\left.+\int_{|x|\geq 2r}|\nabla u_n|^q\,\mathrm{d}x+\int_{|x|\geq 2r}b(x)|u_n|^q\,\mathrm{d}x\right\}\\ &\leq 2\big((2^N-1)\omega_N\big)^{1/N}\left\{\left(\int_{r\leq|x|\leq 2r}|u|^{p^*}\,\mathrm{d}x\right)^{1/p^*}\|u\|^{p-1}\right.\\ &\qquad\left.+\left(\int_{r\leq|x|\leq 2r}|u|^{q^*}\,\mathrm{d}x\right)^{1/q^*}\|u\|^{q-1}\right\}.\end{aligned}$$

Therefore, by inequalities (2.11) and (2.17) it follows that

$$\begin{aligned}(2.18)\qquad &\limsup_{n\to\infty}\left\{\int_{|x|\geq 2r}|\nabla u_n|^p\,\mathrm{d}x+\int_{|x|\geq 2r}a(x)|u_n|^p\,\mathrm{d}x\right.\\ &\qquad\left.+\int_{|x|\geq 2r}|\nabla u_n|^q\,\mathrm{d}x+\int_{|x|\geq 2r}b(x)|u_n|^q\,\mathrm{d}x\right\}<\epsilon.\end{aligned}$$

Combining inequalities (2.12) and (2.18), we deduce that

$$(2.19)\qquad \limsup_{n\to\infty}\int_{|x|\geq 2r}g(x,u_n)u_n\,\mathrm{d}x=0.$$

Now we use the dominated convergence theorem together with the fact that g has subcritical growth to infer that

$$(2.20)\qquad \limsup_{n\to\infty}\int_{|x|\leq 2r}g(x,u_n)u_n\,\mathrm{d}x=\int_{|x|\leq 2r}g(x,u)u\,\mathrm{d}x;$$

and since $\int_{\mathbb{R}^N}g(x,u_n)u_n\,\mathrm{d}x<\infty$, by the choice of $r>R>1$ and from equalities (2.19) and (2.20), we obtain

$$(2.21)\qquad \lim_{n\to\infty}\int_{\mathbb{R}^N}g(x,u_n)u_n\,\mathrm{d}x=\int_{\mathbb{R}^N}g(x,u)u\,\mathrm{d}x.$$

It remains to show that the norm sequence $(\|u_n\|)_{n\in\mathbb{N}}\subset\mathbb{R}$ is such that $\|u_n\|\to\|u\|\in\mathbb{R}$ as $n\to\infty$. Using Hölder's inequality and making some computations, it follows that

$$\begin{aligned}&\left\{\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\,\mathrm{d}x\right)^{(p-1)/p}-\left(\int_{\mathbb{R}^N}|\nabla u|^p\,\mathrm{d}x\right)^{(p-1)/p}\right\}\\ &\qquad\times\left\{\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\,\mathrm{d}x\right)^{1/p}-\left(\int_{\mathbb{R}^N}|\nabla u|^p\,\mathrm{d}x\right)^{1/p}\right\}\\ &\qquad+\left\{\left(\int_{\mathbb{R}^N}a(x)|u_n|^p\,\mathrm{d}x\right)^{(p-1)/p}-\left(\int_{\mathbb{R}^N}a(x)|u|^p\,\mathrm{d}x\right)^{(p-1)/p}\right\}\end{aligned}$$

$$
\begin{aligned}
&\times \left\{ \left(\int_{\mathbb{R}^N} a(x)|u_n|^p \,\mathrm{d}x \right)^{1/p} - \left(\int_{\mathbb{R}^N} a(x)|u|^p \,\mathrm{d}x \right)^{1/p} \right\} \\
&+ \left\{ \left(\int_{\mathbb{R}^N} |\nabla u_n|^q \,\mathrm{d}x \right)^{(q-1)/q} - \left(\int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x \right)^{(q-1)/q} \right\} \\
&\times \left\{ \left(\int_{\mathbb{R}^N} |\nabla u_n|^q \,\mathrm{d}x \right)^{1/q} - \left(\int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x \right)^{1/q} \right\} \\
&+ \left\{ \left(\int_{\mathbb{R}^N} b(x)|u_n|^q \,\mathrm{d}x \right)^{(q-1)/q} - \left(\int_{\mathbb{R}^N} b(x)|u|^q \,\mathrm{d}x \right)^{(q-1)/q} \right\} \\
&\times \left\{ \left(\int_{\mathbb{R}^N} b(x)|u_n|^q \,\mathrm{d}x \right)^{1/q} - \left(\int_{\mathbb{R}^N} b(x)|u|^q \,\mathrm{d}x \right)^{1/q} \right\} \\
&- \int_{\mathbb{R}^N} \big(g(x,u_n) - g(x,u)\big)(u_n - u) \,\mathrm{d}x \\
\leq{}& \big(J'(u_n) - J'(u)\big)(u_n - u) = o(1).
\end{aligned}
$$

We remark that all the terms between curly brackets in the previous expression have the same signals; therefore, by the limit (2.21) we get

$$
\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \,\mathrm{d}x &= \int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x, \\
\lim_{n\to\infty} \int_{\mathbb{R}^N} a(x)|u_n|^p \,\mathrm{d}x &= \int_{\mathbb{R}^N} a(x)|u|^p \,\mathrm{d}x,
\end{aligned}
$$

and also

$$
\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^q \,\mathrm{d}x &= \int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x, \\
\lim_{n\to\infty} \int_{\mathbb{R}^N} b(x)|u_n|^q \,\mathrm{d}x &= \int_{\mathbb{R}^N} b(x)|u|^q \,\mathrm{d}x.
\end{aligned}
$$

This implies that

$$
\lim_{n\to\infty} \|u_n\|_{1,p}^p = \|u\|_{1,p}^p \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_{1,q}^q = \|u\|_{1,q}^q.
$$

Moreover, $u_n \rightharpoonup u$ weakly in E as $n \to \infty$; and finally, $u_n \to u$ strongly in E as $n \to \infty$. For the details, see DiBenedetto [24, Proposition V.11.1]. □

LEMMA 2.5. *Suppose that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset E$ and a function $u \in E$ such that $u_n \to u$ in E and $J'(u_n) \to 0$ as $n \to \infty$. Then there exists a subsequence, still denoted in the same way, such that $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$*

Proof. See Assunção, Carrião and Miyagaki [8] or Benmouloud, Echarghaoui and Sbaï [14]. □

Using Lemmas 2.1, 2.2, 2.3, 2.4, and 2.5 we conclude that there exists $u \in E$ which is a critical point for the functional J. Moreover, this critical point is a positive ground state solution to the auxiliary problem (2.4), that is, $J(u) = c > 0$ and $J'(u) = 0$.

3. Estimate for the solution to the auxiliary problem

In this section, we show that the solution to the auxiliary problem (2.4) obtained in the previous section verifies an important estimate. To do this, we use several lemmas.

LEMMA 3.1. *For $R > 1$, every positive ground state solution u to problem (2.4) verifies the estimate*

$$\|u\|_{1,p}^p + \|u\|_{1,q}^q \leq \frac{dkp}{p-1}.$$

Proof. Combining inequalities (2.6), (2.9) and (2.10), it follows that

$$\frac{(p-1)}{kp}\big\{\|u\|_{1,p}^p + \|u\|_{1,q}^q\big\} \leq J(u) - \frac{1}{\theta}J'(u)u = J(u) = c \leq d.$$

The conclusion of the lemma follows immediately. □

We remark that the boundedness of the norm of the ground state solution to problem (2.4) shown in Lemma 3.1 depends only on the potential functions a_∞ and b_∞, on the nonlinearity f and on the constant θ; it is independent on the constant $R > 1$.

The next lemma is a crucial step to establish an important estimate involving the norm of the solution to the auxiliary problem (2.4) in the space $L^\infty(\mathbb{R}^N)$. To prove it, we adapt the arguments by Alves and Souto [1]; see also Gilbarg and Trudinger [28, Section 8.6], Brézis and Kato [17], Pucci and Servadei [35], and Bastos, Miyagaki and Vieira [11].

LEMMA 3.2. *Suppose that $p, r \in \mathbb{R}$ verify the inequality $pr > N$. Let $H\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|H(x,s)| \leq h(x)|s|^{p-2}s$ for all $s > 0$ with the function $h\colon \mathbb{R}^N \to \mathbb{R}$ so that $h \in L^r(\mathbb{R}^N)$ and let $A, B\colon \mathbb{R}^N \to \mathbb{R}$ be nonnegative functions. Suppose also that $v \in E \subset D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$ is a weak solution to the problem*

$$-\Delta_p v - \Delta_q v + A(x)|v|^{p-2}v + B(x)|v|^{q-2}v = H(x,v), \quad x \in \mathbb{R}^N. \tag{3.1}$$

Then there exists a constant $M_1 = M_1(N, p, q, r, \|h\|_{L^r(\mathbb{R}^N)}) > 0$, which does not depend on the functions A and B, such that

$$\|v\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \max\big\{\|v\|_{L^{p^*}(\mathbb{R}^N)}, K, KL_v, 1\big\},$$

where K and L_v are defined by (3.5) and by (3.6), respectively.

Proof. Let $\beta > 1$; for every $m \in \mathbb{N}$ we define the subsets

$$A_m \equiv \{x \in \mathbb{R}^N : 1 < |v(x)|^{\beta-1} \leq m\};$$
$$B_m \equiv \{x \in \mathbb{R}^N : |v(x)|^{\beta-1} > m\};$$
$$C_m \equiv \{x \in \mathbb{R}^N : |v(x)|^{\beta-1} \leq 1\}.$$

We also define the sequence of functions $(v_m)_{m\in\mathbb{N}} \subset D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$ by

$$v_m(x) \equiv \begin{cases} |v(x)|^{p(\beta-1)}v(x), & \text{if } x \in A_m; \\ m^p v(x), & \text{if } x \in B_m; \\ |v(x)|^{q(\beta-1)}v(x), & \text{if } x \in C_m. \end{cases}$$

It is easy to verify that for every $x \in \mathbb{R}^N$ we have

$$v_m(x) \leq \max\{|v(x)|^{p(\beta-1)+1}, |v(x)|^{q(\beta-1)+1}\}.$$

Additionally, simple computations show that

$$\nabla v_m(x) = \begin{cases} (p(\beta-1)+1)|v(x)|^{p(\beta-1)}\nabla v(x), & \text{if } x \in A_m; \\ m^p \nabla v(x), & \text{if } x \in B_m; \\ (q(\beta-1)+1)|v(x)|^{q(\beta-1)}\nabla v(x), & \text{if } x \in C_m. \end{cases}$$

Furthermore, $(v_m)_{m\in\mathbb{N}} \subset E$. Indeed,

$$\begin{aligned} &\int_{\mathbb{R}^N} a(x)|v_m|^p \,\mathrm{d}x \\ &\quad \leq \int_{A_m} a(x)\big(|v|^{p-1}v\big)m^{p(p-1)+p} \,\mathrm{d}x + \int_{B_m} a(x)|v|^{p-1}v m^{p(p-1)+p} \,\mathrm{d}x \\ &\qquad + \int_{C_m} a(x)\big(|v|^{p-1}v\big) \,\mathrm{d}x \\ &\quad \leq m^{p^2} \int_{\mathbb{R}^N} a(x)|v|^{p-1}v \,\mathrm{d}x < +\infty. \end{aligned}$$

And in a similar way, we have

$$\int_{\mathbb{R}^N} b(x)|v_m|^q \,\mathrm{d}x = m^{pq} \int_{\mathbb{R}^N} b(x)|v|^{q-1}v \,\mathrm{d}x < +\infty.$$

Multiplying both sides of the differential equation (3.1) by the test function v_m and integrating the left-hand side with the help of the divergence theorem, we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \cdot \nabla v_m \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v \cdot \nabla v_m \,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^N} A(x)|v|^{p-2}v v_m \,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|v|^{q-2}v v_m \,\mathrm{d}x \\ &\quad = \int_{\mathbb{R}^N} H(x,v)v_m \,\mathrm{d}x. \end{aligned}$$

Using the definition of the function v_m, we obtain

$$
\begin{aligned}
(3.2)\quad & \big(p(\beta-1)+1\big)\bigg\{\int_{A_m} |\nabla v|^p |v|^{p(\beta-1)}\,\mathrm{d}x + \int_{A_m} |\nabla v|^q |v|^{p(\beta-1)}\,\mathrm{d}x\bigg\} \\
&\quad + \big(q(\beta-1)+1\big)\bigg\{\int_{C_m} |\nabla v|^p |v|^{q(\beta-1)}\,\mathrm{d}x + \int_{C_m} |\nabla v|^q |v|^{q(\beta-1)}\,\mathrm{d}x\bigg\} \\
&= \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x \\
&\quad - m^p\bigg\{\int_{B_m} |\nabla v|^p\,\mathrm{d}x + \int_{B_m} |\nabla v|^q\,\mathrm{d}x\bigg\} \\
&\le \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|v|^{p-2} v v_m\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|v|^{q-2} v v_m\,\mathrm{d}x.
\end{aligned}
$$

Now we define another sequence of functions $(w_m)_{m\in\mathbb{N}}\subset E$ by

$$
w_m(x) = \begin{cases} |v(x)|^{\beta-1}v(x), & \text{if } x\in A_m\cup C_m; \\ m v(x), & \text{if } x\in B_m. \end{cases}
$$

Direct computations show that

$$
\nabla w_m(x) = \begin{cases} \beta|v(x)|^{\beta-1}\nabla v(x), & \text{if } x\in A_m\cup C_m; \\ m\nabla v(x), & \text{if } x\in B_m. \end{cases}
$$

Using the hypothesis $2\le q\le p<N$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla w_m|^p\,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|w_m|^p\,\mathrm{d}x - \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} A(x)|v|^{p-2} v v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w_m|^q\,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|w_m|^q\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x - \int_{\mathbb{R}^N} B(x)|v|^{q-2} v v_m\,\mathrm{d}x \\
&\le \beta^p \int_{A_m\cup C_m} |\nabla v|^p |v|^{p(\beta-1)}\,\mathrm{d}x + \beta^p \int_{A_m\cup C_m} |\nabla v|^q |v|^{q(\beta-1)}\,\mathrm{d}x \\
&\quad - \big(p(\beta-1)+1\big) \\
&\quad \times \bigg\{\int_{A_m} |\nabla v|^p |v|^{p(\beta-1)}\,\mathrm{d}x + \int_{A_m} |\nabla v|^q |v|^{p(\beta-1)}\,\mathrm{d}x\bigg\} \\
&\quad + \int_{A_m} B(x)\big(|v|^{q\beta} - |v|^{p(\beta-1)+q}\big)\,\mathrm{d}x \\
&\quad + \int_{C_m} A(x)\big(|v|^{p\beta} - |v|^{p+q(\beta-1)}\big)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \big(m^q - m^p\big) \int_{B_m} B(x)|v|^q \,\mathrm{d}x - \big(q(\beta-1)+1\big) \\
&\times \left\{ \int_{C_m} |\nabla v|^p |v|^{q(\beta-1)} \,\mathrm{d}x + \int_{C_m} |\nabla v|^q |v|^{q(\beta-1)} \,\mathrm{d}x \right\} \\
&+ \big(m^q - m^p\big) \int_{B_m} |\nabla v|^q \,\mathrm{d}x.
\end{aligned}
$$

And after we get rid of the nonpositive terms, we can regroup the expressions to obtain

$$
\begin{aligned}
&\int_{R^N} |\nabla w_m|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|w_m|^p \,\mathrm{d}x + \int_{R^N} |\nabla w_m|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|w_m|^q \,\mathrm{d}x \\
&\quad = \big(\beta^p - \big(p(\beta-1)+1\big)\big) \int_{A_m} |\nabla v|^p |v|^{p(\beta-1)} \,\mathrm{d}x \\
&\qquad + \beta^p \int_{C_m} |\nabla v|^p |v|^{p(\beta-1)} \,\mathrm{d}x \\
&\qquad + \big(\beta^q - \big(q(\beta-1)+1\big)\big) \int_{C_m} |\nabla v|^q |v|^{q(\beta-1)} \,\mathrm{d}x \\
&\qquad + \beta^q \int_{A_m} |\nabla v|^q |v|^{q(\beta-1)} \,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \cdot \nabla v_m \,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|v|^{p-2} v v_m \,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} |\nabla v|^{q-2} \nabla v \cdot \nabla v_m \,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|v|^{q-2} v v_m \,\mathrm{d}x.
\end{aligned}
$$

So, using inequality (3.2) we deduce that

$$
\begin{aligned}
&\int_{R^N} |\nabla w_m|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|w_m|^p \,\mathrm{d}x + \int_{R^N} |\nabla w_m|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|w_m|^q \,\mathrm{d}x \\
&\quad \leq \left(\frac{\beta^p}{q(\beta-1)+1} \right) \left\{ \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \cdot \nabla v_m \,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|v|^{p-2} v v_m \,\mathrm{d}x \right. \\
&\qquad \left. + \int_{\mathbb{R}^N} |\nabla v|^{q-2} \nabla v \cdot \nabla v_m \,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|v|^{q-2} v v_m \,\mathrm{d}x \right\} \\
&\qquad + \beta^p \int_{C_m} |\nabla v|^p |v|^{p(\beta-1)} \,\mathrm{d}x + \beta^q \int_{A_m} |\nabla v|^q |v|^{q(\beta-1)} \,\mathrm{d}x.
\end{aligned}
$$

Now we estimate some integrals that appear in the previous inequality. First, by definition of A_m we have

$$
\begin{aligned}
&\int_{A_m} |\nabla v|^q |v|^{q(\beta-1)} \,\mathrm{d}x \\
&\quad = \int_{A_m} \frac{|\nabla v|^{q-2}}{[p(\beta-1)+1]|v|^{(p-q)(\beta-1)}} \nabla v \cdot \nabla v_m \,\mathrm{d}x
\end{aligned}
$$

$$\le \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|v|^{p-2}vv_m\,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|v|^{q-2}vv_m\,\mathrm{d}x.$$

In a similar way, by definition of C_m we have

$$\begin{aligned}\int_{C_m} &|\nabla v|^p|v|^{p(\beta-1)}\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|v|^{p-2}vv_m\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|v|^{q-2}vv_m\,\mathrm{d}x.\end{aligned}$$

Using these inequalities, we deduce that

$$\begin{aligned}\int_{R^N} &|\nabla w_m|^p\,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|w_m|^p\,\mathrm{d}x + \int_{R^N} |\nabla w_m|^q\,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|w_m|^q\,\mathrm{d}x \\ &\le \Big(\beta^p + \frac{\beta^p}{q(\beta-1)+1}\Big) \\ &\quad \times \Big\{\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|v|^{p-2}vv_m\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|v|^{q-2}vv_m\,\mathrm{d}x\Big\} \\ &\le 2\beta^p\Big\{\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|v|^{p-2}vv_m\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v\cdot\nabla v_m\,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|v|^{q-2}vv_m\,\mathrm{d}x\Big\} \\ &= 2\beta^p\int_{\mathbb{R}^N} H(x,v)v_m\,\mathrm{d}x.\end{aligned}$$

Using the Sobolev inequality (2.5) and the hypothesis $H(x,s)\le h(x)|s|^{p-1}$, we obtain

$$\begin{aligned}\Big(\int_{A_m\cup C_m} &|w_m|^{p^*}\,\mathrm{d}x\Big)^{p/p^*} \\ &\le \Big(\int_{\mathbb{R}^N} |w_m|^{p^*}\,\mathrm{d}x\Big)^{p/p^*} \\ &\le S\int_{\mathbb{R}^N} |\nabla w_m|^p\,\mathrm{d}x \\ &\le S\Big\{\int_{\mathbb{R}^N} |\nabla w_m|^p\,\mathrm{d}x + \int_{\mathbb{R}^N} a(x)|w_m|^p\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{\mathbb{R}^N} |\nabla w_m|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} b(x)|w_m|^q \,\mathrm{d}x \Big\} \\
&\leq 2S\beta^p \int_{\mathbb{R}^N} H(x,v) v_m \,\mathrm{d}x \\
&\leq 2S\beta^p \int_{\mathbb{R}^N} h(x)|v|^{p-1} v_m \,\mathrm{d}x \\
&= 2S\beta^p \Big\{ \int_{A_m} h(x)|v|^{p-2}v|v|^{p(\beta-1)}v \,\mathrm{d}x + \int_{B_m} h(x)|v|^{p-2}v\, m^p v \,\mathrm{d}x \\
&\quad + \int_{C_m} h(x)|v|^{p-2}v|v|^{q(\beta-1)}v \,\mathrm{d}x \Big\} \\
&\leq 2S\beta^p \Big\{ \int_{\mathbb{R}^N} h(x)|v|^{p\beta} \,\mathrm{d}x + \int_{\mathbb{R}^N} h(x)|v|^{p} \,\mathrm{d}x \Big\},
\end{aligned}
$$

where in the last passage we used the definitions of the functions v_m and w_m, together with the facts that in B_m we have $|w_m|^p \leq |v|^{p\beta}$ and in C_m we have $|v|^{p+q(\beta-1)} \leq |v|^p$.

Passing to the limit as $m \to \infty$ and using Lebesgue's dominated convergence theorem, it follows that

$$
\Big(\int_{\mathbb{R}^N} |v|^{p^*\beta} \,\mathrm{d}x \Big)^{p/p^*} \leq 2S\beta^p \Big\{ \int_{\mathbb{R}^N} h(x)|v|^{p\beta} \,\mathrm{d}x + \int_{\mathbb{R}_N} h(x)|v|^{p} \,\mathrm{d}x \Big\}.
$$

Applying Hölder's inequality to both terms on the right-hand side of the previous inequality, we obtain

$$
\int_{\mathbb{R}^N} h(x)|v|^{p\beta} \,\mathrm{d}x \leq \|h\|_{L^r(\mathbb{R}^N)} \|v\|^{p\beta}_{L^{p\beta r'}(\mathbb{R}^N)}
$$

and

$$
\int_{\mathbb{R}_N} h(x)|v|^{p} \,\mathrm{d}x \leq \|h\|_{L^r(\mathbb{R}^N)} \|v\|^{p}_{L^{p r'}(\mathbb{R}^N)};
$$

hence

$$
\begin{aligned}
&\|v\|^{p\beta}_{L^{p^*\beta}(\mathbb{R}^N)} \\
&\quad \leq 2S\|h\|_{L^r(\mathbb{R}^N)} \beta^p \big\{ \|v\|^{p\beta}_{L^{p\beta r'}(\mathbb{R}^N)} + \|v\|^{p}_{L^{p r'}(\mathbb{R}^N)} \big\} \\
&\quad \leq 2S\|h\|_{L^r(\mathbb{R}^N)} \beta^p \big\{ \max\big\{ \|v\|^{p\beta}_{L^{p\beta r'}(\mathbb{R}^N)}, 1 \big\} + \max\big\{ \|v\|^{p}_{L^{p r'}(\mathbb{R}^N)}, 1 \big\} \big\} \\
&\quad = C_1^p \beta^p \max\big\{ \|v\|^{p\beta}_{L^{p\beta r'}(\mathbb{R}^N)}, \max\big\{ \|v\|^{p}_{L^{p r'}(\mathbb{R}^N)}, 1 \big\} \big\},
\end{aligned}
$$

where $C_1^p = C_1^p(N,p,q,r,\|h\|_{L^r(\mathbb{R}^N)}) \equiv 4S\|h\|_{L^r(\mathbb{R}^N)} > 0$.

Writing $\beta = \sigma^j$ for $j \in \mathbb{N}$ we deduce that

$$
\begin{aligned}
&\|v\|_{L^{p^*\sigma^j}(\mathbb{R}^N)} \\
&\quad \leq C_1^{1/\sigma^j} \sigma^{j/\sigma^j} \max\big\{ \|v\|_{L^{p\sigma^j r'}(\mathbb{R}^N)}, \max\big\{ \|v\|^{1/\sigma^j}_{L^{p r'}(\mathbb{R}^N)}, 1 \big\} \big\}.
\end{aligned} \tag{3.3}
$$

Choosing $\sigma = p^*/pr' > 1$, from inequality (3.3) with $j=1$ we obtain

$$\|v\|_{L^{p^*\sigma}(\mathbb{R}^N)} \le C_1^{1/\sigma}\sigma^{1/\sigma}\max\big\{\|v\|_{L^{p^*}(\mathbb{R}^N)}, \max\big\{\|v\|^{1/\sigma}_{L^{pr'}(\mathbb{R}^N)}, 1\big\}\big\};$$

and from inequality (3.3) with $j=2$ together with the previous inequality we obtain

$$\begin{aligned}
&\|v\|_{L^{p^*\sigma^2}(\mathbb{R}^N)}\\
&\quad\le C_1^{1/\sigma^2}\sigma^{2/\sigma^2}\max\big\{\|v\|_{L^{p^*\sigma}(\mathbb{R}^N)}, \max\big\{\|v\|^{1/\sigma^2}_{L^{pr'}(\mathbb{R}^N)}, 1\big\}\big\}\\
&\quad\le C_1^{1/\sigma^2}\sigma^{2/\sigma^2}\\
&\qquad\times\max\big\{C_1^{1/\sigma}\sigma^{1/\sigma}\max\big\{\|v\|_{L^{p^*}(\mathbb{R}^N)}, \max\big\{\|v\|^{1/\sigma}_{L^{pr'}(\mathbb{R}^N)}, 1\big\}\big\},\\
&\qquad\max\big\{\|v\|^{1/\sigma^2}_{L^{pr'}(\mathbb{R}^N)}, 1\big\}\big\}\\
&\quad\le C_1^{1/\sigma+1/\sigma^2}\sigma^{1/\sigma+2/\sigma^2}\\
&\qquad\times\max\Big\{\|v\|_{L^{p^*}(\mathbb{R}^N)}, \max\big\{(C_1^{1/\sigma}\sigma^{1/\sigma})^{-1}, 1\big\}, \max\big\{(C_1^{1/\sigma}\sigma^{1/\sigma})^{-1}, 1\big\}\\
&\qquad\times\max\big\{\|v\|^{1/\sigma}_{L^{pr'}(\mathbb{R}^N)}, \|v\|^{1/\sigma^2}_{L^{pr'}(\mathbb{R}^N)}, 1\big\}\Big\}.
\end{aligned}$$

Proceeding in this way, for $j \in \mathbb{N}$ we obtain

$$\|v\|_{L^{p^*\sigma^j}(\mathbb{R}^N)} \le C_1^{s_j}\sigma^{t_j}\max\big\{\|v\|_{L^{p^*}(\mathbb{R}^N)}, K_j, K_jL_j\big\}, \tag{3.4}$$

where $s_j \equiv 1/\sigma + 1/\sigma^2 + \cdots + 1/\sigma^j$; $t_j \equiv 1/\sigma + 2/\sigma^2 + \cdots + j/\sigma^j$;

$$K_j \equiv \begin{cases} 1, & \text{if } j=1;\\ \max_{1\le i\le j-1}\{C_1^{-s_i}\sigma^{-t_i}, 1\}, & \text{if } j\ge 2;\end{cases}$$

and

$$L_j \equiv \max_{1\le i\le j}\big\{\|v\|^{1/\sigma^i}_{L^{pr'}(\mathbb{R}^N)}, 1\big\}.$$

Since $\sigma > 1$, we have $\lim_{j\to\infty} s_j = 1/(\sigma-1)$ and $\lim_{j\to\infty} t_j = \sigma/(\sigma-1)^2$; hence,

$$\lim_{j\to\infty} K_j \equiv K = \begin{cases} (C_1^{1/(\sigma-1)}\sigma^{\sigma/(\sigma-1)^2})^{-1} & \text{if } C_1 \le 1;\\ (C_1^{1/\sigma}\sigma^{1/\sigma})^{-1} & \text{if } C_1 > 1;\end{cases} \tag{3.5}$$

and

$$\lim_{j\to\infty} L_j \equiv L_v = \begin{cases} 1, & \text{if } \|v\|_{L^{pr'}(\mathbb{R}^N)} \le 1;\\ \|v\|^{1/(\sigma-1)}_{L^{pr'}(\mathbb{R}^N)}, & \text{if } \|v\|_{L^{pr'}(\mathbb{R}^N)} > 1.\end{cases} \tag{3.6}$$

Using the fact that $v \in E \subset D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, applying Hölder's inequality we deduce that $L_v < +\infty$.

Finally, passing to the limit as $j \to \infty$ and using inequality (3.4) we obtain

$$\begin{aligned}
\|v\|_{L^\infty(\mathbb{R}^N)} &= \lim_{j\to\infty} \|v\|_{L^{p^*\sigma^j}(\mathbb{R}^N)} \\
&\leq C_1^{1/(\sigma-1)} \sigma^{\sigma/(\sigma-1)^2} \max\big\{\|v\|_{L^{p^*}(\mathbb{R}^N)}, K, KL_v, 1\big\} \\
&\equiv M_1 \max\big\{\|v\|_{L^{p^*}(\mathbb{R}^N)}, K, KL_v, 1\big\},
\end{aligned} \tag{3.7}$$

where $M_1 = M_1(N, p, q, r, \|h\|_{L^r(\mathbb{R}^N)})$. This conclude the proof of the lemma. $\square$

LEMMA 3.3. *Let the number $R > 1$ be given; then there exist a constant $M_2 = M_2(N, p, q, r, a_\infty, b_\infty, \theta, c_0)$ such that any positive ground state solution $u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$ to the auxiliary problem (2.4) verifies the inequality*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq M_2.$$

Proof. Consider $R > 1$ and let $u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$ be a positive ground state solution to the auxiliary problem (2.4). Now we define the function $H\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$H(x,t) \equiv \begin{cases} f(t), & \text{if } |x| \leq R \text{ or if } |x| > R \text{ and } f(t) \leq \frac{a(x)}{k}|t|^{p-2}t; \\ 0, & \text{if } |x| > R \text{ and } f(t) > \frac{a(x)}{k}|t|^{p-2}t. \end{cases}$$

We also define the functions $A, B\colon \mathbb{R}^N \to \mathbb{R}$ by

$$A(x) = \begin{cases} a(x), & \text{if } |x| \leq R \text{ or if } |x| > R \text{ and } f(u(x)) \leq \frac{a(x)}{k}u(x); \\ (1-\frac{1}{k})a(x), & \text{if } |x| > R \text{ and } f(u(x)) > \frac{a(x)}{k}u(x), \end{cases}$$

and $B(x) = b(x)$.

Considering these functions and using $v \in E$ as a test function, we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|u|^{p-2}uv \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} B(x)|u|^{q-2}uv \,\mathrm{d}x - \int_{\mathbb{R}^N} H(x,u)v \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} a(x)|u|^{p-2}uv \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} b(x)|u|^{q-2}uv \,\mathrm{d}x - \int_{\mathbb{R}^N} g(x,u)v \,\mathrm{d}x.
\end{aligned}$$

From hypothesis (f_1), we have $|H(x,t)| \leq |f(t)| \leq c_1|t|^{p^*-1}$ for $|t|$ small enough; from hypothesis (f_2) we have $|H(x,t)| \leq |f(t)| \leq c_2|t|^{\tau-1}$ for $|t|$ big enough with $\tau \in (p, p^*)$. Combining both cases, for every $t \in \mathbb{R}^+$ and for every $\tau \in (p, p^*)$ we obtain $|H(x,t)| \leq |f(t)| \leq c_0|t|^{p^*-1}$. Then, it follows that $|H(x,u)| \leq c_0|u(x)|^{\tau-p}|u(x)|^{p-1} = h(x)|u(x)|^{p-1}$, where $h(x) \equiv c_0|u(x)|^{\tau-p}$.

Direct computations show that $h \in L^r(\mathbb{R}^N)$ for $r = p^*/(\tau - p)$. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |h(x)|^r \,\mathrm{d}x \\
&\quad \le c_0^{p^*/(\tau-p)} \int_{\mathbb{R}^N} |u|^{p^*} \,\mathrm{d}x \\
&\quad \le c_0^{p^*/(\tau-p)} S^{p^*/p} \bigg(\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x \bigg)^{p^*/p} \\
&\quad \le c_0^{p^*/(\tau-p)} S^{p^*/p} \bigg\{ \int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} a(x)|u|^p \,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} b(x)|u|^q \,\mathrm{d}x \bigg\}^{p^*/p} \\
&\quad \le c_0^{p^*/(\tau-p)} S^{p^*/p} \big\{ \|u\|_{1,p}^p + \|u\|_{1,q}^q \big\}^{p^*/p} < +\infty.
\end{aligned}$$

In this way, any positive ground state solution $u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$ to the auxiliary problem (2.4) verifies the hypothesis of Lemma 3.2. Concluding the argument, from inequality (2.5) and from Lemma 3.1 we have

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \le S^{1/p} \big\{ \|u\|_{1,p}^p + \|u\|_{1,q}^q \big\}^{1/p} \le \bigg(\frac{Sdkp}{p-1} \bigg)^{1/p}.$$

Finally, combining estimate (3.7) with the previous inequality we obtain

$$\begin{aligned}
\|u\|_{L^\infty(\mathbb{R}^N)} &\le M_1 \max\big\{ \|u\|_{L^{p^*}(\mathbb{R}^N)}, K, KL_u, 1 \big\} \\
&\le M_1 \max\bigg\{ \bigg(\frac{Sdkp}{p-1} \bigg)^{1/p}, K, KL_u, 1 \bigg\} \\
&\equiv M_2,
\end{aligned}$$

where $M_2 = M_2(N, p, q, r, a_\infty, b_\infty, \theta, c_0)$. The lemma is proved. $\square$

LEMMA 3.4. *Suppose that $R_0 \ge R > 1$ and let $u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$ be a positive ground state solution to the auxiliary problem* (2.4)*. Then u verifies the inequality*

$$u(x) \le M_2 \frac{R^{(N-p)/(p-1)}}{|x|^{(N-p)/(p-1)}}$$

for every $|x| \ge R > 1$.

Proof. Given $R_0 \ge R > 1$, we define the function $v\colon \mathbb{R}^N \backslash \{0\} \to \mathbb{R}$ by

$$v(x) \equiv M_2 \frac{R_0^{(N-p)/(p-1)}}{|x|^{(N-p)/(p-1)}}.$$

By hypothesis, $u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$ is a positive ground state solution to the auxiliary problem (2.4); therefore, we can apply Lemma 3.3 to deduce that $\|u\|_{L^\infty(\mathbb{R}^N)} \leq M_2$. This implies that if $|x| = R_0$, then $\|u\|_{L^\infty(\mathbb{R}^N)} \leq v(x)$. Now we define the function $w\colon \mathbb{R}^N \backslash \{0\} \to \mathbb{R}$ by

$$w(x) = \begin{cases} 0, & \text{if } |x| \leq R_0; \\ (u-v)^+, & \text{if } |x| \geq R_0. \end{cases}$$

In this way, $w \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$; moreover, $w \in E$ because $u, v \in E$.

To complete the proof of the lemma, we will show that $(u-v)^+ = 0$ for $|x| \geq R_0$. To accomplish this goal, we use the hypotheses on the potential functions a and b; we will also use the function $w \in E$ as a test function to obtain

$$\begin{aligned}
(3.8) \quad & \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla w \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^{q-2} \nabla u \cdot \nabla w \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} g(x,u) w \, \mathrm{d}x - \int_{\mathbb{R}^N} a(x)|u|^{p-2} u w \, \mathrm{d}x - \int_{\mathbb{R}^N} b(x) |u|^{q-2} u w \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N \backslash B_{R_0}(0) \wedge f(t) \leq a(x)|t|^{p-2} t/k} f(u) w \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N \backslash B_{R_0}(0) \wedge f(t) > a(x)|t|^{p-2} t/k} \frac{a(x)}{k} |u|^{p-2} u w \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} a(x) |u|^{p-2} u w \, \mathrm{d}x - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} b(x) |u|^{q-2} u w \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N \backslash B_{R_0}(0) \wedge f(t) \leq a(x)|t|^{p-2} t/k} \frac{a(x)}{k} |u|^{p-2} u w \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N \backslash B_{R_0}(0) \wedge f(t) > a(x)|t|^{p-2} t/k} \frac{a(x)}{k} |u|^{p-2} u w \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} a(x) |u|^{p-2} u w \, \mathrm{d}x - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} b(x) |u|^{q-2} u w \, \mathrm{d}x \\
&= \left(\frac{1}{k} - 1 \right) \int_{\mathbb{R}^N \backslash B_{R_0}(0)} a(x) |u|^{p-2} u w \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N \backslash B_{R_0}(0)} b(x) |u|^{q-2} u w \, \mathrm{d}x \\
&\leq 0
\end{aligned}$$

because u is a positive function and w is a nonnegative function, while $k > 1$.

Using the radially symmetric form of the operator $\Delta_m u$, we have

$$\int_{\mathbb{R}^N \backslash B_{R_0}(0)} |\nabla v|^{m-2} \nabla v \cdot \nabla \phi \, \mathrm{d}x = 0$$

for $m \in \{p, q\}$ and for every function $\phi \in E$. Therefore,

$$\begin{aligned}(3.9)\quad &\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \cdot \nabla w \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v \cdot \nabla w \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N \setminus B_{R_0}(0)} |\nabla v|^{p-2}\nabla v \cdot \nabla w \,\mathrm{d}x + \int_{\mathbb{R}^N \setminus B_{R_0}(0)} |\nabla v|^{q-2}\nabla v \cdot \nabla w \,\mathrm{d}x \\ &= 0.\end{aligned}$$

Defining the subsets

$$\widetilde{A} \equiv \big\{x \in \mathbb{R}^N : |x| \geq R_0 \text{ and } u(x) > v(x)\big\}$$

and

$$\widetilde{B} \equiv \big\{x \in \mathbb{R}^N : |x| < R_0 \text{ or } u(x) \leq v(x)\big\},$$

we have $w(x) = u(x) - v(x)$ for $x \in \tilde{A}$ and $w(x) = 0$ for $x \in \tilde{B}$. Using inequality (3.8) and equation (3.9) we get

$$\begin{aligned}(3.10)\qquad 0 &\geq \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla w \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla w \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \cdot \nabla w \,\mathrm{d}x - \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v \cdot \nabla w \,\mathrm{d}x \\ &= \int_{\widetilde{A}} \big[|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big] \cdot (\nabla u - \nabla v) \,\mathrm{d}x \\ &\quad + \int_{\widetilde{A}} \big[|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big] \cdot (\nabla u - \nabla v) \,\mathrm{d}x.\end{aligned}$$

Denoting by $\langle \cdot, \cdot \rangle : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ the standard scalar product, given $p \geq 2$ there exists a positive constant $c_p \in \mathbb{R}^+$ such that for every $x, y \in \mathbb{R}^N$ it is valid the inequality

$$(3.11)\qquad \big\langle |x|^{p-2}x - |y|^{p-2}y, x - y\big\rangle \geq c_p \|x - y\|^p.$$

For the proof, we refer the reader to Simon [37]. From inequalities (3.10) and (3.11), it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla w|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w|^q \,\mathrm{d}x \\ &= \int_{\widetilde{A}} |\nabla u - \nabla v|^p \,\mathrm{d}x \int_{\widetilde{A}} |\nabla u - \nabla v|^q \,\mathrm{d}x \\ &\leq c_p^{-1} \int_{\widetilde{A}} \big[|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big] \cdot (\nabla u - \nabla v) \,\mathrm{d}x \\ &\quad + c_q^{-1} \int_{\widetilde{A}} \big[|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big] \cdot (\nabla u - \nabla v) \,\mathrm{d}x \\ &\leq 0.\end{aligned}$$

From this inequality, we deduce that each term on the left-hand side of the previous inequality must be zero, that is, w is constant in $\mathbb{R}^N$. But we already know that $w(x)=0$ in the ball $B_{R_0}(0)$; therefore, $w(x)=0$ for every $x\in\mathbb{R}^N$. This implies that $(u-v)^+=0$ for $|x|\geq R_0$ and $u(x)\leq v(x)$ for every $x\in\mathbb{R}^N$. The proof of the lemma is complete. □

4. Obtaining the solution of the original problem

In this section, we finally show that the solution to the auxiliary problem (2.4) obtained in Section 2 is in fact a solution to problem (1.1).

Proof of Theorem 1.1. From Lemmas 2.3 and 2.4, the auxiliary problem (2.4) has a positive ground state solution $u\in D^{1,p}(\mathbb{R}^N)\cap D^{1,q}(\mathbb{R}^N)$. To accomplish our goal we need to show that for every $x\in B_R^c(0)$ the function u verifies the inequality

$$f(u)\leq\frac{a(x)}{k}|u|^{p-2}u.$$

From Lemma 3.4 and by the first inequality in (1.2), if $|x|\geq R$, then

$$\begin{aligned}\frac{f(u)}{|u|^{p-2}u}&\leq c_0\frac{|u|^{p^*-2}}{|u|^{p-2}}\leq c_0\left\{M_2\frac{(R^{p/(p-1)})^{(N-p)/p}}{(|x|^{p/(p-1)})^{(N-p)/p}}\right\}^{p^*-p}\\&=c_0M_2^{p^*-p}\frac{R^{p^2/(p-1)}}{|x|^{p^2/(p-1)}}.\end{aligned}$$

Now we define the constant

$$\Lambda^*\equiv c_0kM_2^{p^*-p}.$$

Considering $\Lambda\geq\Lambda^*$, it follows from the hypothesis (P_3) that

$$\frac{f(u)}{|u|^{p-2}u}\leq\frac{\Lambda^*}{k}\frac{R^{p^2/(p-1)}}{|x|^{p^2/(p-1)}}\leq\frac{\Lambda}{k}\frac{R^{p^2/(p-1)}}{|x|^{p^2/(p-1)}}\leq\frac{a(x)}{k}.$$

The proof of the theorem is complete. □

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